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Kyoto University
On the parity of poly-Euler numbers

By

YASUO OHNO* and YOSHITAKA SASAKI**

Abstract

Poly-Euler numbers are introduced in [9] via special values of an L-function as a generalization of the Euler numbers. In this article, poly-Euler numbers with negative index are mainly treated, and the parity of them is shown as the main theorem. Furthermore the divisibility of poly-Euler numbers are also discussed.

§ 1. Introduction

For every integer $k$, we define poly-Euler numbers $E_n^{(k)} \ (n = 0, 1, 2, \ldots)$, which is introduced as a generalization of the Euler number, by

$$
\frac{\text{Li}_k(1-e^{-4t})}{4t(\cosh t)} = \sum_{n=0}^{\infty} \frac{E_n^{(k)}}{n!} t^n.
$$

Here,

$$
\text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (|x| < 1, \ k \in \mathbb{Z})
$$

is the $k$-th polylogarithm. When $k = 1$, $E_n^{(1)}$ is the Euler number defined by

$$
\frac{1}{\cosh t} = \sum_{n=0}^{\infty} \frac{E_n^{(1)}}{n!} t^n.
$$
The reason why we refer to $E_n^{(k)}$'s as “poly-Euler numbers” will be mentioned in the next section from the point of view of the relation between the poly-Bernoulli number and Arakawa-Kaneko’s zeta-function. In this article, we treat some number theoretical properties of poly-Euler numbers with negative index ($k \leq 0$). Our main theorem is Theorem 3.1 described in Section 3, which mentions the parity of poly-Euler numbers can be determined definitely. In Section 4, we discuss the divisibility of poly-Euler numbers via congruence relations of them. Tables 1 and 2 cited at the end of this article are the lists of numerical values of poly-Euler numbers. General properties of poly-Euler numbers including the case of positive index are treated in [8].

§2. The poly-Bernoulli numbers and Arakawa-Kaneko’s zeta-function

For every integer $k$, the poly-Bernoulli numbers $\mathbb{B}_n^{(k)}$ and the modified poly-Bernoulli numbers $C_n^{(k)}$ introduced by Kaneko [4] are defined by

\begin{align}
\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} &= \sum_{n=0}^{\infty} \frac{\mathbb{B}_n^{(k)}}{n!} t^n \\
\frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} &= \sum_{n=0}^{\infty} \frac{C_n^{(k)}}{n!} t^n,
\end{align}

respectively. When $k = 1$, the above generating functions become

$$
\frac{te^t}{e^t - 1} \quad \text{and} \quad \frac{t}{e^t - 1},
$$

respectively. Therefore $\mathbb{B}_n^{(k)}$ and $C_n^{(k)}$ are generalizations of the classical Bernoulli numbers. Some number theoretic properties of the poly-Bernoulli number were given by Kaneko [4], Arakawa and Kaneko [2] and others. Furthermore the combinatorial interpretations of $\mathbb{B}_n^{(-k)}$ were given by Brewbaker [3] and Launois [6]. Recently, Shikata [10] gives the alternative proof of the result of Brewbaker.

It is known that the poly-Bernoulli numbers are special values of Arakawa-Kaneko’s zeta-function. Arakawa and Kaneko [1] introduced a zeta-function:

$$
\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} dt \quad (k \geq 1).
$$

We refer to the above function as Arakawa-Kaneko’s zeta-function. Arakawa-Kaneko’s zeta-function satisfies $\xi_1(s) = s\zeta(s + 1)$, and

$$
\xi_k(-n) = \sum_{l=0}^{n} \binom{n}{l} (n)_l \mathbb{B}_l^{(k)} = (-1)^n C_n^{(k)}
$$

for any non-positive integer $n$, where $\zeta(s)$ is the Riemann zeta-function. Hence Arakawa-Kaneko’s zeta-function is a kind of generalization of the Riemann zeta-function. Furthermore, we should mention that Arakawa-Kaneko’s zeta-function is applied to research
on multiple zeta values. For example, Kaneko and Ohno [5] showed a duality property of multiple zeta-star values by using this property.

From the point of view mentioned above, poly-Euler numbers should be defined as special values of an $L$-function generalized by using the method of Arakawa and Kaneko. Moreover it can be reasonably expected that such $L$-function has nice properties and applications similar to Arakawa-Kaneko’s zeta-function.

The Euler number $E_n$ is the generalized Bernoulli number associated with the Dirichlet character of conductor 4, and the $L$-function is

\begin{equation}
L(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{e^t + e^{-t}} \, dt
\end{equation}

satisfies $L(-n) = E_n/2$ for any non-negative integer $n$. The second author gave in [9] a general method for defining $L$-functions that have similar properties to Arakawa-Kaneko’s zeta-function. By using the method, a generalization of $L(s)$ is given by

\begin{equation}
L_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\text{Li}_k(1 - e^{-4t})}{4(e^t + e^{-t})} \, dt \quad (k \geq 1).
\end{equation}

It is natural to define poly-Euler numbers as special values at non-positive integers of the above $L$-function. Therefore, we define poly-Euler numbers as above (1.1).

We should mention that the above generalized $L$-function is applicable to research on multiple $L$ values. In fact, the second author [9] treated the generalization of Dirichlet $L$-functions for general Dirichlet characters and showed such generalized $L$-functions can be written in terms of multiple $L$-functions.

Remark. Although $L_1(s)$ does not reduce to $L(s)$ ($L_1(s) = sL(s+1)$), the above definition (2.3) is optimal as an analogue of Arakawa-Kaneko’s zeta-function. In fact, we have $\xi_1(s) = s\zeta(s+1)$.

§ 3. The parity of poly-Euler numbers

In this section, we determine the parity of poly-Euler numbers $(n+1)E_n^{(-k)}$.

**Theorem 3.1.** For any non-negative integer $k$, $(n+1)E_n^{(-k)}$ is even (odd, respectively) integer, when $n$ is odd (even, respectively).

**Proof of Theorem 3.1.** We first review an explicit formula:

**Lemma 3.2** (Ohno-Sasaki [8]). For any non-negative integers $k$ and $n$, we have

\begin{equation}
(n+1)E_n^{(-k)} = (-1)^k \sum_{l=0}^k (-1)^l \binom{k}{l} \sum_{m=1, m: \text{odd}}^{n+1} \binom{n+1}{m} (4l+2)^{n+1-m},
\end{equation}
where the symbol \(\left\{ \binom{k}{l} \right\}\) is the Stirling number of the second kind defined by the recurrence relation

\[
\left\{ \binom{k+1}{l} \right\} = \left\{ \binom{k}{l-1} \right\} + l \left\{ \binom{k}{l} \right\}
\]

with

\[
\left\{ \binom{0}{0} \right\} = 1, \quad \left\{ \binom{k}{0} \right\} = \left\{ \binom{0}{l} \right\} = 0 \quad (k, l \neq 0)
\]

for any integers \(k\) and \(l\).

See [8] for the detailed proof of Lemma 3.2.

Remark. We easily see that the right-hand side of (3.1) is an integer, since the Stirling numbers are integers. Furthermore this fact indicates that the denominator of poly-Euler number \(E_n^{(-k)}\) is at most \(n+1\).

We prove Theorem 3.1 by using the above lemma. Note that

\[
\sum_{m=1}^{n+1} \binom{n+1}{m} (4l + 2)^{n+1-m} \equiv \begin{cases} 
1 \pmod{2} & n : \text{even}, \\
0 \pmod{2} & n : \text{odd}
\end{cases}
\]

for any non-negative integer \(l\). Hence, when \(n\) is odd, we have Theorem 3.1 immediately from Lemma 3.2.

On the other hand, when \(n\) is even, we have

\[
(n+1)E_n^{(-k)} \equiv \sum_{l=0}^{k} \frac{k!}{l!} \left\{ \binom{k}{l} \right\} = \left\{ \binom{k}{0} \right\} + \left\{ \binom{k}{1} \right\} = \left\{ \binom{k+1}{1} \right\} = 1 \pmod{2}
\]

for any non-negative integer \(k\). Here, we have used the recurrence relation (3.2). Thus the proof of Theorem 3.1 is completed. \(\square\)

§ 4. Congruence relations of poly-Euler numbers

In the previous section, we definitely determined the parity of poly-Euler numbers. In this section, we treat the divisibility of poly-Euler numbers via congruence relations of them. In particular, we consider the case of

\[
(n+1)E_n^{(-k)} \pmod{n+1},
\]

which allows us to evaluate whether \(E_n^{(-k)}\) is an integer. In [8], we treat the case when \(n+1\) is an odd prime. Hence we discuss the composite cases here.
Theorem 4.1. For any non-negative integer $k$, we have

\[ 6E_5^{(-k)} \equiv \begin{cases} 
4 \pmod{6} & \text{if } k \text{ is even,} \\
0 \pmod{6} & \text{if } k \text{ is odd.} 
\end{cases} \]

Proof of Theorem 4.1. From Lemma 3.2, we have

\[ (4.1) \quad 6E_5^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l l! \begin{pmatrix} k \\ l \end{pmatrix} \sum_{j=0}^{2} \binom{6}{2j+1} (4l+2)^{5-2j}. \]

We see that $l! \equiv 0 \pmod{6}$ for $l \geq 3$ and

\[ \sum_{j=0}^{2} \binom{6}{2j+1} (4l+2)^{5-2j} \equiv 2(4l+2)^3 \pmod{6} \]

Thus (4.1) becomes

\[ 6E_5^{(-k)} \equiv (-1)^k 4 \left( \begin{pmatrix} k \\ 0 \end{pmatrix} + \begin{pmatrix} k \\ 2 \end{pmatrix} \right) \pmod{6} \]

\[ \equiv \begin{cases} 
4 \pmod{6} & \text{if } k \equiv 0 \pmod{3}, \\
0 \pmod{6} & \text{if } k \equiv 1 \pmod{3}, \\
2 \pmod{6} & \text{if } k \equiv 2 \pmod{3}. 
\end{cases} \]

Thus (4.1) becomes

\[ 6E_5^{(-k)} \equiv (-1)^k 4 \left( \begin{pmatrix} k \\ 2 \end{pmatrix} \right) \pmod{6} \]

\[ \equiv \begin{cases} 
1 \pmod{6} & \text{if } k \text{ is even,} \\
3 \pmod{6} & \text{if } k \text{ is odd.} 
\end{cases} \]

Therefore, we claim

\[ (4.2) \quad (-1)^k \begin{pmatrix} k \\ 2 \end{pmatrix} \equiv \begin{cases} 
1 \pmod{6} & \text{if } k \text{ is even,} \\
3 \pmod{6} & \text{if } k \text{ is odd.} 
\end{cases} \]

Using the expression $\begin{pmatrix} k \\ 2 \end{pmatrix} = 2^{k-1} - 1 \ (k \geq 2)$, we have

\[ \begin{pmatrix} k \\ 2 \end{pmatrix} = 2^{k-1} - 1 = \sum_{j=0}^{k-2} 2^j = 1 + \sum_{j=1, \ j:\ \text{odd}}^{k-2} 2^j + \sum_{j=2, \ j:\ \text{even}}^{k-2} 2^j \]

\[ \equiv 1 + 2[(k-1)/2] + 4[(k-2)/2] \pmod{6}, \]

which gives (4.2). Thus we have Theorem 4.1. \qed

By the same way as above, we can also show the following theorem:
Theorem 4.2. For any non-negative integer $k$, we have

\[ 12E_{11}^{(-k)} \equiv \begin{cases} 4 \pmod{12} & \text{if } k \text{ is even;} \\ 0 \pmod{12} & \text{if } k \text{ is odd.} \end{cases} \]

In general, to understand the prime factors of the numerator of $E_n^{(-k)}$ is proper. In [8], we prove a congruence relation

\[ (n + 1)E_n^{(-k)} \equiv 0 \pmod{p} \]

holds for any odd prime $p$, odd positive integer $n$ and non-positive integer $k$ satisfying $k \equiv p - 2 \pmod{p - 1}$. The second assertion of Theorem 4.1 is also given by combining the above congruence relation with Theorem 3.1.

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References

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