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On the 3-divisibility of class numbers of certain quadratic fields

By

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Abstract

This is an announcement of the original paper [11]. Let m_1, m_2 and m_3 be square-free integers. First, we obtain that there exist infinitely many fundamental discriminants D with $\gcd(m_1m_2, D) = 1$ such that the class numbers of $\mathbb{Q}(\sqrt{m_1D})$ and $\mathbb{Q}(\sqrt{m_2D})$ are both divisible by 3. This is a generalization of the result of T. Komatsu[14]. Secondly, we obtain that there exist infinitely many positive fundamental discriminants D with $\gcd(m_1m_2, D) = 1$ such that the class numbers of $\mathbb{Q}(\sqrt{m_1D})$ and $\mathbb{Q}(\sqrt{m_2D})$ are both indivisible by 3. Especially, there exist infinitely many positive fundamental discriminants D with $\gcd(m_1m_2m_3, D) = 1$ such that the class numbers of real quadratic fields $\mathbb{Q}(\sqrt{m_1D})$, $\mathbb{Q}(\sqrt{m_2D})$ and $\mathbb{Q}(\sqrt{m_3D})$ are indivisible by 3. These are generalizations of the result of D. Byeon[4]. For the result of the indivisibility case, we obtain an application to the Iwasawa invariants concerning Greenberg's Conjecture.

§ 1. Introduction

For a given positive integer n, there are infinitely many quadratic fields whose class numbers are divisible by n. In the imaginary case, such results are obtained by T. Nagell[18], N. C. Ankeny and S. Chowla[1], R. A. Mollin[17], etc. In the real case, Y. Yamamoto[23], P. J. Weinberger[22], etc. gave the same results. All the proofs of them were given by constructing such quadratic fields explicitly. Many results of the divisibility of class numbers of quadratic fields are known for the case where $n = 3$ particularly. We begin with a result of T. Komatsu.

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Theorem 1.1 (Komatsu, [14, Theorem A]). *Fix a non-zero integer t. Then, there exist infinitely many fundamental discriminants D such that the class numbers of $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ are both divisible by 3.*

Remark. The result for the case where $t = -1$ is known by [13] earlier. In Theorem 1.1, we can take both positive and negative integers for D.

When $t = -3$, Theorem 1.1 follows from Scholz inequality.

Theorem 1.2 (Scholz, [21]). *Let r and s be 3-ranks of the ideal class groups of a real quadratic field $\mathbb{Q}(\sqrt{D})$ and an imaginary quadratic field $\mathbb{Q}(\sqrt{-3D})$ respectively. Then, we have*

$$r \leq s \leq r + 1.$$

We denote the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$ by $h(D)$. From Theorem 1.2, for a positive integer D, if $3 \mid h(D)$, then $3 \mid h(-3D)$. Since there are infinitely many real quadratic fields whose class numbers are divisible by 3, there exist infinitely many positive fundamental discriminants D such that the class numbers of $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{-3D})$ are both divisible by 3. As a generalization of Theorem 1.1, we obtain the following.

Theorem 1.3. *Let m_1 and m_2 be distinct non-zero square-free integers. Then, there exist infinitely many fundamental discriminants D with $\gcd(m_1 m_2, D) = 1$ such that the class numbers of $\mathbb{Q}(\sqrt{m_1 D})$ and $\mathbb{Q}(\sqrt{m_2 D})$ are both divisible by 3.*

Remark. We can take both positive and negative integers for D. In Theorem 1.1, the condition $\gcd(t, D) = 1$ is not assumed.

On the other hand, concerning the indivisibility of class numbers of quadratic fields, D. Byeon proved the following.

Theorem 1.4 (Byeon, [4, Theorem 1.1]). *Let t be a square-free integer. Then, there exist infinitely many positive fundamental discriminants D with a positive inferior limit density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ are both indivisible by 3.*

As a generalization of Theorem 1.4, we obtain the following.

Theorem 1.5. *Let m_1, m_2 and m_3 be square-free positive integers.*
(1) *There exist infinitely many positive fundamental discriminants D with a positive inferior limit density such that $\gcd(m_1 m_2 m_3, D) = 1$ and the class numbers of real quadratic fields $\mathbb{Q}(\sqrt{m_1 D})$, $\mathbb{Q}(\sqrt{m_2 D})$ and $\mathbb{Q}(\sqrt{m_3 D})$ are indivisible by 3.*

(2) *There exist infinitely many positive fundamental discriminants D with a positive inferior limit density such that* $\gcd(m_1m_2, D) = 1$ *and the class numbers of quadratic fields* $\mathbb{Q}(\sqrt{m_1D})$ *and* $\mathbb{Q}(\sqrt{-m_2D})$ *are both indivisible by* 3.

In Theorem 1.5, we can obtain an application to the Iwasawa invariants.

This paper is organized as follows. In Section 2, we sketch the outline of the proof of Theorem 1.3 by constructing an explicit cubic polynomial which gives an unramified cyclic cubic extension of the quadratic field. In Section 3, we sketch the outline of the proof of Theorem 1.5 by using the result of J. Nakagawa and K. Horie[19]. Their detailed proofs are stated in [11]. Finally, in Section 4, we state an application to the Iwasawa invariants related to Greenberg's Conjecture.

§ 2. Outline of the proof of Theorem 1.3

In this section, we sketch the outline of the proof of Theorem 1.3. The method of the proof is based on the one in [14].

Let m_1 and m_2 be distinct non-zero square-free integers. We consider only the case where $4 \nmid m_1m_2$. (The case where $4 \mid m_1m_2$ can be shown by a similar way. For detail, see [11].) We assume that m_2 is odd. Let l be a prime number which is inert in the extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ and satisfies the condition

$$\left(\frac{m_1}{l}\right) = \left(\frac{m_2}{l}\right) = 1.$$

Note that there exists such prime number l from Chebotarev density theorem. We take integers n_i $(i = 1, 2)$ satisfying the following conditions:

$$n_i \equiv \begin{cases} \pm 3^2(4m_i - 3) \bmod 3^6 \text{ if } m_i \equiv 1 \bmod 3 \\ \pm 3^2(4m_i + 12) \bmod 3^6 \text{ if } m_i \equiv 2 \bmod 3 \\ \pm 4m_i \bmod 27 \text{ if } m_i \equiv 3 \bmod 9 \\ \pm m_i \bmod 27 \text{ if } m_i \equiv 6 \bmod 9 \end{cases}$$

and

$$m_in_i^2 \equiv 1 \bmod l.$$

Furthermore, we assume that n_1 is even if m_1 is odd and that n_2 is odd. By using the Chinese remainder theorem, it is seen that there exists such integer n_i. Now, put $r_1 := m_1n_1^2$, $r_2 := m_2n_2^2$ and $r := r_1r_2$. Let P be the set of prime numbers that is defined by

$$P := \{p : \text{prime} \mid p \neq 3 \text{ and } p \mid r(r-1)(r_1 - r_2)\}.$$

We denote by T the set of integers t which satisfy the conditions:

$$\begin{cases} t \equiv \pm 3 \bmod 81 \\ t \equiv -1 \bmod l \\ t \not\equiv r,\ r_1 \bmod p \quad \text{for any } p \in P \\ 2t \not\equiv 3(r_1+r_2) \bmod q \quad \text{for every prime factors } q \neq 3 \text{ of } m_1m_2. \end{cases}$$

By using the Chinese remainder theorem, we obtain that T is an infinite set. We define three subsets of T as follows. For $r > 0$, let $T_1 := \{t \in T \mid t \geq 2r\}$ and $T_2 := \{t \in T \mid t \leq \text{ Max } \{r_1, r_2\}\}$. For $r < 0$, let $T_3 := \left\{t \in T \mid t \geq \text{Max}\{t_0, \sqrt{-r/3}\}\right\}$, where t_0 is a real number such that $t_0 > \text{Max}\{r_1, r_2\}$ and $2t_0^3 - 3(r_1+r_2)t_0^2 + 6rt_0 - r(r_1+r_2) = 0$. Note that t_0 is uniquely determined. Define

$$D_{r_1,r_2}(X) := \frac{1}{27}(3X^2+r)\{2X^3 - 3(r_1+r_2)X^2 + 6rX - r(r_1+r_2)\}.$$

From the congruence relation of r_1, r_2 and t, it follows that $D_{r_1,r_2}(t)$ is an integer. Let $\mathcal{F}(S)$ denote the family $\{\mathbb{Q}(\sqrt{m_1 D_{r_1,r_2}(t)}) \mid t \in S\}$ for a subset S of $\mathbb{Z}$. For a prime number p and an integer a, the symbol $v_p(a)$ denotes the greatest exponent n such that $p^n \mid a$. Then, we have the following.

Theorem 2.1. *Let m_1 and m_2 be distinct non-zero square-free integers with $4 \nmid m_1m_2$. For every $t \in T$, the class numbers of the quadratic fields $\mathbb{Q}(\sqrt{m_1 D_{r_1,r_2}(t)})$ and $\mathbb{Q}(\sqrt{m_2 D_{r_1,r_2}(t)})$ are both divisible by 3 and $\gcd(m_1m_2/3^{v_3(m_1m_2)}, D_{r_1,r_2}(t)) = 1$. Moreover, the families $\mathcal{F}(T_1)$, $\mathcal{F}(T_2)$ and $\mathcal{F}(T_3)$ each include infinitely many quadratic fields. In particular, if $m_1 > 0$, $m_2 > 0$ and $t \in T_1$(resp. $t \in T_2$), the quadratic fields $\mathbb{Q}(\sqrt{m_1 D_{r_1,r_2}(t)})$ and $\mathbb{Q}(\sqrt{m_2 D_{r_1,r_2}(t)})$ are both real (resp. both imaginary). Furthermore, if $m_1m_2 < 0$ and $t \in T_3$, one of the quadratic fields $\mathbb{Q}(\sqrt{m_1 D_{r_1,r_2}(t)})$ and $\mathbb{Q}(\sqrt{m_2 D_{r_1,r_2}(t)})$ is real and the other one is imaginary.*

This theorem is essential for the proof of the case $4 \nmid m_1m_2$ of Theorem 1.3. In fact, the case $12 \nmid m_1m_2$ of Theorem 1.3 follows from Theorem 2.1 immediately. In the case $3 \mid m_1m_2$, we can show Theorem 1.3 by using Theorem 2.1 as follows. From the congruence relation r_1, r_2 and t, it is seen that $v_3(D_{r_1,r_2}(t)) = 3$. Then, we have

$$\mathbb{Q}\left(\sqrt{m_i D_{r_1,r_2}(t)}\right) = \mathbb{Q}\left(\sqrt{\frac{m_i}{3} \cdot \frac{D_{r_1,r_2}(t)}{3^3}}\right)$$

if $3 \mid m_i$ and

$$\mathbb{Q}\left(\sqrt{m_i D_{r_1,r_2}(t)}\right) = \mathbb{Q}\left(\sqrt{3m_i \cdot \frac{D_{r_1,r_2}(t)}{3^3}}\right)$$

if $3 \nmid m_i$. Putting $m_i' := m_i/3$ (resp. $m_i' := 3m_i$) if $3 \mid m_i$ (resp. $3 \nmid m_i$), we have $\gcd(m_1'm_2', D_{r_1,r_2}(t)/3^3) = 1$. Moreover, we see that the class numbers of

$\mathbb{Q}(\sqrt{m_1' D_{r_1,r_2}(t)/3^3})$ and $\mathbb{Q}(\sqrt{m_2' D_{r_1,r_2}(t)/3^3})$ are both divisible by 3. To prove Theorem 2.1, we need the following.

Proposition 2.2. *For any $t \in T$ and $i = 1, 2$, we have $3 \mid h(m_i D_{r_1,r_2}(t))$.*

Proposition 2.3. (1) *Assume $m_1 > 0$ and $m_2 > 0$. If $t \in T_1$, the quadratic fields $\mathbb{Q}(\sqrt{m_1 D_{r_1,r_2}(t)})$ and $\mathbb{Q}(\sqrt{m_2 D_{r_1,r_2}(t)})$ are both real. If $t \in T_2$, the quadratic fields $\mathbb{Q}(\sqrt{m_1 D_{r_1,r_2}(t)})$ and $\mathbb{Q}(\sqrt{m_2 D_{r_1,r_2}(t)})$ are both imaginary.*
(2) *Assume $m_1 m_2 < 0$. If $t \in T_3$, one of $\mathbb{Q}(\sqrt{m_1 D_{r_1,r_2}(t)})$ and $\mathbb{Q}(\sqrt{m_2 D_{r_1,r_2}(t)})$ is real and the other one is imaginary.*

Proposition 2.4. *We have $\sharp\, \mathcal{F}(T) = \infty$. In particular, we have $\sharp\, \mathcal{F}(T_1) = \infty$, $\sharp\, \mathcal{F}(T_2) = \infty$ and $\sharp\, \mathcal{F}(T_3) = \infty$.*

Proposition 2.2 is obtained as follows. For a fixed $t \in T$, we put $u := t^3 + 3rt$, $w := 3t^2 + r$, $a := u - r_1 w$, $b := u - r_2 w$ and $c := t^2 - r$. Define $f_1(Z) := Z^3 - 3cZ - 2a$ and $f_2(Z) := Z^3 - 3cZ - 2b$. Then we can show that the polynomials $f_i(Z)$ $(i = 1, 2)$ are both irreducible over $\mathbb{Q}$. Let K_{f_i} denote the minimal splitting field of $f_i(Z)$ over $\mathbb{Q}$. Then $\mathbb{Q}(\sqrt{m_i D_{r_1,r_2}(t)})$ is contained in K_{f_i}. By using the result of P. Llorente and E. Nart[16], we see that the cyclic cubic extensions $K_{f_i}/\mathbb{Q}(\sqrt{m_i D_{r_1,r_2}(t)})$ $(i = 1, 2)$ are both unramified. From this and class field theory, we have $3 \mid h(m_i D_{r_1,r_2}(t))$. Proposition 2.3 is proved by checking the sign of the factor $2t^3 - 3(r_1 + r_2)t^2 + 6rt - r(r_1 + r_2)$ of $D_{r_1,r_2}(t)$. We can see this from the sign of the derivative of the one. Proposition 2.4 is shown as follows. Assume $S \neq \phi$ is a subset of T (resp. T_1, T_2 and T_3) such that $F(S)$ is finite. We can choose a_0 from T (resp. T_1, T_2 and T_3) so that $F(S) \subsetneq \mathcal{F}(S \cup \{a_0\})$.

§ 3. Outline of the proof of Theorem 1.5

In this section, we sketch the outline of the proof of Theorem 1.5.

For a given prime number p, there are infinitely many quadratic fields whose class numbers are indivisible by p. In the imaginary case, such results are obtained by P. Hartung[7], K. Horie[8, 9], K. Horie and Y. Onishi[10], W. Kohnen and K. Ono[15], etc. In the real case, K. Ono[20], D. Byeon[2, 3], etc. gave the same results. For the case where $p = 3$, the results of H. Davenport and H. Heilbronn[5] and J. Nakagawa and K. Horie[19] are known. We begin with the result of Nakagawa and Horie[19].

Suppose $0 < X \in \mathbb{R}$. We denote by $S_+(X)$ the set of positive fundamental discriminants $0 < D < X$ of quadratic fields. Similarly, we denote by $S_-(X)$ the set of negative fundamental discriminants $-X < D < 0$ of quadratic fields. Let m and N be positive integers satisfying the following conditions;

$(*)$ If p is an odd prime divisor of $\gcd(m, N)$, then $p^2 \mid N$ and $p^2 \nmid m$.
$(**)$ If N is even, then (i) $4 \mid N$ and $m \equiv 1 \bmod 4$ or (ii) $16 \mid N$ and $m \equiv 8, 12 \bmod 16$.
We define two sets with these m and N.

$$S_+(X, m, N) := \{D \in S_+(X) \mid D \equiv m \bmod N\},$$

$$S_-(X, m, N) := \{D \in S_-(X) \mid D \equiv m \bmod N\}.$$

As a refinement of the result of [5], Nakagawa and Horie proved as follows.

Theorem 3.1 (Nakagawa and Horie, [19])**.**

$$(1) \liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X, m, N) \mid 3 \nmid h(D)\}}{\sharp S_+(X, m, N)} \geq \frac{5}{6},$$

$$(2) \liminf_{X\to\infty} \frac{\sharp\{D \in S_-(X, m, N) \mid 3 \nmid h(D)\}}{\sharp S_-(X, m, N)} \geq \frac{1}{2},$$

$$(3)\ \sharp S_+(X, m, N) \sim \sharp S_-(X, m, N) \sim \frac{3X}{\pi^2 \varphi(N)} \prod_{p \mid N: prime} \frac{q}{p+1},$$

where $\varphi(N)$ is the Euler function, $q = 4$ if $p = 2$ and $q = p$ otherwise.

By using this, we obtain the following.

Theorem 3.2. *Let m_1, m_2 and m_3 be square-free positive integers. Assume that positive integers m and N satisfy $(*)$, $16 \mid N$, $m \equiv 1 \bmod 4$ and $\gcd(mN, m_1m_2m_3) \mid 2^3$. Then, we have*

$$(1) \liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X, m, m_1m_2m_3N) \mid h(m_iD) \not\equiv 0 \bmod 3\ (i = 1, 2, 3)\}}{\sharp S_+(X, m, m_1m_2m_3N)} \geq \frac{1}{3},$$

$$(2) \liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X, m, m_1m_2N) \mid h(m_1D), h(-m_2D) \not\equiv 0 \bmod 3\}}{\sharp S_+(X, m, m_1m_2N)} \geq \frac{1}{3}.$$

The method of the proof of this theorem is based on the one in [4]. Theorem 1.5 follows from Theorem 3.2.

§ 4. Application of Theorem 1.5

In this section, we state an application of Theorem 1.5 to the Iwasawa invariants of the cyclotomic $\mathbb{Z}_3$-extension of a quadratic field. We begin with a result of K. Iwasawa.

Theorem 4.1 (Iwasawa, [12])**.** *Let p be a prime number, k be an algebraic number field of finite degree and K/k be an arbitrary $\mathbb{Z}_p$-extension. If p does not split in k and the class number of k is indivisible by p, then $\lambda_p(K/k) = \mu_p(K/k) = \nu_p(K/k) = 0$, where $\lambda_p(K/k)$, $\mu_p(K/k)$ and $\nu_p(K/k)$ are the Iwasawa invariants of K/k.*

For a non-zero integer D, the symbols $\lambda_p(D)$, $\mu_p(D)$ and $\nu_p(D)$ denote the Iwasawa invariants of the cyclotomic $\mathbb{Z}_p$-extension of the quadratic field $\mathbb{Q}(\sqrt{D})$. From Theorems 3.2 and 4.1, we obtain the following two corollaries.

Corollary 4.2. *Let m_1 and m_2 be square-free positive integers.*
(1) *There exist infinitely many positive fundamental discriminants D with a positive inferior limit density such that* $\gcd(m_1m_2, D) = 1$ *and* $\lambda_3(m_iD) = \mu_3(m_iD) = \nu_3(m_iD) = 0$ $(i = 1, 2)$.
(2) *There exist infinitely many positive fundamental discriminants D with a positive inferior limit density such that* $\gcd(m_1m_2, D) = 1$, $\lambda_3(m_1D) = \mu_3(m_1D) = \nu_3(m_1D) = 0$ *and* $\lambda_3(-m_2D) = \mu_3(-m_2D) = \nu_3(-m_2D) = 0$.

Corollary 4.3. *Let m_1, m_2 and m_3 be distinct square-free positive integers with* $3 \mid (m_1-m_2)(m_2-m_3)(m_3-m_1)$. *Then, there exist infinitely many positive fundamental discriminants D with a positive inferior limit density such that* $\gcd(m_1m_2m_3, D) = 1$ *and* $\lambda_3(m_iD) = \mu_3(m_iD) = \nu_3(m_iD) = 0$ $(i = 1, 2, 3)$.

If k is a totally real field, it is conjectured that the Iwasawa λ_p and μ_p-invariants of the cyclotomic $\mathbb{Z}_p$-extension of k are equal to 0 (Greenberg's Conjecture, [6]). Corollaries 4.2 (1) and 4.3 imply that there are infinitely many pairs or triples of real quadratic fields satisfying Greenberg's Conjecture for $p = 3$ with a positive inferior limit density for a given proportion of discriminants. Corollaries 4.2 and 4.3 are proved by taking numbers N and m which are defined in Theorem 3.2.

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