PERVERSE COHERENT SHEAVES ON BLOW-UP. II. WALL-CROSSING AND BETTI NUMBERS FORMULA

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Abstract. This is the second of series of papers studyig moduli spaces of a certain class of coherent sheaves, which we call *stable perverse coherent sheaves*, on the blow-up $p\colon \widehat{X} \to X$ of a projective surface X at a point 0.

The followings are main results of this paper:

a) We describe the wall-crossing between moduli spaces caused by twisting of the line bundle $\mathcal{O}(C)$ associated with the exceptional divisor C.
b) We give the formula for virtual Hodge numbers of moduli spaces of stable perverse coherent sheaves.

Moreover we also give proofs of the followings which we observed in a special case in [25]:

c) The moduli space of stable perverse coherent sheaves is isomorphic to the usual moduli space of stable coherent sheaves on the *original surface* if the first Chern class is orthogonal to $[C]$.
d) The moduli space becomes isomorphic to the usual moduli space of stable coherent sheaves on the *blow-up* after twisting by $\mathcal{O}(-mC)$ for sufficiently large m.

Therefore usual moduli spaces of stable sheaves on the blow-up and the original surfaces are connected via wall-crossings.

Introduction

This paper is formally a sequel, but is independent of our previous paper [25] except in §1.3. All the rest do not depend on results in [25], though motivation to various definitions come from [25]. The result in §1.3 is independent of other parts of the paper. See also the comment below.

Let $p\colon \widehat{X} \to X$ be the blow-up of a projective surface X at a point 0. Let C be the exceptional divisor. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. A *stable perverse coherent sheaf* E on $\widehat{X}$, with respect to $\mathcal{O}_X(1)$, is

(1) E is a coherent sheaf on $\widehat{X}$,
(2) $\operatorname{Hom}(E, \mathcal{O}_C(-1)) = 0$,

(3) $\operatorname{Hom}(\mathcal{O}_C, E) = 0$,
(4) p_*E is μ-stable with respect to $\mathcal{O}_X(1)$.

As was explained in [25], this definition came from two sources, a work by Bridgeland [2] and one by King [9]. In [25] the latter was explained in detail, and the first will be explained in this paper.

For a given integer $m \in \mathbb{Z}$ and homological data $c \in H^*(\widehat{X})$, we will consider the moduli space $\widehat{M}^m(c)$ of coherent sheaves E with $\operatorname{ch}(E) = c$ such that $E(-mC)$ is stable perverse coherent. This family of moduli spaces interpolates the moduli space $M^X(p_*(c))$ of stable sheaves on X and the moduli space $M^{\widehat{X}}(c)$ on $\widehat{X}$ as explained in the abstract. We assume that $(c_1, p^*\mathcal{O}_X(1))$ and r are coprime, so the μ-stability and μ-semistability are equivalent on X. Then we construct varieties $\widehat{M}^{m,m+1}(c)$ connecting various $\widehat{M}^m(c)$ by the diagram

$$(*) \quad \cdots \swarrow \quad \widehat{M}^m(c) \xrightarrow{\xi_m} \widehat{M}^{m,m+1}(c) \xleftarrow{\xi_m^+} \widehat{M}^{m+1}(c) \xrightarrow{\xi_{m+1}} \widehat{M}^{m+1,m+2}(c) \swarrow \quad \cdots$$

The morphism ξ_m^+ is a kind of 'flip' of ξ_m. (See Proposition 3.36 for the precise statement.) This kind of the diagram appears often in the variation of GIT quotients [27] and moduli spaces of sheaves (by Thaddeus, Ellingsrud-Göttsche, Friedman-Qin and others) when we move ample line bundles.

Furthermore, $\widehat{M}^{m,m+1}(c)$ will be constructed as the Brill-Noether locus in the moduli space $M^X(p_*(c))$ of stable sheaves on X, and the fibers of ξ_m, ξ_m^+ over $F \in M^X(p_*(c))$ are Grassmann varieties consisting of subspaces $V \subset \operatorname{Hom}(\mathcal{O}_C(-m-1), p^*F)$ and $U \subset \operatorname{Hom}(p^*F, \mathcal{O}_C(-m-1))$ of $\dim V = (c_1, [C]) + m$, $\dim U = (c_1, [C]) + m + r$ respectively. The dimensions of spaces of homomorphisms depend on the sheaf F, so ξ_m, ξ_m^+ are *stratified* Grassmann bundles. This looks similar to the picture observed in the context of quiver varieties [18] and exceptional bundles on K3 [29, 14] (see also [22] for an exposition). But there is a sharp distinction between the blowup case and these cases. In the other cases, the spaces of homomorphisms (or extensions) appearing in the fibers of ξ_m and ξ_m^+ are *dual* to each other, and $\dim U = \dim V$, so that two varieties are related by the stratified Mukai flop. However, our spaces $\operatorname{Hom}(\mathcal{O}_C(-m-1), p^*F)$ and $\operatorname{Hom}(p^*F, \mathcal{O}_C(-m-1))$ have different dimensions, and $\dim U \neq \dim V$. (See also Remark 3.30 for another difference.)

Next we consider the formula for (virtual) Hodge numbers. This study was not originally planned when we started this research project, and is motivated by recent works on wall-crossings of Donaldson-Thomas invariants [3, 10]. It turns out to be a simple application of techniques developed for the blow-up formula for virtual Hodge polynomials in [24, Th. 3.13].

Since the formula becomes complicated in higher rank cases, we consider the rank 1 case, where $\widehat{M}^0(c)$ (resp. $\widehat{M}^m(c)$) is the Hilbert scheme $X^{[N]}$ (resp. $\widehat{X}^{[N]}$) of N points in X (resp. $\widehat{X}$ for sufficiently large m depending on N). Then we have the formula for the generating function of Hodge polynomials

$$(**)\quad \sum_{N=0}^{\infty} P_{x,y}(\widehat{M}^m(c))\mathfrak{q}^N = \left(\sum_{N=0}^{\infty} P_{x,y}(X^{[N]})\mathfrak{q}^N\right)\left(\prod_{d=1}^{m}\frac{1}{1-(xy)^d\mathfrak{q}^d}\right),$$

where $c = 1 - N\,\mathrm{pt}$. When $m \to \infty$, the left hand side converges to $\sum_{N=0}^{\infty} P_{x,y}(\widehat{X}^{[N]})\mathfrak{q}^N$ as we have just remarked. Then the above formula is compatible with the *famous* Göttsche formula[1] of Betti numbers of Hilbert schemes of points of surfaces [5]. Thus factors of the infinite product of the Dedekind η-function appear one by one when we cross walls.

Moreover, in this rank 1 case, $\widehat{M}^1(c)$ is isomorphic to the nested Hilbert scheme of N and $N+1$ points in X, where two subschemes differ only at 0. The above formula coincides with Cheah's formula [4, Theorem 3.3.3(5)] in this special case. However our $\widehat{M}^m(c)$ for $m \geq 2$ seems new even in rank 1 case. In particular, they are different from incidence varieties used to define Heisenberg generators in [20, Chap. 8].

In higher rank cases, we have the formula relating virtual Hodge polynomials of $\widehat{M}^m(c)$ and $M^X(p_*(c))$. (See Corollary 5.7.) In the limit $m \to \infty$, the formula converges to the blow-up formula for virtual Hodge polynomials in [24, Th. 3.13]. (See also [24, Rem. 3.14] for earlier works.)

Similar wall-crossing formulae for Donaldson-Thomas invariants in 3-dimensional situation of [2] will be discussed elsewhere ([17, 16]).

Finally let us comment on the quiver description in our first paper [25]. The most of materials in this paper can be worked out in the language of quiver representations. In fact, the constructions of moduli spaces and the diagram $(*)$ are automatic in that setup, and the assertion that the fibers are Grassmann varieties are easy to prove. The only missing is the isomorphism $\widehat{M}^m(c) \cong \widehat{M}^0(ce^{-m[C]})$ induced by the tensor product by $\mathcal{O}(-mC)$. We do not know how to construct the isomorphism explicitly in terms of quivers. This, if it is possible, would

[1] We learned this naming from Atsushi Takahashi.

be given by analog of reflection functors, developed by the first-named author in the context of quiver varieties [21].

The paper is organized as follows. In §1 we study the category of perverse coherent sheaves $\operatorname{Per}(\widehat{X}/X)$ which is the heart of the t-structure in the derived category $\mathbf{D}(\widehat{X})$ of coherent sheaves on $\widehat{X}$, introduced in more general setting in [2]. One of the main results in this section is a simple criterion when a coherent sheaf E is perverse coherent (see Proposition 1.9(1)). In §2 we construct moduli spaces of perverse coherent sheaves in the general context in [2]. One of key observations is that though perverse coherent sheaves are objects in $\mathbf{D}(\widehat{X})$ in general, they are genuine sheaves if we impose the stability and the assumption on the dimension of their supports. This was already observed in [2] in the case of perverse ideal sheaves. Combined with the result in §1 we get the conditions (1)$\sim$(4) in the blow-up case. In §3 we construct the diagram $(*)$. Our tools here are Brill-Noether loci and moduli spaces of coherent systems, which had been used in different settings as we mentioned above. In §4 we show that $\widehat{M}^0(c)$ is an incidence variety in the product of two moduli spaces $M^X(p_*(c)) \times M^X(p_*(c) + n\,\mathrm{pt})$ $(n = (c_1, [C]))$. In §5 we give the formula for virtual Hodge numbers of $\widehat{M}^m(c)$. The proof goes like that of [24, Th. 3.13]. We observe that the formula is universal, i.e. is independent of the surface X, and is enough to compute it in the moduli of framed sheaves. Then we can use a torus action to deduce it from a combinatorial study of fixed points. The combinatorics involves Young diagrams and removable boxes, which is closely related to one appearing in the Pieri formula (but only for the multiplication by e_1 !) for Macdonald polynomials [13, §VI.6].

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Notations. $\mathbf{D}(X)$ denotes the unbounded derived category of coherent sheaves on a variety X. The full subcategory of complexes with bounded cohomology sheaves is denoted by $\mathbf{D}^b(X)$.

We consider a blowup $p\colon \widehat{X} \to X$ of a smooth projective surface X at a point $0 \in X$. But occasionaly we consider a general situation where $p\colon Y \to X$ is a birational morphism of projective varieties such that $\mathbf{R}p_*(\mathcal{O}_Y) = \mathcal{O}_X$ and $\dim p^{-1}(x) \le 1$ for any $x \in X$.

When we write $\mathcal{O}$ without indicating the variety, it means the structure sheaf of $\mathcal{O}_{\widehat{X}}$.

Let $C = p^{-1}(0) \subset \widehat{X}$ denote the exceptional divisor. Let $\mathcal{O}(C)$ denote the line bundle associated with C, and $\mathcal{O}(mC)$ its m^{th} tensor product $\mathcal{O}(C)^{\otimes m}$ when $m > 0$, $(\mathcal{O}(C)^{\otimes -m})^\vee$ if $m < 0$, and $\mathcal{O}$ if $m = 0$.

The structure sheaf of the exceptional divisor C is denoted by $\mathcal{O}_C$. If we twist it by the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ over $C \cong \mathbb{P}^1$, we denote the resulted sheaf by $\mathcal{O}_C(n)$. Since C has the self-intersection number -1, we have $\mathcal{O}_C \otimes \mathcal{O}(C) = \mathcal{O}_C(-1)$.

For $c \in H^*(\widehat{X})$, its degree 0, 2, 4-parts are denoted by r, c_1, ch_2 respectively. If we want to specify c, we denote by $r(c)$, $c_1(c)$, $\mathrm{ch}_2(c)$.

We also use the following notations often:

- $\operatorname{rk} E$ is the rank of a coherent sheaf E.
- $e := \mathrm{ch}(\mathcal{O}_C(-1))$.
- pt is a single point in X or $\widehat{X}$. Its Poincaré dual in $H^4(X)$ or $H^4(\widehat{X})$ is also denoted by the same notation.
- $\chi(E,F) := \displaystyle\sum_{i=-\infty}^{\infty} (-1)^i \dim \operatorname{Ext}^i(E,F) = \sum_{i=-\infty}^{\infty} (-1)^i \dim \operatorname{Hom}(E, F[i])$.
- $h^0(E,F) := \dim \operatorname{Hom}(E,F)$.
- $\chi(E) := \chi(\mathcal{O}_{\widehat{X}}, E)$.
- $h^0(E) := h^0(\mathcal{O}_{\widehat{X}}, E)$.

1. Perverse coherent sheaves on blow-up

1.1. **General situation.** Let $p\colon Y \to X$ be a birational morphism of projective varieties such that $\mathbf{R}p_*\mathcal{O}_Y = \mathcal{O}_X$ and $\dim p^{-1}(x) \le 1$ for any $x \in X$. This is the assumption considered to define *perverse coherent sheaves* in [2]. We set $Z := \{x \in X \mid \dim p^{-1}(x) = 1\}$. Then $p^{-1}(Z)$ is the exceptional locus of p. The example we have in mind is the blowup of a projective surface X at a smooth point $0 \in X$, but we review the arguments in [2] for the completeness in this subsection.

Definition 1.1 ([2, 3.2])**.** Let $\operatorname{Per}(Y/X)$ be the full subcategory of $\mathbf{D}(Y)$ consisting of objects $E \in \mathbf{D}(Y)$ satisfying the following conditions:

(1) $H^i(E) = 0$ for $i \neq -1, 0$,
(2) $R^0p_*(H^{-1}(E)) = 0$ and $R^1p_*(H^0(E)) = 0$,
(3) $\operatorname{Hom}(H^0(E), K) = 0$ for any sheaf K on Y with $\mathbf{R}p_*(K) = 0$.

An object $E \in \operatorname{Per}(Y/X)$ is called a *perverse coherent sheaf*.

By [2, §§2,3] $\operatorname{Per}(Y/X)$ is the heart of a t-structure on $\mathbf{D}(Y)$, and in particular, is an abelian category. This will be reviewed below.

An object $E \in \mathrm{Per}(Y/X)$ satisfies $H^i(\mathbf{R}p_*(E)) = 0$ for $i \neq 0$. Thus $\mathbf{R}p_*(E) \in \mathrm{Coh}(X)$.

Lemma 1.2 (cf. [2, 5.1])**.** (1) *For a coherent sheaf F on X, we have an exact sequence*

$$0 \to R^1p_*(L^{-1}p^*(F)) \to F \to p_*p^*(F) \to 0.$$

Moreover we have $p^*(F) \in \mathrm{Per}(Y/X)$. Furthermore, $F \cong p_*p^*(F)$ if F is torsion free.

(2) *Let E be a coherent sheaf on Y. For a natural homomorphism $\phi\colon p^*p_*(E) \to E$, we have* (i) $\mathbf{R}p_*(\operatorname{Ker}\phi) = 0$, (ii) $p_*(\operatorname{Im}\phi) \to p_*(E)$ *is isomorphic,* (iii) $p_*(\operatorname{Coker}\phi) = 0$, (iv) $R^1p_*(\operatorname{Im}\phi) = 0$, $R^1p_*(E) \cong R^1p_*(\operatorname{Coker}\phi)$*.*

(3) *A coherent sheaf E belongs to* $\mathrm{Per}(Y/X)$ *if and only if $\phi\colon p^*p_*(E) \to E$ is surjective.*

(4) *For a coherent sheaf F on X, we have* $\mathrm{Ext}^1(p^*(F), K) = 0$ *for all $K \in \mathrm{Coh}(Y)$ with $\mathbf{R}p_*(K) = 0$.*

Proof. (1) The first assertion is a consequence of the projection formula $\mathbf{R}p_*(\mathbf{L}p^*(F)) = F$ and the spectral sequence

$$R^pp_*(L^qp^*(F)) \Rightarrow H^{p+q}(\mathbf{R}p_*(\mathbf{L}p^*(F))).$$

We also get $R^1p_*(p^*(F)) = 0$ at the same time. Now we have $\mathrm{Hom}(p^*(F), K) = \mathrm{Hom}(F, p_*(K)) = 0$ for $K \in \mathrm{Coh}(Y)$ with $\mathbf{R}p_*(K) = 0$. Therefore $p^*(F)$ is perverse coherent. For the last assertion we note that $R^1p_*(L^{-1}p^*(F))$ is supported on $p^{-1}(Z)$, and hence is torsion.

(2) We have exact sequences

$$0 \to p_*(\operatorname{Ker}\phi) \to p_*(p^*(p_*(E))) \to p_*(\operatorname{Im}\phi) \to R^1p_*(\operatorname{Ker}\phi) \to 0,$$

$$\begin{tikzcd}
0 \arrow[r] & p_*(\operatorname{Im}\phi) \arrow[r] & p_*(E) \arrow[r] & p_*(\operatorname{Coker}\phi) \arrow[dll] & \\
& R^1p_*(\operatorname{Im}\phi) \arrow[r] & R^1p_*(E) \arrow[r] & R^1p_*(\operatorname{Coker}\phi) \arrow[r] & 0,
\end{tikzcd}$$

where we have used $R^1p_*(p^*(p_*(E))) = 0$ from (1) in the first exact sequence. Since the composition $p_*(E) \to p_*(p^*(p_*(E))) \to p_*(E)$ is the identity, (1) implies that both homomorphisms are isomorphisms. Therefore $p_*(\operatorname{Im}\phi) \to p_*(E)$ is also an isomorphism. We have $\mathbf{R}p_*(\operatorname{Ker}\phi) = 0$ from the first exact sequence.

Since $R^1p_*(\operatorname{Im}\phi) = 0$ follows from $R^1p_*(p^*(p_*(E))) = 0$, the second exact sequence gives $p_*(\operatorname{Coker}\phi) = 0$ and $R^1p_*(E) \cong R^1p_*(\operatorname{Coker}\phi)$.

(3) Suppose $E \in \mathrm{Coh}(Y)$ and $\phi\colon p^*p_*(E) \to E$ is surjective. We have $R^1p_*(E) = 0$ from (2)(iv). We also have $0 \to \mathrm{Hom}(E, K) \to \mathrm{Hom}(p^*(p_*(E)), K)$ and $\mathrm{Hom}(p^*(p_*(E)), K) = \mathrm{Hom}(p_*(E), p_*(K)) = 0$ for a sheaf K with $\mathbf{R}p_*(K) = 0$. Therefore $E \in \mathrm{Per}(Y/X)$.

Conversely suppose $E \in \operatorname{Per}(Y/X) \cap \operatorname{Coh}(Y)$. By (2)(iii),(iv) we have $\mathbf{R}p_*(\operatorname{Coker}\phi) = 0$. By Definition 1.1(3) we have $0 = \operatorname{Hom}(E, \operatorname{Coker}\phi)$, i.e. $\operatorname{Coker}\phi = 0$.

(4) We consider a distinguished triangle $\mathbf{L}^{<0}p^*F \to \mathbf{L}p^*F \to p^*F \to \mathbf{L}^{<0}p^*F[1]$. We apply the functor $\operatorname{Hom}(\bullet, K)$ to get an exact sequence

$$\operatorname{Hom}(\mathbf{L}^{<0}p^*F[1], K[1]) \to \operatorname{Hom}(p^*F, K[1]) \to \operatorname{Hom}(\mathbf{L}p^*F, K[1]).$$

We have

$$\operatorname{Hom}(\mathbf{L}p^*F, K[1]) = \operatorname{Hom}(F, \mathbf{R}p_*(K[1])) = \operatorname{Hom}(F, \mathbf{R}p_*(K)[1]) = 0,$$
$$\operatorname{Hom}(\mathbf{L}^{<0}p^*F[1], K[1]) = \operatorname{Hom}(\mathbf{L}^{<0}p^*F, K) = 0,$$

where the latter follows from the degree reason. We therefore have $\operatorname{Hom}(p^*F, K[1]) = \operatorname{Ext}^1(p^*F, K) = 0$. □

Let

$$\mathcal{C} := \{K \in \operatorname{Coh}(Y) \mid \mathbf{R}p_*(K) = 0\},$$
$$\mathcal{T} := \{E \in \operatorname{Coh}(Y) \mid R^1p_*(E) = 0,\ \operatorname{Hom}(E, K) = 0 \text{ for all } K \in \mathcal{C}\},$$
$$\mathcal{F} := \{E \in \operatorname{Coh}(Y) \mid p_*(E) = 0\}.$$

From the above definition, we have

$$\operatorname{Per}(Y/X) = \{E \in \mathbf{D}(Y) \mid H^i(E) = 0 \text{ for } i \neq 0, -1, H^{-1}(E) \in \mathcal{F}, H^0(E) \in \mathcal{T}\}.$$

Then the definition of $\operatorname{Per}(Y/X)$ is an example of a general construction in [6, §2]:

Lemma 1.3. *$(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\operatorname{Coh}(Y)$ in the sense of* [6, §2].

Proof. We check two assertions: (i) $\operatorname{Hom}(T, F) = 0$ for $T \in \mathcal{T}$, $F \in \mathcal{F}$, (ii) for any $E \in \operatorname{Coh}(Y)$, there exists an exact sequence $0 \to T \to E \to F \to 0$ with $T \in \mathcal{T}$, $F \in \mathcal{F}$.

(i) By Lemma 1.2(3), we have $p^*p_*(T) \twoheadrightarrow T$. Therefore $\operatorname{Hom}(T, F) \subset \operatorname{Hom}(p^*p_*(T), F) = \operatorname{Hom}(p_*(T), p_*(F)) = 0$ for $T \in \mathcal{T}$, $F \in \mathcal{F}$.

(ii) For $E \in \operatorname{Coh} Y$, let us consider the exact sequence $0 \to \operatorname{Im}\phi \to E \to \operatorname{Coker}\phi \to 0$ for ϕ as in Lemma 1.2(2). We have $R^1p_*(\operatorname{Im}\phi) = 0$ and $p_*(\operatorname{Coker}\phi) = 0$ by (2)(iii),(iv). We also have $\operatorname{Hom}(\operatorname{Im}\phi, K) \subset \operatorname{Hom}(p^*p_*(E), K) = \operatorname{Hom}(p_*(E), p_*(K)) = 0$ for $K \in \mathcal{C}$. Therefore $\operatorname{Im}\phi \in \mathcal{T}$, $\operatorname{Coker}\phi \in \mathcal{F}$. □

An exact sequence $0 \to A \to B \to C \to 0$ in $\operatorname{Per}(Y/X)$ is a distinguished triangle $A \to B \to C \to A[1]$ in $\mathbf{D}(Y)$ such that all A, B, $C \in \operatorname{Per}(Y/X)$. Hence it induces an exact sequence

$$0 \to H^{-1}(A) \to H^{-1}(B) \to H^{-1}(C) \to H^0(A) \to H^0(B) \to H^0(C) \to 0$$

in $\mathrm{Coh}(Y)$. From a general theory, if $A \to B \to C \to A[1]$ is a distinguished triangle in $\mathbf{D}(Y)$ such that all A, $C \in \mathrm{Per}(Y/X)$, then $B \in \mathrm{Per}(Y/X)$. We have $\mathrm{Ext}^i_{\mathrm{Per}(Y/X)}(A,B) = \mathrm{Hom}_{\mathbf{D}(Y)}(A, B[i])$ for $A, B \in \mathrm{Per}(Y/X)$, $i = 0, 1$ ([6, Cor. 2.2(c)]). It was proved that $\mathbf{D}^b(Y) \cong \mathbf{D}^b(\mathrm{Per}(Y/X))$ in [28], but we will not use it in this paper.

Remark 1.4. Let $E \in \mathrm{Per}(Y/X) \cap \mathrm{Coh}(Y)$. By Lemma 1.2(3) we have the exact sequence $0 \to \mathrm{Ker}\,\phi \to p^*p_*(E) \to E \to 0$ in the category $\mathrm{Coh}(Y)$. This gives a distinguished triangle $\mathrm{Ker}\,\phi \to p^*p_*(E) \to E \to \mathrm{Ker}\,\phi[1]$. Now notice that $p^*p_*(E)$, E, $\mathrm{Ker}\,\phi[1] \in \mathrm{Per}(Y/X)$. Therefore we have $0 \to p^*p_*(E) \to E \to \mathrm{Ker}\,\phi[1] \to 0$ in the category of $\mathrm{Per}(Y/X)$.

Lemma 1.5. *Let E, F be objects in* $\mathrm{Per}(Y/X)$*, and hence* $\mathbf{R}p_*(E)$*,* $\mathbf{R}p_*(F) \in \mathrm{Coh}(Y)$*.*

(1) *Assume that $H^i(E) = 0$ for $i \neq 0$ and $p_*(E) = 0$. Then $E = 0$.*

(2) *a homomorphism $\xi\colon E \to F$ is injective in* $\mathrm{Per}(Y/X)$ *if and ony if $H^{-1}(E) \to H^{-1}(F)$ is injective in* $\mathrm{Coh}(Y)$ *and* $\mathbf{R}p_*(E) \to \mathbf{R}p_*(F)$ *is injective in* $\mathrm{Coh}(X)$*.*

Proof. (1) Since $\mathbf{R}p_*(E) = 0$ from the assumption and the Definition 1.1(2), we have $\mathrm{Hom}(E,E) = 0$ by Definition 1.1(3). Thus $E = 0$.

(2) We first assume that $E \to F$ is injective in $\mathrm{Per}(Y/X)$. Since $\mathrm{Per}(Y/X)$ is an abelian category, we have an exact sequence

$$0 \to E \to F \to G \to 0, \quad G := \mathrm{Coker}\,\xi \in \mathrm{Per}(Y/X).$$

Hence $H^{-1}(E) \to H^{-1}(F)$ is injective in $\mathrm{Coh}(Y)$ and we have an exact sequence in $\mathrm{Coh}(X)$:

$$0 \to \mathbf{R}p_*(E) \to \mathbf{R}p_*(F) \to \mathbf{R}p_*(G) \to 0,$$

as $\mathbf{R}p_*(E)$, $\mathbf{R}p_*(F)$, $\mathbf{R}p_*(G) \in \mathrm{Coh}(X)$.

Conversely, we assume that $H^{-1}(E) \to H^{-1}(F)$ is injective in $\mathrm{Coh}(Y)$ and $\mathbf{R}p_*(E) \to \mathbf{R}p_*(F)$ is injective in $\mathrm{Coh}(X)$. Let $K \in \mathrm{Per}(Y/X)$ be the kernel of ξ in $\mathrm{Per}(Y/X)$. Then $H^{-1}(K) = 0$ and we have an exact sequence

$$0 \to \mathbf{R}p_*(K) \to \mathbf{R}p_*(E) \to \mathbf{R}p_*(F).$$

Hence $\mathbf{R}p_*(K) = 0$. By (1), we get $K = 0$. $\square$

Lemma 1.6. *Let $E \in \mathrm{Coh}(Y)$ and let F be a subsheaf of $p_*(E)$. Then $F \to p_*(p^*(F))$ is an isomorphism.*

Proof. Consider the composite of $F \to p_*(p^*F) \to p_*(p^*(p_*(E))) \to p_*(E)$. It is equal to the given inclusion $F \hookrightarrow p_*(E)$. Hence $F \to p_*(p^*F)$ is injective. On the other hand, $F \to p_*(p^*(F))$ is surjective by Lemma 1.2(1). So $F \to p_*(p^*(F))$ is an isomorphism. $\square$

1.2. **Blow-up case.** Suppose $p\colon \widehat{X} \to X$ is a blow-up of a projective surface X at a smooth point $0 \in X$.

We first determine the sheaves K appearing the condition (3) in Definition 1.1.

Lemma 1.7. *Let K be a sheaf on $\widehat{X}$.*

(1) *If $p_*(K) = 0$, then there is a filtration*

$$K = F^0 \supset F^1 \supset \cdots \supset F^{s-1} \supset F^s = 0$$

such that $F^k/F^{k+1} \cong \mathcal{O}_C(-1-a_k)$ for $a_k \geq 0$. In particular, we have $K = 0$ if $\operatorname{Hom}(K, \mathcal{O}_C(-1)) = 0$.

(2) *If $\mathbf{R}p_*(K) = 0$, then $K = \mathcal{O}_C(-1)^{\oplus s}$.*

Remark 1.8. The filtration in (1) can be considered as a kind of Harder-Narashimhan filtration. This will be clear in a different proof given in the next subsection.

Proof. (1) We may assume $K \neq 0$. Since $p_*(K) = 0$, K is of pure dimension 1, and hence $c_1(K) = s[C]$ with $s > 0$. Then

$$\chi(K, \mathcal{O}_C(-1)) = \int_{\widehat{X}} \operatorname{ch}(K)^\vee \operatorname{ch}(\mathcal{O}_C(-1)) \operatorname{td} \widehat{X} = -(c_1(K), [C]) = s > 0.$$

Let $\mathbb{C}_0$ be the skyscraper at 0. Since

$$\begin{aligned} &\operatorname{Ext}^2(K, \mathcal{O}_C(-1)) \cong \operatorname{Hom}(\mathcal{O}_C, K)^\vee \cong \operatorname{Hom}(p^*(\mathbb{C}_0), K)^\vee \\ \cong\ &\operatorname{Hom}(\mathbb{C}_0, p_*(K))^\vee = 0 \end{aligned}$$

from the assumption, we must have $\operatorname{Hom}(K, \mathcal{O}_C(-1)) \neq 0$. Take a non-zero homomorphism $\phi\colon K \to \mathcal{O}_C(-1)$. Then we have $p_*(\operatorname{Ker}\phi) = 0$ and $\operatorname{Im}\phi \cong \mathcal{O}_C(-1-a)$ with $a \geq 0$. Applying this procedure to $\operatorname{Ker}\phi$, we get the assertion.

(2) We first note that $\chi(K) = 0$. Let E be a subsheaf of K. Then $H^0(\widehat{X}, E) = 0$, which implies that $\chi(E) \leq 0$. Applying this to $E := \operatorname{Ker}\phi$ in the proof of (1), we get $\chi(\operatorname{Im}\phi) \geq 0$. Then a in (1) must be 0, i.e. $\operatorname{Im}\phi \cong \mathcal{O}_C(-1)$. We also have $\chi(\operatorname{Ker}\phi) = 0$. Repeating this argument, we conclude all $F^k/F^{k+1} \cong \mathcal{O}_C(-1)$. Since $\operatorname{Ext}^1(\mathcal{O}_C(-1), \mathcal{O}_C(-1)) = 0$, we get the assertion. $\square$

Proposition 1.9. (1) *A coherent sheaf E on $\widehat{X}$ belongs to $\operatorname{Per}(\widehat{X}/X)$ if and only if $\operatorname{Hom}(E, \mathcal{O}_C(-1)) = 0$.*

(2) *Let $E \in \operatorname{Per}(\widehat{X}/X) \cap \operatorname{Coh}(\widehat{X})$ and $\phi\colon p^*(p_*(E)) \to E$ be the natural homomorphism. Then $\operatorname{Ker}\phi \cong \operatorname{Ext}^1(E, \mathcal{O}_C(-1))^\vee \otimes \mathcal{O}_C(-1)$, and the exact sequence $0 \to \operatorname{Ker}\phi \to p^*(p_*(E)) \to E \to 0$ obtained from Lemma 1.2(3) is the universal extension of E with respect to $\mathcal{O}_C(-1)$.*

Proof. (1) From Definition 1.1(3) $E \in \operatorname{Per}(\widehat{X}/X)$ satisfies $\operatorname{Hom}(E, \mathcal{O}_C(-1)) = 0$. For the converse, suppose $E \in \operatorname{Coh}(\widehat{X})$ satisfies $\operatorname{Hom}(E, \mathcal{O}_C(-1)) = 0$. By Lemma 1.2(3) it is enough to show that $\phi\colon p^*p_*(E) \to E$ is surjective. By Lemma 1.2(2)(iii) we have $p_*(\operatorname{Coker}\phi) = 0$. Since $\operatorname{Hom}(\operatorname{Coker}\phi, \mathcal{O}_C(-1)) \subset \operatorname{Hom}(E, \mathcal{O}_C(-1)) = 0$ from the assumption, we have $\operatorname{Coker}\phi = 0$ from Lemma 1.7(1).

(2) Consider $\phi\colon p^*p_*(E) \to E$. By Lemma 1.2(2),(3) this is surjective and $\operatorname{Ker}\phi$ satisfies $\mathbf{R}p_*(\operatorname{Ker}\phi) = 0$. By Lemma 1.7(2), $\operatorname{Ker}\phi = \mathcal{O}_C(-1)^{\oplus s}$ for some $s \in \mathbb{Z}_{\geq 0}$. We also have $\operatorname{Ext}^1(E, \mathcal{O}_C(-1)) \cong \operatorname{Hom}(\operatorname{Ker}\phi, \mathcal{O}_C(-1))$ by Lemma 1.2(4). □

Lemma 1.10. *If $E \in \operatorname{Per}(\widehat{X}/X) \cap \operatorname{Coh}(\widehat{X})$, then $R^1p_*(E(C)) = 0$.*

Proof. By the exact sequence $0 \to \mathcal{O}_{\widehat{X}} \to \mathcal{O}_{\widehat{X}}(C) \to \mathcal{O}_C(-1) \to 0$, we have $\mathbf{R}p_*(\mathcal{O}_{\widehat{X}}(C)) = \mathcal{O}_X$. From the projection formula we have

$$\mathbf{R}p_*(\mathcal{O}_{\widehat{X}}(C) \otimes \mathbf{L}p^*(p_*(E))) \cong \mathbf{R}p_*(\mathcal{O}_{\widehat{X}}(C)) \otimes p_*(E) \cong p_*(E).$$

The spectral sequence as in the proof of Lemma 1.2(2) implies $R^1p_*(\mathcal{O}_{\widehat{X}}(C) \otimes p^*(p_*(E))) = 0$. As $p^*p_*(E) \to E$ is surjective by the assumption, we have the conclusion. □

Lemma 1.11. *Let $E \in \operatorname{Coh}(\widehat{X})$. Then $p_*(E)$ does not contain a 0-dimensional subsheaf at 0 if and only if $\operatorname{Hom}(\mathcal{O}_C, E) = 0$.*

Proof. We have

$$\operatorname{Hom}(\mathcal{O}_C, E) \cong \operatorname{Hom}(p^*\mathbb{C}_0, E) \cong \operatorname{Hom}(\mathbb{C}_0, p_*(E)).$$

Now the assertion is clear. □

1.3. Perverse coherent sheaves and representations of a quiver.

This subsection is a detour. We look at the definition of the perverse coherent sheaves in view of [25]. The result of this subsection will not be used later.

Let $X = \mathbb{C}^2$ and let $\widehat{X}$ be the blowup of X at the origin 0. As a byproduct of the main result of [25], we have an equivalence between the derived category $\mathbf{D}^b_c(\operatorname{Coh}\widehat{X})$ of complexes of coherent sheaves whose homologies have proper supports and the derived category of finite dimensional modules of the quiver $\bullet \underset{d}{\overset{B_1, B_2}{\leftleftarrows\!\!\rightarrow}} \bullet$ with relation $B_1 d B_2 = B_2 d B_1$.

Proposition 1.12. *The abelian category $\{E \in \operatorname{Per}(\widehat{X}/X) \mid E$ has a proper support $\}$ is equivalent to the abelian category of finite dimensional representations of the above quiver with relation $B_1 d B_2 = B_2 d B_1$.*

Proof. Let us first recall how we constructed the equivalence between derived categories in [25]. Let us number the left (resp. right) vertex as 0 (resp. 1). We consider line bundles $L_0 := \mathcal{O}$ and $L_1 := \mathcal{O}(C)$, and homomorphisms between them $s = z_1/z = z_2/w\colon L_0 \to L_1$, z, $w\colon L_1 \to L_0$, where $\widehat{X} = \{(z_1, z_2, [z : w]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid z_1 w = z_2 z\}$. For an object $E \in \mathbf{D}^b_c(\operatorname{Coh}\widehat{X})$, we define

$$V_k = \mathbf{R}(\mathrm{pt})_*(E \otimes L_k), \quad (k = 0, 1)$$

where pt is a projection of $\widehat{X}$ to a point. Then the homomorphisms s, z, w give a structure of a quiver representation. Conversely given a complex of quiver representations, we define a double complex of coherent sheaves on $\widehat{X}$ as

$$\mathcal{A} := \begin{matrix} V_0 \otimes L_1 \\ \oplus \\ V_1 \otimes L_0 \end{matrix} \xrightarrow{\alpha} \mathcal{B} := \begin{matrix} \mathbb{C}^2 \otimes V_0 \otimes L_0 \\ \oplus \\ \mathbb{C}^2 \otimes V_1 \otimes L_0 \end{matrix} \xrightarrow{\beta} \mathcal{C} := \begin{matrix} V_0 \otimes L_0 \\ \oplus \\ V_1 \otimes L_1 \end{matrix},$$

with α, β as in [25, (1.2)]. We assign the degree by $\deg \mathcal{A} = -2$, $\deg \mathcal{B} = -1$, $\deg \mathcal{C} = 0$. Then the associated total complex is an object in $\mathbf{D}^b_c(\operatorname{Coh}(\widehat{X}))$. (For the reader familiar with [25]: We consider the $W = 0$ case. So objects have proper supports, and hence implicitly have the framing.)

Let us start the proof of this proposition. Suppose E is a perverse coherent sheaf on $\widehat{X}$ with a proper support. The corresponding representation satisfies

$$H^i(V_0) = H^i(\widehat{X}, E), \qquad H^i(V_1) = H^i(\widehat{X}, E(C)).$$

We have a spectral sequence $H^i(\widehat{X}, H^j(E)) \Longrightarrow H^{i+j}(\widehat{X}, E)$. Therefore $H^i(V_0) = 0$ for $i \neq 0$ follow from the following vanishing results, which are direct consequence of the definition of perverse coherent sheaves:

$$H^0(\widehat{X}, H^{-1}(E)) = H^0(\mathbb{C}^2, R^0 p_*(H^{-1}(E))) = 0,$$
$$H^1(\widehat{X}, H^0(E)) = H^0(\mathbb{C}^2, R^1 p_*(H^0(E))) = 0.$$

Next consider V_1. We consider the exact sequence of vector bundles

$$0 \to \mathcal{O}_{\widehat{X}}(C) \xrightarrow{\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]} \mathcal{O}_{\widehat{X}}^{\oplus 2} \xrightarrow{[-w\ z]} \mathcal{O}_{\widehat{X}}(-C) \to 0.$$

This exact sequence is preserved under $\bullet \otimes H^{-1}(E)$ as $\mathcal{O}_{\widehat{X}}(-C)$ is locally-free. Therefore we get an exact sequence $0 \to p_*(H^{-1}(E(C))) \to p_*(H^{-1}(E))^{\oplus 2}$. But the right hand side vanish by the assumption. This implies $H^{-1}(V_1) = 0$ as above. We have $H^0(E) \in \operatorname{Per}(\widehat{X}/X)$, and hence we have $R^1 p_*(H^0(E(C))) = 0$ from Lemma 1.10. This gives $H^1(V_1) = 0$. Vanishing of other cohomology groups is trivial.

For the converse we check that the object $E = \{\mathcal{A} \to \mathcal{B} \to \mathcal{C}\} \in \mathbf{D}^b_c(\operatorname{Coh} \widehat{X})$ corresponding to a quiver representation satisfies the conditions in Definition 1.1. The condition (1) follows from the injectivity of α as argued in [25, §5.2]. The condition (2) follows from $H^i(V_0) = 0$ for $i \neq 0$ by the above discussion. Let us consider the condition (3). Note that $H^0(E) = \operatorname{Coker}\beta$. We also know that $\mathcal{O}_C(-1)[1]$ corresponds to the representation $C_0 = \ 0 \underset{\longrightarrow}{\overset{\longleftarrow}{\longleftarrow}} \mathbb{C}$ by [25, Prop. 5.3]. Let us write the corresponding complex as $\{\mathcal{A}' \to \mathcal{B}' \to \mathcal{C}'\}$. Then we apply the proof of [25, Prop. 5.12] to show that (i) $\operatorname{Hom}(H^0(E), \mathcal{O}_C(-1))$ is isomorphic to the space of homomorphisms between monads, where the complex $\{\mathcal{A}' \to \mathcal{B}' \to \mathcal{C}'\}$ is shifted by -1, and (ii) this space of homomorphisms between monads vanishes. $\square$

Let us give another proof of Lemma 1.7 based on the above result.

(1) If $p_*(K) = 0$, $E := K[1]$ is a perverse coherent sheaf. Let us consider the corresponding complex as above. Since K is a sheaf, we have $H^0(E) = 0$, which means β is surjective. By [25, Lem. 5.1] this is equivalent to $\operatorname{codim} T_1 > \operatorname{codim} T_0$ or $(T_0, T_1) = (V_0, V_1)$ for any subrepresentation $T = (T_0, T_1)$ of E. Taking the Harder-Narashimhan filtration with respect to the slope $\theta(S_0, S_1) = (-\dim S_0 + \dim S_1)/(\dim S_0 + \dim S_1)$, we get a filtration $K = Y^0 \supset Y^1 \supset \cdots \supset Y^{N-1} \supset Y^N = 0$ such that Y^k/Y^{k+1} is θ-semistable and $\theta(Y^0/Y^1) < \theta(Y^1/Y^2) < \cdots < \theta(Y^{N-1}/Y^N)$. Taking $(T_0, T_1) = Y^1$, $\operatorname{codim} T_1 > \operatorname{codim} T_0$ means $\theta(Y^0/Y^1) > 0$. Therefore all $\theta(Y^k/Y^{k+1}) > 0$. Looking at the classification of all stable representations in [25, Rem. 2.17], we find that Y^k/Y^{k+1} is (a direct sum of) $C_m = \mathcal{O}_C(-m-1)$ for $m \geq 0$.

(2) If we further have $\mathbf{R}p_*(K) = 0$, we have $V_0 = 0$ for the representation corresponding to $E = K[1]$. Then assertion is obvious.

2. Moduli spaces of semistable perverse coherent sheaves

We return to the general situation considered in §1.1, i.e. $p\colon Y \to X$ is a birational morphism of projective varieties such that $\mathbf{R}p_*(\mathcal{O}_Y) = \mathcal{O}_X$ and $\dim p^{-1}(x) \leq 1$ for all $x \in X$.

2.1. **Stability.** Let $\mathcal{O}_X(1)$ be an ample line bundle on X. We also denote the pull-back $p^*(\mathcal{O}_X(1))$ by $\mathcal{O}_Y(1)$.

Let M be a line bundle on Y. For $E^\bullet \in \mathbf{D}^b(Y)$, we define $a_i(E^\bullet, M) \in \mathbb{Z}$ by the coefficient of the Hilbert polynomial of $E^\bullet$ with respect to M:

$$\chi(E^\bullet \otimes M^{\otimes m}) = \sum_{i=0}^{\dim Y} a_i(E^\bullet, M)\binom{m+i}{i}.$$

We use the same notation for $E \in \mathbf{D}^b(X)$ and a line bundle M on X instead of Y.

If $E^\bullet$ is a coherent sheaf of dimension d and M is ample, then $a_i(E^\bullet, M) = 0$, $i > d$ and $a_d(E^\bullet, M) > 0$. For a perverse coherent sheaf $E^\bullet \in \operatorname{Per}(X/Y)$, we denote $a_i(E^\bullet, \mathcal{O}_X(1)) = a_i(\pi_*(E^\bullet), \mathcal{O}_Y(1))$ by $a_i(E^\bullet)$ if there is no fear of confusion.

Recall that we say a coherent sheaf E of dimension d on X is *(semi)stable* if

$$a_d(E)\chi(F(m))(\leq)a_d(F)\chi(E(m)) \quad \text{for } m \gg 0$$

for any proper subsheaf $0 \neq F \subsetneq E$. Here we adapt the convention for the short-hand notation in [8]. The above means two assertions: semistable if we have '$\leq$', and stable if we have '$<$'. If E is semistable, then it is pure of dimension d: if $0 \neq F \subsetneq E$ is of dimension $< d$, then the right hand side is 0, while the left hand side is positive. Under the assumption that E is pure, the above inequality is equivalent to $\chi(F(m))/a_d(F)(\leq)\chi(E(m))/a_d(E)$, as we automatically has $a_d(F) > 0$.

We say E is μ-*(semi)stable* if it is purely d-dimensional and

$$\frac{a_{d-1}(F)}{a_d(F)}(\leq)\frac{a_{d-1}(E)}{a_d(E)}$$

for any subsheaf $0 \neq F \subset E$ with $a_d(F) < a_d(E)$.

If E is a d-dimensional coherent sheaf on X, then we have the following implications:

$$E \text{ is } \mu\text{-stable} \Longrightarrow E \text{ is stable} \Longrightarrow E \text{ is semistable} \Longrightarrow E \text{ is } \mu\text{-semistable}.$$

Now we return to the situation $p\colon Y \to X$. Let L_0 an ample line bundle on Y. We set $L_l = L_0(l)$.

We consider the following conditions on an object $E^\bullet \in \operatorname{Per}(Y/X)$:

$$\dim p_*(E^\bullet) > \dim Z, \tag{2.1a}$$

$$\chi(E^\bullet(m) \otimes L_l^{\otimes n}) > 0 \quad \text{for } n \gg 0. \tag{2.1b}$$

Definition 2.2. Let $E^\bullet \in \operatorname{Per}(Y/X)$ be an object satisfying (2.1). Then $E^\bullet$ is *(semi)stable* if for any proper subobject $F^\bullet \in \operatorname{Per}(Y/X)$ of $E^\bullet$, we have

$$\chi(F^\bullet(m))(\leq)\frac{\chi(F^\bullet(m) \otimes L_l^{\otimes n})}{\chi(E^\bullet(m) \otimes L_l^{\otimes n})}\chi(E^\bullet(m)) \tag{2.3}$$

for all $n \gg l \gg m \gg 0$.

Remark 2.4. The above definition is suitable for perverse coherent sheaves satisfying (2.1). On the other hand, it is also natural to expect that $\mathcal{O}_C(-m-1)[1]$ (in case of $p\colon \widehat{X} \to X$) is stable in some definition in view of [25].

Note that for $E^\bullet \in \operatorname{Per}(Y/X)$ satisfying (2.1) with $d := \dim p_*(E^\bullet)$, E is (semi)stable if and only if $H^{-1}(E^\bullet) = 0$ and for any subsheaf F of $E := H^0(E^\bullet)$ in $\operatorname{Coh}(Y)$ with $F \in \operatorname{Per}(Y/X)$,

$$\chi(F(m)) < \frac{a_d(F, \mathcal{O}_Y(1))}{a_d(E, \mathcal{O}_Y(1))}\chi(E(m)) \tag{1}$$

or

$$\begin{cases} \chi(F(m)) = \dfrac{a_d(F, \mathcal{O}_Y(1))}{a_d(E, \mathcal{O}_Y(1))}\chi(E(m)), \\ a_d(F, \mathcal{O}_Y(1)) < \dfrac{a_d(F, L_l)}{a_d(E, L_l)} a_d(E, \mathcal{O}_Y(1)) \end{cases} \tag{2}$$

or

(3)
$$\begin{cases} \chi(F(m)) = \dfrac{a_d(F, \mathcal{O}_Y(1))}{a_d(E, \mathcal{O}_Y(1))}\chi(E(m)), \\ a_d(F, \mathcal{O}_Y(1)) = \dfrac{a_d(F, L_l)}{a_d(E, L_l)} a_d(E, \mathcal{O}_Y(1)), \\ \chi(F(m)\otimes L_l^{\otimes n}) (\le) \dfrac{a_d(F, \mathcal{O}_Y(1))}{a_d(E, \mathcal{O}_Y(1))}\chi(E(m)\otimes L_l^{\otimes n}) \quad \text{for } n \gg l \gg m \gg 0. \end{cases}$$

Remark 2.5. If $d = \dim Y$, then a_d is essentially the rank, so we have

$$a_d(F, \mathcal{O}_Y(1)) = \frac{a_d(F, L_l)}{a_d(E, L_l)} a_d(E, \mathcal{O}_Y(1)).$$

Therefore the second case (2) does not occur.

Lemma 2.6. *Let $E^\bullet \in \operatorname{Per}(Y/X)$ be an object satisfying* (2.1). *Then*

(1) *If $E^\bullet$ is semistable, then $E^\bullet \in \operatorname{Coh}(Y)$.*

(2) *If $E^\bullet$ is semistable, then $p_*(E^\bullet)$ is semistable.*

(3) *Suppose further $E^\bullet = E \in \operatorname{Coh}(Y)$. Then E is* (*semi*)*stable if and only if* (2.3) *holds for any proper subsheaf $F^\bullet = F$ of E in* $\operatorname{Coh}(Y)$ *which is also in* $\operatorname{Per}(Y/X)$.

(4) *Suppose further $E^\bullet = E \in \operatorname{Coh}(Y)$ and $p_*(E)$ is stable. Then E is stable.*

Proof. (1) First note that we have $\chi(E^\bullet(m)\otimes L_l^{\otimes n}) > 0$ for $n \gg 0$ by (2.1b). On the other hand, as $0 \neq \mathbf{R}p_*(E^\bullet) \in \operatorname{Coh}(X)$ from (2.1a) and the perversity, we have $\chi(E^\bullet(m)) > 0$ for $m \gg 0$.

Next note that $E^\bullet$ contains a subobject $H^{-1}(E^\bullet)[1]$. Assume that $H^{-1}(E^\bullet)[1] \neq 0$. Then $\chi(H^{-1}(E^\bullet)[1](m)) = \chi(R^1 p_*(H^{-1}(E^\bullet))(m)) \geq 0$ and $\chi(H^{-1}(E^\bullet)[1](m) \otimes L_l^{\otimes n}) = -\chi(H^{-1}(E^\bullet)(m) \otimes L_l^{\otimes n}) < 0$, which means that $E^\bullet$ is not semistable.

(2) Suppose F_1 is a subsheaf of $p_*(E)$. Note that we can apply Lemma 1.6 to $F_1 \subset p_*(E)$ thanks to (1). Therefore $p_*(p^*(F_1)) = F_1$. Let α be the composition of $p^*(F_1) \to p^*p_*(E) \to E$. We have $p^*(F_1)$, $E \in \operatorname{Per}(Y/X)$ by Lemma 1.2(1) and the assumption respectively. We have $H^{-1}(p^*(F_1)) = H^{-1}(E) = 0$. We also have that $F_1 = p_*(p^*(F_1)) \to p_*(E)$ is injective by the assumption. Therefore the homomorphism α is injective in the category $\operatorname{Per}(Y/X)$ by Lemma 1.5(2). Therefore we have the inequality (2.3) for $F^\bullet := p^*(F_1)$. It means that

$$\chi(F_1(m)) \leq \frac{\chi(p^*(F_1)(m) \otimes L_l^{\otimes n})}{\chi(E^\bullet(m) \otimes L_l^{\otimes n})} \chi(p_*(E)(m)).$$

From the note after Definition 2.2, we must have

$$\chi(F_1(m)) \leq \frac{a_d(p^*(F_1), \mathcal{O}_Y(1))}{a_d(E^\bullet, \mathcal{O}_Y(1))} \chi(p_*(E)(m)).$$

Since $a_d(E^\bullet, \mathcal{O}_Y(1)) = a_d(p_*(E^\bullet), \mathcal{O}_X(1))$, $a_d(p^*(F_1), \mathcal{O}_Y(1)) = a_d(F_1, \mathcal{O}_X(1))$, this inequality says $p_*(E^\bullet)$ is semistable.

(3) The 'only if' part is clear: if $F \subset E$ is a subsheaf, then E/F is also in $\operatorname{Per}(Y/X)$ obviously from the definition. Therefore $0 \to F \to E \to E/F \to 0$ is also exact in $\operatorname{Per}(Y/X)$, hence F is a subobject of E in $\operatorname{Per}(Y/X)$.

Let us show the 'if' part for the semistability. Suppose $F^\bullet$ is a subobject of E, and let $E/F^\bullet$ be the quotient in $\operatorname{Per}(Y/X)$. Then we have an exact sequence in $\operatorname{Coh}(Y)$:

$$\begin{tikzcd} 0 \arrow[r] & H^{-1}(F^\bullet) \arrow[r] & H^{-1}(E) = 0 \arrow[r] & H^{-1}(E/F^\bullet) \arrow[dll] & \\ & H^0(F^\bullet) \arrow[r] & E \arrow[r] & H^0(E/F^\bullet) \arrow[r] & 0, \end{tikzcd}$$

Therefore we have $H^{-1}(F^\bullet) = 0$ and an exact sequence in $\operatorname{Coh}(Y)$:

$$0 \to H^{-1}(E/F^\bullet) \to H^0(F^\bullet) \to F' \to 0,$$

where $F' = \operatorname{Im}(H^0(F^\bullet) \to E)$. Note that $F' \in \operatorname{Per}(Y/X)$. Take the direct image with respect to p to get

$$0 \to p_*(H^0(F^\bullet)) \to p_*(F') \to R^1 p_*(H^{-1}(E/F^\bullet)) \to 0.$$

Therefore we have

$$\begin{aligned}\chi(F^\bullet(m)\otimes L_l^{\otimes n}) &= \chi(F'(m)\otimes L_l^{\otimes n}) + \chi(H^{-1}(E/F^\bullet)(m)\otimes L_l^{\otimes n}))\\ &\ge \chi(F'(m)\otimes L_l^{\otimes n}),\\ \chi(p_*(F')(m)) &= \chi(p_*(H^0(F^\bullet))(m)) + \chi(R^1p_*(H^{-1}(E/F^\bullet))(m))\\ &\ge \chi(F^\bullet(m)).\end{aligned} \tag{2.7}$$

So the inequality (2.3) for F' implies one for $F^\bullet$.

Finally show the 'if' part for the stability. Assume that the strict inequality in (2.3) holds for a proper subsheaf F. Suppose that the equality holds for a subobject $F^\bullet \subset E$ in $\operatorname{Per}(Y/X)$. From the above discussion, $F' = E$ follows from the assumption. Therefore $H^0(E/F^\bullet) = 0$. Moreover the inequalities in (2.7) must be the equalities, so we must have $H^{-1}(E/F^\bullet) = 0$. Therefore $E = F^\bullet$.

(4) To test the stability of E, it is enough to check the inequality for a subsheaf $F \subset E$ such that $F \in \operatorname{Per}(Y/X)$ by (3). We have $p_*(F) \subset p_*(E)$. We may assume $p_*(F) \neq 0$ by Lemma 1.5(1). If $p_*(F) \neq p_*(E)$, then the stability of $p_*(E)$ implies the strict inequality for (2.3). Here we have used (2.1a) so that the leading coefficient of $\chi(E(m)\otimes L_l^{\otimes n})$ is $a_d(E, \mathcal{O}_Y) = a_d(p_*(E), \mathcal{O}_X)$.

Therefore we may assume $p_*(F) = p_*(E)$. Let C be the cokernel of $F \to E$ in $\operatorname{Coh}(Y)$. Then C is perverse coherent and $\mathbf{R}p_*(C) = 0$. Therefore $C = 0$ by Lemma 1.5(1). Hence $F = E$. □

If $E^\bullet = E \in \operatorname{Per}(Y/X) \cap \operatorname{Coh}(Y)$, we have the following implications:

$p_*(E)$ is stable $\Longrightarrow$ E is stable $\Longrightarrow$ E is semistable $\Longrightarrow$ $p_*(E)$ is semistable.

Lemma 2.8. *Suppose that Y is a nonsingular surface,* $\dim p_*(E^\bullet) = 2$, *and* $E^\bullet = E \in \operatorname{Coh}(Y) \cap \operatorname{Per}(Y/X)$ *satisfies the condition* (2.1). *Then E is semistable if and only if the followings hold*:

a) $p_*(E^\bullet)$ *is semistable,*

b)

$$(c_1(F), c_1(L_0)) \ge \frac{\operatorname{rk} F}{\operatorname{rk} E}(c_1(E), c_1(L_0))$$

for any subsheaf $F \subset E$ *such that* $F \in \operatorname{Per}(X/Y)$ *and* $\chi(F(m)) = \operatorname{rk} F\chi(E^\bullet(m))/\operatorname{rk} E^\bullet$.

Moreover, E is stable if and only if the followings hold:

c) $p_*(E)$ *is semistable,*

d) $p_*(E)$ *is stable, or the strict inequality in above* b) *holds.*

This is a consequence of the note after Definition 2.2 and Lemma 2.6.

2.2. **Construction of moduli spaces.** Thanks to the discussion in the previous subsection, we can work entirely in the category of coherent sheaves.

For a perverse coherent sheaf E, let $h(x, y)$ be the polynomial such that $\chi(E(m) \otimes L_0^{\otimes n}) = h(m, n)$. Then $\chi(E(m) \otimes L_l^{\otimes n}) = \chi(E(m + ln) \otimes L_0^{\otimes n}) = h(m + ln, n)$. We call h the Hilbert polynomial of the perverse coherent sheaf E.

The following is the main result in this subsection.

Theorem 2.9. *Let $\mathcal{X} \to S$ be a flat family of projective schemes and $p\colon \mathcal{Y} \to \mathcal{X}$ a family of birational maps over S such that $\dim p^{-1}(x) \leq 1$ for all $x \in \mathcal{X}$ and $\mathbf{R}p_*(\mathcal{O}_{\mathcal{Y}}) = \mathcal{O}_{\mathcal{X}}$. Let $\mathcal{O}_{\mathcal{X}}(1)$ be a relatively ample line bundle on $\mathcal{X}/S$ and $\mathcal{L}_0$ a relatively ample line bundle on $\mathcal{Y}/S$. Then there is a coarse moduli scheme $\overline{M}^p_{\mathcal{Y}/\mathcal{X}/S}(h)$ parametrizing S-equivalence classes of semistable perverse coherent sheaves E on $\mathcal{X}_s$, $s \in S$ with the Hilbert polynomial h. Moreover, $\overline{M}^p_{\mathcal{Y}/\mathcal{X}/S}(h)$ is a projective scheme over S. There is an open subscheme $M^p_{\mathcal{Y}/\mathcal{X}/S}(h) \subset \overline{M}^p_{\mathcal{Y}/\mathcal{X}/S}(h)$ parametrizing isomorphism classes of stable perverse coherent sheaves.*

For simplicity, we treat the absolute case $p\colon Y \to X$.

Our construction of the moduli space of semistable perverse coherent sheaves is a modification of that of usual moduli spaces by Simpson [26] (see also [8, 15]). This idea was already appeared in [2]. However we modify the arguments in many places, so we need to recall almost all steps of the usual proof.

Definition 2.10. Let λ be a nonnegative rational number.

(1) Let E be a coherent sheaf of dimension d on X. Then E is *of type* λ (with respect to the semi-stability), if the following two conditions hold:

a) E is of pure dimension d,
b) For all subsheaf F of E we have

$$a_{d-1}(F) \leq \frac{a_d(F)}{a_d(E)} a_{d-1}(E) + \lambda.$$

Note that this is equivalent to the μ-stability if $\lambda = 0$.

(2) For a perverse coherent sheaf E on Y with $E \in \mathrm{Coh}(Y)$, E is *of type* λ, if $p_*(E)$ is of type λ.

Since the set of type λ coherent sheaves on X is bounded (see e.g., [8, 3.3.7]) and $p^*(p_*(E)) \to E$ is surjective for a perverse coherent sheaf of type λ, we get the following.

Lemma 2.11. *The set of type λ perverse coherent sheaves on X with a fixed Hilbert polynomial is bounded.*

From Langer's important result [11, Cor. 3.4] (see also [8, 3.3.1]), we have the following estimate for the dimension of sections for F on X of type λ:

$$\frac{h^0(F)}{a_d(F)} \leq \frac{1}{d!}\left[\frac{a_{d-1}(F)}{a_d(F)} + \lambda + c\right]_+^d, \tag{2.12}$$

where c depends only on $(X, \mathcal{O}_X(1))$, d, $a_d(F)$ and $[x]_+ := \max\{x, 0\}$.

Definition 2.13. Let $\mathcal{U} \equiv \mathcal{U}(\lambda, h)$ be the set of pairs $(E' \subset E)$ such that E is a perverse coherent sheaf of type λ with the Hilbert polynomial h, $E' \in \mathrm{Per}(X/Y)$, and $E'' := E/E'$ satisfies

$$\chi(E(m))\frac{a_d(E'')}{a_d(E)} \geq \chi(E''(m)) \quad \text{for } m \gg 0. \tag{2.14}$$

The inequality means $p_*(E')$ *destabilizes* $p_*(E)$ in a weak sense (i.e. '=' is allowed).

Since the set of E is bounded, by Grothendieck's boundedness theorem, the set $\mathcal{U}$ of such pairs $(E' \subset E)$ is also bounded. Hence there is an integer $m(\lambda)$ which depends on h and λ such that if $m \geq m(\lambda)$ and $(E' \subset E) \in \mathcal{U}$,

$$H^0(E'(m)) \otimes \mathcal{O}_Y \to E'(m) \text{ is surjective and} \tag{2.15a}$$

$$H^i(E'(m)) = 0 \quad \text{for } i > 0, \tag{2.15b}$$

and for $F \in \mathrm{Coh}(X)$ of type λ

$$H^0(F(m)) \otimes \mathcal{O}_X \to F(m) \text{ is surjective and} \tag{2.16a}$$

$$H^i(F(m)) = 0 \quad \text{for } i > 0. \tag{2.16b}$$

In particular, we apply (2.15) to $(E = E)$ to have that the above two conditions hold for E.

Furthermore, since the set of Hilbert polynomials of E' is finite, we may assume that $m(\lambda)$ satisfies also the followings: for all $m \geq m(\lambda)$, we can choose sufficiently large integers $l(m)$ and $n(m) \gg l(m)$ such that

(2.17a)
$$\frac{\chi(E(m))}{\chi(E(m) \otimes L_{l(m)}^{\otimes n(m)})} \geq \frac{\chi(E'(m))}{\chi(E'(m) \otimes L_{l(m)}^{\otimes n(m)})}$$
$$\iff \frac{\chi(E(m))}{\chi(E(m) \otimes L_l^{\otimes n})} \geq \frac{\chi(E'(m))}{\chi(E'(m) \otimes L_l^{\otimes n})} \quad \text{for all } n \gg l \gg m,$$

$$\chi(E'(m) \otimes L_{l(m)}^{\otimes n(m)}) = h^0(E'(m) \otimes L_{l(m)}^{\otimes n(m)}) \tag{2.17b}$$

hold for $(E' \subset E) \in \mathcal{U}$.

For $m \geq m(0)$ let V_m be a vector space of dimension $h(m,0)$. Let $\mathfrak{Q} := \mathrm{Quot}^{h[m]}_{V_m \otimes \mathcal{O}_Y/Y}$ be the quot-scheme parametrizing all quotients $V_m \otimes \mathcal{O}_Y \twoheadrightarrow F$ (in $\mathrm{Coh}(X)$) with the Hilbert polynomial $h[m]$, where $h[m](x,y) = h(m+x,y)$. Let $V_m \otimes \mathcal{O}_{\mathfrak{Q}\times Y} \twoheadrightarrow \widetilde{E}(m)$ be the universal quotient sheaf on $\mathfrak{Q}\times Y$. Let $\mathfrak{Q}^{ss}$ be the open subscheme of $\mathfrak{Q}$ consisting of quotients $f\colon V_m \otimes \mathcal{O}_Y \to E(m)$ such that

(1) the canonical map $V_m \to H^0(E(m))$ is an isomorphism and
(2) E is a semi-stable sheaf.

Note that $E(m)$ is automatically in $\mathrm{Per}(Y/X)$. Other conditions clearly define the *open* subscheme.

By the above discussion all E appearing as $(E' \subset E) \in \mathcal{U}$ together with a choice of basis of $H^0(E(m))$ gives a closed point in $\mathcal{Q}$ if $m \geq m(\lambda)$. In particular, we can construct the moduli scheme as a quotient of $\mathcal{Q}^{ss}$.

In order to take the quotient via the GIT, we use a Grassmann embedding of $\mathcal{Q}$ as follows. Let $n \gg l \gg m$. Set $W := H^0(L_l^{\otimes n})$. Let $\mathfrak{G}(l,n) := \mathrm{Gr}(V_m \otimes W, h(m+ln,n))$ be the Grassmannian parametrizing $h(m+ln,n)$-dimensional quotient spaces of $V_m \otimes W$. For a quotient $(V_m \otimes \mathcal{O}_Y \to E(m)) \in \mathfrak{Q}$ its kernel F satisfies

(1) $H^0(F \otimes L_l^{\otimes n}) \otimes \mathcal{O}_Y \to F \otimes L_l^{\otimes n}$ is surjective and
(2) $H^i(F \otimes L_l^{\otimes n}) = 0$, $i > 0$

for sufficiently large n. Hence we get a quotient vector space $V_m \otimes W \to H^0(E(m) \otimes L_l^{\otimes n})$. Thus we get a morphism $\mathfrak{Q} \to \mathfrak{G}(l,n)$, which is a closed immersion. This embedding depends on the choice of $n \gg l \gg m$. We have a natural action of $SL(V_m)$ on $\mathfrak{G}(l,n)$. Let $\mathcal{L} := \mathcal{O}_{\mathfrak{G}(l,n)}(1)$ be the tautological line bundle on $\mathfrak{G}(l,n)$. Then $\mathcal{L}$ has an $SL(V_m)$-linearization. We consider the GIT semi-stability with respect to $\mathcal{L}$. The following is well-known (cf. [8, 4.4.5])

Proposition 2.18. *Let $\alpha\colon V_m \otimes W \twoheadrightarrow A$ be a quotient corresponding to a point of $\mathfrak{G}(l,n)$. Then it is GIT (semi)stable with respect to $\mathcal{L}$ if and only if*

$$\frac{\dim[\alpha(V' \otimes W)]}{\dim V'} (\geq) \frac{\dim[\alpha(V_m \otimes W)]}{\dim V_m}$$

for all non-zero proper subspaces V' of V_m.

We prepare several estimates in order to compare the semistability of E and that of the corresponding point in the Grassmann variety.

Lemma 2.19. *Let E be a d-dimensional sheaf with the Hilbert polynomial h and E' a subsheaf of E. Then if we take a sufficiently large*

$l \gg m$ depending on h, m, we have

$$\left|\frac{a_d(E', L_l)}{a_d(E, L_l)} - \frac{a_d(E')}{a_d(E)}\right| \le \frac{1}{3h(m,0)a_d(E)!}.$$

In particular, we may suppose that $l(m)$ in the above (2.17) satisfies this condition.

Proof. We have $a_d(E) \ge a_d(E') \ge 0$ and $a_d(E/E', L_l) \ge l^d a_d(E/E')$. Since $a_d(E/E', L_l) = a_d(E, L_l) - a_d(E', L_l)$ and $a_d(E/E') = a_d(E) - a_d(E')$, we have $a_d(E, L_l) - l^d a_d(E) \ge a_d(E', L_l) - l^d a_d(E') \ge 0$. Hence we have

$$\begin{aligned}\left|\frac{a_d(E', L_l)}{a_d(E, L_l)} - \frac{a_d(E')}{a_d(E)}\right| &\le \left|\frac{a_d(E', L_l) - l^d a_d(E')}{a_d(E, L_l)}\right| + \left|\frac{l^d a_d(E')}{a_d(E, L_l)} - \frac{a_d(E')}{a_d(E)}\right| \\ &\le 2\left|1 - \frac{l^d a_d(E)}{a_d(E, L_l)}\right|.\end{aligned}$$

If we take a sufficiently large l depending on h, this can be made smaller than an arbitrary given number. □

We consider a set of pairs

$$\mathcal{F} := \{(V_m \otimes \mathcal{O}_Y \twoheadrightarrow E(m)) \in \mathfrak{Q}\} \times \{V' \subset V_m\}.$$

Let $\alpha\colon V_m \otimes W \twoheadrightarrow H^0(E(m) \otimes L_l^{\otimes n})$ be the corresponding point in $\mathfrak{G}(l, n)$. We set $E'(m) := \operatorname{Im}(V' \otimes \mathcal{O}_Y \to E(m))$. Since $\mathcal{F}$ is a bounded set, $E'(m)$ satisfies

$$\alpha(V' \otimes W) = H^0(E'(m) \otimes L_l^{\otimes n}) \tag{2.20a}$$

$$H^i(E'(m) \otimes L_l^{\otimes n}) = 0 \quad \text{for } i > 0 \tag{2.20b}$$

for a sufficiently large n which depends on m and l. Then we have $\dim[\alpha(V' \otimes W)] = \chi(E'(m) \otimes L_l^{\otimes n})$.

Lemma 2.21. (1) $\dim V' \le h^0(E'(m))$.

(2) *Take $l \gg m$ as in Lemma 1.7. If we take n sufficiently large depending on h, m, l, we have*

$$\left|\frac{\dim[\alpha(V' \otimes W)]}{\dim[\alpha(V_m \otimes W)]} - \frac{a_d(E')}{a_d(E)}\right| < \frac{1}{2\dim V_m\, a_d(E)!}.$$

Proof. (1) We have a natural homomorphism $V' \to H^0(E'(m))$. If we compose an injective homomorphism $H^0(E'(m)) \to H^0(E(m)) = V_m$, it becomes equal to the given inclusion $V' \subset V_m$, so it is injective. The assertion follows.

(2) We have

$$\left|\frac{\chi(E'(m) \otimes L_l^{\otimes n})}{\chi(E(m) \otimes L_l^{\otimes n})} - \frac{a_d(E')}{a_d(E)}\right| < \frac{1}{2\dim V_m\, a_d(E)!}$$

for this sufficiently large n by Lemma 2.19. Thus the assertion follows from the above conditions (1),(2). $\square$

We replace $n(m) \gg l(m) \gg m$ in (2.17) if necessary so that they also satisfy the assertion in this lemma.

Proposition 2.22. *There is an integer $m_1(\geq m(0))$ such that for all $m \geq m_1$, $\mathfrak{Q}^{ss}$ is contained in $\mathfrak{G}(l,n)^{ss}$, where $l = l(m), n = n(m)$.*

Proof. We first take $m \geq m(0)$.

Suppose $E \in \mathfrak{Q}^{ss}$, i.e. E is semistable, and take $V' \subset V_m$. From Lemma 2.21(1),(2) we have

(2.23)
$$\begin{aligned}
&\dim V_m \dim[\alpha(V' \otimes W)] - \dim V' \dim[\alpha(V_m \otimes W)] \\
&\geq h^0(E(m)) \dim[\alpha(V' \otimes W)] - h^0(E'(m)) \dim[\alpha(V_m \otimes W)] \\
&\geq \left(h^0(E(m)) \frac{a_d(E')}{a_d(E)} - h^0(E'(m)) - \frac{1}{2a_d(E)!} \right) \dim[\alpha(V_m \otimes W)].
\end{aligned}$$

Since $p_*(E)$ is semistable, in the same way as in [8, 4.4.1], we see that there is an integer m_3 which depends on h such that for $m \geq m_3$ and a subsheaf E' of E,

$$\frac{h^0(E'(m))}{a_d(E')} \leq \frac{h^0(E(m))}{a_d(E)} \tag{2.24}$$

and the equality holds, if and only if

$$\frac{\chi(E'(m+m'))}{a_d(E')} = \frac{\chi(E(m+m'))}{a_d(E)} \tag{2.25}$$

for all m'. More precisely we apply the argument in [8, 4.4.1] to $p_*(E') \subset p_*(E)$.

We take $m_1 := \max\{m_3, m(0)\}$ so that both (2.23,2.24) hold for $m \geq m_1$.

If the inequality in (2.24) is strict, the last expression of (2.23) is positive. Therefore α is stable.

So we may assume the equality in (2.24) holds. Then $p_*(E')$ is also semistable by (2.25), and we may assume (2.15a,b) holds for E'. So we have $h^0(E'(m)) = \chi(E'(m))$. Therefore the middle expression of (2.23) is equal to

$$\chi(E(m))\chi(E'(m) \otimes L_l^{\otimes n}) - \chi(E'(m))\chi(E(m) \otimes L_l^{\otimes n}).$$

This is nonnegative by the semistability of E. Therefore our claim holds. $\square$

Proposition 2.26. *There is an integer m_2 such that for all $m \geq m_2$, $\mathfrak{Q}^{ss}$ is a closed subscheme of $\mathfrak{G}(l,n)^{ss}$, where $n = n(m)$, $l = l(m)$.*

We choose an $m(\geq m_1)$ so that $h(m)/a_d(h) > 1$. We shall prove that $\mathfrak{Q}^{ss} \to \mathfrak{G}(l,n)^{ss}$ is proper. Let $(R, \mathfrak{m})$ be a discrete valuation ring and its maximal ideal, and K the quotient field of R. We set $T := \operatorname{Spec}(R)$ and $U := \operatorname{Spec}(K)$. Let $U \to \mathfrak{Q}^{ss}$ be a morphism such that $U \to \mathfrak{Q}^{ss} \to \mathfrak{G}(l,n)^{ss}$ is extended to a morphism $T \to \mathfrak{G}(l,n)^{ss}$. Since $\mathfrak{Q}$ is a closed subscheme of $\mathfrak{G}(l,n)$, there is a morphism $T \to \mathfrak{Q}$, *i.e.,* there is a flat family of quotients:

$$V_m \otimes \mathcal{O}_Y \otimes \mathcal{O}_T \to \mathcal{E}(m) \to 0.$$

Let $\alpha\colon V_m \otimes W \otimes R \to p_{T*}(\mathcal{E}(m) \otimes L^{\otimes n})$ be the quotient of $V_m \otimes W \otimes R$ corresponding to the morphism $T \to \mathfrak{G}(l,n)^{ss}$. We set $E := \mathcal{E} \otimes R/\mathfrak{m}$.

Claim 1. $V_m \to H^0(E(m))$ is injective.

Proof. We set $V' := \operatorname{Ker}(V_m \to H^0(E(m)))$. Then $\alpha(V' \otimes W) = 0$. Hence we get

$$\begin{aligned} 0 &\leq \dim V_m \dim[\alpha(V' \otimes W)] - \dim V' \dim[\alpha(V_m \otimes W)] \\ &= -\dim V' \dim[\alpha(V_m \otimes W)] \leq 0. \end{aligned}$$

Therefore $V' = 0$. $\square$

Claim 2. There is a rational number λ which depends on h such that E is of type λ.

Proof. By [26, Lem. 1.17] (see also [8, 4.4.2]) there is a purely d-dimensional sheaf F with the Hilbert polynomial $h(x,0)$ and a homomorphism $p_*(E) \to F$ whose kernel is a coherent sheaf of dimension less than d. Note that the assumption in [26, Lem. 1.17] that $p_*(E)$ can be deformed to a pure sheaf is satisfied by our definition of E. We shall first check that F is of type λ. We need to check the inequality in Definition 2.10(1b) for the maximal destabilizing subsheaf of F. Let $F \to F''$ be the corresponding quotient, which is semistable. We set $E' := \operatorname{Ker}(p_*(E) \to F'')$ and $E'' := \operatorname{Im}(p_*(E) \to F'')$. Since F'' is semistable, (2.12) gives

$$\frac{1}{d!}\left[m + \frac{a_{d-1}(F'')}{a_d(F'')} + c\right]_+^d \geq \frac{h^0(F''(m)))}{a_d(F'')} \geq \frac{h^0(E''(m))}{a_d(E'')}, \tag{2.27}$$

where we have used $h^0(E''(m)) \leq h^0(F''(m))$ and $a_d(E'') = a_d(F'')$ in the second inequality.

We note that $V_m \to H^0(p_*(E)(m))$ is injective by Claim 1. We set $V' := V_m \cap H^0(E'(m))$. Then

$$h^0(E''(m)) \geq \dim V_m - \dim V'. \tag{2.28}$$

Let $E_1(m)$ be the image of $V' \otimes \mathcal{O}_Y \to E(m)$. Then E_1 comes from $(E, V') \in \mathcal{F}$. (We have denoted the corresponding sheaf by E' above, but we change the notation as it is already used for a different sheaf.) Since E'' is purely d-dimensional and $(p_*(E_1) + E')/E'$ is supported on Z, we have $p_*(E_1) \subset E'$. Therefore

$$a_d(E_1) \leq a_d(E') = a_d(E) - a_d(E''). \tag{2.29}$$

We write $\varepsilon := 1/2h(m,0)\, a_d(E)!$ the constant appearing in Lemma 2.21(2) for brevity. Then

$$\begin{aligned}
\frac{h^0(E''(m))}{a_d(E'')} &\geq \frac{\dim V_m - \dim V'}{a_d(E'')} && \text{(by (2.28))}\\
&\geq \frac{\dim V_m}{a_d(E'')}\left(1 - \frac{\dim[\alpha(V' \otimes W)]}{\dim[\alpha(V_m \otimes W)]}\right) && \text{(by the semistability of } \alpha)\\
&\geq \frac{\dim V_m}{a_d(E'')}\left(1 - \frac{a_d(E_1)}{a_d(E)} - \varepsilon\right) && \text{(by Lemma 2.21(2))}\\
&\geq \frac{\dim V_m}{a_d(E'')}\left(\frac{a_d(E'')}{a_d(E)} - \varepsilon\right) && \text{(by (2.29))}\\
&\geq \dim V_m \left(\frac{1}{a_d(E)} - \varepsilon\right) && \text{(as } a_d(E'') \geq 1).
\end{aligned}$$

There is a rational number λ_1 and an integer $m_4 \geq \lambda_1 - a_{d-1}(E)/a_d(E)$ which depend on $h(x,0)$ such that

$$\dim V_m \left(\frac{1}{a_d(E)} - \varepsilon\right) = \frac{\dim V_m}{a_d(E)} - \frac{1}{2a_d(E)!} \geq \frac{1}{d!}\left(m + \frac{a_{d-1}(E)}{a_d(E)} - \lambda_1\right)^d$$

for $m \geq m_4$. Combining this with the above inequality and (2.27), we get

$$\frac{1}{d!}\left(m + \frac{a_{d-1}(F'')}{a_d(F'')} + c\right) \geq 0$$

and

$$\frac{a_{d-1}(E)}{a_d(E)} - \lambda_1 \leq \frac{a_{d-1}(F'')}{a_d(F'')} + c \tag{2.30}$$

for $m \geq m_4$. Hence F is of type $\lambda := (\lambda_1 + c)a_d(E)$.

We set $m_2 := \max\{m_4, m(\lambda)\}$ and take $m \geq m_2$. We consider $V_m = H^0(p_*(E)(m)) \to H^0(F(m))$ and let V' be the kernel. Then

$J := \mathrm{Im}(V' \otimes \mathcal{O}_Y \to E(m))$, restricted to $Y \setminus p^{-1}(Z)$, is of dimension less than d. Hence we get $a_d(J) = 0$. By Lemma 2.21(2) (applied to $E' := J$) and Proposition 2.18, we have $V' = 0$. Thus $H^0(\pi_*(E)(m)) \to H^0(F(m))$ is injective. But both have dimension equal to $h(m, 0)$, and hence they are isomorphic. Since $H^0(F(m)) \otimes \mathcal{O}_X \to F(m)$ is surjective, $p_*(E) \to F$ must be surjective. As they have the same Hilbert polynomials, they are isomorphic. Therefore $p_*(E)$ is of pure dimension d, of type λ and $V_m \to H^0(E(m))$ is an isomorphism. Thus we complete the proof of Claim 2. □

Proof of Proposition 2.26. Finally we need to show that E is semistable. Then it gives the lifting $T \to \mathfrak{Q}^{ss}$ and finish the proof that $\mathfrak{Q}^{ss} \to \mathfrak{G}(l, n)^{ss}$ is proper.

Assume that there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

such that $E_1 \in \mathrm{Per}(X/Y)$ and E_1 destabilizes the semistability. Then (2.14) is satisfied, so $(E_1 \subset E) \in \mathcal{U}$, so E_1 satisfies (2.15). Since $\alpha \in \mathfrak{G}(l, n)$ corresponding to E is semistable, we have the inequality in Proposition 2.18 for $V' := H^0(E_1(m)) \subset H^0(E) = V_m$. But by (2.20) the inequality is equivalent to

$$\frac{\chi(E_1(m))}{\chi(E_1(m) \otimes L_{l(m)}^{\otimes n(m)})} \leq \frac{\chi(E(m))}{\chi(E(m) \otimes L_{l(m)}^{\otimes n(m)})},$$

which means that E_1 is not a destabilizing subsheaf. Therefore E is semistable. □

By standard arguments, we see that $SL(V_m)s$, $s \in \mathfrak{Q}^{ss}$ is a closed orbit if and only if the corresponding semistable perverse coherent sheaf E is isomorphic to $\bigoplus_i E_i$, where E_i are stable perverse coherent sheaves. This completes the proof of Theorem 2.9.

3. Wall-crossing

Hereafter we only consider the case when $p\colon Y = \widehat{X} \to X$ is the blow-up of a point 0 in a nonsingular projective surface X. Let $\mathcal{O}_X(1)$ be an ample line bundle on X and let $\widehat{M}^m(c)$ be the moduli space of objects E such that $E(-mC)$ is stable perverse coherent with Chern character $c \in H^*(\widehat{X})$. We say E is *m-stable* if this stability condition is satisfied. When $m = 0$ this was denoted by $M^p_{Y/X/\mathbb{C}}(c)$ in Theorem 2.9.

We assume that $r(c) > 0$ and $\gcd((c_1, p^*\mathcal{O}_X(1)), r(c)) = 1$, then μ-stability and μ-semistability (and hence also (semi)stability) are equivalent. Then E is stable perverse coherent if and only if $E \in \mathrm{Coh}(\widehat{X}) \cap$

$\operatorname{Per}(\widehat{X}/X)$ and $p_*(E)$ is μ-stable by Lemma 2.6. In particular, we have $\overline{M}^p_{Y/X/\mathbb{C}}(c) = M^p_{Y/X/\mathbb{C}}(c)$ in the notation in Theorem 2.9. This assumption is *essential* to compare moduli spaces on $\widehat{X}$ and X. See Lemma 2.8 that the relation of stabilities is delicate if we do not assume the condition.

In case of framed sheaves on $\widehat{\mathbb{P}}^2 = \widehat{\mathbb{C}}^2 \cup \ell_\infty$, moduli spaces corresponding to $\widehat{M}^m(c)$ for various m are constructed by GIT quotients of the *common* variety with respect to various choices of polarizations in the quiver description. From a general construction by Thaddeus [27], we can construct a diagram $(*)$ in Introduction, which induces a flip $\widehat{M}^m(c) \dashrightarrow \widehat{M}^{m+1}(c)$ under some mild assumptions. Unfortunately our spaces $\widehat{M}^m(c)$ and $\widehat{M}^{m+1}(c)$ are not quotients of a *common* space. Therefore we must construct the space $\widehat{M}^{m,m+1}(c)$ and the diagram *by hand*. This will be done in this section. We also study the fibers of ξ_m and ξ_m^+. Under a condition (= [27, (4.4)]) Thaddeus showed that fibers are weighted projective spaces. This condition (even in the framed case) is *not* satisfied, but we will show that the fibers are Grassmanns.

We have an isomorphism $\widehat{M}^m(c) \cong \widehat{M}^0(ce^{-m[C]})$ given by $E \mapsto E(-mC)$, twisting by the line bundle $\mathcal{O}_{\widehat{X}}(-mC)$. Therefore we only need to consider the case $m = 0$. But we also use $\widehat{M}^m(c)$ to simplify the notation, and make the change of moduli spaces apparent.

3.1. **A distinguished chamber – torsion free sheaves on blow-down.** As is explained above, we restrict ourselves to the case $m = 0$ in this subsection.

By the definition of $\widehat{M}^0(c)$ we have a morphism

$$\begin{array}{cccc} \xi\colon & \widehat{M}^0(c) & \to & M^X(p_*(c)) \\ & E & \mapsto & p_*(E), \end{array} \tag{3.1}$$

where $M^X(p_*(c))$ is the moduli space of μ-stable sheaves on X.

Here $p_*(c)$ is defined so that it is compatible with the Riemann-Roch formula. So it is twisted from the usual push-forward homomorphism as $p_*(c) = p_*^{\mathrm{usual}}(c \operatorname{td} \widehat{X})(\operatorname{td} X)^{-1}$. In particular, we have $p_*(e) = p_*(\operatorname{ch}(\mathcal{O}_C(-1))) = 0$. This convention will be used throughout this paper.

Lemma 3.2. *Let* $E \in \widehat{M}^0(c)$. *Then we have* $\operatorname{Hom}(E, \mathcal{O}_C(-1)) = \operatorname{Ext}^2(E, \mathcal{O}_C(-1)) = 0$ *and* $\operatorname{Hom}(\mathcal{O}_C, E) = \operatorname{Ext}^2(\mathcal{O}_C, E) = 0$. *In particular,* $\chi(E, \mathcal{O}_C(-1)) = \chi(\mathcal{O}_C, E) = -(c_1(E), [C]) \le 0$. *(cf.* [25, Lem. 7.3]*).*

Proof. By the Serre duality, we have $\operatorname{Ext}^2(E, \mathcal{O}_C(-1)) = \operatorname{Hom}(\mathcal{O}_C, E)^\vee$ and $\operatorname{Ext}^2(\mathcal{O}_C, E) = \operatorname{Hom}(E, \mathcal{O}_C(-1))^\vee$. Then the assertions follow from the definition of stable perverse coherent sheaves. □

We first consider the case $(c_1, [C]) = 0$.

Proposition 3.3 (cf. [25, Prop. 7.4])**.** *The morphism $\xi\colon \widehat{M}^0(c) \to M^X(p_*(c))$ is an isomorphism if $(c_1, [C]) = 0$.*

Proof. We have $\dim \operatorname{Ext}^1(E, \mathcal{O}_C(-1)) = \chi(E, \mathcal{O}_C(-1)) = -(c_1(E), [C]) = 0$. Therefore we have $E = p^* p_*(E)$ by Proposition 1.9(2). □

Besides the morphism $\xi\colon \widehat{M}^0(c) \to M^X(p_*(c))$, we have another natural morphism:

Lemma 3.4. *We have a morphism*

$$\begin{array}{cccc} \eta\colon & \widehat{M}^0(c) & \to & M^X(p_*(c) + n\,\mathrm{pt}) \\ & E & \mapsto & p_*(E(C)), \end{array}$$

where $n = (c_1, [C])$.

Proof. From Lemma 1.10 the direct image sheaf $p_*(E(C))$ has the Chern character $p_*(\operatorname{ch}(E)e^{[C]}) = p_*(ce^{[C]}) = p_*(c) + n\,\mathrm{pt}$. Therefore it is enough to show that $p_*(E(C))$ is μ-stable.

As $\operatorname{Hom}(\mathcal{O}_C, E(C)) = \operatorname{Hom}(\mathcal{O}_C(1), E) = 0$ from $\operatorname{Hom}(\mathcal{O}_C, E) = 0$, $p_*(E(C))$ is torsion free by Lemma 1.11.

Consider $p_*(E) \to p_*(E(C))$. This is an isomorphism outside the point 0. Therefore the kernel is 0 since $p_*(E)$ is torsion free by the assumption. Since $p_*(E)$ is μ-stable and $p_*(E(C))/p_*(E)$ is 0-dimensional, $p_*(E(C))$ is also μ-stable. □

3.2. **The morphism to the Uhlenbeck compactification downstairs.** Let $M_0^X(p_*(c))$ be the Uhlenbeck compactifiction, that is $\bigsqcup M_{lf}^X(p_*(c) + m\,\mathrm{pt}) \times S^m X$, where $M_{lf}^X(p_*(c) + m\,\mathrm{pt})$ is the moduli space of μ-stable *locally free* sheaves on X. Then J. Li [12] defined a scheme structure which is projective, and there is a projective morphism $\pi\colon M^X(p_*(c))_{\mathrm{red}} \to M_0^X(p_*(c))$ sending E to $(E^{\vee\vee}, \operatorname{Supp}(E^{\vee\vee}/E))$. In [24, F.11] the authors defined a projective morphism $\widehat{\pi}$ from the moduli space of torsion-free sheaves on $\widehat{X}$ to $M_0^X(p_*(c))$. One of essential ingredients of the construction was a morphism to $M^X(p_*(ce^{-m[C]}))$ for sufficiently large m. Since we have the natural morphism $\widehat{M}^m(c) \to M^X(ce^{-m[C]})$ by the construcion in the previous subsection, we can apply the same method to define a projective morphism

$$\begin{array}{llcl} (3.5) & & & \\ \widehat{\pi}\colon & \widehat{M}^m(c)_{\mathrm{red}} & \to & M_0^X(p_*(c)) \\ & E & \mapsto & (p_*(E)^{\vee\vee}, \mathrm{Supp}(p_*(E)^{\vee\vee}/p_*(E)) + \mathrm{Supp}(R^1p_*(E)))\,. \end{array}$$

3.3. **Smoothness.**

Lemma 3.6. *Let $E \in \widehat{M}^m(c)$. We have an injective homomorphism*

$$\mathrm{Hom}(E, E \otimes K_{\widehat{X}}) \hookrightarrow \mathrm{Hom}(p_*(E)^{\vee\vee}, p_*(E)^{\vee\vee} \otimes K_X).$$

Proof. Since we have $\mathrm{Hom}(E, E \otimes K_{\widehat{X}}) \cong \mathrm{Hom}(E(-mC), E(-mC) \otimes K_{\widehat{X}})$ and $p_*(E)^{\vee\vee} \cong p_*(E(-mC))^{\vee\vee}$, we may assume $m = 0$.

If E is perverse coherent, Lemma 1.2(3) implies that the natural homomorphism $p^*p_*(E) \to E$ induces an injection

$$\mathrm{Hom}(E, E\otimes K_{\widehat{X}}) \hookrightarrow \mathrm{Hom}(p^*p_*(E), E\otimes K_{\widehat{X}}) \cong \mathrm{Hom}(p_*(E), p_*(E(C))\otimes K_X)).$$

We compose it with the induced homomorphism from $p_*(E(C)) \hookrightarrow p_*(E(C))^{\vee\vee} = p_*(E)^{\vee\vee}$ to replace the right most term by $\mathrm{Hom}(p_*(E), p_*(E)^{\vee\vee}\otimes K_X)$. Let us consider the exact sequence $0 \to p_*(E) \to p_*(E)^{\vee\vee} \to Q \to 0$. Since $p_*(E)^{\vee\vee}$ is torsion free, we have $\mathrm{Hom}(Q, p_*(E)^{\vee\vee} \otimes K_X) = 0$. We have $\mathrm{Ext}^1(Q, p_*(E)^{\vee\vee} \otimes K_X) \cong \mathrm{Ext}^1(p_*(E)^{\vee\vee\vee}, Q)^\vee = 0$ as Q is 0-dimensional. Therefore we have

$$\mathrm{Hom}(p_*(E)^{\vee\vee}, p_*(E)^{\vee\vee} \otimes K_X) \cong \mathrm{Hom}(p_*(E), p_*(E)^{\vee\vee} \otimes K_X). \qquad \square$$

Corollary 3.7. *If $(\mathcal{O}_X(1), K_X) < 0$, then $\widehat{M}^m(c)$ is nonsingular of dimension $2r\Delta(c) - (r^2-1)\chi(\mathcal{O}_X) + h^1(\mathcal{O}_X)$, where $\Delta(c) := \int_{\widehat{X}} c_2 - (r-1)/(2r)c_1^2$.*

In general, the number $2r\Delta(c) - (r^2-1)\chi(\mathcal{O}_X) + h^1(\mathcal{O}_X)$ is called the *expected dimension* of $\widehat{M}^m(c)$, and denoted by $\exp\dim \widehat{M}^m(c)$. If any irreducible component of $\widehat{M}^m(c)$ has dimension equal to $\exp\dim \widehat{M}^m(c)$, then we say $\widehat{M}^m(c)$ has the expected dimension. By the results of Donaldson, Zuo, Gieseker-Li, O'Grady (see [8, §9]) there exists a constant Δ_0 depending only on X, $\mathcal{O}_X(1)$ and $r(c)$ such that $M^X(c)$ is irreducible, normal, locally of complete intersection, and of expected dimension for $\Delta(c) \geq \Delta_0$. The argument is applicable to $\widehat{M}^m(c)$.

Proposition 3.8. *There exists a constant Δ_0 such that $\widehat{M}^m(c)$ is irreducible, normal and of expected dimension if $\Delta(c) \geq \Delta_0$.*

3.4. **Evaluation homomorphisms.** This subsection is the technical heart of this paper. Starting from a stable perverse coherent sheaf and a vector subspace in the space of homomorphisms from $\mathcal{O}_C(-1)$ or to $\mathcal{O}_C$, we construct a new stable perverse coherent sheaf. It will become a key to analyze the change of stability conditions.

Lemma 3.9. (1) *Let* $E \in \operatorname{Per}(\widehat{X}/X) \cap \operatorname{Coh}(\widehat{X})$ *such that* $p_*(E)$ *is torsion free, and let* $V \subset \operatorname{Hom}(\mathcal{O}_C(-1), E)$ *be a subspace. Then the evaluation homomorphism induces an exact sequence* (*in the category* $\operatorname{Coh}(\widehat{X})$)

$$0 \to V \otimes \mathcal{O}_C(-1) \xrightarrow{\mathrm{ev}} E \to E' := \operatorname{Coker}(\mathrm{ev}) \to 0, \tag{3.10}$$

and $\operatorname{Coker}(\mathrm{ev}) \in \operatorname{Per}(\widehat{X}/X)(\cap \operatorname{Coh}(\widehat{X}))$.

(2) *Let* $E' \in \operatorname{Per}(\widehat{X}/X) \cap \operatorname{Coh}(\widehat{X})$ *and let* $V' \subset \operatorname{Ext}^1(E', \mathcal{O}_C(-1))$ *be a subspace. Then the associated extension* (*in* $\operatorname{Coh}(\widehat{X})$)

$$0 \to (V')^\vee \otimes \mathcal{O}_C(-1) \to E \to E' \to 0$$

defines $E \in \operatorname{Per}(\widehat{X}/X)(\cap \operatorname{Coh}(\widehat{X}))$.

(3) *Let* $F \in \operatorname{Per}(\widehat{X}/X) \cap \operatorname{Coh}(\widehat{X})$ *and let* $U \subset \operatorname{Hom}(F, \mathcal{O}_C)$ *be a subspace. Then the evaluation homomorphism induces an exact sequence* (*in* $\operatorname{Coh}(\widehat{X})$)

$$0 \to F' := \operatorname{Ker}(\mathrm{ev}) \to F \xrightarrow{\mathrm{ev}} U^\vee \otimes \mathcal{O}_C \to 0, \tag{3.11}$$

and $F' \in \operatorname{Per}(\widehat{X}/X)(\cap \operatorname{Coh}(\widehat{X}))$.

(4) *Let* $F' \in \operatorname{Per}(\widehat{X}/X) \cap \operatorname{Coh}(\widehat{X})$ *and let* $U' \subset \operatorname{Ext}^1(\mathcal{O}_C, F')$ *be a subspace. The associated extension* (*in* $\operatorname{Coh}(\widehat{X})$)

$$0 \to F' \to F \to U' \otimes \mathcal{O}_C \to 0$$

defines $F \in \operatorname{Per}(\widehat{X}/X)(\cap \operatorname{Coh}(\widehat{X}))$ *satisfying* $\operatorname{Hom}(\mathcal{O}_C, F) \cong \operatorname{Hom}(\mathcal{O}_C, F')$.

In (1) we have an exact sequence in $\operatorname{Per}(\widehat{X}/X)$:

$$0 \to E \to E' \to V \otimes \mathcal{O}_C(-1)[1] \to 0.$$

This corresponds to the inclusion $V \subset \operatorname{Hom}(\mathcal{O}_C(-1), E) \cong \operatorname{Ext}^1(\mathcal{O}_C(-1)[1], E)$. This makes sense without the assumption that $p_*(E)$ is torsion free, but E' may not be a sheaf in general. Similarly in (2) we have an exact sequence in $\operatorname{Per}(\widehat{X}/X)$:

$$0 \to E \to E' \to (V')^\vee \otimes \mathcal{O}_C(-1)[1] \to 0$$

corresponding to the inclusion $V' \subset \operatorname{Ext}^1(E', \mathcal{O}_C(-1)) = \operatorname{Hom}(E', \mathcal{O}_C(-1)[1])$. In (3), (4), the natural exact sequences in $\operatorname{Coh}(\widehat{X})$ are also exact sequences in $\operatorname{Per}(\widehat{X}/X)$.

In the following there are two ways to prove the assertion. One is working in the category $\operatorname{Coh}(\widehat{X})$ and check the condition in Proposition 1.9(1) to show that sheaves are perverse coherent. The other is working in the category $\operatorname{Per}(\widehat{X}/X)$ and check the condition $H^{-1}(\ )=0$ to show that objects are, in fact, sheaves. We will give proofs of (2),(3) in the first way, and ones of (1),(4) in the second way. We leave other proofs as an exercise for a reader.

Proof of Lemma 3.9. (1) Let $0\to E\to E'\to V\otimes\mathcal{O}_C(-1)[1]\to 0$ be the extension in $\operatorname{Per}(\widehat{X}/X)$ corresponding to $V\subset\operatorname{Ext}^1(\mathcal{O}_C(-1)[1],E)$. Then we have $\mathbf{R}p_*(E)\cong\mathbf{R}p_*(E')$. Applying $\mathbf{R}p_*(\ )$ to an exact sequence $0\to H^{-1}(E')[1]\to E'\to H^0(E')\to 0$ in $\operatorname{Per}(\widehat{X}/X)$, we get an injective homomorphism $R^1p_*(H^{-1}(E'))\to\mathbf{R}p_*(E')\cong\mathbf{R}p_*(E)\cong p_*(E)$. But $R^1p_*(H^{-1}(E'))$ is a torsion, so we have $R^1p_*(H^{-1}(E'))=0$ from our assumption that $p_*(E)$ is torsion free. Therefore $H^{-1}(E')\cong\mathcal{O}_C(-1)^{\oplus s}$ by Lemma 1.7(2).

From the first exact sequence, we get a long exact sequence

$$0\to\operatorname{Hom}(\mathcal{O}_C(-1)[1],E)\to\operatorname{Hom}(\mathcal{O}_C(-1)[1],E')\to V\to\operatorname{Ext}^1(\mathcal{O}_C(-1)[1],E')$$

The first term is 0 as it is $\operatorname{Ext}^{-1}(\mathcal{O}_C(-1),E)$. The right most arrow is injective by our choice. Therfore $\operatorname{Hom}(\mathcal{O}_C(-1)[1],E')=0$. Therefore s must be 0. This shows E' is a sheaf.

(2) Applying $\operatorname{Hom}(\bullet,\mathcal{O}_C(-1))$ to the given exact sequence, we get

$$0\to\operatorname{Hom}(E,\mathcal{O}_C(-1))\to V'\to\operatorname{Ext}^1(E',\mathcal{O}_C(-1)).$$

By the construction the right most arrow is injective. Hence $\operatorname{Hom}(E,\mathcal{O}_C(-1))=0$. Therefore $E\in\operatorname{Per}(\widehat{X}/X)$.

(3) We consider the following exact sequences in $\operatorname{Coh}(\widehat{X})$:

$$\begin{gathered}0\to\operatorname{Ker}(\mathrm{ev})\to F\to\operatorname{Im}(\mathrm{ev})\to 0,\\ 0\to\operatorname{Im}(\mathrm{ev})\to U^\vee\otimes\mathcal{O}_C\to\operatorname{Coker}(\mathrm{ev})\to 0.\end{gathered}\tag{3.12}$$

Applying $\mathbf{R}p_*(\mathcal{O}_{\widehat{X}}(C)\otimes\bullet)$ to the second exact sequence, we have $\mathbf{R}p_*(\operatorname{Im}(\mathrm{ev})\otimes\mathcal{O}_{\widehat{X}}(C))=0$, $\mathbf{R}p_*(\operatorname{Coker}(\mathrm{ev})\otimes\mathcal{O}_{\widehat{X}}(C))=0$. By Lemma 1.7(2), we have $\operatorname{Im}(\mathrm{ev})\cong\mathcal{O}_C^{\oplus a}$, $\operatorname{Coker}(\mathrm{ev})\cong\mathcal{O}_C^{\oplus b}$ for some $a,b\in\mathbb{Z}_{\ge 0}$.

Applying $\operatorname{Hom}(\bullet,\mathcal{O}_C)$ to (3.12), we get

$$0\to\operatorname{Hom}(\operatorname{Im}(\mathrm{ev}),\mathcal{O}_C)\to\operatorname{Hom}(F,\mathcal{O}_C),$$
$$0\to\operatorname{Hom}(\operatorname{Coker}(\mathrm{ev}),\mathcal{O}_C)\to U\to\operatorname{Hom}(\operatorname{Im}(\mathrm{ev}),\mathcal{O}_C).$$

As the composition of $U\to\operatorname{Hom}(\operatorname{Im}(\mathrm{ev}),\mathcal{O}_C)\to\operatorname{Hom}(F,\mathcal{O}_C)$ is the natural inclusion, the left homomorphism is injective. So $\operatorname{Hom}(\operatorname{Coker}(\mathrm{ev}),\mathcal{O}_C)=0$. But as we already observed $\operatorname{Coker}(\mathrm{ev})=\mathcal{O}_C^{\oplus b}$, this means $\operatorname{Coker}(\mathrm{ev})=0$. Hence ev is surjective.

Applying $\mathbf{R}\operatorname{Hom}(\bullet, \mathcal{O}_C(-1))$ to the first exact sequence of (3.12), we get

$$0 \to \operatorname{Hom}(\operatorname{Ker}(\mathrm{ev}), \mathcal{O}_C(-1)) \to \operatorname{Ext}^1(\operatorname{Im}(\mathrm{ev}), \mathcal{O}_C(-1)).$$

But as $\operatorname{Im}(\mathrm{ev}) \cong U^\vee \otimes \mathcal{O}_C$, the latter space is 0, hence $\operatorname{Hom}(\operatorname{Ker}(\mathrm{ev}), \mathcal{O}_C(-1)) = 0$. Therefore $\operatorname{Ker}(\mathrm{ev}) \in \operatorname{Per}(\widehat{X}/X)$.

(4) Noticing $\mathcal{O}_C \in \operatorname{Per}(\widehat{X}/X)$, we consider the extension in the category $\operatorname{Per}(\widehat{X}/X)$ instead of $\operatorname{Coh}(\widehat{X})$. Then $H^{-1}(F) = 0$ as $H^{-1}(F') = 0 = H^{-1}(\mathcal{O}_C \otimes U)$. Therefore $F \in \operatorname{Coh}(\widehat{X})$. So the extension is also an exact sequence in $\operatorname{Coh}(\widehat{X})$. □

Lemma 3.13. *In the following* (a) $(a = 1, 2, 3, 4)$ *we suppose* E, E', F, F' *are as in the corresponding Lemma 3.9*(a).

(1) *If* E *is stable, then so is* E'.
(2) *If* E' *is stable, then so is* E.
(3) *If* F *is stable, then so is* F'.
(4) *If* F' *is stable, then so is* F.

Proof. (1) As $p_*(E) \cong p_*(E')$ from the exact sequence (3.10) and $\mathbf{R}p_*(\mathcal{O}_C(-1)) = 0$, the assertion is clear.

(2) The same argument as in (1).

(3) From the exact sequence $0 \to F' := \operatorname{Ker}(\mathrm{ev}) \to F \to U^\vee \otimes \mathcal{O}_C \to 0$ we get an exact sequence

$$0 \to p_*(F') \to p_*(F) \to U^\vee \otimes \mathbb{C}_0 \to 0. \tag{3.14}$$

As $p_*(F)$ is torsion free, so is $p_*(F')$. Note that $p_*(F')$ and $p_*(F)$ have the same μ, as they differ only at 0. Therefore $p_*(F')$ is also μ-stable, and hence F' is stable.

(4) Since F' is stable, $p_*(F')$ is torsion free, so we have $\operatorname{Hom}(\mathcal{O}_C, F') = 0$ by Lemma 1.11. Therefore $\operatorname{Hom}(\mathcal{O}_C, F) = 0$ by the last assertion in Lemma 3.9(4), so $p_*(F)$ is also torsion free. We have the exact sequence (3.14). Then by the same argument as in Lemma 3.4, the μ-stability of $p_*(F')$ and the torsion freeness of $p_*(F)$ implies the μ-stability of $p_*(F)$. □

3.5. **Stable sheaves becoming unstable after the wall-crossing.**

Note that

$$\begin{aligned} E \in \widehat{M}^0(c) \quad &\Longrightarrow \quad \operatorname{Hom}(E, \mathcal{O}_C(-1)) = 0, \quad \operatorname{Hom}(\mathcal{O}_C, E) = 0, \\ E \in \widehat{M}^1(c) \quad &\Longrightarrow \quad \operatorname{Hom}(E, \mathcal{O}_C(-2)) = 0, \quad \operatorname{Hom}(\mathcal{O}_C(-1), E) = 0. \end{aligned}$$

Note also that

$$\begin{aligned} \operatorname{Hom}(E, \mathcal{O}_C(-1)) = 0 \quad &\Longrightarrow \quad \operatorname{Hom}(E, \mathcal{O}_C(-2)) = 0, \\ \operatorname{Hom}(\mathcal{O}_C(-1), E) = 0 \quad &\Longrightarrow \quad \operatorname{Hom}(\mathcal{O}_C, E) = 0. \end{aligned}$$

The following proposition says that these are the only conditions which are altered under the wall-crossing.

Proposition 3.15. (1) *Suppose E^- is 0-stable, but not 1-stable, i.e. E^- is stable perverse coherent, but $E^-(-C)$ is not. Then $V := \operatorname{Hom}(\mathcal{O}_C(-1), E^-) \neq 0$ and the evaluation homomorphism gives rise an exact sequence*

$$0 \to V \otimes \mathcal{O}_C(-1) \to E^- \to E' \to 0$$

such that E' is both 0-stable and 1-stable. Moreover the induced homomorphism $V \to \operatorname{Ext}^1(E', \mathcal{O}_C(-1))$ is injective.

Conversely if E' is both 0 and 1-stable and E^- is the extension corresponding to a nonzero subspace V of $\operatorname{Ext}^1(E', \mathcal{O}_C(-1))$ as above. Then E^- is 0-stable, but not 1-stable, and V is naturally identified with $\operatorname{Hom}(\mathcal{O}_C(-1), E^-)$.

These give a bijection

$$\begin{aligned} &\{E^- \in \widehat{M}^0(c) \setminus \widehat{M}^1(c) \mid \dim \operatorname{Hom}(\mathcal{O}_C(-1), E^-) = i\} \\ \longleftrightarrow\ &\{(E', V) \mid E' \in \widehat{M}^0(c-ie) \cap \widehat{M}^1(c-ie), V \in \operatorname{Gr}(i, \operatorname{Ext}^1(E', \mathcal{O}_C(-1))\}. \end{aligned}$$

(2) *Suppose $E^+ \in \widehat{M}^1(c) \setminus \widehat{M}^0(c)$. Then $U := \operatorname{Hom}(E^+, \mathcal{O}_C(-1)) \neq 0$ and the evaluation homomorphism gives rise an exact sequence*

$$0 \to E' \to E^+ \to U^\vee \otimes \mathcal{O}_C(-1) \to 0$$

such that E' is both 0-stable and 1-stable. Moreover it induces an injection $U^\vee \to \operatorname{Ext}^1(\mathcal{O}_C(-1), E')$.

Conversely if E' is both 0 and 1-stable and E^+ is the extension corresponding to a nonzero subspace $U^\vee$ of $\operatorname{Ext}^1(\mathcal{O}_C(-1), E')$. Then E^+ is 1-stable, but not 0-stable, and $U^\vee$ is naturally identified with $\operatorname{Hom}(E^+, \mathcal{O}_C(-1))^\vee$.

These give a bijection

$$\begin{aligned} &\{E^+ \in \widehat{M}^1(c) \setminus \widehat{M}^0(c) \mid \dim \operatorname{Hom}(E^+, \mathcal{O}_C(-1)) = i\} \\ \longleftrightarrow\ &\{(E', U^\vee) \mid E' \in \widehat{M}^0(c-ie) \cap \widehat{M}^1(c-ie), U^\vee \in \operatorname{Gr}(i, \operatorname{Ext}^1(\mathcal{O}_C(-1), E')\}. \end{aligned}$$

Proof. (1) By Lemma 3.9(1) we can consruct the exact sequence as in the statement. Then E' is stable by Lemma 3.13(1). Note also that we have $\operatorname{Hom}(\mathcal{O}_C(-1), E') = 0$ from the exact sequence and our choice of V. Therefore $p_*(E'(-C))$ is torsion free by Lemma 1.11.

Next consider $0 \to V \otimes \mathcal{O}_C \to E^-(-C) \to E'(-C) \to 0$. We have an injective homomorphism

$$0 \to \operatorname{Hom}(E'(-C), \mathcal{O}_C(-1)) \to \operatorname{Hom}(E^-(-C), \mathcal{O}_C(-1)) \cong \operatorname{Hom}(E^-, \mathcal{O}_C(-2)).$$

But $\operatorname{Hom}(E^-, \mathcal{O}_C(-2)) = 0$ as $\operatorname{Hom}(E^-, \mathcal{O}_C(-1)) = 0$ from the assumption. Therefore $E'(-C) \in \operatorname{Per}(\widehat{X}/X)$. Since $p_*(E')$ is μ-stable and $p_*(E')/p_*(E'(-C))$ is 0-dimensional, $p_*(E'(-C))$ is also μ-stable. This shows $E'(-C)$ is stable, and hence E' is both 0 and 1-stable. As $E(-C)$ is not stable by the assumption, we have $V \neq 0$. The injection $V \to \operatorname{Ext}^1(E', \mathcal{O}_C(-1))$ comes from the long exact sequence associated with the given exact sequence, together with $\operatorname{Hom}(E^-, \mathcal{O}_C(-1)) = 0$.

Let us show the converse. From Lemma 3.13(2), E^- is stable. As $\operatorname{Hom}(\mathcal{O}_C(-1), E^-) \neq 0$, E^- is not 1-stable. Moreover we have a natural isomorphism $V \cong \operatorname{Hom}(\mathcal{O}_C(-1), E^-)$ induced from the given exact sequence together with $\operatorname{Hom}(\mathcal{O}_C(-1), E') = 0$.

It is also clear that these constructions give a bijection.

(2) As $E^+(-C)$ is stable by the assumption, $E^+(-C) \in \operatorname{Per}(\widehat{X}/X)$. Therefore we can apply Lemma 3.13(3) for $F = E^+(-C)$ with $U = \operatorname{Hom}(F, \mathcal{O}_C) \cong \operatorname{Hom}(E^+, \mathcal{O}_C(-1))$. Then the corresponding exact sequence $0 \to E' \to E^+ \to U^\vee \otimes \mathcal{O}_C(-1) \to 0$ defines E' such that $E'(-C)$ is stable, i.e. E' is 1-stable.

Note $\operatorname{Hom}(E', \mathcal{O}_C(-1)) = 0$ from the exact sequence and our choice of U. Then $E' \in \operatorname{Per}(\widehat{X}/X)$ by Proposition 1.9(1). By Lemma 3.4 we have $p_*(E')$ is μ-stable, as $E'(-C)$ is stable. Therefore E' is also 0-stable. The injection $U^\vee \to \operatorname{Ext}^1(\mathcal{O}_C(-1), E')$ is induced from the given exact sequence and $\operatorname{Hom}(\mathcal{O}_C(-1), E^+) = 0$.

Let us show the converse. From Lemma 3.13(4) applied to $F' := E'(-C)$ with $U' := \operatorname{Ext}^1(\mathcal{O}_C, E'(-C))$, $E^+(-C)$ is stable, i.e. E^+ is 1-stable. A natural isomorphism $U^\vee \cong \operatorname{Hom}(E^+, \mathcal{O}_C(-1))^\vee$ is induced from the given exact sequence and $\operatorname{Hom}(E', \mathcal{O}_C(-1)) = 0$. □

Proposition 3.16. *Let $E^- \in \widehat{M}^m(c)$ (resp. $E^+ \in \widehat{M}^{m+1}(c)$) and suppose that its image under $\widehat{\pi}$ in* (3.5) *has the multiplicity N at 0 in its symmetric product part. If $m > N$, then E^- (resp. E^+) is $(m+1)$-stable (resp. m-stable).*

Proof. Suppose that E^- is m-stable, but not $(m+1)$-stable for some $m \geq 0$. From Proposition 3.15(1) $i := \dim \operatorname{Hom}(\mathcal{O}_C(-m-1), E^-) > 0$ and we have an exact sequence

$$0 \to p_*(E^-) \to p_*(E') \to \mathbb{C}_0^{\oplus im} \to R^1p_*(E^-) \to R^1p_*(E') \to 0.$$

As E^-, E' are m-stable, and hence $\operatorname{Hom}(\mathcal{O}_C(-m), E')$, $\operatorname{Hom}(\mathcal{O}_C(-m), E^-)$ are 0. Therefore we have $\operatorname{Hom}(\mathcal{O}_C, E^-) = 0 = \operatorname{Hom}(\mathcal{O}_C, E')$. By

Lemma 1.11 $p_*(E^-)$, $p_*(E')$ are torsion free. Therefore $p_*(E^-) \to p_*(E^-)^{\vee\vee}$, $p_*(E') \to p_*(E')^{\vee\vee}$ are injective. From the above exact sequence, we have $p_*(E^-)^{\vee\vee} \cong p_*(E')^{\vee\vee}$. Therefore we have an exact sequence

$$0 \to p_*(E')/p_*(E^-) \to p_*(E^-)^{\vee\vee}/p_*(E^-) \to p_*(E')^{\vee\vee}/p_*(E') \to 0.$$

We get

$$\begin{aligned}\mathrm{len}_0(p_*(E^-)^{\vee\vee}/p_*(E^-)) &\geq \mathrm{len}_0(p_*(E')^{\vee\vee}/p_*(E')) + im - \mathrm{len}_0(R^1p_*(E^-)) \\ &\geq m - \mathrm{len}_0(R^1p_*(E^-)),\end{aligned}$$

where len_0 is the length of the stalk at 0. This inequality is impossible if $m > \mathrm{len}_0(p_*(E^-)^{\vee\vee}/p_*(E^-)) + \mathrm{len}_0(R^1p_*(E^-))$. From the definition of $\widehat{\pi}$, we get the assertion. The proof for E^+ is the same. $\square$

3.6. **Brill-Noether locus and moduli of coherent systems.** Motivated by Proposition 3.15, we introduce the Brill-Noether locus:

Definition 3.17 (Brill-Noether locus)**.** We set

$$\begin{aligned}\widehat{M}^m(c)_i &:= \{E^- \in \widehat{M}^m(c) \mid \dim \mathrm{Hom}(\mathcal{O}_C(-m-1), E^-) = i\}, \\ \widehat{M}^{m+1}(c)^i &:= \{E^+ \in \widehat{M}^{m+1}(c) \mid \dim \mathrm{Hom}(E^+, \mathcal{O}_C(-m-1)) = i\},\end{aligned}$$

When we replace '$= i$' by '$\geq i$' in the right hand side, the corresponding moduli spaces are denoted by the left hand side with 'i' replaced by '$\geq i$'.

The scheme structures on $\widehat{M}^m(c)_i$, $\widehat{M}^{m+1}(c)^i$ are defined as in [14, 5.5] (cf. [1, Ch. IV]). Let us briefly explain an essential point. Let $\mathcal{E}^-$ be a universal family over $\widehat{X} \times \widehat{M}^m(c)$ and let f be the projection to $\widehat{M}^m(c)$. Then we construct an exact sequence

(3.18)

$$0 \to \mathrm{Hom}_f(\mathcal{O}_C(-m-1), \mathcal{E}^-) \to \mathcal{F}_0 \xrightarrow{\rho} \mathcal{F}_1 \to \mathrm{Ext}^1_f(\mathcal{O}_C(-m-1), \mathcal{E}^-) \to 0$$

such that $\mathcal{F}_0$, $\mathcal{F}_1$ are vector bundles. Then we define $\widehat{M}^m(c)_{\geq i}$ to be the zero locus of $\bigwedge^{\mathrm{rk}\,\mathcal{F}_0+1-i}\rho$. Moreover $\widehat{M}^m(c)_i$ is the open subscheme $\widehat{M}^m(c)_{\geq i} \setminus \widehat{M}^m(c)_{\geq i+1}$ of $\widehat{M}^m(c)_{\geq i}$.

In Proposition 3.15 we have $E^- \in \widehat{M}^0(c)_i$, $E^+ \in \widehat{M}^1(c)^i$. Therefore Proposition 3.15 says that when we change the stability condition from 0 to 1, $\widehat{M}^0(c)_{\geq 1}$ is replaced by $\widehat{M}^1(c)^{\geq 1}$, and $\widehat{M}^0(c)_0 \cong \widehat{M}^0(c) \cap \widehat{M}^1(c) \cong \widehat{M}^1(c)^0$ is preserved. We have a set-theoretical diagram

$$\begin{tikzcd}[column sep=small] \widehat{M}^0(c) \arrow[dr] & & \widehat{M}^1(c) \arrow[dl] \\ & \bigsqcup_i \widehat{M}^0(c-ie)_0 \cong \bigsqcup_i \widehat{M}^1(c-ie)^0 & \end{tikzcd} \tag{3.19}$$

and fibers over $E' \in \widehat{M}^0(c-ie)_0$ of the left and right arrows are Grassmann $\operatorname{Gr}(i, \operatorname{Ext}^1(E', \mathcal{O}_C(-1)))$ and $\operatorname{Gr}(i, \operatorname{Ext}^1(\mathcal{O}_C(-1), E'))$ respectively. This is similar to $(*)$, but we need to endow the target $\bigsqcup_i \widehat{M}^0(c-ie)_0$ with a scheme structure.

Let us introduce moduli spaces of coherent systems in order to study Brill-Noether loci more closely.

Definition 3.20. Let $\widehat{M}(c,n)$ be the moduli space of coherent systems $(E,$
$V \subset \operatorname{Hom}(\mathcal{O}_C(-1), E))$ such that $E \in \widehat{M}^0(c)$ and $\dim V = n$.

The construction is standard: $\widehat{M}(c,n)$ is constructed as a closed subscheme of a suitable Grassmannian bundle over $\widehat{M}^0(c)$. We have a natural morphism $q_1 \colon \widehat{M}(c,n) \to \widehat{M}^0(c)$. We have a universal family $\mathcal{V}$ which is a rank n vector subbundle of $q_1^*(\mathcal{F}_0)$ contained in $\operatorname{Ker}(q_1^*\rho)$, where $\mathcal{F}_1$ and ρ are as in (3.18).

For $(E,V) \in \widehat{M}(c,n)$, we set $E' := \operatorname{Coker}(\operatorname{ev}\colon V \otimes \mathcal{O}_C(-1) \to E)$. By Lemma 3.13(1), we have $E' \in \widehat{M}^0(c-ne)$. Thus we get a morphism $q_2 \colon \widehat{M}(c,n) \to \widehat{M}^0(c-ne)$.

Therefore we have the following diagram:

$$\begin{tikzcd} & \widehat{M}(c,n) \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \widehat{M}^0(c) & & \widehat{M}^0(c-ne) \end{tikzcd} \tag{3.21}$$

Conversely suppose that $E' \in \widehat{M}^0(c-ne)$ and an n-dimensional subspace $V^\vee \subset \operatorname{Ext}^1(E', \mathcal{O}_C(-1))$ are given. Then we can consider the corresponding extension (3.10). By Lemma 3.13(2), we have $E \in \widehat{M}^0(c)$. Moreover, the exact sequence (3.10) induces an injection $V \to \operatorname{Hom}(\mathcal{O}_C(-1), E)$. Thus $(E,V) \in \widehat{M}(c,n)$. This gives an isomorphism from $\widehat{M}(c,n)$ to the moduli space of 'dual' coherent systems $(E', V^\vee \subset \operatorname{Ext}^1(E', \mathcal{O}_C(-1)))$ such that $E' \in \widehat{M}^0(c-ne)$, $\dim V^\vee = n$.

Note $\operatorname{Hom}(E', \mathcal{O}_C(-1)) = 0 = \operatorname{Ext}^2(E', \mathcal{O}_C(-1))$ and $\dim \operatorname{Ext}^1(E', \mathcal{O}_C(-1))$ $= (c_1, [C]) + n$ by Lemma 3.2. This is a constant independent of $E' \in \widehat{M}^0(c-ne)$. Let $\mathcal{E}'$ be an universal family over $\widehat{X} \times \widehat{M}^0(c-ne)$ and let f be the projection to $\widehat{M}^0(c-ne)$. By the above observation $\operatorname{Ext}^1_f(\mathcal{E}', \mathcal{O}_C(-1))$ is a vector bundle of rank $(c_1,[C])+n$ over $\widehat{M}^0(c-ne)$.

Therefore

Lemma 3.22. *The projection q_2 identifies $\widehat{M}(c,n)$ with the Grassmann bundle $\operatorname{Gr}(n, \operatorname{Ext}^1_f(\mathcal{E}', \mathcal{O}_C(-1)))$ of n-dimensional subspaces associated* *with the vector bundle* $\mathrm{Ext}^1_f(\mathcal{E}', \mathcal{O}_C(-1))$ *over* $\widehat{M}^0(c-ne)$. *In particular, we have*

$$\dim \widehat{M}(c,n) = \dim \widehat{M}^0(c-ne) + n(c_1,[C]),$$
$$\exp\dim \widehat{M}(c,n) = \exp\dim \widehat{M}^0(c) - n(n+r+(c_1,[C])).$$

If $(\mathcal{O}_X(1), K_X) < 0$, *then* $\widehat{M}(c,n)$ *is smooth and of expected dimension, provided it is nonempty.*

Proposition 3.23. *Let us consider the diagram in* (3.21).

(1) *The image of* $q_1 \colon \widehat{M}(c,n) \to \widehat{M}^0(c)$ *is the Brill-Noether locus* $\widehat{M}^0(c)_{\geq n}$.

(2) *The morphism* $q_1 \colon \widehat{M}(c,n) \to \widehat{M}^0(c)_{\geq n}$ *becomes an isomorphism if we restrict it to the open subscheme* $q_1^{-1}(\widehat{M}^0(c)_n)$.

(3) $\widehat{M}^0(c)_n$ *is a* $\mathrm{Gr}(n, n+(c_1,[C]))$*-bundle over* $\widehat{M}^0(c-ne)_0$ *via the restriction of* q_2.

(4) *Suppose* $\widehat{M}^0(c-ne)$ *is irreducible. Then* $\overline{\widehat{M}^0(c)_n} = \widehat{M}^0(c)_{\geq n}$.

(5) *Suppose that* $\widehat{M}^0(c)$ *and* $\widehat{M}^0(c-ne)$ *are irreducible and of expected dimension. Suppose further that* $\widehat{M}^0(c-ne)$ *is normal. Then the Brill-Noether locus* $\widehat{M}^0(c)_{\geq n} = \overline{\widehat{M}^0(c)_n}$ *is Cohen-Macauley and normal.*

Proof. (1) is clear.

(2) We use the following facts: (i) $\widehat{M}(c,n) \to \widehat{M}^0(c)$ is projective, (ii) we have an exact sequence

$$\mathbb{C} \to V^\vee \otimes V \xrightarrow{g} \mathrm{Ext}^1(E', E) \to \mathrm{Ext}^1(E,E),$$

with $V = \mathrm{Hom}(\mathcal{O}_C(-1), E)$, (iii) the Zariski tangent space of $\widehat{M}(c,n)$ at (E,V) is $\operatorname{coker} g = \mathrm{Ext}^1(E',E)/(V^\vee \otimes V)$. (See [7].)

(3) follows from Lemma 3.22 and (1).

(4) From the assumption and Lemma 3.22 $\widehat{M}(c,n)$ is irreducible. Then the assertion follows from (1).

(5) From the assumption $\widehat{M}^0(c)$ is a local complete intersection ([8, Th. 4.5.8]), and hence Cohen-Macauley. Since the determinantal subvariety $\widehat{M}^0(c)_{\geq n}$ has the correct codimension $\operatorname{codim}_{\widehat{M}^0(c)}(\widehat{M}^0(c)_{\geq n}) = n(n+r+(c_1,C))$, it is also Cohen-Macauley. From the assumption and Lemma 3.22 $\widehat{M}(c,n)$ is normal. Therefore $\widehat{M}^0(c)_{\geq n}$ is also normal. □

Remark 3.24. If we take $\Delta(c-ne) \geq \Delta_0$, where Δ_0 is as in Proposition 3.8, $\widehat{M}^0(c)$, $\widehat{M}^0(c-ne)$, are irreducible, normal and of expected dimension.

Lemma 3.25. *Suppose $(\mathcal{O}_X(1), K_X) < 0$. If $2(c_1, [C]) > 2r\Delta - r - 1 - (r^2-1)\chi(\mathcal{O}_X) + h^1(\mathcal{O}_X)$, then $\widehat{M}^0(c)_{\geq 1} = \emptyset$.*

Proof. We may asume that $(c_1, [C]) \geq 0$.

Suppose that $\widehat{M}^0(c - ne) \neq \emptyset$ for $n > 0$. Then

$$\begin{aligned}
0 \leq\ & \dim \widehat{M}^0(c-ne) \\
&= 2r\Delta(c) - n(n + r + 2(c_1, [C])) - (r^2-1)\chi(\mathcal{O}_X) + h^1(\mathcal{O}_X) \\
&\leq 2r\Delta(c) - (1 + r + 2(c_1, [C])) - (r^2-1)\chi(\mathcal{O}_X) + h^1(\mathcal{O}_X).
\end{aligned}$$

The result follows. □

Next we consider the corresponding study for another Brill-Noether locus $\widehat{M}^1(c)^i$ appearing in the other side of the wall.

Definition 3.26. Let $\widehat{N}(c, n)$ be the moduli of coherent systems $(E, U \subset \operatorname{Hom}(E, \mathcal{O}_C(-1)))$ such that $E \in \widehat{M}^1(c)$ and $\dim U = n$.

We have a natural morphism $q_1' \colon \widehat{N}(c, n) \to \widehat{M}^1(c)$.

For $(E, U) \in \widehat{N}(c, n)$, we set $E' := \operatorname{Ker}(E \to U^\vee \otimes \mathcal{O}_C(-1))$. By Lemma 3.13(3), we have $E' \in \widehat{M}^1(c - ne)$. We thus have the diagram:

$$\begin{tikzcd}
& \widehat{N}(c,n) \arrow[dl, "q_1'"'] \arrow[dr, "q_2'"] & \\
\widehat{M}^1(c) & & \widehat{M}^1(c-ne).
\end{tikzcd} \tag{3.27}$$

Conversely suppose that $E' \in \widehat{M}^1(c - ne)$ and an n-dimensional subspace $U^\vee \subset \operatorname{Ext}^1(\mathcal{O}_C(-1), E')$ are given. Then we can consider the associated exact sequence

$$0 \to E' \to E \to U^\vee \otimes \mathcal{O}_C(-1) \to 0 \tag{3.28}$$

by Lemma 3.13(4), we have $E \in \widehat{M}^1(c)$. Moreover (3.28) induces an injection $U \subset \operatorname{Hom}(E, \mathcal{O}_C(-1))$. Therefore $(E, U) \in \widehat{N}(c, n)$.

Note $\operatorname{Hom}(\mathcal{O}_C(-1), E') = 0 = \operatorname{Ext}^2(\mathcal{O}_C(-1), E')$ and $\dim \operatorname{Ext}^1(\mathcal{O}_C(-1), E') = (c_1(E'), [C]) + \operatorname{rk} E'$ by Lemma 3.2. If $\mathcal{E}'$ denotes an universal sheaf over $\widehat{M}^1(c - ne)$, then $\operatorname{Ext}^1_f(\mathcal{O}_C(-1), \mathcal{E}')$ is a vector bundle of rank $(c_1, [C]) + n + r$ over $\widehat{M}^1(c - ne)$.

We have

Lemma 3.29. *The projection q_2' identifies $\widehat{N}(c, n)$ with the Grassmann bundle $\operatorname{Gr}(n, \operatorname{Ext}^1_f(\mathcal{O}_C(-1), \mathcal{E}'))$ of n-dimensional subspaces associated with the vector bundle $\operatorname{Ext}^1_f(\mathcal{O}_C(-1), \mathcal{E}')$ over $\widehat{M}^1(c-ne)$. In particular,* *we have*

$$\dim \widehat{N}(c,n) = \dim \widehat{M}^1(c-ne) + n(r+(c_1,[C])),$$
$$\exp\dim \widehat{N}(c,n) = \exp\dim \widehat{M}^1(c) - n(n+(c_1,[C])).$$

If $((\mathcal{O}_X(1), K_X) < 0$, *then* $\widehat{N}(c,n)$ *is smooth and of expected dimension, provided it is nonempty.*

We have the statements corresponding to Proposition 3.23. Since they are very similar, we omit them.

Remark 3.30. As already mentioned in the introduction, a similar Grassmann bundle structure has been observed in the contexts of quiver varieties [18] and an exceptional bundle on K3 [29, 14] (see also [22] for an exposition). The moduli spaces of coherent systems in [29] and Hecke correspondences [18] play the same role connecting two moduli spaces with different Chern classes. However, there is a sharp distinction between the above blowup case and the other cases, which can be considered as the (-2)-curve. Namely the Grassmann bundle is defined only on a Brill-Noether locus, as both Hom and Ext survive in general for the other cases.

3.7. **Contraction of the Brill-Noether locus.** Consider $\widehat{M}^0(c)$ and set $n := (c_1,[C])$, $e := \operatorname{ch}(\mathcal{O}_C(-1))$, $c_\perp := c + ne$. Then we have $(c_\perp,[C]) = 0$. Therefore we have $\widehat{M}^0(c_\perp) \cong M^X(p_*(c_\perp)) = M^X(p_*(c))$ by Proposition 3.3. Therefore ξ in (3.1) can be considered as $\xi\colon \widehat{M}^0(c) \to \widehat{M}^0(c_\perp)$. Explicitly it is given by $\xi(E) = p^*(p_*(E))$.

Proposition 3.31. *Suppose* $n := (c_1,[C]) \geq 0$. *Let* ξ *be as in* (3.1).

(1) $\xi(\widehat{M}^0(c))$ *is identified with the Brill-Noether locus* $\widehat{M}^0(c_\perp)_{\geq n}$ *via the above isomorphism. In particular,* $\xi(\widehat{M}^0(c))$ *is a Cohen-Macauley and normal subscheme of* $M^X(p_*(c))$, *provided* $\widehat{M}^0(c_\perp)$, $\widehat{M}^0(c)$ *are irreducible and of expected dimension, and* $\widehat{M}^0(c)$ *is normal.*

(2) ξ *is an immersion on* $\widehat{M}^0(c)_0$.

(3) *Each Brill-Noether stratum* $\widehat{M}^0(c_\perp)_{n+i}$ *is isomorphic to* $\widehat{M}^0(c-ie)_0$, *so* $\widehat{M}^0(c_\perp)_{\geq n}$ *can be considered as a scheme structure on* $\bigsqcup_i \widehat{M}^0(c-ie)_0$ *requested in* (3.19).

(4) ξ *maps* $\widehat{M}^0(c)_i$ *to* $\widehat{M}^0(c_\perp)_{n+i}$, *and it can be identified with the Grassmann bundle* $\widehat{M}^0(c)_i \to \widehat{M}^0(c-ie)_0$ *in* (3.19) *under the isomorphism in* (3).

Proof. We consider the following diagram:

$$\begin{tikzcd} & \widehat{M}(c_\perp, n) \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \widehat{M}^0(c_\perp) \arrow[dr, "\cong"'] & & \widehat{M}^0(c) \arrow[dl, "\xi"] \\ & M^X(p_*(c)) & \end{tikzcd}$$

By Lemma 3.22 q_2 is the Grassmann bundle of n-planes in a vector bundle of rank $(c_1, [C]) = n$. Therefore q_2 is an isomorphism. Therefore the image of ξ is identified with the image of q_1. Hence (1) follows from Proposition 3.23(1).

Moreover q_1 is an immersion over $q_1^{-1}(\widehat{M}^0(c_\perp)_n)$ by Proposition 3.23(1). Via the isomorphism q_2 it is identified with $\widehat{M}^0(c)_0$. Hence we get (2).

(3) is also proved in a similar way. We consider $\widehat{M}(c_\perp, n+i)$ and the diagram

$$\begin{tikzcd} & \widehat{M}(c_\perp, n+i) \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \widehat{M}^0(c_\perp) & & \widehat{M}^0(c-ie) \end{tikzcd}$$

Then q_2 is again an isomorphism in this case also, and we have $\widehat{M}^0(c_\perp)_{n+i} \cong \widehat{M}(c_\perp, n+i)_0 \cong \widehat{M}^0(c-ie)_0$. $\square$

Let us constract the contraction in the other side of the wall. We consider the diagram with a yet undefined morphism ξ^+:

$$\begin{tikzcd} & \widehat{N}(c_\perp e^{[C]}, n') \arrow[dl, "q_1'"'] \arrow[dr, "q_2'"] & \\ \widehat{M}^0(c_\perp) \cong \widehat{M}^1(c_\perp e^{[C]}) \arrow[dr, "\cong"'] & & \widehat{M}^1(c) \arrow[dl, dotted, "\xi^+"] \\ & M^X(p_*(c)) & \end{tikzcd}$$

where $n' = (c_1, [C]) + r$, which is equal to the rank of the vector bundle $\operatorname{Ext}^1_f(\mathcal{O}_C(-1), \mathcal{E}')$ over $\widehat{M}^1(c)$. Therefore q_2' is an isomorphism. Hence we can define ξ^+ so that the diagram commutes.

Proposition 3.32. *Suppose $n' := (c_1, [C]) + r \geq 0$.*

(1) *$\xi^+(\widehat{M}^1(c))$ is identified with the Brill-Noether locus $\widehat{M}^0(c_\perp)^{\geq n'}$ via the above isomorphism. In particular, $\xi^+(\widehat{M}^1(c))$ is a Cohen-Macauley and normal subscheme of $M^X(p_*(c))$, provided $\widehat{M}^0(c_\perp)$, $\widehat{M}^1(c)$ are irreducible and of expected dimension, and $\widehat{M}^1(c)$ is normal.*

(2) *ξ^+ is an immersion on $\widehat{M}^1(c)^0$.*

(3) *Each Brill-Noether stratum* $\widehat{M}^0(c_\perp)^{n'+i}$ *is isomorphic to* $\widehat{M}^1(c-ie)^0$*, so* $\widehat{M}^0(c_\perp)^{\geq n'}$ *can be considered as a scheme structure on* $\bigsqcup_i \widehat{M}^1(c-ie)^0$ *requested in* (3.19)*.*

(4) ξ^+ *maps* $\widehat{M}^1(c)^i$ *to* $\widehat{M}^0(c_\perp)^{n'+i}$*, and it can be identified with the Grassmann bundle* $\widehat{M}^1(c)^i \to \widehat{M}^1(c-ie)^0$ *in* (3.19) *under the isomorphism in* (3)*.*

The proof is the same as one for Proposition 3.31, as we have the commutative diagram. We just describe how $E \in \widehat{M}^1(c)$ is mapped under the diagram:

$$\begin{tikzcd}[column sep=small]
 & & (0 \to E \arrow[dll, mapsto] \to E' \to U^\vee \otimes \mathcal{O}_C(-1) \to 0) \arrow[drr, mapsto, "\cong"'] & & \\
E'(-C) \longleftrightarrow E' \arrow[drr, mapsto, "\cong"'] & & & & E \arrow[dll, mapsto] \\
 & & p_*(E'(-C)) & &
\end{tikzcd}$$

We finally need to show that the targets of ξ and ξ^+ are the same.

Proposition 3.33. *Suppose that* $(c_1(c_\perp), [C]) = 0$*. Then* $\widehat{M}^0(c_\perp)^{\geq n+r} = \widehat{M}^0(c_\perp)_{\geq n}$*. In fact, the both Brill-Noether loci are identified with*

$$\{F \in M^X(p_*(c_\perp)) \mid \dim \operatorname{Hom}(F, \mathbb{C}_0) \geq n+r\}$$

under the isomorphism $\widehat{M}^0(c_\perp) \cong M^X(p_*(c_\perp))$*.*

Proof. Let $\mathcal{F}$ be a universal family over $X \times M^X(p_*(c_\perp))$. Let $0 \to \mathcal{V} \to \mathcal{W} \to \mathcal{F} \to 0$ be a locally free resolution.

Then $\widehat{M}^0(c_\perp)_{\geq n}$ is defined by the zero locus of $\bigwedge^{-\chi(\mathcal{O}_C(-1), p^*(\mathcal{V}))+1-n}\rho$, where

$$\begin{tikzcd}
 & 0 \arrow[r] & \operatorname{Hom}_f(\mathcal{O}_C(-1), p^*(\mathcal{F})) \arrow[dll] \\
\operatorname{Ext}^1_f(\mathcal{O}_C(-1), p^*(\mathcal{V})) \arrow[r, "\rho"'] & \operatorname{Ext}^1_f(\mathcal{O}_C(-1), p^*(\mathcal{W})) \arrow[r] & \operatorname{Ext}^1_f(\mathcal{O}_C(-1), p^*(\mathcal{F})) \arrow[dll] \\
0. & &
\end{tikzcd}$$

On the other hand, $\widehat{M}^0(c_\perp)^{\geq n+r}$ is defined by the zero locus of $\bigwedge^{-\chi(p^*(\mathcal{W}), \mathcal{O}_C)+1-n-r}\rho'$, where

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_f(p^*(\mathcal{F}), \mathcal{O}_C) \arrow[r] & \operatorname{Hom}_f(p^*(\mathcal{W}), \mathcal{O}_C) \arrow[r, "\rho'"] & \operatorname{Hom}_f(p^*(\mathcal{V}), \mathcal{O}_C) \arrow[dll] \\
 & \operatorname{Ext}^1_f(p^*(\mathcal{F}), \mathcal{O}_C) \arrow[r] & 0 &
\end{tikzcd}$$

The transpose of ρ is given by

$$\operatorname{Ext}^1_f(p^*(\mathcal{W}), \mathcal{O}_C(-2)) \to \operatorname{Ext}^1_f(p^*(\mathcal{V}), \mathcal{O}_C(-2)),$$

which is naturally isomorphic to ρ'. Moreover, the projection formula shows that ρ' is equal to

$$\mathrm{Hom}_f(\mathcal{W}, \mathbb{C}_0) \to \mathrm{Hom}_f(\mathcal{V}, \mathbb{C}_0),$$

which implies the isomorphisms among Brill-Noether loci as in the assertion. □

3.8. **Ample line bundles on moduli spaces.** If both $\widehat{M}^m(c)$ and $\widehat{M}^{m+1}(c)$ would be GIT quotients of a *common* variety for the stability conditions separated by a single wall, they are flip provided $\xi_m \colon \widehat{M}^m(c) \to \widehat{M}^{m,m+1}(c)$ would be a small contraction ([27]). As we do not know how to construct this picture in our setting, we prove this statement *directly*. Moreover the smallness condition is related to the dimension of the moduli spaces, and hence we do not expect such a result unless we assume $(\mathcal{O}_X(1), K_X) < 0$ or c_2 is sufficiently large. Instead of assuming these kinds of conditions, we produce a line bundle which is relatively ample on $\widehat{M}^m(c)$, but not on $\widehat{M}^{m+1}(c)$, where we consider the spaces relative to the Uhlenbeck compactification $M_0^X(p_*(c))$.

We continue to assume $\gcd(r, (c_1, p^*\mathcal{O}_X(1))) = 1$. For $d \in K(X)$ with $\mathrm{rk}(d) = r$ and $c_1(d) = c_1(p_*(c))$, there is a class $\alpha_d \in K(X)$ such that $\mathrm{rk}\,\alpha_d = 0$ and $\chi(d \otimes \alpha_d) = 1$.

Let p_X, p_M be the projections from $X \times M^X(d)$ to the first and second factors respectively. If we twist a universal bundle $\mathcal{E}$ by a line bundle L over the moduli space $M^X(d)$, we have $\det p_{M!}(\mathcal{E} \otimes p_M^* L \otimes p_X^*\alpha) = \det p_{M!}(\mathcal{E} \otimes p_X^*\alpha) \otimes L^{\otimes \chi(d\otimes\alpha)}$ for $\alpha \in K(X)$. Therefore for $\alpha = \alpha_d$ we can normalize a universal family $\mathcal{E}_d$ on $X \times M^X(d)$ so that $\det p_{M!}(\mathcal{E}_d \otimes p_X^*\alpha) = \mathcal{O}_{M^X(d)}$.

Lemma 3.34. *Let* $\beta \in K(X)$ *be a class with* $\mathrm{rk}\,\beta = -1$. *Then* $\det p_{M!}(\mathcal{E}_d \otimes p_X^*\beta)$ *is relatively ample over* $M_0^X(d)$.

Remark 3.35. For $\beta' := \beta - \chi(d \otimes \beta)\alpha_d$, we have $\chi(d \otimes \beta') = 0$, which means that $\det p_{M!}(\mathcal{E} \otimes \beta')$ does not depend on the choice of the universal family $\mathcal{E}$.

Proof. By Simpson's construction of the moduli space, $N_{\mathcal{E}} := \det p_{M!}(\mathcal{E}(n+m))^{\chi(d(n))} \otimes \det p_{M!}(\mathcal{E}(m))^{-\chi(d(n+m))}$ is ample for $n \gg m \gg 0$, where $\mathcal{E}$ is a universal family. Since N does not depend on the choice of the universal family, we may assume that $N_{\mathcal{E}} = N_{\mathcal{E}_d}$. We set $\gamma := \chi(d(n))\mathcal{O}_X(n+m) - \chi(d(n+m))\mathcal{O}_X(m)$. Then $\mathrm{rk}\,\gamma < 0$ and $\beta \in \mathbb{Q}_{>0}\gamma + \mathbb{Q}h + \mathbb{Q}\alpha_d$, where $h \in K(X)$ is a class such that $\det p_{M!}(\mathcal{E} \otimes h)$ descends to a determinant line bundle on $M_0^X(d)$. (See [8, §8.2].) Therefore $\det p_{M!}(\mathcal{E}_d \otimes \beta)$ is relatively ample over $M_0^X(d)$. □

Suppose $d = p_*(c)$ and take β with $\operatorname{rk}\beta = -1$ as above, and we normalize the universal family as above.

Proposition 3.36. *We set* $L_t := \det p_{\widehat{M}!}\left(\mathcal{E} \otimes p_X^*(\beta + t\mathcal{O}_C(-1))\right)$.

(1) *If* $m-1 < t < m$*, then* L_t *is relatively ample over* $M_0^X(d)$*.*

(2) *Assume that* $\widehat{M}^m(c) \neq \widehat{M}^{m+1}(c)$*. Then* L_t *is not relatively ample over* $M_0^X(d)$ *for* $t \geq m$*.*

(3) *Assume that* $\widehat{M}^m(c) \neq \widehat{M}^{m-1}(c)$*. Then* L_t *is not relatively ample over* $M_0^X(d)$ *for* $t \leq m-1$*.*

Proof. (1) We note that $\mathcal{O}_{\widehat{X}}(-mC) = \mathcal{O}_{\widehat{X}} - m\mathcal{O}_C(-1) - \frac{m(m+1)}{2}\mathbb{C}_p$, where p is a point in C. Hence $c \otimes (\beta + m\mathcal{O}_C(-1))) = c(-mC) \otimes (\beta - \frac{m(m+1)}{2}\mathbb{C}_p)$. Since $p_*(E(-mC)) \in M^X(d - r\frac{m(m+1)}{2}\mathbb{C}_0)$ for $E \in \widehat{M}^m(c)$, L_m is the pull-back of a relatively ample line bundle on $M^X(d - r\frac{m(m+1)}{2}\mathbb{C}_0)$ by the previous lemma. In the same way, we see that L_{m-1} also the pull-back of a relatively ample line bundle on $M^X(d - r\frac{m(m-1)}{2}\mathbb{C}_0)$. Since $\widehat{M}^m(c) \to M^X(d - r\frac{m(m+1)}{2}\mathbb{C}_0) \times M^X(d - r\frac{m(m-1)}{2}\mathbb{C}_0)$ is an embedding, $aL_m + bL_{m-1}$ is relatively ample for $a, b > 0$.

(2) By $\widehat{M}^m(c) \to M^X(d - r\frac{m(m+1)}{2}\mathbb{C}_0)$ the Grassmann bundle structures of the Brill-Noether loci $\widehat{M}^m(c)_i$ are contracted. From the assumption, the Brill-Noether locus $\widehat{M}^m(c)_{\geq 1}$ is nonempty, so L_m is not relatively ample. The proof of (3) is the same. □

This completes our construction of the diagram $(*)$ in the introduction.

3.9. **Another distinguished chamber – torsion free sheaves on blow-up.**

Proposition 3.37 (cf. [25, Prop. 7.1])**.** *Fix* $c \in H^*(\widehat{X})$*. There exists* m_0 *such that if* $m \geq m_0$*,* $\widehat{M}^m(c)$ *is the moduli space of* $(p^*H - \varepsilon C)$*-stable torsion free sheaves on* $\widehat{X}$ *for sufficiently small* $\varepsilon > 0$*.*

If $(\mathcal{O}_X(1), K_X) < 0$*, then we can take*

$$m_0 = -(c_1, [C]) + r\Delta - \frac{1}{2}(r + 1 + (r^2-1)\chi(\mathcal{O}_X) - h^1(\mathcal{O}_X)) + 1.$$

Proof. Consider the projective morphism $\widehat{\pi}\colon \widehat{M}^m(c) \to M_0^X(p_*(c))$ in (3.5), where $M_0^X(p_*(c))$ is the Uhlenbeck compactification on X. From Proposition 3.16, there exists m_0 such that if $m \geq m_0$ and $E \in \widehat{M}^m(c)$, we have $E \in \widehat{M}^{m+1}(c)$, i.e. $\widehat{M}^m(c) \cong \widehat{M}^{m+1}(c) \cong \widehat{M}^{m+2}(c) \cong \cdots$. If $(\mathcal{O}_X(1), K_X) < 0$, then m_0 can be explicitly given by Lemma 3.25.

Suppose that E is torsion free and $(p^*H - \varepsilon C)$-stable. Then $p_*(E(-mC))$ is μ-stable for sufficiently large m. The torsion freeness of E implies

$\operatorname{Hom}(\mathcal{O}_C(-mC), E) = 0$ for any m. On the other hand, we have $\operatorname{Hom}(E(-mC), \mathcal{O}_C)$ is zero for $m \gg 0$, as $\mathcal{O}_{\widehat{X}}(-C)$ is relatively ample with respect to $p\colon \widehat{X} \to X$. Therefore E is m-stable for sufficiently large m. From the above discussion, E is m_0-stable.

Conversely suppose that E is m_0-stable. Then E is m-stable for any $m \geq m_0$. In particular, $\operatorname{Hom}(\mathcal{O}_C, E(-mC)) = 0$ for $m \geq m_0$. Suppose that E is not torsion free, and let $0 \neq T \subset E$ be its torsion part. Then as $\mathcal{O}_{\widehat{X}}(-C)$ is relatively ample, we have $p_*(T(-mC)) \neq 0$ for $m \gg 0$. Since $p_*(T(-mC))$ is supported at 0, we have

$$0 \neq \operatorname{Hom}(\mathbb{C}_0, p_*(T(-mC))) = \operatorname{Hom}(\mathcal{O}_C, T(-mC)) \subset \operatorname{Hom}(\mathcal{O}_C, E(-mC)).$$

This is a contradiction. Therefore E is torsion free. Since $p_*(E(-mC))$ is μ-stable for any $m \geq m_0$, E is $(p^*H - \varepsilon C)$-stable for sufficiently small ε. □

3.10. **The distinguished chamber – revisited.** In this subsection we assume $0 \geq (c_1, [C]) > -r$ and study moduli spaces $\widehat{M}^1(c)$ under this assumption. We can twist sheaves by a line bundle $\mathcal{O}(C)$, and this condition is satisfied. But it also changes the stability condition, so studying only $\widehat{M}^1(c)$ means that we are choosing a certain chamber.

The case $(c_1, [C]) = 0$ was already discussed in §3.1. (Strictly speaking we studied $\widehat{M}^0(c)$.) So we consider the case $0 > (c_1, [C]) > -r$.

Proposition 3.38. *Suppose* $0 < n := -(c_1, [C]) < r$. *We have a diagram*

$$\begin{tikzcd} & \widehat{N}(c,n) \arrow[dl, "q_1'"'] \arrow[dr, "q_2'"] & \\ \widehat{M}^1(c) & & \widehat{M}^1(c-ne) \end{tikzcd}$$

such that (i) $\widehat{M}^1(c) = \widehat{M}^1(c)^{\geq n}$, (ii) q_1' *is surjective and isomorphism over the open subscheme* $\widehat{M}^1(c)^n$, (iii) q_2' *is a* $\operatorname{Gr}(n,r)$*-bundle.*

If $\widehat{M}^1(c)$, $\widehat{M}^1(c-ne)$ *are irreducible and of expected dimension, then* q_1' *is birational.*

We have $(c_1(c-ne), [C]) = (c_1, [C]) + n = 0$. Therefore $\widehat{M}^1(c-ne)$ becomes $M^X(p_*(c))$ after crossing a single wall.

Proof. Let $E \in \widehat{M}^1(c)$. We have $\chi(E, \mathcal{O}_C(-1)) = -(c_1, [C]) = n > 0$ by our assumption. As $\operatorname{Ext}^2(E, \mathcal{O}_C(-1)) = \operatorname{Hom}(\mathcal{O}_C, E)^\vee = 0$ by the stability of E, we have $\dim \operatorname{Hom}(E, \mathcal{O}_C(-1)) \geq n$. This shows (i).

We consider $q_1'\colon \widehat{N}(c,n) \to \widehat{M}^1(c)$ as in (3.27). From the above observation, it is surjective. Moreover it is an isomorphism over $\widehat{M}^1(c)^n$. (ii) follows.

We have $q_2'\colon \widehat{N}(c,n) \to \widehat{M}^0(c - n\operatorname{ch}(\mathcal{O}_C))$. By Lemma 3.29 it is the Grassmann bundle $\operatorname{Gr}(n, \operatorname{Ext}^1_f(\mathcal{O}_C, \mathcal{E}'))$ of n-dimensional subspaces in $\operatorname{Ext}^1_f(\mathcal{O}_C, \mathcal{E}')$ over $\widehat{M}^0(c - n\operatorname{ch}(\mathcal{O}_C))$, which is of rank $(c_1(E'), [C]) = (c_1, [C]) + n = r$. Therefore we have (iii). $\square$

4. Moduli spaces as incidence varieties

Recall that we have a morphism

$$\begin{array}{cccc} \xi \times \eta\colon & \widehat{M}^0(c) & \to & M^X(p_*(c)) \times M^X(p_*(c) + n\,\mathrm{pt}) \\ & E & \mapsto & (p_*(E), p_*(E(C))), \end{array}$$

where $n = (c_1, [C])$. (See §3.1.)

The purpose of this section is to prove the following:

Theorem 4.1. *The morphism $\xi \times \eta$ identifies $\widehat{M}^0(c)$ with the incidence variety $L(p_*(c) + n\,\mathrm{pt}, n)$ with $n = (c_1, [C])$, where*

$$L(c', n) := \{(F, U) \mid F \in M^X(c'), U \subset \operatorname{Hom}(F, \mathbb{C}_0), \dim U = n\}$$

for $c' \in H^*(X)$.

Remark 4.2. If $c' = 1 - N\,\mathrm{pt}$, $n = 1$, then $L(c', 1) = \{(I, U) \mid I \in X^{[N]}, U \subset \operatorname{Hom}(I, \mathbb{C}_0), \dim U = 1\} \subset X^{[N+1]} \times X^{[N]}$ is called the *nested Hilbert scheme*, and has been studied by various people. Here $X^{[N]}$ is the Hilbert scheme of N points in X.

The variety $L(c', m)$ is the quotient of the moduli of framed sheaves $(F, F \to \mathbb{C}_0^{\oplus m})$ by the action of $GL(m)$. We have a projective morphism $\sigma\colon L(c', m) \to M^X(c')$ by sending (F, U) to F. For $(F, U) \in L(c', m)$, we set $F' := \operatorname{Ker}(F \to U^\vee \otimes \mathbb{C}_0)$. It is easy to see that $F \to U^\vee \otimes \mathbb{C}_0$ is surjective. Moreover F' is a μ-stable sheaf. Thus we also have a morphism $\varsigma\colon L(c', n) \to M^X(c' - n\,\mathrm{pt})$ by sending (F, U) to F'. By the same argument as in the case of $\widehat{M}(c, n)$, we have an isomorphism from $L(c', n)$ to the moduli space of 'dual' coherent system $(F', U^\vee \subset \operatorname{Ext}^1(\mathbb{C}_0, F'))$ with $F' \in M^X(c' - n\,\mathrm{pt})$, $\dim U^\vee = n$.

Consider

$$\begin{array}{cccc} \sigma \times \varsigma\colon & L(c', n) & \to & M^X(c') \times M^X(c' - n\,\mathrm{pt}) \\ & (F, U) & \mapsto & (F, F'). \end{array}$$

Lemma 4.3. *The morphism $\sigma \times \varsigma$ is a closed immersion.*

For this purpose, it is sufficient to prove that

(1) $\sigma \times \varsigma$ is injective and
(2) $d(\sigma \times \varsigma)_*$ is injective.

From the μ-stability of F, F' and $\mu(F) = \mu(F')$, the following holds from a standard argument.

Lemma 4.4. $\operatorname{Hom}(F, F) \cong \operatorname{Hom}(F', F') \cong \operatorname{Hom}(F', F) \cong \mathbb{C}$.

Proof of (1). Assume that $(F_1, U_1), (F_2, U_2) \in L(c', m)$ satisfy $F_1 \cong F_2$ and $F_1' \cong F_2'$, where $F_\alpha' := \operatorname{Ker}(F_\alpha \to {U_\alpha}^\vee \otimes \mathbb{C}_0)$ for $\alpha = 1, 2$. Since $\operatorname{Hom}(F_1', F_2') \cong \operatorname{Hom}(F_1', F_2) \cong \operatorname{Hom}(F_1, F_2)$ by the previous lemma, we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1' & \longrightarrow & F_1 & \longrightarrow & U_1^\vee \otimes \mathbb{C}_0 & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & F_2' & \longrightarrow & F_2 & \longrightarrow & U_2^\vee \otimes \mathbb{C}_0 & \longrightarrow & 0
\end{array}$$

Hence $(F_1, U_1) \cong (F_2, U_2)$. $\square$

Proof of (2). The Zariski tangent space of $L(c', n)$ at (F, U) is

$$\operatorname{Ext}^1(F, F')/\operatorname{End}(U)$$

and the obstruction for an infinitesimal lifting belongs to

$$\operatorname{Ext}^2(F, F') \cong \operatorname{Hom}(F', F \otimes K_Y)^\vee,$$

where $\operatorname{End}(U) \to \operatorname{Ext}^1(F, F')$ is the homomorphism given by the composition of $\operatorname{End}(U) \to U^\vee \otimes \operatorname{Hom}(F, \mathbb{C}_0) = \operatorname{Hom}(F, U^\vee \otimes C_0)$ and $\operatorname{Hom}(F, U^\vee \otimes \mathbb{C}_0) \to \operatorname{Ext}^1(F, F')$. Therefore the assertion follows from the following lemma. $\square$

Lemma 4.5.

$$d(\sigma \times \varsigma)_* : \operatorname{Ext}^1(F, F')/\operatorname{End}(U) \to \operatorname{Ext}^1(F', F') \oplus \operatorname{Ext}^1(F, F)$$

is injective.

Proof. We have the following exact and commutative diagram:

$$\begin{array}{ccccc}
0 & & & & \\
\Big\downarrow & & & & \\
U^\vee \otimes \operatorname{Hom}(F, \mathbb{C}_0)/\operatorname{End}(U) & \longrightarrow & \operatorname{Ext}^1(F, F')/\operatorname{End}(U) & \longrightarrow & \operatorname{Ext}^1(F, F) \\
\Big\downarrow & & \Big\downarrow & & \\
\operatorname{Hom}(F', U^\vee \otimes \mathbb{C}_0) & \xrightarrow{\alpha} & \operatorname{Ext}^1(F', F') & &
\end{array}$$

Since $\operatorname{Hom}(F', F') \to \operatorname{Hom}(F', F)$ is isomorphic, α is injective, which implies the assertion. $\square$

Obviously we have an isomorphism

$$\begin{array}{ccc}
L(c', n) & \xrightarrow{\cong} & N(p^*(c')e^{[C]}, n) \\
(F, U) & \mapsto & (p^*(F)(C), U),
\end{array}$$

where $N(p^*(c')e^{[C]}, n)$ is as in Definition 3.26 and we have used Proposition 3.3.

We also have a morphism

$$\begin{array}{ccc} N(p^*(c')e^{[C]}, n) & \xrightarrow{\cong} & \widehat{M}^0(p^*(c') - n\,\mathrm{ch}(\mathcal{O}_C)) \\ (E(C), U) & \mapsto & E' := \mathrm{Ker}(E \to U^\vee \otimes \mathcal{O}_C), \end{array}$$

which is essentially q_2' in (3.27). As $\mathrm{rk}\,\mathrm{Ext}^1(\mathcal{O}_C, E') = n$ by Lemma 3.2, this morphism is an isomorphism by Lemma 3.29.

As $p^*(c') - n\,\mathrm{ch}(\mathcal{O}_C) = c$, this completes the proof of Theorem 4.1.

Remark 4.6. If $n \gg \dim M_H^p(p^*(c) - n\mathcal{O}_C)$, then we also have an embedding $M_H^p(p^*(c) - n\mathcal{O}_C) \to L(c, n) \to M_H(c - n\,\mathrm{pt})$.

5. Betti numbers

In this section, we prove the formula $(**)$ in the introduction and its higher rank generalization.

5.1. **Framed moduli spaces.** We consider $p\colon \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ the blow-up of the projective plane at $0 = [1 : 0 : 0]$. Let $\ell_\infty = \{[0 : z_1 : z_2]\}$ and denote its inverse image $p^{-1}(\ell_\infty)$ by the same notation ℓ_∞ for brevity. Following [25] we consider the framed moduli space of framed coherent sheaves (E, Φ) on $\widehat{\mathbb{P}}^2 = \widehat{\mathbb{C}}^2 \cup \ell_\infty$ with $\mathrm{ch}(E) = c$, where E is assumed to be locally free along ℓ_∞, the framing Φ is a trivialization $\Phi\colon E|_{\ell_\infty} \xrightarrow{\cong} \mathcal{O}_{\ell_\infty}^{\oplus r}$ over ℓ_∞, and finally E satisfies

$$\mathrm{Hom}(E, \mathcal{O}_C(-m-1)) = 0, \qquad \mathrm{Hom}(\mathcal{O}_C(-m), E) = 0.$$

This space was written as $\widehat{M}_\zeta(r, k, n)$ in [25], where $r(c) = r$, $(c_1(c), [C]) = -k$, $\Delta(c) := \int_{\widehat{\mathbb{P}}^2} c_2(c) - (r-1)c_1(c)^2/(2r) = n$, and the parameter $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$ satisfying $0 > m\zeta_0 + (m+1)\zeta_1 \gg -1$. But we use the same notation $\widehat{M}^m(c)$ as in the ordinary moduli space for brevity. We hope this does not make any confusion. Also we set $X = \mathbb{C}^2$, $\widehat{X} = \widehat{\mathbb{C}}^2$. This convention applies to the other moduli spaces: $M^X(c')$ denotes the framed moduli space of torsion free sheaves on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$, $M_0^X(c')$ denotes the Uhlenbeck partial compactification, i.e. $M_0^X(c') = M_{lf}^X(c') \sqcup M_{lf}^X(c' + \mathrm{pt}) \times \mathbb{C}^2 \sqcup M_{lf}^X(c' + 2\,\mathrm{pt}) \times S^2(\mathbb{C}^2) \sqcup \cdots$, where $M_{lf}^X(c')$ denotes the framed moduli space of *locally free* sheaves on $\mathbb{P}^2$.

A modification of the construction of the moduli space in §2 to the framed moduli space is standard, and is omitted. Otherwise, we can use the quiver description in [25] to construct the framed moduli space. We also have a projective morphism $\widehat{\pi}\colon \widehat{M}^m(c) \to M_0(p_*(c))$, where M_0

denote the Uhlenbeck partial compactification of the framed moduli space on $\mathbb{P}^2$. (See [20, Chapters 2, 3] or [24, §3].)

As is mentioned in the beginning of §3, we may assume $m = 0$ for most purposes.

5.2. **Universality of the blow-up formula.** We consider the framed moduli spaces and ordinary moduli spaces of m-stable sheaves simultaneously. So $p\colon \widehat{X} \to X$ be the blowup of either a projective surface or $\mathbb{C}^2$ at the point 0. We define a stratification of $M^X(c)$, $\widehat{M}^m(c)$ as in [24, F.4]: Let $\iota\colon X \setminus \{0\} \to X$ be the inclusion. We define

$$M^X(c)_k := \{E \in M^X(c) \mid \Delta(\iota_*(E|_{X\setminus\{0\}})) = \Delta(c) - k\},$$
$$\widehat{M}^m(c)_k := \{E \in \widehat{M}^m(c) \mid \Delta(\iota_*(E|_{\widehat{X}\setminus C})) = \Delta(c) - k\},$$

where we identified $\widehat{X} \setminus C$ with $X \setminus \{0\}$. Then [24, Cor. F.22] shows that we have the following equalities in the Grothendieck group of $\mathbb{C}$-varieties when $m = \infty$:

$$\sum_{c'} [M^X(c')]\mathfrak{q}^{\Delta(c')} = \Big(\sum_{c'} [M^X(c')_0]\mathfrak{q}^{\Delta(c')}\Big)\Big(\sum_n [\mathfrak{Q}(r,n)]\mathfrak{q}^n\Big),$$
$$\sum_{c} [\widehat{M}^m(c)]\mathfrak{q}^{\Delta(c)} = \Big(\sum_{c'} [M^X(c')_0]\mathfrak{q}^{\Delta(c')}\Big)\Big(\sum_n [\widehat{\mathfrak{Q}}^m(r,k,n)]\mathfrak{q}^n\Big),$$

where c', c runs over all $H^*(X)$, $c \in H^*(\widehat{X})$ with fixed $r(c) = r(c') = r$ and $c_1(c) = c_1$, $c_1(c') = p^*c + k[C]$. Here $\mathfrak{Q}(r,n)$, $\widehat{\mathfrak{Q}}^{m=\infty}(r,k,n)$ are certain quot-schemes, which are independent of surfaces. Moreover they are the *same* for framed moduli spaces and ordinary moduli spaces. These equalities in the Grothendieck group of $\mathbb{C}$-varieties imply the corresponding equalities for virtual Hodge polynomials. If varieties are smooth and projective (e.g. the rank 1 case or $(\mathcal{O}_X(1), K_X) < 0$), then virtual Hodge polynomials are equal to Hodge polynomials.

From the proof of [24, Cor. F.22], the same result holds for *finite* m. In particular, in order to prove the formula $(**)$ or its higher rank generalization, it is enough to prove it for framed moduli spaces. So we only consider framed moduli spaces in the rest of this section.

5.3. **A combinatorial description of fixed points.** We have an $(r+2)$-dimensional torus $\widetilde{T} = T^r \times (\mathbb{C}^*)^2$ action on $\widehat{M}^0(c)$, $M^X(p_*(c))$. The first factor T^r acts by the change of framing, and the second factor $(\mathbb{C}^*)^2$ acts via the action on the base space $\mathbb{P}^2$ given by

$$[z_0 : z_1 : z_2] \mapsto [z_0 : t_1 z_1 : t_2 z_2],$$

and the induced action on $\widehat{\mathbb{P}}^2$.

The purpose of this subsection is to classify the fixed points in $\widehat{M}^0(c)$. As in the case of $\widehat{M}^m(c)$ for $m \gg 0$ and $M^X(c)$ ([23] or [24, §3]), a framed sheaf (E,Φ) is fixed by the first factor T^r if and only if it decomposes into a direct sum $E = E_1 \oplus \cdots \oplus E_r$ into rank 1 sheaves. So we first assume that the rank r is 1, and $\widetilde{T} = \mathbb{C}^* \times (\mathbb{C}^*)^2$, but the first factor $\mathbb{C}^*$ acts trivially. By Theorem 4.1 we have

$$\begin{aligned}\widehat{M}^0(c) &\cong L(p_*(c) + n\,\mathrm{pt}, n)\\ &= \{(F,\Phi,U) \mid (F,\Phi) \in M^X(p_*(c)+n\,\mathrm{pt}), U \subset \operatorname{Hom}(F,\mathbb{C}_0), \dim U = n\}\end{aligned}$$

with $n = (c_1,[C])$, and it is an incidence variety in $M^X(p_*(c)+n\,\mathrm{pt}) \times M^X(p_*(c))$. As we are assuming that the rank is 1, it is the product $X^{[N+n]} \times X^{[N]}$ of Hilbert scheme of points in $X = \mathbb{C}^2$, where $p_*(c) + n\,\mathrm{pt} = 1 - N\,\mathrm{pt}$. Also recall $X^{[N]}$ is the set of all ideals in the polynomial ring $\mathbb{C}[x,y]$ such that $\dim \mathbb{C}[x,y]/I = N$. So we have

$$\begin{aligned}&\widehat{M}^0(c)\\ &\cong \{(I',I) \in X^{[N+n]} \times X^{[N]} \mid I' \subset I \subset \mathbb{C}[x,y] \text{ a flag of ideals, } I/I' \cong \mathbb{C}^n\},\end{aligned}$$

where $\mathbb{C}^n$ is the n-dimensional vector space with the trivial $\mathbb{C}[x,y]$-module structure.

As the isomorphism is $\widetilde{T}$-equivariant, a fixed point is mapped to a fixed point. The torus fixed points in $X^{[N]}$ are monomial ideals I in $\mathbb{C}[x,y]$, and are in bijection to Young diagrams with N boxes as in [20, Chap. 5]. Moreover, the box at the coordinate (a,b) corresponds to the 1-dimensional weight space $\mathbb{C}x^a y^b \pmod{I}$ of weight $t_1^{-a}t_2^{-b}$.

Therefore the fixed points in $\widehat{M}^0(c)$ correspond to pairs (I,I') of monomial ideals such that $I/I' \cong \mathbb{C}^n$. Let Y be the Young diagram corresponding to I'. Its boxes correspond to weight spaces of $\mathbb{C}[x,y]/I'$. Then $I/I' \subset \mathbb{C}[x,y]/I'$ is a direct sum of weight spaces, so corresponds to a subset S of boxes in Y. Moreover, as I/I' must be the trivial $\mathbb{C}[x,y]$-module, so it must be contained in

$$\operatorname{Ker} \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] : \mathbb{C}[x,y]/I' \to \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}[x,y]/I'.$$

Therefore S must be consisting of removable boxes. Here recall a box in a Young diagram Y at the coordinate (a,b) is *removable* if there are no boxes above and right of (a,b). In terms of a monomial ideal I' corresponding to Y, removable boxes correspond to weight spaces contained in $\operatorname{Ker}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

Conversely if (Y,S) is given, then we set I, I' be the monomial ideals corresponding to $Y\setminus S$, Y respectively. Then $I' \subset I$, and I/I' is a trivial $\mathbb{C}[x,y]$-module.

For an arbitrary rank case we have r-tuples of such pairs (Y_α, S_α) corresponding to each factor E_α $(\alpha = 1, \dots, r)$.

Lemma 5.1. *The torus fixed points in $\widehat{M}^0(c)$ are in bijection to r-tuples of pairs (Y_α, S_α) of a Young diagram Y_α and a set S_α consisting of removable boxes such that $\sum_\alpha \#S_\alpha = (c_1, [C])$, $\sum_\alpha |Y_\alpha| = -\int_{\widehat{X}} \operatorname{ch}_2 + \frac{1}{2}(c_1, [C])$.*

We mark a box in S_α and call it a *marked box*. (See Figure 1.)

As fixed points are isolated, so the class of $\widehat{M}^m(c)$ in the Grothendieck group of $\mathbb{C}$-varieties is a polynomial in the class of $\mathbb{C}$. In particular, it is determined by its Poincaré polynomial. Therefore we will discuss only on Poincaré polynomials hereafter.

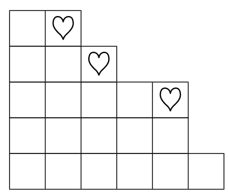

FIGURE 1. Young diagram and marked removable boxes

5.4. **Tangent space – rank 1 case.** We first state the weight decomposition of the tangent space in the rank 1 case.

Let (Y, S) be a pair of a Young diagram and marked removable boxes corresponding to a torus fixed point (E, Φ) in $\widehat{M}^0(c)$. We call a box in Y *irrelevant* if

a) the upmost box in the column is marked, and
b) the rightmost box in the row is marked.

In Figure 2 the boxes with $\heartsuit$ are marked removable boxes, and the boxes with $\heartsuit$ or $\spadesuit$ are irrelevant boxes. We call a box *relevant* if it is not irrelevant. Then

Proposition 5.2. *We have*

$$\operatorname{ch} T_{(E,\Phi)} \widehat{M}^0(c) = \sum_s \left(t_1^{-l_Y(s)} t_2^{a_{Y\setminus S}(s)+1} + t_1^{l_{Y\setminus S}(s)+1} t_2^{-a_Y(s)} \right),$$

where the summation runs over all relevant boxes s in Y, and $Y \setminus S$ is the Young diagram obtained by removing all marked boxes from Y.

The proof will be given in more general higher rank cases in Proposition 5.5.

We have $\widehat{M}^0(ce^{-m[C]}) \cong \widehat{M}^m(c)$, so we may assume $c_1 = 0$. Then

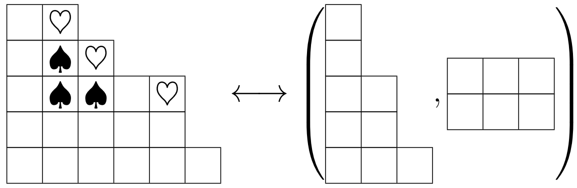

FIGURE 2. Marked removable boxes $\heartsuit$ and a pair of Young diagrams

Corollary 5.3. *Let* $c_N = 1 - N\,\mathrm{pt}$. *The Poincaré polynomial of* $\widehat{M}^m(c_N)$ *is given by*

$$\sum t^{2(N+m-l(Y))}$$

where the summation runs over all Young diagrams with m *marked removable boxes with* $|Y| = N + m(m+1)/2$*, and* $l(Y)$ *is the number of columns in* Y*.*

Proof. By the same argument as in [20, Cor. 5.10], it is enough to count the dimension of sum of weight spaces which satisfy either of the followings:

(1) the weight of t_2 is negative,
(2) the weight of t_2 is 0 and the weight of t_1 is negative.

The second possibility cannot happen. Therefore it is number of relevant boxes with $a_Y(s) > 0$. This is equal to $|Y| - m(m-1)/2 - l(Y) = N + m - l(Y)$. □

5.5. **A combinatorial bijection.** In [23, §3] we parametrized torus fixed points in the Hilbert schemes of points on the blowup $\widehat{\mathbb{C}}^2$ via a pair of partitions. The parametrization in the previous subsection must be related to this parametrization in the limit $m \to \infty$. This will be done in this subsection.

Let us consider two sets A, B consisting of

(1) pairs of Young diagrams Y and sets S of m marked removable boxes such that $|Y| - m(m+1)/2 = N$,
(2) pairs of Young diagrams (Y^1, Y^2) such that Y^2 has at most m columns and $|Y^1| + |Y^2| = N$

respectively. Note that m is fixed here, so it must be included in the set B if we move it. We construct a bijection between A and B.

Take a Young diagram with marked boxes from A. We define a Young diagram Y^1 by removing all columns containing marked boxes from Y. (And we shift columns to the left to fill out empty columns.) We define another Young diagram Y^2 as follows. We first define a

Young diagram Y' consisting of columns removed from Y when we got Y^1. Then we remove all the irrelevant boxes from Y'. (And we move boxes to down to fill out empty spots.) Call the resulted Young diagram Y^2. See Figure 2 where the boxes with $\heartsuit$ are marked removable boxes, and the boxes with $\heartsuit$ or $\spadesuit$ are irrelevant boxes. This Y^2 is a Young diagram which has at most m columns and $|Y^1| + |Y^2| = N$. Thus we have a map from A to B.

Conversely from $(Y^1, Y^2) \in B$ we can construct a Young diagram Y with marked removable boxes by the reverse procedure. Namely we add m boxes to the first (=leftmost) column of Y^2, $m-1$ boxes to the second column, ... Put markings on the top box in each column of Y^2. Then merge two Young diagrams to Y keeping columns.

Corollary 5.4. *Let $C_N = 1 - N\,\mathrm{pt}$. The Poincaré polynomial of $\widehat{M}^m(c_N)$ is given by*

$$P_t(\widehat{M}^m(c_N)) = \sum t^{2(|Y^1|+|Y^2|-l(Y^1))},$$

where the summation runs over all pairs of Young diagrams $(Y^1, Y^2) \in B$. Therefore its generating function is

$$\sum_{N=0}^{\infty} P_t(\widehat{M}^m(c_N))\mathfrak{q}^N = \left(\prod_{d=1}^{\infty} \frac{1}{1 - t^{2d-2}\mathfrak{q}^d}\right)\left(\prod_{d=1}^{m} \frac{1}{1 - t^{2d}\mathfrak{q}^d}\right).$$

Let $m \to \infty$. Then $\widehat{M}^m(c_N)$ becomes the Hilbert schemes $(\widehat{\mathbb{C}}^2)^{[N]}$ of points on $\widehat{\mathbb{C}}^2$ by Proposition 3.37 for $m \gg 0$. From the above formula we get

$$\sum_N P_t((\widehat{\mathbb{C}}^2)^{[N]})\mathfrak{q}^N = \left(\prod_{d=1}^{\infty} \frac{1}{1 - t^{2d-2}\mathfrak{q}^d}\right)\left(\prod_{d=1}^{\infty} \frac{1}{1 - t^{2d}\mathfrak{q}^d}\right).$$

This is nothing but Göttsche's formula for Betti numbers of $(\widehat{\mathbb{C}}^2)^{[N]}$. (See e.g., [20].)

5.6. **Tangent space – general case.** We consider general case. Let (Y_α, S_α), (Y_β, S_β) be two pairs of Young diagrams and marked removable boxes. Let (E_α, Φ_α), (E_β, Φ_β) be the corresponding framed perverse coherent sheaves of rank 1.

For a given pair $(s, s') \in S_\alpha \times S_\beta$, we consider the box u (resp. u') which is the same row as in s (resp. s') and the same column as in s' (resp. s). We have

(1) If $a'(s) \le a'(s')$ (i.e., s' sits higher than or equal to s), then $u \in Y_\beta$, $u' \notin Y_\alpha \setminus S_\alpha$.

(2) If $a'(s) > a'(s')$ (i.e., s' sits lower than s), then $u \notin Y_\beta$, $u' \in Y_\alpha \setminus S_\alpha$.

We say u or u' is *irrelevant* accordingly. We say a box (in $Y_\alpha \setminus S_\alpha$ or Y_β) is *relevant* otherwise.

u'			s'
s			u

u			s
s'			u'

FIGURE 3. The irrelevant box is u in the first case, and u' in the second case.

Proposition 5.5. *We have*

$$\operatorname{ch} \operatorname{Ext}^1(E_\alpha, E_\beta(-\ell_\infty)) = {\sum_{s\in Y_\alpha\setminus S_\alpha}}' t_1^{-l_{Y_\beta}(s)} t_2^{a_{Y_\alpha\setminus S_\alpha}(s)+1} + {\sum_{t\in Y_\beta}}' t_1^{l_{Y_\alpha\setminus S_\alpha}(t)+1} t_2^{-a_{Y_\beta}(t)},$$

where the summation runs over all relevant boxes $s \in Y_\alpha \setminus S_\alpha$, $t \in Y_\beta$.

Proof. The space $\operatorname{Ext}^1(E_\alpha, E_\beta(-\ell_\infty))$ is a weight space of the tangent space of $\widehat{M}^0(c)$ at a $\widetilde{T}$-fixed point $(E,\Phi) \cong (E_1,\Phi_1)\oplus\cdots\oplus(E_r,\Phi_r)$. Since $\widehat{M}^0(c)$ and $L(c',n)$ are isomorphic by Theorem 4.1, the tangent space $\operatorname{Ext}^1(E, E(-\infty))$ of $\widehat{M}^0(c)$ at (E,Φ) is isomorphic to the tangent space of $L(c',n)$ at $((F,\Phi),U)$ corresponding to (E,Φ). In the genuine moduli space of sheaves case, the latter was given $\operatorname{Ext}^1(F,F')/\operatorname{End}(U)$ where $F' := \operatorname{Ker}(F \to U^\vee \otimes \mathbb{C}_0)$. (See the proof of Lemma 4.3(1).) In the framed case, it is modified as $\operatorname{Ext}^1(F, F'(-\ell_\infty))/\operatorname{End}(U)$. Since the isomorphism $\widehat{M}^0(c) \cong L(c',n)$ is $\widetilde{T}$-equivariant, the weight spaces at fixed points must be respected, so $\operatorname{Ext}^1(E_\alpha, E_\beta(-\ell_\infty))$ is isomorphic to $\operatorname{Ext}^1(F_\alpha, F'_\beta(-\ell_\infty))/\operatorname{Hom}(U_\alpha, U_\beta)$, where (F_α, U_α) corresponds to the summand E_α.

If (F_α, U_α) corresponds to a marked Young diagram (Y_α, S_α), then the T^2-character of $\operatorname{Ext}^1(F_\alpha, F'_\beta(-\ell_\infty))$ was computed in [23, §2]:

$$\operatorname{ch} \operatorname{Ext}^1(F_\alpha, F'_\beta(-\ell_\infty)) = \sum_{s\in Y_\alpha\setminus S_\alpha} t_1^{-l_{Y_\beta}(s)} t_2^{a_{Y_\alpha\setminus S_\alpha}(s)+1} + \sum_{t\in Y_\beta} t_1^{l_{Y_\alpha\setminus S_\alpha}(t)+1} t_2^{-a_{Y_\beta}(t)},$$

where we should notice that F_α corresponds to the Young diagram $Y_\alpha \setminus S_\alpha$, while F'_β corresponds to Y_β.

On the other hand, we have

$$\operatorname{ch} \operatorname{Hom}(S_\alpha, S_\beta) = \sum_{s\in S_\alpha,\ s'\in S_\beta} t_1^{l'(s)-l'(s')} t_2^{a'(s)-a'(s')}.$$

For a given pair $(s,s')\in S_\alpha\times S_\beta$, we consider the boxes u and u' explained as above. Then we have

$$l_{Y_\alpha\setminus S_\alpha}(u)+1=l'(s)-l'(s'),\qquad -a_{Y_\beta}(u)=a'(s)-a'(s'),$$
$$-l_{Y_\beta}(u')=l'(s)-l'(s'),\qquad a_{Y_\alpha\setminus S_\alpha}(u')+1=a'(s)-a'(s').$$

Therefore we substract the box u from Y_β, or u' from $Y_\alpha\setminus S_\alpha$ according to $u\in Y_\beta$ or $u'\in Y_\alpha\setminus S_\alpha$ to get the assertion. □

Corollary 5.6. *The Poincaré polynomial of $\widehat{M}^0(c)$ is given by*

$$P_t(\widehat{M}^0(c))=\sum_{(\vec{m},\vec{Y^1},\vec{Y^2})}\prod_{\alpha=1}^{r}t^{2(r|Y_\alpha^1|+r|Y_\alpha^2|-\alpha l(Y_\alpha^1))}\prod_{\alpha<\beta}t^{(m_\alpha-m_\beta)(m_\alpha-m_\beta-1)},$$

where the summation runs over r-tuples $(\vec{m},\vec{Y^1},\vec{Y^2})=((m_1,Y_1^1,Y_1^2),\dots,(m_r,Y_r^1,Y_r^2))$ of triples of nonnegative integers and two Young diagrams such that $\sum_\alpha m_\alpha=(c_1,[C])$, the number of columns of Y_α^2 is at most m_α $(\alpha=1,\dots,r)$, and $\sum_\alpha|Y_\alpha^1|+|Y_\alpha^2|=\Delta(c)-1/(4r)\sum_{\alpha,\beta}(m_\alpha-m_\beta)^2$. Here $\Delta(c)=\int_{\widehat{X}}[-\operatorname{ch}_2+1/(2r)\,c_1^2]$.

And their generating function (for fixed r, c_1) is given by

$$\sum_c P_t(\widehat{M}^0(c))\mathfrak{q}^{\Delta(c)}=\sum_{\substack{m_\alpha\ge 0\\ \sum m_\alpha=(c_1,[C])}}\prod_{\alpha=1}^{r}\left(\prod_{d=1}^{\infty}\frac{1}{1-t^{2(rd-\alpha)}\mathfrak{q}^d}\times\prod_{d=1}^{m_\alpha}\frac{1}{1-t^{2rd}\mathfrak{q}^d}\right)\times t^{-2\langle\vec{m},\rho\rangle}(t^{2r}\mathfrak{q})^{(\vec{m},\vec{m})/2},$$

where $\langle\vec{m},\rho\rangle=\sum_{\alpha<\beta}(m_\alpha-m_\beta)/2$, $(\vec{m},\vec{m})=1/(2r)\sum_{\alpha,\beta}(m_\alpha-m_\beta)^2$.

Proof. The torus fixed points in $\widehat{M}^0(c)$ is parametrized by r-tuples $((Y_1,S_1),\dots,(Y_r,S_r))$ of pairs of Young diagrams with marked removable boxes with $\sum_\alpha|S_\alpha|=(c_1,[C])$, $\sum_\alpha|Y_\alpha|=-\int_{\widehat{X}}\operatorname{ch}_2+\frac{1}{2}(c_1,[C])$. Moreover such r-tuples correspond to r-tuples of triples of nonnegative intergers and two Young diagrams $((m_1,Y_1^1,Y_1^2),\dots,(m_r,Y_r^1,Y_r^2))$ as above by §5.5, where $m_\alpha=|S_\alpha|$.

As in [24, Th. 3.8] we take a one parameter subgroup $\lambda\colon\mathbb{C}^*\to\widetilde{T}$ with

$$\lambda(t)=(t^{N_1},t^{N_2},t^{n_1},\dots,t^{n_r})$$

and

$$N_2\gg n_1>n_2>\cdots>n_r\gg N_1>0.$$

Then we compute the dimension of negative weight spaces of the tangent space at each fixed point. Thus we count those weight spaces such that

(1) weight of t_2 is negative,

(2) weight of t_2 is zero and weight of e_1 is negative,
(3) weight of t_2, e_1 are zero and weight of e_2 is negative,
(4) weight of t_2, e_1, e_2 are zero and weight of e_3 is negative,
...
$(r+1)$ weight of t_2, e_1, e_2, $\dots$, e_{r-1} are zero and weight of e_r is negative,
$(r+2)$ weight of t_2, e_1, e_2, $\dots$, e_r are zero and weight of t_1 is negative.

We have decomposition of the tangent space

$$T_{(E,\Phi)}\widehat{M}^0 = \bigoplus_{\alpha,\beta} \operatorname{Ext}^1(E_\alpha, E_\beta(-\ell_\infty)).$$

The T^r-weight of the summand $\operatorname{Ext}^1(E_\alpha, E_\beta(-\ell_\infty))$ is given by $e_\beta e_\alpha^{-1}$. Therefore in the summand $\alpha = \beta$, the total dimension of negative weight spaces is $2(|Y_\alpha^1| + |Y_\alpha^2| - l(Y_\alpha^1))$ as in the rank 1 case (see Corollary 5.4). In the summand $\alpha < \beta$, we compute the total dimension of weight spaces whose t_2-weight is nonpositive. It is given by

$$2\left[|Y_\beta| - \#\{(s,s') \in S_\alpha \times S_\beta \mid a'(s) \le a'(s')\}\right]$$

by the same argument as in Corollary 5.3. In the summand $\alpha > \beta$, we get

$$2\left[|Y_\beta| - l(Y_\beta) - \#\{(s,s') \in S_\alpha \times S_\beta \mid a'(s) < a'(s')\}\right].$$

We combine the last term for $\alpha < \beta$ and the corresponding term for $\alpha \leftrightarrow \beta$ to have

$$\#\{(s,s') \in S_\alpha \times S_\beta \mid a'(s) \le a'(s')\} + \#\{(s',s) \in S_\beta \times S_\alpha \mid a'(s') < a'(s)\} \\ = m_\alpha m_\beta.$$

We also note

$$|Y_\beta| = |Y_\beta^1| + |Y_\beta^2| + \frac{1}{2} m_\beta(m_\beta + 1), \qquad l(Y_\beta) = l(Y_\beta^1) + m_\beta.$$

So in total we have

$$2\sum_{\alpha=1}^{r}\left[r(|Y_\alpha^1| + |Y_\alpha^2|) - \alpha l(Y_\alpha^1) + \frac{r-1}{2} m_\alpha(m_\alpha+1)\right] - 2\sum_{\alpha<\beta}(m_\alpha + m_\alpha m_\beta)$$
$$= 2\sum_{\alpha=1}^{r}\left[r(|Y_\alpha^1| + |Y_\alpha^2|) - \alpha l(Y_\alpha^1)\right] + \sum_{\alpha<\beta}(m_\alpha - m_\beta)(m_\alpha - m_\beta - 1).$$

From this we get the formula. □

For general $\widehat{M}^m(c)$, we just need to apply the formula for $\widehat{M}^0(ce^{-m[C]})$ and replace m_α by $m + k_\alpha$:

Corollary 5.7.

$$\sum_{c\colon\ r,\ (c_1,[C])\text{ fixed}} P_t(\widehat{M}^m(c))\mathfrak{q}^{\Delta(c)}$$
$$= \sum_{\substack{k_\alpha\geq -m\\ k_1+\cdots+k_r=(c_1,[C])}} \prod_{\alpha=1}^{r}\left(\prod_{d=1}^{\infty}\frac{1}{1-t^{2(rd-\alpha)}\mathfrak{q}^d}\times\prod_{d=1}^{m+k_\alpha}\frac{1}{1-t^{2rd}\mathfrak{q}^d}\right)$$
$$\times\, t^{-2\langle\vec{k},\rho\rangle}(t^{2r}\mathfrak{q})^{(\vec{k},\vec{k})/2}.$$

In the limit $m\to\infty$, we recover the formula [24, Cor. 3.10].

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