J. ALGEBRAIC GEOMETRY
00 (XXXX) 000–000
S 1056-3911(XX)0000-0

NON-VANISHING THEOREM FOR LOG CANONICAL PAIRS

OSAMU FUJINO

Abstract

We obtain a correct generalization of Shokurov's non-vanishing theorem for log canonical pairs. It implies the base point free theorem for log canonical pairs. We also prove the rationality theorem for log canonical pairs. As a corollary, we obtain the cone theorem for log canonical pairs. We do not need Ambro's theory of quasi-log varieties.

Contents

1. Introduction

The following theorem is the main theorem of this paper. It is a generalization of Shokurov's non-vanishing theorem for log canonical pairs. This new non-vanishing theorem greatly simplifies the proof of the fundamental theorems for log canonical pairs. In this paper, we do not need Ambro's framework of *quasi-log varieties* in [A].

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

Theorem 1.1 (Non-vanishing theorem)**.** *Let X be a normal projective variety and B an effective $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let L be a nef Cartier divisor on X. Assume that $aL-(K_X+B)$ is ample for some $a>0$. Then the base locus of the linear system $|mL|$ contains no lc*

Received 2009/12/1, final version.

centers of (X, B) for every $m \gg 0$, that is, there is a positive integer m_0 such that $|mL|$ contains no lc centers of (X, B) for every $m \geq m_0$.

By this new non-vanishing theorem, we can easily obtain the base point free theorem for log canonical pairs.

Theorem 1.2 (Base point free theorem)**.** *Let X be a normal projective variety and B an effective $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let L be a nef Cartier divisor on X. Assume that $aL - (K_X + B)$ is ample for some $a > 0$. Then the linear system $|mL|$ is base point free for every $m \gg 0$.*

Theorem 1.2 is a special case of [A, Theorem 5.1]. We can also prove the rationality theorem for log canonical pairs without any difficulties. It is a special case of [A, Theorem 5.9].

Theorem 1.3 (Rationality theorem)**.** *Let (X, B) be a projective log canonical pair such that $a(K_X + B)$ is Cartier for a positive integer a. Let H be an ample Cartier divisor on X. Assume that $K_X + B$ is not nef. We put*

$$r = \max\{t \in \mathbb{R} \,|\, H + t(K_X + B) \text{ is nef}\}.$$

Then r is a rational number of the form u/v $(u, v \in \mathbb{Z})$ where $0 < v \leq a(\dim X + 1)$.

As a corollary, we obtain the cone theorem for log canonical pairs. It is a formal consequence of the rationality and the base point free theorems. It is a special case of [A, Theorem 5.10].

Theorem 1.4 (Cone theorem)**.** *Let (X, B) be a projective log canonical pair. Then we have*

(i) *There are (countably many) rational curves $C_j \subset X$ such that $0 < -(K_X + B) \cdot C_j \leq 2 \dim X$, and*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+B)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

(ii) *For any $\varepsilon > 0$ and ample $\mathbb{Q}$-divisor H,*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+B+\varepsilon H)\geq 0} + \sum_{\mathit{finite}} \mathbb{R}_{\geq 0}[C_j].$$

(iii) *Let $F \subset \overline{NE}(X)$ be a $(K_X + B)$-negative extremal face. Then there is a unique morphism $\varphi_F : X \to Z$ such that $(\varphi_F)_* \mathcal{O}_X \simeq \mathcal{O}_Z$, Z is projective, and an irreducible curve $C \subset X$ is mapped to a point by φ_F if and only if $[C] \in F$. The map φ_F is called the* contraction *of F.*

(iv) *Let F and φ_F be as in* (iii)*. Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Z on Z such that $L \simeq \varphi_F^* L_Z$.*

In [A], Ambro did not discuss any generalization of Shokurov's non-vanishing theorem. Instead, he introduced the notion of *quasi-log varieties* and proved the base point free theorem for quasi-log varieties by induction on the dimension. His approach is natural but demands some very complicated and powerful vanishing and torsion-free theorems for reducible varieties. For details of the theory of quasi-log varieties, see [F2] and [F4].

All the results in this paper are stated and investigated in [F2] following [A]. So, the contribution of this paper is to give a correct formulation of Shokurov's non-vanishing theorem for log canonical pairs and prove it in a simple manner. Once we obtain correct formulations of vanishing and non-vanishing theorems for log canonical pairs (see Theorems 2.2, 2.5, and Theorem 1.1), there are no difficulties to obtain the fundamental theorems for log canonical pairs. I hope that this paper will supply a new method to study linear systems on log canonical pairs.

We summarize the contents of this paper. In Section 2, we collect some preliminary results on vanishing and torsion-free theorems. We prove the basic properties of lc centers. This section contains no new results. In Section 3, we give a proof of the non-vanishing theorem. This section is the main part of this paper. Our proof is short and very easy to understand. Shokurov's concentration method is the main ingredient of Section 3. Section 4 is devoted to the proof of the base point free theorem. The reader will be surprised since our proof is very easy and understand that our non-vanishing theorem is powerful. In Section 5, we prove the rationality theorem for log canonical pairs. The proof is essentially the same as the one for klt pairs. We need only the vanishing theorem given in Section 2 (cf. Theorem 2.2) to obtain the rationality theorem. In the final section: Section 6, we give a proof of the cone theorem. The reader who understands [F3] can read this paper without any difficulties.

We close this introduction with the following notation.

Notation. Let X be a normal variety and B an effective $\mathbb{Q}$-divisor such that K_X+B is $\mathbb{Q}$-Cartier. Then we can define the *discrepancy* $a(E,X,B)\in\mathbb{Q}$ for every prime divisor E *over* X. If $a(E,X,B)\geq -1$ (resp. > -1) for every E, then (X,B) is called *log canonical* (resp. *kawamata log terminal*). We sometimes abbreviate log canonical (resp. kawamata log terminal) to *lc* (resp. *klt*).

Assume that (X,B) is log canonical. If E is a prime divisor over X such that $a(E,X,B)=-1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X,B), where $c_X(E)$ is the closure of the image of E on X.

Let (X,B) be a log canonical pair and M an effective $\mathbb{Q}$-divisor on X. The *log canonical threshold* of (X,B) with respect to M is defined by

$$c = \sup\{t \in \mathbb{R} \,|\, (X, B + tM) \text{ is log canonical}\}.$$

We can easily check that c is a rational number and that $(X, B + cM)$ is lc but not klt.

Let (X, B) be a log canonical pair. Then a *stratum* of (X, B) denotes X itself or an lc center of (X, B).

Let Y be a smooth variety and T a simple normal crossing divisor on Y. Then a *stratum* of T means an lc center of the pair (Y, T).

Let r be a rational number. The integral part $\llcorner r \lrcorner$ is the largest integer $\leq r$ and the fractional part $\{r\}$ is defined by $r - \llcorner r \lrcorner$. We put $\ulcorner r \urcorner = -\llcorner -r \lrcorner$ and call it the round-up of r. For a $\mathbb{Q}$-divisor $D = \sum_{i=1}^{r} d_i D_i$, where D_i is a prime divisor for every i and $D_i \neq D_j$ for $i \neq j$, we call D a *boundary* $\mathbb{Q}$-divisor if $0 \leq d_i \leq 1$ for every i. We note that $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. We put $\llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i$, $\ulcorner D \urcorner = \sum \ulcorner d_i \urcorner D_i$, $\{D\} = \sum \{d_i\} D_i$, $D^{<1} = \sum_{d_i < 1} d_i D_i$, and $D^{=1} = \sum_{d_i = 1} D_i$.

We write $\mathrm{Bs}|L|$ to denote the *base locus* of the linear system $|L|$.

Acknowledgments. The author was partially supported by The Inamori Foundation and by the Grant-in-Aid for Young Scientists (A) $\sharp$20684001 from JSPS. He would like to thank Takeshi Abe for discussions.

2. On vanishing and torsion-free theorems

In this section, we collect some preliminary results for the reader's convenience. The next theorem is a very special case of [A, Theorem 3.2].

Theorem 2.1 (Torsion-freeness and vanishing theorem)**.** *Let Y be a smooth projective variety and B a boundary $\mathbb{Q}$-divisor such that* $\mathrm{Supp} B$ *is simple normal crossing. Let $f : Y \to X$ be a projective morphism and L a Cartier divisor on Y such that $H \sim_{\mathbb{Q}} L - (K_Y + B)$ is f-semi-ample.*

(i) *Every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some stratum of (Y, B).*
(ii) *Assume that $H \sim_{\mathbb{Q}} f^* H'$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H' on X. Then $H^p(X, R^q f_* \mathcal{O}_Y(L)) = 0$ for every $p > 0$ and $q \geq 0$.*

The proof of Theorem 2.1 is not difficult. For a short and almost self-contained proof, see [F3]. See also [F2, Chapter 2] for a thorough treatment. As an application of Theorem 2.1, we prepare the following powerful vanishing theorem. It will play basic roles for the study of log canonical pairs.

Theorem 2.2 (cf. [A, Theorem 4.4])**.** *Let X be a normal projective variety and B a boundary $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let D be a Cartier divisor on X. Assume that $D - (K_X + B)$ is ample. Let $\{C_i\}$ be* any *set of lc centers of the pair* (X, B)*. We put* $W = \bigcup C_i$ *with the reduced scheme structure. Then we have*

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(D)) = 0, \quad H^i(X, \mathcal{O}_X(D)) = 0,$$

and

$$H^i(W, \mathcal{O}_W(D)) = 0$$

for every $i > 0$*, where* $\mathcal{I}_W$ *is the defining ideal sheaf of* W *on* X*. In particular, the restriction map*

$$H^0(X, \mathcal{O}_X(D)) \to H^0(W, \mathcal{O}_W(D))$$

is surjective.

Proof. Let $f : Y \to X$ be a resolution such that $\mathrm{Supp} f_*^{-1}B \cup \mathrm{Exc}(f)$, where $\mathrm{Exc}(f)$ is the exceptional locus of f, is a simple normal crossing divisor. We can further assume that $f^{-1}(W)$ is a simple normal crossing divisor on Y. We can write

$$K_Y + B_Y = f^*(K_X + B).$$

Let T be the union of the irreducible components of $B_Y^{=1}$ that are mapped into W by f. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A - T) \to \mathcal{O}_Y(A) \to \mathcal{O}_T(A) \to 0,$$

where $A = \ulcorner -(B_Y^{<1}) \urcorner$. Note that A is an effective f-exceptional divisor. We obtain the following long exact sequence

$$\begin{aligned} 0 \to f_*\mathcal{O}_Y(A - T) \to f_*\mathcal{O}_Y(A) \to f_*\mathcal{O}_T(A) \\ \xrightarrow{\delta} R^1 f_*\mathcal{O}_Y(A - T) \to \cdots . \end{aligned}$$

Since

$$A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) = -(K_Y + B_Y) \sim_{\mathbb{Q}} -f^*(K_X + B),$$

every non-zero local section of $R^1 f_*\mathcal{O}_Y(A - T)$ contains in its support the f-image of some stratum of $(Y, \{B_Y\} + B_Y^{=1} - T)$ by Theorem 2.1 (i). On the other hand, $W = f(T)$. Therefore, the connecting homomorphism δ is the zero map. Thus, we have a short exact sequence

$$0 \to f_*\mathcal{O}_Y(A - T) \to \mathcal{O}_X \to f_*\mathcal{O}_T(A) \to 0.$$

So, we obtain $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ and $f_*\mathcal{O}_Y(A - T) \simeq \mathcal{I}_W$, the defining ideal sheaf of W. The isomorphism $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ plays crucial roles in this paper. Thus we write it as a lemma.

Lemma 2.3. *We have* $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$*. It obviously implies that* $f_*\mathcal{O}_T \simeq \mathcal{O}_W$*.*

Since

$$f^*D + A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) \sim_{\mathbb{Q}} f^*(D - (K_X + B)),$$

and

$$f^*D + A - (K_Y + \{B_Y\} + B_Y^{=1}) \sim_{\mathbb{Q}} f^*(D - (K_X + B)),$$

we have

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(D)) \simeq H^i(X, f_*\mathcal{O}_Y(A - T) \otimes \mathcal{O}_X(D)) = 0$$

and

$$H^i(X, \mathcal{O}_X(D)) \simeq H^i(X, f_*\mathcal{O}_Y(A) \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$ by Theorem 2.1 (ii). By the long exact sequence

$$\begin{aligned}\cdots \to H^i(X, \mathcal{O}_X(D)) &\to H^i(W, \mathcal{O}_W(D)) \\ &\to H^{i+1}(X, \mathcal{I}_W \otimes \mathcal{O}_X(D)) \to \cdots,\end{aligned}$$

we have $H^i(W, \mathcal{O}_W(D)) = 0$ for every $i > 0$. We finish the proof. □

As a corollary, we can easily check the following result (cf. [A, Propositions 4.7 and 4.8]).

Theorem 2.4. *Let X be a normal projective variety and B an effective $\mathbb{Q}$-divisor such that (X, B) is log canonical. Then we have the following properties.*

(1) *(X, B) has at most finitely many lc centers.*
(2) *An intersection of two lc centers is a union of lc centers.*
(3) *Any union of lc centers of (X, B) is semi-normal.*
(4) *Let $x \in X$ be a closed point such that (X, B) is lc but not klt at x. Then there is a unique minimal lc center W_x passing through x, and W_x is normal at x.*

Proof. We use the notation in the proof of Theorem 2.2. (1) is obvious. (3) is also obvious by Lemma 2.3 since T is a simple normal crossing divisor. Let C_1 and C_2 be two lc centers of (X, B). We fix a closed point $P \in C_1 \cap C_2$. For the proof of (2), it is enough to find an lc center C such that $P \in C \subset C_1 \cap C_2$. We put $W = C_1 \cup C_2$. By Lemma 2.3, we obtain $f_*\mathcal{O}_T \simeq \mathcal{O}_W$. This means that $f : T \to W$ has connected fibers. We note that T is a simple normal crossing divisor on Y. Thus, there exist irreducible components T_1 and T_2 of T such that $T_1 \cap T_2 \cap f^{-1}(P) \neq \emptyset$ and that $f(T_i) \subset C_i$ for $i = 1, 2$. Therefore, we can find an lc center C with $P \in C \subset C_1 \cap C_2$. We finish the proof of (2). Finally, we will prove (4). The existence and the uniqueness of the minimal lc center follow from (2). We take the unique minimal lc center $W = W_x$ passing through x. By Lemma 2.3, we have $f_*\mathcal{O}_T \simeq \mathcal{O}_W$. By shrinking W around x, we can assume that every stratum of T dominates W. Thus, $f : T \to W$

factors through the normalization W^ν of W. Since $f_*\mathcal{O}_T \simeq \mathcal{O}_W$, we obtain that $W^\nu \to W$ is an isomorphism. So, we obtain (4). □

We close this section with the following very useful vanishing theorem. It is a special case of [A, Theorem 4.4]. For details, see [F2, Theorem 3.39]. Here, we give a quick reduction to Theorem 2.1 (ii) by using [BCHM] for the reader's convenience.

Theorem 2.5. *Let (X, B) be a projective lc pair and W a minimal lc center of (X, B). Let D be a Cartier divisor on W such that $D-(K_X+B)|_W$ is ample. Then $H^i(W, \mathcal{O}_W(D)) = 0$ for every $i > 0$.*

Proof. By Hacon (cf. [KK, Theorem 3.1]), we can make a projective birational morphism $f : Y \to X$ such that $K_Y + B_Y = f^*(K_X + B)$ and that (Y, B_Y) is dlt. It is an application of the results in [BCHM]. For the definition and the basic properties of *dlt* pairs, see [KM, Section 2.3] and [F1]. We take an lc center V of (Y, B_Y) such that $f(V) = W$ and put $K_V + B_V = (K_Y + B_Y)|_V$. Then (V, B_V) is dlt (cf. [F1, Section 3.9]) and $K_V + B_V \sim_{\mathbb{Q}} f^*((K_X + B)|_W)$. Let $g : Z \to V$ be a resolution such that $K_Z + B_Z = g^*(K_V + B_V)$ and that $\mathrm{Supp} B_Z$ is simple normal crossing. Then we have $K_Z + B_Z \sim_{\mathbb{Q}} h^*((K_X + B)|_W)$, where $h = f \circ g$. Since

$$h^*(D - (K_X + B)|_W) \sim_{\mathbb{Q}} h^*D + \ulcorner -(B_Z^{<1}) \urcorner - (K_Z + B_Z^{=1} + \{B_Z\}),$$

we obtain

$$H^i(W, h_*\mathcal{O}_Z(h^*D + \ulcorner -(B_Z^{<1}) \urcorner)) = 0$$

for every $i > 0$ by Theorem 2.1 (ii). We note that

$$h_*\mathcal{O}_Z(h^*D + \ulcorner -(B_Z^{<1}) \urcorner) \simeq f_*\mathcal{O}_V(f^*D)$$

by the projection formula since $\ulcorner -(B_Z^{<1}) \urcorner$ is effective and g-exceptional. We note that $\mathcal{O}_W(D)$ is a direct summand of $f_*\mathcal{O}_V(f^*D) \simeq \mathcal{O}_W(D) \otimes f_*\mathcal{O}_V$ since W is normal (cf. Theorem 2.4 (4)). Therefore, we have $H^i(W, \mathcal{O}_W(D)) = 0$ for every $i > 0$. □

Remark 2.6. We can prove Theorem 2.5 without using [BCHM]. For the original argument, see [A, Theorem 4.4] and [F2, Theorem 3.39]. It depends on the theory of mixed Hodge structures (cf. [F2, Chapter 2]).

3. Non-vanishing theorem

In this section, we prove the non-vanishing theorem, which is the main theorem of this paper. The proof given here is very easy.

Proof of Theorem 1.1. Let W be a minimal lc center of (X, B). If $L|_W$ is numerically trivial, then we have

$$h^0(W, \mathcal{O}_W(L)) = \chi(W, \mathcal{O}_W(L)) = \chi(W, \mathcal{O}_W) = h^0(W, \mathcal{O}_W) = 1$$

by [Kl, Chapter II §2 Theorem 1] and the vanishing theorem (see Theorem 2.5). Therefore, $L|_W$ is linearly trivial since $L|_W$ is numerically trivial. In particular, $|mL|_W|$ is free for every $m > 0$. On the other hand,

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(W, \mathcal{O}_W(mL))$$

is surjective for every $m \geq a$ by Theorem 2.2. Thus, $\mathrm{Bs}|mL|$ does not contain W for every $m \geq a$.

Assume that $L|_W$ is not numerically trivial. Let $x \in W$ be a general smooth point. If l is a sufficiently large integer, then we can find an effective Cartier divisor N on W such that $N \sim b(lL-(K_X+B))|_W$ with $\mathrm{mult}_x N > b \dim W$ for some positive divisible integer b by Shokurov's concentration method. See, for example, [KM, 3.5 Step 2]. If b is sufficiently large and divisible, then $\mathcal{I}_W \otimes \mathcal{O}_X(b(lL-(K_X+B)))$ is generated by global sections and $H^1(X, \mathcal{I}_W \otimes \mathcal{O}_X(b(lL-(K_X+B)))) = 0$ since $lL-(K_X+B)$ is ample, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. By using the following short exact sequence

$$\begin{aligned} 0 &\to H^0(X, \mathcal{I}_W \otimes \mathcal{O}_X(b(lL-(K_X+B)))) \\ &\to H^0(X, \mathcal{O}_X(b(lL-(K_X+B)))) \\ &\to H^0(W, \mathcal{O}_W(b(lL-(K_X+B)))) \to 0, \end{aligned}$$

we can find an effective $\mathbb{Q}$-divisor M on X with the following properties.

(i) $M|_W$ is an effective $\mathbb{Q}$-divisor such that $\mathrm{mult}_x M|_W > \dim W$.
(ii) $M \sim_{\mathbb{Q}} lL-(K_X+B)$ for some positive large integer l.
(iii) $(X, B+M)$ is lc outside W.

We take the log canonical threshold c of (X, B) with respect to M. Then $(X, B+cM)$ is lc but not klt. By the above construction, we have $0 < c < 1$. We replace (X, B) with $(X, B+cM)$, a with $a-ac+cl$. Then we have that

$$(a-ac+cl)L-(K_X+B+cM) \sim_{\mathbb{Q}} (1-c)(aL-(K_X+B))$$

is ample. Moreover, we can find a smaller lc center W' of $(X, B+cM)$ contained in W. By repeating this process, we reach the situation where $L|_W$ is numerically trivial.

Anyway, we proved that $\mathrm{Bs}|mL|$ contains no lc centers of (X, B) for every $m \gg 0$. $\square$

4. Base point free theorem

We give a proof of Theorem 1.2. Our proof is much easier than Ambro's proof for quasi-log varieties.

Proof of Theorem 1.2. If L is numerically trivial, then

$$h^0(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$$

by [Kl, Chapter II §2 Theorem 1] and the vanishing theorem (cf. Theorem 2.2). Thus, L is linearly trivial. In this case, $|mL|$ is free for every m. So, from now on, we can assume that L is not numerically trivial.

We assume that (X, B) is klt. Let $x \in X$ be a general smooth point. Then we can find an effective $\mathbb{Q}$-divisor M on X such that

$$M \sim_{\mathbb{Q}} lL - (K_X + B)$$

for some large integer l and that $\mathrm{mult}_x M > n = \dim X$. It is well known as Shokurov's concentration method. See, for example, [KM, 3.5 Step 2]. Let c be the log canonical threshold of (X, B) with respect to M. By construction, we have $0 < c < 1$. Then

$$(a - ac + cl)L - (K_X + B + cM) \sim_{\mathbb{Q}} (1 - c)(aL - (K_X + B))$$

is ample. Therefore, by replacing B with $B + cM$, a with $a - ac + cl$, we can assume that (X, B) is lc but not klt.

From now on, we assume that (X, B) is lc but not klt and that L is not numerically trivial. By Theorem 1.1, we can take general members $D_1, \cdots, D_{n+1} \in |p^{m_1}L|$ for some prime integer p and a positive integer m_1. Since $D_1, \cdots, D_{n+1}$ are general, $(X, B + D_1 + \cdots + D_{n+1})$ is lc outside $\mathrm{Bs}|p^{m_1}L|$. It is easy to see that $(X, B + D)$, where $D = D_1 + \cdots + D_{n+1}$, is not lc at the generic point of any irreducible component of $\mathrm{Bs}|p^{m_1}L|$. Let c be the log canonical threshold of (X, B) with respect to D. Then $(X, B + cD)$ is lc but not klt, and $0 < c < 1$. We note that

$$(c(n+1)p^{m_1} + a)L - (K_X + B + cD) \sim_{\mathbb{Q}} aL - (K_X + B)$$

is ample. By construction, there exists an lc center of $(X, B + cD)$ contained in $\mathrm{Bs}|p^{m_1}L|$. By Theorem 1.1, we can find $m_2 > m_1$ such that $\mathrm{Bs}|p^{m_2}L| \subsetneq \mathrm{Bs}|p^{m_1}L|$. By noetherian induction, there exists m_k such that $\mathrm{Bs}|p^{m_k}L| = \emptyset$. Let p' be a prime integer such that $p' \neq p$. Then, by the same argument, we can prove $\mathrm{Bs}|p'^{m'_{k'}}L| = \emptyset$ for some positive integer $m'_{k'}$. So, there exists a positive number m_0 such that $|mL|$ is free for every $m \geq m_0$. $\square$

5. Rationality theorem

Here, we prove the rationality theorem for log canonical pairs. Before we start the proof, we recall the following lemmas.

Lemma 5.1 (cf. [KM, Lemma 3.19])**.** *Let $P(x,y)$ be a non-trivial polynomial of degree $\leq n$ and assume that P vanishes for all sufficiently large integral solutions of $0 < ay - rx < \varepsilon$ for some fixed positive integer a and positive ε for some $r \in \mathbb{R}$. Then r is rational, and in reduced form, r has denominator $\leq a(n+1)/\varepsilon$.*

For the proof, see [KM, Lemma 3.19].

Lemma 5.2. *Let C be a projective variety and D_1, D_2 Cartier divisors on X. Consider the Hilbert polynomial*

$$P(u_1,u_2) = \chi(C, \mathcal{O}_C(u_1D_1 + u_2D_2)).$$

It is a polynomial in u_1 and u_2 of total degree $\leq \dim C$ (cf. [Kl, Theorem (Snapper)]*). If D_1 is ample, then $P(u_1,u_2)$ is a non-trivial polynomial. It is because $P(u_1,0) = h^0(C, \mathcal{O}_C(u_1D)) \not\equiv 0$ if u_1 is sufficiently large.*

Proof of Theorem 1.3. By using mH with various large m in place of H, we can assume that H is very ample (cf. [KM, 3.4 Step 1]). We put $\omega = K_X + B$ for simplicity. For each $(p,q) \in \mathbb{Z}^2$, let $L(p,q)$ denote the base locus of the linear system $|M(p,q)|$ on X (with reduced scheme structure), where $M(p,q) = pH + qa\omega$. By definition, $L(p,q) = X$ if and only if $|M(p,q)| = \emptyset$.

Claim 1 (cf. [KM, Claim 3.20])**.** *Let ε be a positive number. For (p,q) sufficiently large and $0 < aq - rp < \varepsilon$, $L(p,q)$ is the same subset of X. We call this subset L_0. We let $I \subset \mathbb{Z} \times \mathbb{Z}$ be the set of (p,q) for which $0 < aq - rp < 1$ and $L(p,q) = L_0$. We note that I contains all sufficiently large (p,q) with $0 < aq - rp < 1$.*

For the proof, see [KM, Claim 3.20].

Claim 2. *We assume that r is not rational or that r is rational and has denominator $> a(n+1)$ in reduced form, where $n = \dim X$. Then, for (p,q) sufficiently large and $0 < aq - rp < 1$, $\mathcal{O}_X(M(p,q))$ is generated by global sections at the generic point of any lc center of (X,B).*

Proof of Claim 2. We note that $M(p,q) - \omega = pH + (qa-1)\omega$. If $aq - rp < 1$ and (p,q) is sufficiently large, then $M(p,q) - \omega$ is ample. Let C be an lc center of (X,B). Then $P_C(p,q) = \chi(C, \mathcal{O}_C(M(p,q)))$ is a non-trivial polynomial of degree at most $\dim C \leq \dim X$ by Lemma 5.2. By Lemma 5.1, there exists (p,q) such that $P_C(p,q) \neq 0$ and that (p,q) sufficiently large and $0 < aq - rp < 1$. By the ampleness of $M(p,q) - \omega$, $P_C(p,q) = \chi(C, \mathcal{O}_C(M(p,q))) = h^0(C, \mathcal{O}_C(M(p,q)))$ and $H^0(X, \mathcal{O}_X(M(p,q))) \to H^0(C, \mathcal{O}_C(M(p,q)))$ is surjective by Theorem 2.2. Therefore, $\mathcal{O}_X(M(p,q))$ is generated by global sections at the generic point of C. By combining this with Claim 1, $\mathcal{O}_X(M(p,q))$

is generated by global sections at the generic point of any lc center of (X, B) if (p, q) is sufficiently large with $0 < aq - rp < 1$. So, we obtain Claim 2. □

Note that $\mathcal{O}_X(M(p, q))$ is not generated by global sections because $M(p, q)$ is not nef. Therefore, $L_0 \neq \emptyset$. Let $D_1, \cdots, D_{n+1}$ be general members of $|M(p_0, q_0)|$ with $(p_0, q_0) \in I$. Then $K_X + B + \sum_{i=1}^{n+1} D_i$ is not lc at the generic point of any irreducible component of L_0 and is lc outside L_0. Let c be the log canonical threshold of (X, B) with respect to $D = \sum_{i=1}^{n+1} D_i$. Then, we have $0 < c < 1$ and that $K_X + B + cD$ is lc but not klt. Note that $c > 0$ by Claim 2. Thus, the lc pair $(X, B + cD)$ has some lc centers contained in L_0. Let C be an lc center contained in L_0. We consider $K_X + B + cD = \omega + cD \sim_{\mathbb{Q}} c(n+1)p_0 H + (1 + c(n+1)q_0 a)\omega$. We put $\omega' = K_X + B + cD$ for simplicity. Thus we have $pH + qa\omega - \omega' \sim_{\mathbb{Q}} (p - c(n+1)p_0)H + (qa - (1 + c(n+1)q_0 a))\omega$. If p and q are large enough and $0 < aq - rp \leq aq_0 - rp_0$, then $pH + qa\omega - \omega'$ is ample. It is because

$$
\begin{aligned}
&(p - c(n+1)p_0)H + (qa - (1 + c(n+1)q_0 a))\omega \\
&= (p - (1 + c(n+1))p_0)H + (qa - (1 + c(n+1))q_0 a)\omega + p_0 H + (q_0 a - 1)\omega.
\end{aligned}
$$

Suppose that r is not rational. There must be arbitrarily large (p, q) such that $0 < aq - rp < \varepsilon = aq_0 - rp_0$ and $\chi(C, \mathcal{O}_C(M(p, q))) \neq 0$ by Lemma 5.1 because $P_C(p, q) = \chi(C, \mathcal{O}_C(M(p, q)))$ is a non-trivial polynomial of degree at most $\dim C$ by Lemma 5.2. Since $M(p, q) - \omega'$ is ample by $0 < aq - rp < aq_0 - rp_0$, we have $h^0(C, \mathcal{O}_C(M(p, q))) = \chi(C, \mathcal{O}_C(M(p, q))) \neq 0$ by the vanishing theorem (cf. Theorem 2.2). By the vanishing theorem: Theorem 2.2, the restriction morphism

$$H^0(X, \mathcal{O}_X(M(p, q))) \to H^0(C, \mathcal{O}_C(M(p, q)))$$

is surjective because $M(p, q) - \omega'$ is ample. We note that C is an lc center of $(X, B + cD)$. Thus C is not contained in $L(p, q)$. Therefore, $L(p, q)$ is a proper subset of $L(p_0, q_0) = L_0$, giving the desired contradiction. So now we know that r is rational.

We next suppose that the assertion of the theorem concerning the denominator of r is false. Choose $(p_0, q_0) \in I$ such that $aq_0 - rp_0$ achieves its maximum value, which we can assume has the form d/v. If $0 < aq - rp \leq d/v$ and (p, q) is sufficiently large, then $\chi(C, \mathcal{O}_C(M(p, q))) = h^0(C, \mathcal{O}_C(M(p, q)))$ since $M(p, q) - \omega'$ is ample. There exists sufficiently large (p, q) in the strip $0 < aq - rp < 1$ with $\varepsilon = 1$ for which $h^0(C, \mathcal{O}_C(M(p, q))) = \chi(C, \mathcal{O}_C(M(p, q))) \neq 0$ by Lemma 5.1 since $\chi(C, \mathcal{O}_C(M(p, q)))$ is a non-trivial polynomial of degree at most $\dim C$ by Lemma 5.2. Note that $aq - rp \leq d/v = aq_0 - rp_0$ holds automatically for $(p, q) \in I$. Since $H^0(X, \mathcal{O}_X(M(p, q))) \to H^0(C, \mathcal{O}_C(M(p, q)))$

is surjective by the ampleness of $M(p,q)-\omega'$, we obtain the desired contradiction by the same reason as above. So, we finish the proof. $\square$

6. Cone theorem

In this final section, we give a proof of the cone theorem for log canonical pairs.

Proof of Theorem 1.4. The estimate $\leq 2\dim X$ in (i) can be proved by Kawamata's argument in [Ka] with the aid of [BCHM]. For details, see [F2, Subsection 3.1.3] or [F5, Section 18]. The other statements in (i) and (ii) are formal consequences of the rationality theorem. For the proof, see [KM, Theorem 3.15]. The statements (iii) and (iv) are obvious by Theorem 1.2 and the statements (i) and (ii). See Steps 7 and 9 in [KM, 3.3 The Cone Theorem]. $\square$

References

[A] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233.

[BCHM] C. Birkar, P. Cascini, C. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type, to appear in J. Amer. Math. Soc.

[F1] O. Fujino, What is log terminal?, in *Flips for* 3*-folds and* 4*-folds* (Alessio Corti, ed.), 29–62, Oxford University Press, 2007.

[F2] O. Fujino, Introduction to the log minimal model program for log canonical pairs, preprint (2009).

[F3] O. Fujino, On injectivity, vanishing and torsion-free theorems for algebraic varieties, Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), no. 8, 95–100.

[F4] O. Fujino, Introduction to the theory of quasi-log varieties, to appear in the Schiermonnikoog volume.

[F5] O. Fujino, Fundamental theorems for the log minimal model program, preprint (2009).

[Ka] Y. Kawamata, On the length of an extremal rational curve, Invent. Math. **105** (1991), no. 3, 609–611.

[Kl] S. L. Kleiman, Toward a numerical theory of ampleness, Ann. of Math. (2) **84** (1966), 293–344.

[KK] J. Kollár, S. Kovács, Log canonical singularities are Du Bois, preprint (2009).

[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties,* Cambridge Tracts in Mathematics, Vol. **134**, 1998.

[S] V. V. Shokurov, The nonvanishing theorem, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 635–651.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502 Japan
E-mail address: fujino@math.kyoto-u.ac.jp