# ESTIMATION OF ARITHMETIC LINEAR SERIES

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ABSTRACT. In this paper, we introduce arithmetic linear series and give a general way to estimate them based on Yuan's idea. As an application, we consider an arithmetic analogue of the algebraic restricted volumes.

## INTRODUCTION

In the paper [5], Lazarsfeld and Mustaţă propose general and systematic usage of Okounkov's idea ([9], [10]) in order to study asymptotic behavior of linear series on an algebraic variety. It is a very simple way, but it yields a lot of consequences, like Fujita's approximation theorem. Yuan [11] generalized this way to the arithmetic situation, and he established the arithmetic Fujita's approximation theorem, which was also proved by Chen [2] independently. In this paper, we introduce arithmetic linear series and give a general way to estimate them based on Yuan's idea. As an application, we consider an arithmetic analogue of the algebraic restricted volumes.

**Arithmetic linear series.** Let $X$ be a $d$-dimensional projective arithmetic variety and $\overline{L}$ a continuous hermitian invertible sheaf on $X$. Let $K$ be a subset of $H^0(X, L)$. The convex lattice hull $\mathrm{CL}(K)$ of $K$ is defined to be

$$\mathrm{CL}(K) := \{x \in \langle K\rangle_{\mathbb{Z}} \mid \exists m \in \mathbb{Z}_{>0}\ \ mx \in m * K\},$$

where $\langle K\rangle_{\mathbb{Z}}$ is the $\mathbb{Z}$-submodule generated by $K$ and

$$m * K = \{x_1 + \cdots + x_m \mid x_1, \ldots, x_m \in K\}.$$

We call $K$ an *arithmetic linear series* of $\overline{L}$ if

(1) $K = \mathrm{CL}(K)$,
(2) $-x \in K$ for all $x \in K$, and
(3) $K \subseteq B_{\sup}(\overline{L}) := \{s \in H^0(X, L)_{\mathbb{R}} \mid \|s\|_{\sup} \leq 1\}$.

In the case where $K = B_{\sup}(\overline{L}) \cap H^0(X, L)$, it is said to be *complete*. One of main results of this paper is a uniform estimation of the number of points in the arithmetic linear series in terms of the number of valuation vectors.

**Theorem A.** *Let $\nu$ be the valuation attached to a good flag over a prime $p$ (see Conventions and terminology 9 for the valuation attached to the flag, and Subsection 1.4 for the definition of a good flag over a prime). If $K \neq \{0\}$, then we have*

$$\begin{aligned}|\#\nu(K \setminus \{0\}) \log p - \log \#(K)| \\ \leq \left(\log(4p \operatorname{rk}\langle K\rangle_{\mathbb{Z}}) + \frac{\sigma(\overline{L}) + \log(2p \operatorname{rk}\langle K\rangle_{\mathbb{Z}})}{\log p} \log(4) \operatorname{rk} H^0(\mathcal{O}_X)\right) \operatorname{rk}\langle K\rangle_{\mathbb{Z}},\end{aligned}$$

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*where* $\sigma(\overline{L})$ *is given by*

$$\sigma(\overline{L}) := \inf_{\overline{A}\,:\,ample} \frac{\widehat{\deg}(\widehat{c}_1(\overline{A})^{d-1}\cdot \widehat{c}_1(\overline{L}))}{\deg(A_{\mathbb{Q}}^{d-1})}.$$

The ideal for the proof of the above theorem is essentially the same as Yuan's paper [11], in which he treated only the complete arithmetic linear series in my sense. A new point is the usage of convex lattices, that is, a general observation for arithmetic linear series. By this consideration, we obtain several advantages in applications. For example, we have the following theorem, which is a stronger version of [11, Theorem 3.3]. The arithmetic Fujita's approximation theorem is an immediate consequence of it.

**Theorem B.** *Let* $\overline{L}$ *be a big continuous hermitian invertible sheaf on* $X$*. For any positive* $\epsilon$*, there is a positive integer* $n_0 = n_0(\epsilon)$ *such that, for all* $n \geq n_0$*,*

$$\liminf_{k\to\infty} \frac{\log \#\operatorname{CL}(V_{k,n})}{n^d k^d} \geq \frac{\widehat{\operatorname{vol}}(\overline{L})}{d!} - \epsilon,$$

*where* $V_{k,n} = \{s_1 \otimes \cdots \otimes s_k \in H^0(X, knL) \mid s_1, \ldots, s_k \in \hat{H}^0(X, n\overline{L})\}$ *and* $\operatorname{CL}(V_{k,n})$ *is the convex lattice hull of* $V_{k,n}$ *in* $H^0(X, knL)$ *(cf. for details, see Subsection* 1.2*).*

**Arithmetic analogue of restricted volume.** For further applications, let us consider an arithmetic analogue of the restricted volume on algebraic varieties. Let $Y$ be a $d'$-dimensional arithmetic subvariety of $X$, that is, $Y$ is an integral closed subscheme of $X$ such that $Y$ is flat over $\operatorname{Spec}(\mathbb{Z})$. Let $\overline{L}$ be a continuous hermitian invertible sheaf on $X$. We denote

$$\operatorname{Image}(H^0(X,L) \to H^0(Y, L|_Y))$$

by $H^0(X|Y, L)$. Let $\|\cdot\|_{\mathrm{sup,quot}}^{X|Y}$ be the the quotient norm of $H^0(X|Y,L)\otimes_{\mathbb{Z}}\mathbb{R}$ induced by the surjective homomorphism

$$H^0(X,L)\otimes_{\mathbb{Z}}\mathbb{R} \to H^0(X|Y,L)\otimes_{\mathbb{Z}}\mathbb{R}$$

and the norm $\|\cdot\|_{\sup}$ on $H^0(X,L)\otimes_{\mathbb{Z}}\mathbb{R}$. We define $\hat{H}^0_{\mathrm{quot}}(X|Y,\overline{L})$ and $\widehat{\operatorname{vol}}_{\mathrm{quot}}(X|Y,\overline{L})$ to be

$$\hat{H}^0_{\mathrm{quot}}(X|Y,\overline{L}) := \left\{ s \in H^0(X|Y,L) \mid \|s\|_{\mathrm{sup,quot}}^{X|Y} \leq 1 \right\} \quad \text{and}$$

$$\widehat{\operatorname{vol}}_{\mathrm{quot}}(X|Y,\overline{L}) := \limsup_{m\to\infty} \frac{\log \#\hat{H}^0_{\mathrm{quot}}(X|Y,m\overline{L})}{m^{d'}/d'!}.$$

Note that $\hat{H}^0_{\mathrm{quot}}(X|Y,\overline{L})$ is an arithmetic linear series of $\overline{L}\big|_Y$. A continuous hermitian invertible sheaf $\overline{L}$ is said to be $Y$*-effective* if there is $s \in \hat{H}^0(X,\overline{L})$ with $s|_Y \neq 0$. Moreover, $\overline{L}$ is said to be $Y$*-big* if there are $n$, $\overline{A}$ and $\overline{M}$ such that $n$ is a positive integer, $\overline{A}$ is an ample $C^\infty$-hermitian invertible sheaf, $\overline{M}$ is a $Y$-effective continuous hermitian invertible sheaf and $n\overline{L} = \overline{A} + \overline{M}$. The semigroup consisting of isomorphism classes of $Y$-big continuous hermitian invertible sheaves is denoted by $\widehat{\operatorname{Big}}(X;Y)$. Then we have the following theorem, which is a generalization of [1] and [11, Theorem 2.7 and Theorem B].

**Theorem C.** (1) *If* $\overline{L}$ *is a* $Y$*-big continuous hermitian invertible sheaf on* $X$*, then* $\widehat{\operatorname{vol}}_{\mathrm{quot}}\left(X|Y,\overline{L}\right) > 0$ *and*

$$\widehat{\operatorname{vol}}_{\mathrm{quot}}\left(X|Y,\overline{L}\right) = \lim_{m\to\infty} \frac{\log \#\hat{H}^0_{\mathrm{quot}}(X|Y,m\overline{L})}{m^{d'}/d'!}.$$

*In particular,* $\widehat{\operatorname{vol}}_{\mathrm{quot}}(X|Y,n\overline{L}) = n^{d'}\widehat{\operatorname{vol}}_{\mathrm{quot}}(X|Y,\overline{L})$*.*

(2) $\widehat{\mathrm{vol}}_{\mathrm{quot}}\left(X|Y,-\right)^{\frac{1}{d'}}$ *is concave on* $\widehat{\mathrm{Big}}(X;Y)$*, that is,*

$$\widehat{\mathrm{vol}}_{\mathrm{quot}}\left(X|Y,\overline{L}+\overline{M}\right)^{\frac{1}{d'}} \geq \widehat{\mathrm{vol}}_{\mathrm{quot}}\left(X|Y,\overline{L}\right)^{\frac{1}{d'}} + \widehat{\mathrm{vol}}_{\mathrm{quot}}\left(X|Y,\overline{M}\right)^{\frac{1}{d'}}$$

*holds for any* $Y$*-big continuous hermitian invertible sheaves* $\overline{L}$ *and* $\overline{M}$ *on* $X$*.*

(3) *If* $\overline{L}$ *is a* $Y$*-big continuous hermitian invertible sheaf on* $X$*, then, for any positive number* $\epsilon$*, there is a positive integer* $n_0 = n_0(\epsilon)$ *such that, for all* $n \geq n_0$*,*

$$\liminf_{k\to\infty} \frac{\log \#\,\mathrm{CL}\left(\{s_1\otimes\cdots\otimes s_k \mid s_1,\ldots,s_k \in \hat{H}^0_{\mathrm{quot}}(X|Y,n\overline{L})\}\right)}{n^{d'}k^{d'}} \geq \frac{\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y,\overline{L})}{d'!} - \epsilon,$$

*where the convex lattice hull is considered in* $H^0(X|Y,knL)$*.*

(4) *If* $X_{\mathbb{Q}}$ *is smooth over* $\mathbb{Q}$ *and* $\overline{A}$ *is an ample* $C^\infty$*-hermitian invertible sheaf on* $X$*, then*

$$\begin{aligned}\widehat{\mathrm{vol}}_{\mathrm{quot}}\left(X|Y,\overline{A}\right) &= \widehat{\mathrm{vol}}(Y,\left.\overline{A}\right|_Y)\\ &= \lim_{m\to\infty}\frac{\log\#\,\mathrm{Image}(\hat{H}^0(X,m\overline{A})\to H^0(X|Y,mA))}{m^{d'}/d'!}.\end{aligned}$$

Let $C^0(X)$ be the set of real valued continuous functions $f$ on $X(\mathbb{C})$ such that $f$ is invariant under the complex conjugation map on $X(\mathbb{C})$. We denote the group of isomorphism classes of continuous hermitian invertible sheaves on $X$ by $\widehat{\mathrm{Pic}}(X;C^0)$. Let $\overline{\mathcal{O}} : C^0(X) \to \widehat{\mathrm{Pic}}(X;C^0)$ be the homomorphism given by

$$\overline{\mathcal{O}}(f) = (\mathcal{O}_X, \exp(-f)|\cdot|_{can}).$$

$\widehat{\mathrm{Pic}}_{\mathbb{R}}(X;C^0)$ is defined to be

$$\widehat{\mathrm{Pic}}_{\mathbb{R}}(X;C^0) := \frac{\widehat{\mathrm{Pic}}(X;C^0)\otimes\mathbb{R}}{\left\{\sum_i \overline{\mathcal{O}}(f_i)\otimes x_i \mid f_i\in C^0(X), x_i\in\mathbb{R}\ (\forall i),\ \sum_i x_i f_i = 0\right\}}.$$

Let $\gamma : \widehat{\mathrm{Pic}}(X;C^0) \to \widehat{\mathrm{Pic}}_{\mathbb{R}}(X;C^0)$ be the natural homomorphism given by the composition of homomorphisms

$$\widehat{\mathrm{Pic}}(X;C^0) \to \widehat{\mathrm{Pic}}(X;C^0)\otimes\mathbb{R} \to \widehat{\mathrm{Pic}}_{\mathbb{R}}(X;C^0).$$

Let $\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y)$ be the cone in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(X;C^0)$ generated by $\{\gamma(\overline{L}) \mid \overline{L} \in \widehat{\mathrm{Big}}(X;Y)\}$. Note that $\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y)$ is an open set in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(X;C^0)$ in the strong topology, that is, $\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y)\cap W$ is an open set in $W$ in the usual topology for any finite dimensional vector subspace $W$ of $\widehat{\mathrm{Pic}}_{\mathbb{R}}(X;C^0)$. The next theorem guarantees that

$$\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y,-) : \widehat{\mathrm{Big}}(X;Y) \to \mathbb{R}$$

extends to a continuous function $\widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y,-) : \widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y) \to \mathbb{R}$, which can be considered as a partial generalization of [6] and [7].

**Theorem D.** *There is a unique positive valued continuous function*

$$\widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y,-) : \widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y) \to \mathbb{R}$$

*with the following properties:*

(1) *The following diagram is commutative:*

$$
\begin{tikzcd}
\widehat{\mathrm{Big}}(X;Y) \arrow[r, "\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y,-)"] \arrow[d, "\gamma"'] & \mathbb{R} \\
\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y) \arrow[ur, "\widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y-)"'] &
\end{tikzcd}
$$

(2) $\widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y,-)^{\frac{1}{d'}}$ *is positively homogeneous and concave on* $\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y)$*, that is,*

$$
\begin{cases}
\widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y,\lambda x)^{\frac{1}{d'}} = \lambda \widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y,x)^{\frac{1}{d'}} \\
\widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y,x+y)^{\frac{1}{d'}} \geq \widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y,x)^{\frac{1}{d'}} + \widehat{\mathrm{vol}}''_{\mathrm{quot}}(X|Y,y)^{\frac{1}{d'}}
\end{cases}
$$

*hold for all* $\lambda \in \mathbb{R}_{>0}$ *and* $x, y \in \widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y)$.

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**Conventions and terminology.** We fix several conventions and terminology of this paper.

1. Let $M$ be a $\mathbb{Z}$-module, and let $A$ be a *sub-semigroup* of $M$, that is, $x + y \in A$ holds for all $x, y \in A$. If $0 \in A$, then $A$ is called a *sub-monoid* of $M$. The *saturation* $\mathrm{Sat}(A)$ of $A$ in $M$ is defined by

$$\mathrm{Sat}(A) := \{x \in M \mid nx \in A \text{ for some positive integer } n\}.$$

It is easy to see that $\mathrm{Sat}(A)$ is a sub-semigroup of $M$. If $A = \mathrm{Sat}(A)$, then $A$ is said to be *saturated* .

2. Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$, and let $V$ be a vector space over $\mathbb{K}$. A subset $C$ of $V$ is called a convex set in $V$ if $tx + (1-t)y \in C$ for all $x, y \in C$ and $t \in \mathbb{K}$ with $0 \leq t \leq 1$. For a subset $S$ of $V$, it is easy to see that the subset given by

$$\{t_1 s_1 + \cdots + t_r s_r \mid s_1, \ldots, s_r \in S,\ t_1, \ldots, t_r \in \mathbb{K}_{\geq 0},\ t_1 + \cdots + t_r = 1\}$$

is a convex set. It is called the *convex hull by* $S$ and is denoted by $\mathrm{Conv}_{\mathbb{K}}(S)$. Note that $\mathrm{Conv}_{\mathbb{K}}(S)$ is the smallest convex set containing $S$. A function $f : C \to \mathbb{R}$ on a convex set $C$ is said to be *concave over* $\mathbb{K}$ if $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$ holds for any $x, y \in C$ and $t \in \mathbb{K}$ with $0 \leq t \leq 1$.

3. Let $\mathbb{K}$ and $V$ be the same as in the above 2. A subset $C$ of $V$ is called a *cone* in $V$ if the following conditions are satisfied:

(a) $x + y \in C$ for any $x, y \in C$.
(b) $\lambda x \in C$ for any $x \in C$ and $\lambda \in \mathbb{K}_{>0}$.

Note that a cone is a sub-semigroup of $V$. Let $S$ be a subset of $V$. The smallest cone containing $S$, that is,

$$\{\lambda_1 a_1 + \cdots + \lambda_r a_r \mid a_1, \ldots, a_r \in S,\ \lambda_1, \ldots, \lambda_r \in \mathbb{K}_{>0}\}$$

is denoted by $\mathrm{Cone}_{\mathbb{K}}(S)$. It is called the *cone generated by* $S$.

4. Let $\mathbb{K}$ and $V$ be the same as in the above 2. The strong topology on $V$ means that a subset $U$ of $V$ is open set in this topology if and only if, for any finite dimensional vector subspace $W$ of $V$ over $\mathbb{K}$, $U \cap W$ is open in $W$ in the usual topology.

It is easy to see that a linear map of vector spaces over $\mathbb{K}$ is continuous in the strong topology. Moreover, a surjective linear map of vector spaces over $\mathbb{K}$ is an open map in the strong topology. In fact, let $f : V \to V'$ be a surjective homomorphism of vector spaces over $\mathbb{K}$, $U$ an open set of $V$, and $W'$ a finite dimensional vector subspace of $V'$ over $\mathbb{K}$. Then we can find a vector subspace $W$ of $V$ over $\mathbb{K}$ such that $f$ induces the isomorphism $f|_W : W \to W'$. If we set $\widetilde{U} = \bigcup_{t \in \operatorname{Ker}(f)} (U + t)$, then $\widetilde{U}$ is open and $f(W \cap \widetilde{U}) = W' \cap f(U)$, as required.

Let $V'$ be a vector subspace of $V$ over $\mathbb{K}$. Then the induced topology of $V'$ from $V$ coincides with the strong topology of $V'$. Indeed, let $U'$ be an open set of $V'$ in the strong topology. We can easily construct a linear map $f : V \to V'$ such that $V' \hookrightarrow V \xrightarrow{f} V'$ is the identity map. Thus $f^{-1}(U')$ is an open set in $V$, and hence $U' = f^{-1}(U')\big|_{V'}$ is an open set in the induced topology.

5. A closed integral subscheme of an arithmetic variety is called an *arithmetic subvariety* if it is flat over $\operatorname{Spec}(\mathbb{Z})$.

6. Let $X$ be an arithmetic variety. We denote the group of isomorphism classes of continuous hermitian (resp. $C^\infty$-hermitian) invertible sheaves by $\widehat{\operatorname{Pic}}(X; C^0)$ (resp. $\widehat{\operatorname{Pic}}(X; C^\infty)$). $\widehat{\operatorname{Pic}}(X; C^\infty)$ is often denoted by $\widehat{\operatorname{Pic}}(X)$ for simplicity. An element of $\widehat{\operatorname{Pic}}_{\mathbb{Q}}(X; C^0) := \widehat{\operatorname{Pic}}(X; C^0) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\widehat{\operatorname{Pic}}_{\mathbb{Q}}(X; C^\infty) := \widehat{\operatorname{Pic}}(X; C^\infty) \otimes_{\mathbb{Z}} \mathbb{Q}$) is called a continuous hermitian (resp. $C^\infty$-hermitian) $\mathbb{Q}$-invertible sheaf.

7. A $C^\infty$-hermitian invertible sheaf $\overline{A}$ on a projective arithmetic variety $X$ is said to be *ample* if $A$ is ample on $X$, the first Chern form $c_1(\overline{A})$ is positive on $X(\mathbb{C})$ and, for a sufficiently large integer $n$, $H^0(X, nA)$ is generated by the set

$$\{s \in H^0(X, nA) \mid \|s\|_{\sup} < 1\}$$

as a $\mathbb{Z}$-module. Note that, for $\overline{A}, \overline{L} \in \widehat{\operatorname{Pic}}(X; C^\infty)$, if $\overline{A}$ is ample, then there is a positive integer $m$ such that $m\overline{A} + \overline{L}$ is ample.

8. Let $\overline{L}$ be a continuous hermitian invertible sheaf on a projective arithmetic variety $X$. Then $B_{\sup}(\overline{L})$ is defined to be

$$B_{\sup}(\overline{L}) = \{s \in H^0(X, L)_{\mathbb{R}} \mid \|s\|_{\sup} \leq 1\}.$$

Note that $\hat{H}^0(X, \overline{L}) = H^0(X, L) \cap B_{\sup}(\overline{L})$.

9. Let $A$ be a noetherian integral domain and $t \notin A^\times$. As $\bigcap_{n \geq 0} t^n A = \{0\}$, for $a \in A \setminus \{0\}$, we can define $\operatorname{ord}_{tA}(a)$ to be

$$\operatorname{ord}_{tA}(a) = \max\{n \in \mathbb{Z}_{\geq 0} \mid a \in t^n A\}.$$

Let $\{0\} = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d$ be a chain of prime ideals of $A$. Let $A_i = A/P_i$ for $i = 0, \ldots, d$, and let $\rho_i : A_{i-1} \to A_i$ be natural homomorphisms as follows.

$$A = A_0 \xrightarrow{\rho_1} A_1 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{d-1}} A_{d-1} \xrightarrow{\rho_d} A_d.$$

We assume that $P_d$ is a maximal ideal, and that $P_iA_{i-1} = \mathrm{Ker}(\rho_i)$ is a principal ideal of $A_{i-1}$ for every $i = 1, \ldots, d$, that is, there is $t_i \in A_{i-1}$ with $P_iA_{i-1} = t_iA_{i-1}$. For $a \neq 0$, the *valuation vector* $(\nu_1(a), \ldots, \nu_d(a))$ of $a$ is defined in the following way:

$$a_1 := a \quad \text{and} \quad \nu_1(a) := \mathrm{ord}_{t_1A_0}(a_1).$$

If $a_1 \in A_0, a_2 \in A_1, \ldots, a_i \in A_{i-1}$ and $\nu_1(a), \ldots, \nu_i(a) \in \mathbb{Z}_{\geq 0}$ are given, then

$$a_{i+1} := \rho_i(a_it_i^{-\nu_i(a)}) \quad \text{and} \quad \nu_{i+1}(a) := \mathrm{ord}_{t_{i+1}A_i}(a_{i+1}).$$

Note that the valuation vector $(\nu_1(a), \ldots, \nu_d(a))$ does not depend on the choice of $t_1, \ldots, t_d$.

Let $X$ be a noetherian integral scheme and

$$Y_{\cdot} : Y_0 = X \supset Y_1 \supset Y_2 \supset \cdots \supset Y_d$$

a chain of integral subschemes of $X$. We say $Y_{\cdot}$ a *flag* if $Y_d$ consists of a closed point $y$ and $Y_{i+1}$ is locally principal at $y$ in $Y_i$ for all $i = 0, \ldots, d-1$. Let $A = \mathcal{O}_{X,y}$ and $P_i$ the defining prime ideal of $Y_i$ in $A$. Then we have a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d$ of prime ideals as above, so that we obtain the valuation vector $(\nu_1(a), \ldots, \nu_d(a))$ for each $a \in A \setminus \{0\}$. It is called the *valuation vector* attached to the flag $Y_{\cdot}$ and is denoted by $\nu_{Y_{\cdot}}(a)$ or $\nu(a)$. Let $L$ be an invertible sheaf on $X$ and $\omega$ a local basis of $L$ at $y$. For each $s \in H^0(X, L)$, we can find $a_s \in A$ with $s = a_s\omega$. Then $\nu_{Y_{\cdot}}(a_s)$ is denoted by $\nu_{Y_{\cdot}}(s)$. Note that $\nu_{Y_{\cdot}}(s)$ does not depend on the choice of $\omega$.

## 1. Preliminaries

1.1. **Open cones.** Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$, and let $V$ be a vector space over $\mathbb{K}$. A cone in $V$ is said to be *open* if it is an open set in $V$ in the strong topology (see Conventions and terminology 4).

**Proposition 1.1.1.** *Let $C$ be a cone in $V$. Then we have the following:*

(1) *$C$ is open if and only if, for any $a \in C$ and $x \in V$, there is $\delta_0 \in \mathbb{K}_{>0}$ such that $a + \delta_0x \in C$.*
(2) *Let $f : V \to V'$ be a surjective homomorphism of vector spaces over $\mathbb{K}$.*
  (2.1) *If $C$ is open in $V$, then $f(C)$ is also open in $V'$.*
  (2.2) *If $C + \mathrm{Ker}(f) \subseteq C$, then $f^{-1}(f(C)) = C$.*

*Proof.* (1) If $C$ is open, then the condition in (1) is obviously satisfied. Conversely we assume that, for any $a \in C$ and $x \in V$, there is $\delta_0 \in \mathbb{K}_{>0}$ such that $a + \delta_0x \in C$. First let see the following claim:

**Claim 1.1.1.1.** *For any $a \in C$ and $x \in V$, there is $\delta_0 \in \mathbb{K}_{>0}$ such that $a + \delta x \in C$ holds for all $\delta \in \mathbb{K}$ with $|\delta| \leq \delta_0$.*

By our assumption, there are $\delta_1, \delta_2 \in \mathbb{K}_{>0}$ such that $a + \delta_1x, a + \delta_2(-x) \in C$. For $\delta \in \mathbb{K}$ with $-\delta_2 \leq \delta \leq \delta_1$, if we set $\lambda = (\delta + \delta_1)/(\delta_1 + \delta_2)$, then $0 \leq \lambda \leq 1$ and $\delta = \lambda\delta_1 + (1-\lambda)(-\delta_2)$. Thus

$$\lambda(b + \delta_1x) + (1-\lambda)(b + \delta_2(-x)) = b + \delta x \in C.$$

Therefore, if we put $\delta_0 = \min\{\delta_1, \delta_2\}$, then the assertion of the claim follows. □

Let $W$ be a finite dimensional vector subspace of $V$ over $\mathbb{K}$ and $a \in W \cap C$. Let $e_1, \ldots, e_n$ be a basis of $W$. Then, by the above claim, there is $\delta_0 \in \mathbb{K}_{>0}$ such that $a/n + \delta e_i \in C$ holds for all $i$ and all $\delta \in \mathbb{K}$ with $|\delta| \leq \delta_0$. We set

$$U = \{x_1e_1 + \cdots + x_ne_n \mid |x_1| < \delta_0, \ldots, |x_n| < \delta_0\}.$$

It is sufficient to see that $a + U \subseteq C$. Indeed, if $x = x_1e_1 + \cdots + x_ne_n \in U$, then

$$a + x = \sum_{i=1}^{n}(a/n + x_ie_i) \in C.$$

(2) (2.1) follows from the fact that $f$ is an open map (cf. Conventions and terminology 4). Clearly $f^{-1}(f(C)) \supseteq C$. Conversely let $x \in f^{-1}(f(C))$. Then there are $a \in C$ with $f(x) = f(a)$. Thus we can find $u \in \operatorname{Ker}(f)$ such that $x-a = u$ because $f(x-a) = 0$. Hence

$$x = a + u \in C + \operatorname{Ker}(f) \subseteq C.$$

■

To proceed with further arguments, we need the following two lemmas.

**Lemma 1.1.2.** *Let $S$ and $T$ be subsets of $V$. Then*

$$\operatorname{Cone}_{\mathbb{K}}(S+T) \subseteq \operatorname{Cone}_{\mathbb{K}}(S) + \operatorname{Cone}_{\mathbb{K}}(T),$$

*where $S+T = \{s+t \mid s \in S,\ t \in T\}$. Moreover, if "$a \in \mathbb{Z}_{\geq 0}, t \in T \Longrightarrow at \in T$" holds, then $\operatorname{Cone}_{\mathbb{K}}(S+T) = \operatorname{Cone}_{\mathbb{K}}(S) + \operatorname{Cone}_{\mathbb{K}}(T)$.*

*Proof.* The first assertion is obvious. Let $x \in \operatorname{Cone}_{\mathbb{K}}(S) + \operatorname{Cone}_{\mathbb{K}}(T)$. Then there are $s_1, \ldots, s_r \in S$, $t_1, \ldots, t_{r'} \in T$, $\lambda_1, \ldots, \lambda_r \in \mathbb{K}_{>0}$ and $\mu_1, \ldots, \mu_{r'} \in \mathbb{K}_{>0}$ such that $x = \lambda_1s_1 + \cdots + \lambda_rs_r + \mu_1t_1 + \cdots + \mu_{r'}t_{r'}$. We choose a positive integer $N$ with $N\lambda_1 > \mu_1 + \cdots + \mu_{r'}$. Then

$$\begin{aligned} x = \left(\lambda_1 - \frac{\mu_1 + \cdots + \mu_{r'}}{N}\right)(s_1 + 0) + \lambda_2(s_2+0) + \cdots + \lambda_r(s_r+0) \\ + (\mu_1/N)(s_1 + Nt_1) + \cdots + (\mu_{r'}/N)(s_1 + Nt_{r'}) \in \operatorname{Cone}_{\mathbb{K}}(S+T) \end{aligned}$$

because $0, Nt_1, \ldots, Nt_{r'} \in T$. ■

**Lemma 1.1.3.** *Let $P$ be a vector space over $\mathbb{Q}$, $x_1, \ldots, x_r \in P$, $b_1, \ldots, b_m \in \mathbb{Q}$ and $A$ a $(r \times m)$-matrix whose entries belong to $\mathbb{Q}$. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}$ with $(\lambda_1, \ldots, \lambda_r)A = (b_1, \ldots, b_m)$. If $x := \lambda_1x_1 + \cdots + \lambda_rx_r \in P$, then there are $\lambda'_1, \ldots, \lambda'_r \in \mathbb{Q}_{\geq 0}$ such that $x = \lambda'_1x_1 + \cdots + \lambda'_rx_r$ and $(\lambda'_1, \ldots, \lambda'_r)A = (b_1, \ldots, b_m)$. Moreover, if $\lambda_i$'s are positive, then we can choose positive $\lambda'_i$'s.*

*Proof.* If $\lambda_i = 0$, then

$$x = \sum_{j \neq i} \lambda_jx_j \quad \text{and} \quad (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_r)A' = (b_1, \ldots, b_m),$$

where $A'$ is the $(r-1) \times n$-matrix obtained by deleting the $i$-th row from $A$. Thus we may assume that $\lambda_i > 0$ for all $i$. Let $e_1, \ldots, e_n$ be a basis of $\langle x_1, \ldots, x_r, x\rangle_{\mathbb{Q}}$. We set $x_i = \sum_j c_{ij}e_j$ and $x = \sum_j d_je_j$ ($c_{ij} \in \mathbb{Q}$, $d_j \in \mathbb{Q}$). Then $d_j = \sum_i \lambda_ic_{ij}$. Let $C = (c_{ij})$ and we consider linear maps $f_{\mathbb{Q}} : \mathbb{Q}^r \to \mathbb{Q}^{m+n}$ and $f_{\mathbb{R}} : \mathbb{R}^r \to \mathbb{R}^{m+n}$ given by

$$f_{\mathbb{Q}}(s_1, \ldots, s_r) = (s_1, \ldots, s_r)(A, C) \quad \text{and} \quad f_{\mathbb{R}}(t_1, \ldots, t_r) = (t_1, \ldots, t_r)(A, C).$$

Then $f_{\mathbb{R}}(\lambda_1, \ldots, \lambda_r) = (b_1, \ldots, b_m, d_1, \ldots, d_n)$, that is,

$$(b_1, \ldots, b_m, d_1, \ldots, d_n) \in f_{\mathbb{R}}(\mathbb{R}^r) \cap \mathbb{Q}^{m+n}.$$

Note that $f_{\mathbb{R}}(\mathbb{R}^r) \cap \mathbb{Q}^{m+n} = f_{\mathbb{Q}}(\mathbb{Q}^r)$ because

$$\mathbb{Q}^{m+n}/f_{\mathbb{Q}}(\mathbb{Q}^r) \to (\mathbb{Q}^{m+n}/f_{\mathbb{Q}}(\mathbb{Q}^r)) \otimes_{\mathbb{Q}} \mathbb{R}$$

is injective and

$$(\mathbb{Q}^{m+n}/f_{\mathbb{Q}}(\mathbb{Q}^r)) \otimes_{\mathbb{Q}} \mathbb{R} = (\mathbb{Q}^{m+n} \otimes_{\mathbb{Q}} \mathbb{R})/(f_{\mathbb{Q}}(\mathbb{Q}^r) \otimes_{\mathbb{Q}} \mathbb{R}) = \mathbb{R}^{m+n}/f_{\mathbb{R}}(\mathbb{R}^r).$$

Therefore there is $(e_1, \ldots, e_r) \in \mathbb{Q}^r$ with $f_{\mathbb{Q}}(e_1, \ldots, e_r) = (b_1, \ldots, b_m, d_1, \ldots, d_n)$, and hence

$$\begin{cases} f_{\mathbb{Q}}^{-1}(b_1, \ldots, b_m, d_1, \ldots, d_n) = f_{\mathbb{Q}}^{-1}(0) + (e_1, \ldots, e_r), \\ f_{\mathbb{R}}^{-1}(b_1, \ldots, b_m, d_1, \ldots, d_n) = f_{\mathbb{R}}^{-1}(0) + (e_1, \ldots, e_r). \end{cases}$$

In particular, $f_{\mathbb{Q}}^{-1}(b_1, \ldots, b_m, d_1, \ldots, d_n)$ is dense in $f_{\mathbb{R}}^{-1}(b_1, \ldots, b_m, d_1, \ldots, d_n)$. Thus, as $(\lambda_1, \ldots, \lambda_r) \in f_{\mathbb{R}}^{-1}(b_1, \ldots, b_m, d_1, \ldots, d_n) \cap \mathbb{R}^r_{>0}$, we have

$$f_{\mathbb{Q}}^{-1}(b_1, \ldots, b_m, d_1, \ldots, d_n) \cap \mathbb{R}^r_{>0} \neq \emptyset,$$

that is, we can find $(\lambda'_1, \ldots, \lambda'_r) \in \mathbb{Q}^r_{>0}$ with $f_{\mathbb{Q}}(\lambda'_1, \ldots, \lambda'_r) = (b_1, \ldots, b_m, d_1, \ldots, d_n)$. Hence

$$x = \lambda'_1 x_1 + \cdots + \lambda'_r x_r \quad \text{and} \quad (\lambda'_1, \ldots, \lambda'_r)A = (b_1, \ldots, b_m).$$

■

Next we consider the following proposition.

**Proposition 1.1.4.** *Let $P$ be a vector space over $\mathbb{Q}$ and $V = P \otimes_{\mathbb{Q}} \mathbb{R}$. Let $C$ be a cone in $P$. Then we have the following:*

(1) $\operatorname{Cone}_{\mathbb{R}}(C) \cap P = C$.
(2) *If $C$ is open, then $\operatorname{Cone}_{\mathbb{R}}(C)$ is also open.*
(3) *If $D$ is a cone in $P$ with $0 \in D$, then $\operatorname{Cone}_{\mathbb{R}}(C + D) = \operatorname{Cone}_{\mathbb{R}}(C) + \operatorname{Cone}_{\mathbb{R}}(D)$.*

*Proof.* (1) Clearly $C \subseteq \operatorname{Cone}_{\mathbb{R}}(C) \cap P$. We assume that $x \in \operatorname{Cone}_{\mathbb{R}}(C) \cap P$. Then, by Lemma 1.1.3, there are $\omega_1, \ldots, \omega_r \in C$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}_{>0}$ with $x = \lambda_1\omega_1 + \cdots + \lambda_r\omega_r$, which means that $x = \lambda_1\omega_1 + \cdots + \lambda_r\omega_r \in C$.

(2) First let us see the following: for $a \in C$ and $x \in P$, there is $\delta_0 \in \mathbb{Q}_{>0}$ such that $a + \delta x \in \operatorname{Cone}_{\mathbb{R}}(C)$ for all $\delta \in \mathbb{R}$ with $|\delta| \leq \delta_0$. Indeed, by our assumption, there is $\delta_0 \in \mathbb{Q}_{>0}$ such that $a \pm \delta_0 x \in C$. For $\delta \in \mathbb{R}$ with $|\delta| \leq \delta_0$, if we set $\lambda = (\delta + \delta_0)/2\delta_0$, then $0 \leq \lambda \leq 1$ and $\delta = \lambda\delta_0 + (1-\lambda)(-\delta_0)$. Thus $b + \delta x = \lambda(b + \delta_0 x) + (1-\lambda)(b - \delta_0 x) \in \operatorname{Cone}_{\mathbb{R}}(C)$.

By (1) in Proposition 1.1.1, it is sufficient to see that, for $a' \in \operatorname{Cone}_{\mathbb{R}}(C)$ and $x' \in V$, there is a positive $\delta' \in \mathbb{R}_{>0}$ with $a' + \delta' x' \in \operatorname{Cone}_{\mathbb{R}}(C)$. We set $a' = \lambda_1 a_1 + \cdots + \lambda_r a_r$ ($a_1, \ldots, a_r \in C$, $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{>0}$) and $x' = \mu_1 x_1 + \cdots + \mu_n x_n$ ($x_1, \ldots, x_n \in P$, $\mu_1, \ldots, \mu_n \in \mathbb{R}$). We choose $\lambda \in \mathbb{Q}$ such that $0 < \lambda < \lambda_1$. By the above claim, there is $\delta_0 \in \mathbb{Q}_{>0}$ such that $(\lambda/n)a_1 + \delta x_j \in \operatorname{Cone}_{\mathbb{R}}(C)$ for all $j$ and all $\delta \in \mathbb{R}$ with $|\delta| \leq \delta_0$. We choose $\delta' \in \mathbb{R}_{>0}$ such that $|\delta' \mu_j| \leq \delta_0$ for all $j$. Then

$$a' + \delta' x = (\lambda_1 - \lambda)a_1 + \sum_{i \geq 2} \lambda_i a_i + \sum_{j=1}^{n} ((\lambda/n)a_1 + \delta' \mu_j x_j) \in \operatorname{Cone}_{\mathbb{R}}(C),$$

as required.

(3) follows from Lemma 1.1.2. ■

Let $M$ be a $\mathbb{Z}$-module and $A$ a sub-semigroup of $M$. $A$ is said to be *open* if, for any $a \in A$ and $x \in M$, there is a positive integer $n$ such that $na + x \in A$. For example, let $X$ be a projective arithmetic variety and $\widehat{\operatorname{Amp}}(X)$ the sub-semigroup of $\widehat{\operatorname{Pic}}(X; C^{\infty})$ consisting of ample $C^{\infty}$-hermitian invertible sheaves on $X$. Then $\widehat{\operatorname{Amp}}(X)$ is open as a sub-semigroup of $\widehat{\operatorname{Pic}}(X; C^{\infty})$ (cf. Conventions and terminology 7).

**Proposition 1.1.5.** *Let $\iota : M \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ be the natural homomorphism, and let $A$ be sub-semigroups of $M$. Then we have the following.*

(1) $\operatorname{Cone}_{\mathbb{Q}}(\iota(A)) = \{(1/n)\iota(a) \mid n \in \mathbb{Z}_{>0},\ a \in A\}$.

(2) $\operatorname{Sat}(A) = \iota^{-1}(\operatorname{Cone}_{\mathbb{Q}}(\iota(A))$ (*see Conventions and terminology* 1 *for the saturation* $\operatorname{Sat}(A)$ *of* $A$ *in* $M$).

(3) *If $A$ is open, then $\operatorname{Cone}_{\mathbb{Q}}(\iota(A))$ is an open set in $M \otimes_{\mathbb{Z}} \mathbb{Q}$.*

(4) *If $B$ is a sub-monoid of $M$, then*
$$\operatorname{Cone}_{\mathbb{Q}}(\iota(A+B)) = \operatorname{Cone}_{\mathbb{Q}}(\iota(A)) + \operatorname{Cone}_{\mathbb{Q}}(\iota(B)).$$

(5) *Let $f : A \to \mathbb{R}$ be a function on A. If there is a positive real number $e$ such that $f(na) = n^e f(a)$ for all $n \in \mathbb{Z}_{>0}$ and $a \in A$, then there is a unique function $\tilde{f} : \operatorname{Cone}_{\mathbb{Q}}(\iota(A)) \to \mathbb{R}$ with the following properties:*

(5.1) $\tilde{f} \circ \iota = f$.

(5.2) $\tilde{f}(\lambda x) = \lambda^e f(x)$ *for all* $\lambda \in \mathbb{Q}_{>0}$ *and* $x \in \operatorname{Cone}_{\mathbb{Q}}(\iota(A))$.

*Proof.* (1) Let $x \in \operatorname{Cone}_{\mathbb{Q}}(\iota(A))$. Then there are $n, m_1, \ldots, m_r \in \mathbb{Z}_{>0}$ and $a_1, \ldots, a_r \in A$ such that $x = (m_1/n)\iota(a_1) + \cdots + (m_r/n)\iota(a_r)$. Thus, if we set $a = m_1 a_1 + \cdots + m_r a_r \in A$, then $x = (1/n)\iota(a)$. The converse is obvious.

(2) Clearly $\iota^{-1}(\operatorname{Cone}_{\mathbb{Q}}(\iota(A))$ is saturated, and hence
$$\operatorname{Sat}(A) \subseteq \iota^{-1}(\operatorname{Cone}_{\mathbb{Q}}(\iota(A)).$$
Conversely we assume that $x \in \iota^{-1}(\operatorname{Cone}_{\mathbb{Q}}(\iota(A))$. Then, by (1), there are $n \in \mathbb{Z}_{>0}$ and $a \in A$ such that $\iota(x) = (1/n)\iota(a)$. Thus, as $\iota(nx - a) = 0$, there is $n' \in \mathbb{Z}_{>0}$ such that $n'(nx - a) = 0$, which means that $n'nx \in A$, as required.

(3) By (1) in Proposition 1.1.1, it is sufficient to show that, for any $a' \in \operatorname{Cone}_{\mathbb{Q}}(\iota(A))$ and $x' \in M \otimes \mathbb{Q}$, there is $\delta \in \mathbb{Q}_{>0}$ such that $a' + \delta x' \in \operatorname{Cone}_{\mathbb{Q}}(\iota(A))$. We can choose $a \in A$, $x \in M$ and positive integers $n_1$ and $n_2$ such that $a' = (1/n_1)\iota(a)$ and $x' = (1/n_2)\iota(x)$. By our assumption, there is a positive integer $n$ such that $na + x \in A$. Thus
$$nn_1 a' + n_2 x' = \iota(na + x) \in \iota(A),$$
which yields $a' + (n_2/nn_1)x' \in \operatorname{Cone}_{\mathbb{Q}}(\iota(A))$.

(4) By virtue of Lemma 1.1.2,
$$\operatorname{Cone}_{\mathbb{R}}(\iota(A+B)) = \operatorname{Cone}_{\mathbb{R}}(\iota(A) + \iota(B)) = \operatorname{Cone}_{\mathbb{R}}(\iota(A)) + \operatorname{Cone}_{\mathbb{R}}(\iota(B)).$$

(5) First let us see the uniqueness of $\tilde{f}$. Indeed, if it exists, then
$$\tilde{f}((1/n)\iota(a)) = (1/n)^e \tilde{f}(\iota(a)) = (1/n)^e f(a).$$
By the above observation, in order to define $\tilde{f} : \operatorname{Conv}_{\mathbb{Q}}(\iota(A)) \to \mathbb{R}$, it is sufficient to show that if $(1/n)\iota(a) = (1/n')\iota(a')$ ($n, n' \in \mathbb{Z}_{>0}$ and $a, a' \in A$), then $(1/n)^e f(a) = (1/n')^e f(a')$. As $\iota(n'a - na') = 0$, there is $m \in \mathbb{Z}_{>0}$ such that $mn'a = mna'$. Thus
$$(mn')^e f(a) = f((mn')a) = f((mn)a') = (mn)^e f(a),$$
which implies that $(1/n)^e f(a) = (1/n')^e f(a')$. Finally let us see (5.2). We choose positive integers $n, n_1, n_2$ and $a \in A$ such that $\lambda = n_1/n_2$ and $x = (1/n)\iota(a)$. Then
$$\begin{aligned} \tilde{f}(\lambda x) &= \tilde{f}((1/n_2 n)\iota(n_1 a)) = (1/n_2 n)^e f(n_1 a) = (1/n_2 n)^e n_1^e f(a) \\ &= \lambda^e (1/n)^e f(a) = \lambda^e \tilde{f}(x). \end{aligned}$$
■

1.2. **Convex lattice.** Let $M$ be a finitely generated free $\mathbb{Z}$-module. Let $K$ be a subset of $M$. The $\mathbb{Z}$-submodule generated by $K$ in $M$ and the convex hull of $K$ in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ are denoted by $\langle K \rangle_{\mathbb{Z}}$ and $\mathrm{Conv}_{\mathbb{R}}(K)$ respectively. For a positive integer $m$, the $m$-fold sum $m * K$ of elements in $K$ is defined to be

$$m * K = \{x_1 + \cdots + x_m \mid x_1, \ldots, x_m \in K\}.$$

We say $K$ is a *convex lattice* if

$$\langle K \rangle_{\mathbb{Z}} \cap \frac{1}{m}(m * K) \subseteq K, \quad \text{that is,} \quad m\langle K \rangle_{\mathbb{Z}} \cap (m * K) \subseteq mK$$

holds for all $m \geq 1$. Moreover, $K$ is said to be *symmetric* if $-x \in K$ for all $x \in K$. Note that if $K$ is symmetric, then $\mathrm{Conv}_{\mathbb{R}}(K)$ is also symmetric.

**Proposition 1.2.1.** *Let $K$ be a subset of $M$. Then we have the following:*

(1) $\langle K \rangle_{\mathbb{Z}} \cap \displaystyle\bigcup_{m=1}^{\infty} \frac{1}{m}(m * K) = \langle K \rangle_{\mathbb{Z}} \cap \mathrm{Conv}_{\mathbb{R}}(K)$.

(2) *The following are equivalent:*
  (2.1) *$K$ is a convex lattice.*
  (2.2) $K = \langle K \rangle_{\mathbb{Z}} \cap \mathrm{Conv}_{\mathbb{R}}(K)$.
  (2.3) *There are a $\mathbb{Z}$-submodule $N$ of $M$ and a convex set $\Delta$ in $M_{\mathbb{R}}$ such that $K = N \cap \Delta$.*

*Proof.* (1) Obviously $\langle K \rangle_{\mathbb{Z}} \cap \bigcup_{m=1}^{\infty}(1/m)(m*K) \subseteq \langle K \rangle_{\mathbb{Z}} \cap \mathrm{Conv}_{\mathbb{R}}(K)$. We assume that $x \in \langle K \rangle_{\mathbb{Z}} \cap \mathrm{Conv}_{\mathbb{R}}(K)$. Then there are $a_1, \ldots, a_l \in K$ and $\mu_1, \ldots, \mu_l \in \mathbb{R}_{\geq 0}$ such that $x = \mu_1 a_1 + \cdots + \mu_l a_l$ and $\mu_1 + \cdots + \mu_l = 1$. As $x \in \langle K \rangle_{\mathbb{Z}} \subseteq M$, by using Lemma 1.1.3, we can find $\lambda_1, \ldots, \lambda_l \in \mathbb{Q}_{\geq 0}$ such that $\lambda_1 + \cdots + \lambda_l = 1$ and $x = \lambda_1 a_1 + \cdots + \lambda_l a_l$. We set $\lambda_i = d_i/m$ for $i = 1, \ldots, l$. Then, as $d_1 + \cdots + d_l = m$, we have

$$x = \frac{d_1 x_1 + \cdots + d_l x_l}{m} \in \langle K \rangle_{\mathbb{Z}} \cap \frac{1}{m}(m * K).$$

(2) (2.1) $\Longrightarrow$ (2.2) : Since $K$ is a convex lattice, by (1), $K = \langle K \rangle_{\mathbb{Z}} \cap \mathrm{Conv}_{\mathbb{R}}(K)$.
(2.2) $\Longrightarrow$ (2.3) is obvious.
(2.3) $\Longrightarrow$ (2.1) : First of all, note that $\langle K \rangle_{\mathbb{Z}} \subseteq N$ and $\mathrm{Conv}_{\mathbb{R}}(K) \subseteq \Delta$. Thus

$$\langle K \rangle_{\mathbb{Z}} \cap \frac{1}{m}(m * K) \subseteq \langle K \rangle_{\mathbb{Z}} \cap \mathrm{Conv}_{\mathbb{R}}(K) \subseteq N \cap \Delta = K.$$

■

Let $K$ be a subset of $M$. Then, by the above proposition,

$$\langle K \rangle_{\mathbb{Z}} \cap \bigcup_{m=1}^{\infty} \frac{1}{m}(m * K) = \{x \in \langle K \rangle_{\mathbb{Z}} \mid \exists m \in \mathbb{Z}_{>0} \ \ mx \in m * K\}$$

is a convex lattice, so that it is called the *convex lattice hull* of $K$ and is denoted by $\mathrm{CL}(K)$. Note that the convex lattice hull of $K$ is the smallest convex lattice containing $K$. Let $f : M \to M'$ be an injective homomorphism of finitely generated free $\mathbb{Z}$-modules. Then it is easy to see that $f(\mathrm{CL}(K)) = \mathrm{CL}(f(K))$. Finally we consider the following lemma. Ideas for the proof of the lemma can be found in Yuan's paper [11, §2.3].

**Lemma 1.2.2.** *Let $M$ be a finitely generated free $\mathbb{Z}$-module and $r : M \to N$ a homomorphism of finitely generated $\mathbb{Z}$-modules. For a symmetric finite subset $K$ of $M$, we have the following estimation:*

(1.2.2.1) $$\log \#r(K) \geq \log \#(K) - \log \#(\mathrm{Ker}(r) \cap (2 * K)).$$

$$\log \#r(K) \leq \log \#(2 * K) - \log \#(\mathrm{Ker}(r) \cap K). \tag{1.2.2.2}$$

*Moreover, if $\Delta$ is a bounded and symmetric convex set in $M_{\mathbb{R}}$ and $a$ is a real number with $a \geq 1$, then*

$$0 \leq \log \#(M \cap a\Delta) - \log \#(M \cap \Delta) \leq \log(\lceil 2a \rceil) \operatorname{rk} M. \tag{1.2.2.3}$$

*Proof.* Let $t \in r(K)$ and fix $s_0 \in K$ with $r(s_0) = t$. Then, for any $s \in r^{-1}(t) \cap K$,

$$s - s_0 = s + (-s_0) \in \mathrm{Ker}(r) \cap (2 * K).$$

Thus

$$\#(r^{-1}(t) \cap K) \leq \#(\mathrm{Ker}(r) \cap (2 * K)).$$

Therefore,

$$\#(K) = \sum_{t \in r(K)} \#(r^{-1}(t) \cap K) \leq \#(r(K))\#(\mathrm{Ker}(r) \cap (2 * K)),$$

as required.

We set $S = K + \mathrm{Ker}(r) \cap K$. Then $r(S) = r(K)$ and $S \subseteq 2 * K$. Moreover, for all $t \in r(S)$,

$$\#(\mathrm{Ker}(r) \cap K) \leq \#(S \cap r^{-1}(t)).$$

Indeed, if we choose $s_0 \in K$ with $r(s_0) = t$, then

$$s_0 + \mathrm{Ker}(r) \cap K \subseteq S \cap r^{-1}(t).$$

Therefore,

$$\begin{aligned} \#(2 * K) \geq \#(S) &= \sum_{t \in r(S)} \#(r^{-1}(t) \cap S) \geq \#(r(S))\#(\mathrm{Ker}(r) \cap K) \\ &= \#(r(K))\#(\mathrm{Ker}(r) \cap K) \end{aligned}$$

as required.

We set $n = \lceil 2a \rceil$. Applying (1.2.2.1) to the case where $K = M \cap (n/2)\Delta$ and $r : M \to M/nM$, we have

$$\begin{aligned} &\log \#(M \cap (n/2)\Delta) - \log \#(nM \cap 2 * ((n/2)\Delta \cap M)) \\ &\qquad\qquad \leq \log \#M/nM = \log(n) \operatorname{rk} M. \end{aligned}$$

Note that $a \leq n/2$ and

$$\begin{aligned} &\#(nM \cap 2 * ((n/2)\Delta \cap M)) \leq \#(nM \cap (n\Delta \cap M)) \\ &\qquad\qquad = \#(nM \cap n\Delta) = \#(M \cap \Delta). \end{aligned}$$

Hence we obtain

$$\begin{aligned} 0 &\leq \log \#(M \cap a\Delta) - \log \#(M \cap \Delta) \\ &\leq \log \#(M \cap (n/2)\Delta) - \log \#(nM \cap 2 * ((n/2)\Delta \cap M)) \leq \log(n) \operatorname{rk} M. \end{aligned}$$

■

1.3. **Concave function and its continuity.** Let $P$ be a vector space over $\mathbb{Q}$ and $V = P \otimes \mathbb{R}$. Let $C$ be a non-empty open convex set in $V$. Let $f : C \cap P \to \mathbb{R}$ be a concave function over $\mathbb{Q}$ (cf. Conventions and terminology 2).

We assume that $P$ is finite dimensional and $d = \dim_{\mathbb{Q}} P$. Let $h$ be an inner product of $V$. For $x \in V$, we denote $\sqrt{h(x,x)}$ by $\|x\|_h$. Moreover, for a positive number $r$ and $x \in V$, we set

$$U(x,r) = \{y \in V \mid \|y - x\|_h < r\}.$$

**Proposition 1.3.1.** *For any $x \in C$, there are positive numbers $\epsilon$ and $L$ such that $U(x,\epsilon) \subseteq C$ and $|f(y) - f(z)| \leq L\|y - z\|_h$ for all $y, z \in U(x,\epsilon) \cap P$. In particular, there is a unique concave and continuous function $\tilde{f} : C \to \mathbb{R}$ such that $\tilde{f}\Big|_{C\cap P} = f$.*

*Proof.* The proof of this proposition is almost same as one of [4, Theorem 2.2], but we need a slight modification because $x$ is not necessarily a point of $P$. Let us begin with the following claim.

**Claim 1.3.1.1.** *$f(t_1x_1 + \cdots + t_rx_r) \geq t_1f(x_1) + \cdots + t_rf(x_r)$ holds for any $x_1, \ldots, x_r \in C \cap P$ and $t_1, \ldots, t_r \in \mathbb{Q}_{\geq 0}$ with $t_1 + \cdots + t_r = 1$.*

We prove it by induction on $r$. In the case where $r = 1, 2$, the assertion is obvious. We assume $r \geq 3$. If $t_1 = 1$, then the assertion is also obvious, so that we may assume that $t_1 < 1$. Then, by using the hypothesis of induction,

$$\begin{aligned} f(t_1x_1 + \cdots + t_rx_r) &= f\left(t_1x_1 + (1-t_1)\left(\frac{t_2}{1-t_1}x_2 + \cdots + \frac{t_r}{1-t_1}x_r\right)\right) \\ &\geq t_1f(x_1) + (1-t_1)f\left(\frac{t_2}{1-t_1}x_2 + \cdots + \frac{t_r}{1-t_1}x_r\right) \\ &\geq t_1f(x_1) + (1-t_1)\left(\frac{t_2}{1-t_1}f(x_2) + \cdots + \frac{t_r}{1-t_1}f(x_r)\right) \\ &= t_1f(x_1) + \cdots + t_rf(x_r). \end{aligned}$$

□

**Claim 1.3.1.2.** *There are $x_1, \ldots, x_{d+1} \in C \cap P$ such that $x$ is an interior point of $\mathrm{Conv}_{\mathbb{R}}(\{x_1, \ldots, x_{d+1}\})$.*

Let us consider the function $\phi : C^d \to \mathbb{R}$ given by

$$\phi(y_1, \ldots, y_d) = \det(y_1 - x, \ldots, y_d - x).$$

Then $(C^d)_\phi = \{(y_1, \ldots, y_d) \in C^d \mid \phi(y_1, \ldots, y_d) \neq 0\}$ is a non-empty open set, so that we can find $(x_1, \ldots, x_d) \in (C^d)_\phi$ with $x_1, \ldots, x_d \in P$. Next we consider

$$\{x - t_1(x_1 - x) - \cdots - t_d(x_d - x) \mid t_1, \ldots, t_d \in \mathbb{R}_{>0}\} \cap C.$$

This is also a non-empty open set in $C$. Thus there are $x_{d+1} \in C \cap P$ and $t_1, \ldots, t_d \in \mathbb{R}_{>0}$ with $x_{d+1} = x - t_1(x_1 - x) - \cdots - t_d(x_d - x)$, so that

$$x = \frac{t_1x_1 + \cdots + t_dx_d + x_{d+1}}{t_1 + \cdots + t_d + 1}.$$

Thus $x$ is an interior point of $\mathrm{Conv}_{\mathbb{R}}(\{x_1, \ldots, x_{d+1}\})$. □

**Claim 1.3.1.3.** *There is a positive number $c_1$ such that $f(y) \geq -c_1$ holds for all $y \in \mathrm{Conv}_{\mathbb{R}}(\{x_1, \ldots, x_{d+1}\}) \cap P$.*

As $y \in \mathrm{Conv}_{\mathbb{R}}(\{x_1, \ldots, x_{d+1}\}) \cap P$, by Lemma 1.1.3, there are $t_1, \ldots, t_{d+1} \in \mathbb{Q}_{\geq 0}$ such that

$$t_1 + \cdots + t_{d+1} = 1 \quad \text{and} \quad y = t_1 x_1 + \cdots + t_{d+1} x_{d+1}.$$

Thus, by Claim 1.3.1.1,

$$\begin{aligned} f(y) &= f(t_1 x_1 + \cdots + t_{d+1} x_{d+1}) \\ &\geq t_1 f(x_1) + \cdots + t_{d+1} f(x_{d+1}) \geq -t_1 |f(x_1)| - \cdots - t_{d+1} |f(x_{d+1})| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \geq -(|f(x_1)| + \cdots + |f(x_{d+1})|), \end{aligned}$$

as required. □

Let us choose a positive number $\epsilon$ and $x_0 \in P$ such that

$$U(x, 4\epsilon) \subseteq \mathrm{Conv}_{\mathbb{R}}(\{x_1, \ldots, x_{d+1}\}$$

and $x_0 \in U(x, \epsilon) \cap P$. Then

$$U(x, \epsilon) \subseteq U(x_0, 2\epsilon) \subseteq U(x_0, 3\epsilon) \subseteq U(x, 4\epsilon) \subseteq \mathrm{Conv}_{\mathbb{R}}(\{x_1, \ldots, x_{d+1}\}).$$

**Claim 1.3.1.4.** *There is a positive number $c_2$ such that $|f(y)| \leq c_2$ holds for all $y \in U(x_0, 3\epsilon) \cap P$.*

As $(2x_0 - y) - x_0 = x_0 - y$, we have $2x_0 - y \in U(x_0, 3\epsilon) \cap P$, and hence

$$f(x_0) = f(y/2 + (2x_0 - y)/2) \geq f(y)/2 + f(2x_0 - y)/2.$$

Therefore,

$$-c_1 \leq f(y) \leq 2f(x_0) - f(2x_0 - y) \leq 2f(x_0) + c_1,$$

as required. □

Let $y, z \in U(x, \epsilon) \cap P$ with $y \neq z$. We choose $a \in \mathbb{Q}$ with

$$\frac{\epsilon/2}{\|z - y\|_h} + 1 \leq a \leq \frac{\epsilon}{\|z - y\|_h} + 1$$

and we set $w = a(z - y) + y$. Then $\epsilon/2 \leq \|w - z\|_h \leq \epsilon$. Thus $w \in U(x_0, 3\epsilon) \cap P$. Moreover, if we put $t_0 = 1/a$, then

$$z = (1 - t_0)y + t_0 w, \quad z - y = t_0(w - y) \quad \text{and} \quad w - z = (1 - t_0)(w - y).$$

As $\|z - y\|_h / \|w - z\|_h = t_0/(1 - t_0)$, we have

$$\begin{aligned} \frac{f(z) - f(y)}{\|z - y\|_h} &= \frac{f((1 - t_0)y + t_0 w) - f(y)}{\|z - y\|_h} \geq \frac{(1 - t_0)f(y) + t_0 f(w) - f(y)}{\|z - y\|_h} \\ &= t_0 \frac{f(w) - f(y)}{\|z - y\|_h} = (1 - t_0) \frac{f(w) - f(y)}{\|w - z\|_h} \\ &= \frac{f(w) - ((1 - t_0)f(y) + t_0 f(w))}{\|w - z\|_h} \geq \frac{f(w) - f(z)}{\|w - z\|_h} \\ &\geq \frac{-2c_2}{\|w - z\|_h} \geq \frac{-2c_2}{\epsilon/2} = \frac{-4c_2}{\epsilon}. \end{aligned}$$

Exchanging $y$ and $z$, we obtain the same inequality, that is,

$$\frac{f(y) - f(z)}{\|y - z\|_h} \geq \frac{-4c_2}{\epsilon}.$$

Therefore, $|f(z) - f(y)| \leq (4c_2/\epsilon)\|y - z\|_h$ for all $y, z \in U(x, \epsilon) \cap P$.

For the last assertion, note the following: Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence on $C \cap P$ such that $x = \lim_{n\to\infty} a_n \in C$. Then, by the first assertion of this proposition, $\{f(a_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence in $\mathbb{R}$, and hence $\tilde{f}(x)$ is defined by $\lim_{n\to\infty} f(a_n)$. ∎

Next we do not assume that $P$ is finite dimensional. Then we have the following corollary.

**Corollary 1.3.2.** *There is a unique concave and continuous function $\tilde{f} : C \to \mathbb{R}$ such that $\tilde{f}\Big|_{C\cap P} = f$.*

*Proof.* It follows from Proposition 1.3.1 and the following facts: If $x \in V$, then there is a finite dimensional vector subspace $Q$ of $P$ over $\mathbb{Q}$ with $x \in Q \otimes \mathbb{R}$. ∎

1.4. **Good flag over a prime.** In this subsection, we observe the existence of good flags over infinitely many prime numbers.

Let $X$ be a $d$-dimensional projective arithmetic variety. Let $\pi : X \to \operatorname{Spec}(R)$ be the Stein factorization of $X \to \operatorname{Spec}(\mathbb{Z})$, where $R$ is an order of some number field $F$. A chain

$$Y_\cdot : Y_0 = X \supset Y_1 \supset Y_2 \supset \cdots \supset Y_d$$

of subschemes of $X$ is called a *good flag* of $X$ over a prime $p$ if the following conditions are satisfied:

(a) $Y_i$'s are integral and $\operatorname{codim}(Y_i) = i$ for $i = 0, \ldots, d$.
(b) There is $P \in \operatorname{Spec}(R)$ such that $R_P$ is normal, $\pi^{-1}(P) = Y_1$ and the residue field $\kappa(P)$ at $P$ is isomorphic to $\mathbb{F}_p$. In particular, $Y_1$ is a Cartier divisor on $X$.
(c) $Y_d$ consists of a rational point $y$ over $\mathbb{F}_p$.
(d) $Y_i$'s are regular at $y$ for $i = 0, \ldots, d$.
(e) There is a birational morphism $\mu : X' \to X$ of projective arithmetic varieties with the following properties:
  (e.1) $\mu$ is isomorphism over $y$.
  (e.2) If $Y'_i$ is the strict transform of $Y_i$, then $Y'_i$ is a Cartier divisor in $Y'_{i-1}$ for $i = 1, \ldots, d$.

**Proposition 1.4.1.** *There are good flags of $X$ over infinitely many prime numbers. More precisely, if we set $\mathcal{S}_{F/\mathbb{Q}} = \{p \in \operatorname{Spec}(\mathbb{Z}) \mid p$ splits completely in $F$ over $\mathbb{Q}\}$, then there is a finite subset $\Sigma$ of $\mathcal{S}_{F/\mathbb{Q}}$ such that we have a good flag over any prime in $\mathcal{S}_{F/\mathbb{Q}} \setminus \Sigma$.*

*Proof.* Let $\mu : Y \to X$ be a generic resolution of singularities of $X$ such that $Y$ is normal. Let $\pi : X \to \operatorname{Spec}(R)$ and $\tilde{\pi} : Y \to \operatorname{Spec}(O_F)$ be the Stein factorizations of $X \to \operatorname{Spec}(\mathbb{Z})$ and $Y \to \operatorname{Spec}(\mathbb{Z})$ respectively. Then we have the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{\ \mu\ } & Y \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\tilde{\pi}} \\ \operatorname{Spec}(R) & \xleftarrow{\ \rho\ } & \operatorname{Spec}(O_F). \end{array}$$

Let us choose a proper closed subset $Z$ of $X$ such that $\mu : Y \setminus \mu^{-1}(Z) \to X \setminus Z$ is an isomorphism. We set $E = \mu^{-1}(Z)$. Let us choose a chain

$$Y'_1 = Y \times_{\operatorname{Spec}(O_F)} \operatorname{Spec}(F) \supset Y'_2 \supset \cdots \supset Y'_{d-1}$$

of smooth subvarieties of $Y \times_{\operatorname{Spec}(O_F)} \operatorname{Spec}(F)$ such that $\operatorname{codim}(Y'_i) = i - 1$ for $i = 1, \ldots, d-1$ and $\dim(Y'_{d-1} \cap (E \times_{\operatorname{Spec}(O_F)} \operatorname{Spec}(F))) \leq 0$. Let $Y_i$ be the Zariski closure of

$Y_i'$ in $Y$. Then there is a non-empty open set $U$ of $\operatorname{Spec}(O_F)$ such that $\rho$ is an isomorphism over $U$, $Y_1 = Y, Y_2, \ldots, Y_{d-1}$ are smooth over $U$ and $Y_{d-1} \cap E$ is either finite or empty over $U$. Let $e$ be the degree of $Y_{d-1} \cap E$ over $U$. Note that $e$ might be zero. If we put

$$\Sigma_1 = \{p \in \operatorname{Spec}(\mathbb{Z}) \mid \text{there is } P \in \operatorname{Spec}(O_F) \setminus U \text{ with } p\mathbb{Z} = P \cap \mathbb{Z}\},$$

then $\Sigma_1$ is a finite set. Let $p \in \mathcal{S}_{F/\mathbb{Q}} \setminus \Sigma_1$ and $P \in \operatorname{Spec}(O_K)$ with $p\mathbb{Z} = P \cap \mathbb{Z}$. Then $P \in U$ and the residue field at $P$ is isomorphic to $\mathbb{F}_p$. By Weil's conjecture for curves,

$$p + 1 - 2g\sqrt{p} \leq \#(Y_{d-1} \otimes \kappa(P))(\mathbb{F}_p),$$

where $g$ is the genus of $Y_{d-1}'$. Thus there is a finite set $\Sigma_2$ such that, if $p \in \mathcal{S}_{F/\mathbb{Q}} \setminus (\Sigma_1 \cup \Sigma_2)$, then $p + 1 - 2g\sqrt{p} > e$, which means that there is $x \in (Y_{d-1} \otimes \kappa(P))(\mathbb{F}_p)$ with $x \notin E$. Since, for $p \in \mathcal{S}_{F/\mathbb{Q}} \setminus (\Sigma_1 \cup \Sigma_2)$,

$$Y \supset Y_1 \otimes \kappa(P) \supset \cdots \supset Y_{d-1} \otimes \kappa(P) \supset \{x\}$$

is a good flag over $p$,

$$X \supset \mu(Y_1 \otimes \kappa(P)) \supset \cdots \supset \mu(Y_{d-1} \otimes \kappa(P)) \supset \{\mu(x)\}$$

is also a good flag over $p$. ■

## 2. Estimation of linear series in terms of valuation vectors

In this section, we consider a generalization of Yuan's paper [11]. Let us begin with the following proposition, which is a key to Theorem 2.2.

**Proposition 2.1.** *Let $X$ be a $d$-dimensional projective arithmetic variety and fix a good flag $Y_\cdot : X \supset Y_1 \supset Y_2 \supset \cdots \supset Y_d$ over a prime $p$. Let $L$ be an invertible sheaf on $X$, $M$ a $\mathbb{Z}$-submodule of $H^0(X, L)$ and $\Delta$ a bounded symmetric convex set in $H^0(X, L)_{\mathbb{R}}$. Let $r : H^0(X, L) \to H^0(Y_1, L|_{Y_1})$ be the natural homomorphism, $M' = M \cap H^0(X, L - Y_1)$ and $\beta = p \operatorname{rk} M$. Then we have the following:*

(2.1.1) $$\#\nu_{Y_1}(r(M \cap \Delta) \setminus \{0\}) \log p \leq \log \#(M \cap 2\beta\Delta) - \log \#(M' \cap \beta\Delta)$$

*and*

(2.1.2) $$\#\nu_{Y_1}(r(M \cap \Delta) \setminus \{0\}) \log p \geq \log \#(M \cap (1/\beta)\Delta) - \log \#(M' \cap (2/\beta)\Delta),$$

*where $\nu_{Y_1}$ is the valuation on $Y_1$ attached to a flag $Y_1 \supset Y_2 \supset \cdots \supset Y_d$.*

*Proof.* Let $V$ be a vector space generated by $r(M \cap \Delta)$ in $H^0(Y_1, L|_{Y_1})$ over $\mathbb{F}_p$. Note that [5, Lemma 1.3] holds if $Y_d$ consists of a rational point over a base field. Thus

$$\begin{aligned}
\#\nu_{Y_1}(r(\Delta \cap M) \setminus \{0\}) \log p &\leq \#\nu_{Y_1}(V \setminus \{0\}) \log p \\
&= \dim_{\mathbb{F}_p}(V) \log p \qquad (\because \text{[5, Lemma 1.3]}) \\
&= \log \#V.
\end{aligned}$$

Let us choose $s_1, \ldots, s_l \in M \cap \Delta$ such that $r(s_1), \ldots, r(s_l)$ forms a basis of $V$. Let $n$ be the rank of $M$ and $\omega_1, \ldots, \omega_n$ a free basis of $M$. Then $V \subseteq \sum_{i=1}^n \mathbb{F}_p r(\omega_i)$ in $H^0(Y_1, L|_{Y_1})$, which implies $l \leq n$. We set

$$S = \left\{ \sum a_i s_i \mid a_i = 0, 1, \ldots, p-1 \ (\forall i) \right\}.$$

Then $S$ maps surjectively to $V$. Moreover $S \subseteq M \cap \beta\Delta$ because $l \leq n$. Thus we get $\#V \leq \#\left(r(M \cap \beta\Delta)\right)$. Note that $\operatorname{Ker}\left(r|_M : M \to H^0(Y_1, L|_{Y_1})\right) = M'$. Therefore, as $2 * (M \cap \beta\Delta) \subseteq M \cap 2\beta\Delta$, by (1.2.2.2),

$$\log \#r(M \cap \beta\Delta) \leq \log \#(M \cap 2\beta\Delta) - \log \#(M' \cap \beta\Delta),$$

which shows (2.1.1).

Let $W$ be a vector space generated by $r(M \cap (1/\beta)\Delta)$ in $H^0(Y_1, L|_{Y_1})$ over $\mathbb{F}_p$. Let us choose $t_1, \dots, t_{l'} \in M \cap (1/\beta)\Delta$ such that $r(t_1), \dots, r(t_{l'})$ forms a basis of $W$. In the same way as before, we have $l' \leq n$. We set

$$T = \left\{ \sum b_i t_i \mid b_i = 0, 1, \dots, p-1 \ (\forall i) \right\}.$$

Then $T \subseteq M \cap \Delta$ and $W = r(T) \subseteq r(M \cap \Delta)$. Thus

$$\begin{aligned}
\#\nu_{Y_1}\left(r(M \cap \Delta) \setminus \{0\}\right) \log p &\geq \#\nu_{Y_1}\left(W \setminus \{0\}\right) \log p \\
&= \dim_{\mathbb{F}_p}(W) \log p \\
&= \log \#W \\
&\geq \log \#r\left(M \cap (1/\beta)\Delta\right).
\end{aligned}$$

On the other hand, as $2 * (M \cap (1/\beta)\Delta) \subseteq M \cap (2/\beta)\Delta$, by (1.2.2.1),

$$\log \#r(M \cap (1/\beta)\Delta) \geq \log \#(M \cap (1/\beta)\Delta) - \log \#(M' \cap (2/\beta)\Delta),$$

as required for (2.1.2). ∎

Let $X$ be a $d$-dimensional projective arithmetic variety and $\overline{L}$ a continuous hermitian invertible sheaf on $X$. A subset $K$ of $H^0(X, L)$ is called an *arithmetic linear series of* $\overline{L}$ if $K$ is a symmetric convex lattice in $H^0(X, L)$ with

$$K \subseteq B_{\sup}(\overline{L}) := \{ s \in H^0(X_{\mathbb{R}}, L_{\mathbb{R}}) \mid \|s\|_{\sup} \leq 1 \}.$$

If $K = \hat{H}^0(X, \overline{L})$ $(= H^0(X, L) \cap B_{\sup}(\overline{L}))$, then $K$ is said to be *complete*. Then we have the following theorem:

**Theorem 2.2.** *Let $\nu$ be the valuation attached to a good flag $Y_\cdot : X \supset Y_1 \supset Y_2 \supset \cdots \supset Y_d$ over a prime $p$. If $K$ is an arithmetic linear series of $\overline{L}$ with $K \neq \{0\}$, then the following estimation*

$$\begin{aligned}
&|\#\nu(K \setminus \{0\}) \log p - \log \#(K)| \\
&\qquad \leq \left( \log\left(4p \operatorname{rk}\langle K \rangle_{\mathbb{Z}}\right) + \frac{\sigma(\overline{L}) + \log\left(2p \operatorname{rk}\langle K \rangle_{\mathbb{Z}}\right)}{\log p} \log(4) \operatorname{rk} H^0(\mathcal{O}_X) \right) \operatorname{rk}\langle K \rangle_{\mathbb{Z}}
\end{aligned}$$

*holds, where $\sigma(\overline{L})$ is given by*

$$\sigma(\overline{L}) := \inf_{\overline{A}\,:\,\text{ample}} \frac{\widehat{\deg}(\widehat{c}_1(\overline{A})^{d-1} \cdot \widehat{c}_1(\overline{L}))}{\deg(A_{\mathbb{Q}}^{d-1})}.$$

*Proof.* We set $\beta = p \operatorname{rk}\langle K \rangle_{\mathbb{Z}}$, $\Delta = \operatorname{Conv}_{\mathbb{R}}(K)$ and $M_k = \langle K \rangle_{\mathbb{Z}} \cap H^0(X, L - kY_1)$ for $k \geq 0$. Then $M_0 = \langle K \rangle_{\mathbb{Z}}$, $K = M_0 \cap \Delta$, $\operatorname{rk} M_k = \operatorname{rk} M_0$ and

$$M_{k+1} = M_k \cap H^0(Y, (L - kY_1) - Y_1).$$

Let $r_k : H^0(X, L - kY_1) \to H^0(Y_1, L - kY_1|_{Y_1})$ be the natural homomorphism for each $k \geq 0$. Note that

$$\#\nu(K \setminus \{0\}) = \sum_{k \geq 0} \#\nu_{Y_1}\left(r_k(M_k \cap \Delta) \setminus \{0\}\right).$$

Thus, by applying Proposition 2.1 to $L - kY_1$, we obtain

$$\begin{aligned}\sum_{k\geq 0} (\log \#(M_k \cap (1/\beta)\Delta) - \log \#(M_{k+1} \cap (2/\beta)\Delta)) \\ \leq \#\nu(K \setminus \{0\}) \log p \leq \\ \sum_{k\geq 0} (\log \#(M_k \cap 2\beta\Delta) - \log \#(M_{k+1} \cap \beta\Delta))\,,\end{aligned}$$

which implies

$$\begin{aligned}\#\nu(K \setminus \{0\}) \log p \leq \log \#(M_0 \cap 2\beta\Delta) \\ + \sum_{k\geq 1} (\log \#(M_k \cap 2\beta\Delta) - \log \#(M_k \cap \beta\Delta))\end{aligned}$$

and

$$\begin{aligned}\#\nu(K \setminus \{0\}) \log p \geq \log \#(M_0 \cap (1/\beta)\Delta) \\ - \sum_{k\geq 1} (\log \#(M_k \cap (2/\beta)\Delta) - \log \#(M_k \cap (1/\beta)\Delta))\,.\end{aligned}$$

By (1.2.2.3),

$$\log \#(M_0 \cap 2\beta\Delta) \leq \log \#(K) + \log(4\beta) \operatorname{rk} M_0$$

and

$$\log \#(M_k \cap 2\beta\Delta) - \log \#(M_k \cap \beta\Delta) \leq \log(4) \operatorname{rk} M_k = \log(4) \operatorname{rk} M_0.$$

Note that $2\beta\Delta \cap M_k \subseteq \hat{H}^0(\overline{L} - kY_1 + \overline{\mathcal{O}}(\log(2\beta)))$ (for the definition of $\overline{\mathcal{O}}(-)$, see Section 4). Thus, if we set

$$S = \left\{ k \geq 1 \mid \hat{H}^0(\overline{L} - kY_1 + \overline{\mathcal{O}}(\log(2\beta))) \neq \{0\} \right\},$$

then

$$\#\nu(K \setminus \{0\}) \log p \leq \log \#(K) + (\log(4\beta) + \#S \log(4)) \operatorname{rk} M_0.$$

Let $\overline{A}$ be an ample $C^\infty$-hermitian invertible sheaf on $X$. If $k \in S$, then

$$\begin{aligned}0 &\leq \widehat{\deg}(\widehat{c}_1(\overline{A})^{d-1} \cdot \widehat{c}_1(\overline{L} - kY_1 + \overline{\mathcal{O}}(\log(2\beta)))) \\ &= \widehat{\deg}(\widehat{c}_1(\overline{A})^{d-1} \cdot \widehat{c}_1(\overline{L})) + \log(2\beta) \deg(A_{\mathbb{Q}}^{d-1}) - k \frac{\deg(A_{\mathbb{Q}}^{d-1}) \log p}{\operatorname{rk} H^0(\mathcal{O}_X)},\end{aligned}$$

which implies that

$$k \leq \frac{(\sigma(\overline{L}) + \log(2\beta)) \operatorname{rk} H^0(\mathcal{O}_X)}{\log p},$$

and hence

$$\#S \leq \frac{(\sigma(\overline{L}) + \log(2\beta)) \operatorname{rk} H^0(\mathcal{O}_X)}{\log p}.$$

Further, by using (1.2.2.3), we can see

$$\log \#(M_0 \cap (1/\beta)\Delta) \geq \log \#(K) - \log(2\beta) \operatorname{rk} M_0$$

and

$$\log \#(M_k \cap (2/\beta)\Delta) - \log \#(M_k \cap (1/\beta)\Delta) \leq \log(4) \operatorname{rk} M_k = \log(4) \operatorname{rk} M_0.$$

Hence, as before, we obtain

$$\#\nu(K \setminus \{0\}) \log p \geq \log \#(K) - (\log(2\beta) + \#S \log(4)) \operatorname{rk} M_0,$$

as required. ■

**Corollary 2.3.** *There is a positive constant $c = c(X, \overline{L})$ depending only on $X$ and $\overline{L}$ with the following property: For a good flag*

$$Y_{\cdot} : X \supset Y_1 \supset Y_2 \supset \cdots \supset Y_d$$

*over a prime $p$, there is a positive constant $m_0 = m_0(p, X_{\mathbb{Q}}, L_{\mathbb{Q}})$ depending only on $p$, $X_{\mathbb{Q}}$ and $L_{\mathbb{Q}}$ such that, if $m \geq m_0$, then*

$$|\#\nu_{Y_{\cdot}}(K \setminus \{0\}) \log p - \log \#(K)| \leq \frac{cm^d}{\log p}$$

*holds for any arithmetic linear series $K$ of $m\overline{L}$, where $\nu_{Y_{\cdot}}$ is the valuation attached to the flag $Y_{\cdot} : X \supset Y_1 \supset Y_2 \supset \cdots \supset Y_d$.*

*Proof.* The problem is an estimation of $C_m$ given by

$$\left(\log\left(4p \operatorname{rk} H^0(mL)\right) + \frac{\sigma(m\overline{L}) + \log\left(2p \operatorname{rk} H^0(mL)\right)}{\log p} \log(4) \operatorname{rk} H^0(\mathcal{O}_X)\right) \operatorname{rk} H^0(mL).$$

First of all, there is a constant $c_1$ depending only on $X_{\mathbb{Q}}$ and $L_{\mathbb{Q}}$ such that

$$\operatorname{rk} H^0(mL) \leq c_1 m^{d-1}$$

for all $m \geq 0$. Thus

$$\begin{aligned} C_m \leq \frac{1}{m}\left(\log\left(4pc_1 m^{d-1}\right)\log(p) + \log\left(2pc_1 m^{d-1}\right)\log(4)\operatorname{rk} H^0(\mathcal{O}_X)\right)\frac{c_1 m^d}{\log p} \\ + \sigma(\overline{L})\log(4)\operatorname{rk} H^0(\mathcal{O}_X)\frac{c_1 m^d}{\log p}. \end{aligned}$$

We can find a positive integer $m_0$ depending only on $p$ and $c_1$ such that if $m \geq m_0$, then

$$\frac{1}{m}\left(\log\left(4pc_1 m^{d-1}\right)\log(p) + \log\left(2pc_1 m^{d-1}\right)\log(4)\operatorname{rk} H^0(\mathcal{O}_X)\right) \leq 1 + \operatorname{rk} H^0(\mathcal{O}_X).$$

Therefore,

$$C_m \leq \left(1 + (1 + \sigma(\overline{L})\log(4))\operatorname{rk} H^0(\mathcal{O}_X)\right) c_1 \frac{m^d}{\log p}$$

for $m \geq m_0$, as required. ■

As an application of Corollary 2.3, we have the following theorem. The arithmetic Fujita's approximation theorem is a straightforward consequence of this result.

**Theorem 2.4.** *Let $\overline{L}$ be a big continuous hermitian invertible sheaf on a projective arithmetic variety $X$. For any positive $\epsilon$, there is a positive integer $n_0 = n_0(\epsilon)$ such that, for all $n \geq n_0$,*

$$\liminf_{k \to \infty} \frac{\log \#(K_{k,n})}{n^d k^d} \geq \frac{\widehat{\operatorname{vol}}(\overline{L})}{d!} - \epsilon,$$

*where $K_{k,n}$ is the convex lattice hull of*

$$V_{k,n} = \{s_1 \otimes \cdots \otimes s_k \mid s_1, \ldots, s_k \in \hat{H}^0(X, n\overline{L})\}$$

*in $H^0(X, knL)$.*

*Proof.* A generalization of this theorem will be proved in Theorem 6.2. ■

## 3. BASE LOCUS OF CONTINUOUS HERMITIAN INVERTIBLE SHEAF

Let $X$ be a projective arithmetic variety and $\overline{L}$ a continuous hermitian invertible sheaf on $X$. We define the *base locus* $\mathrm{Bs}(\overline{L})$ of $\overline{L}$ to be

$$\mathrm{Bs}(\overline{L}) = \mathrm{Supp}\left(\mathrm{Coker}\left(\langle \hat{H}^0(X,\overline{L})\rangle_{\mathbb{Z}} \otimes \mathcal{O}_X \to L\right)\right),$$

that is,

$$\mathrm{Bs}(\overline{L}) = \{x \in X \mid s(x) = 0 \text{ for all } s \in \hat{H}^0(X,\overline{L})\}.$$

Moreover, the *stable base locus* $\mathrm{SBs}(\overline{L})$ is defined to be

$$\mathrm{SBs}(\overline{L}) = \bigcap_{m\geq 1} \mathrm{Bs}(m\overline{L})$$

The following proposition is the basic properties of $\mathrm{Bs}(\overline{L})$ and $\mathrm{SBs}(\overline{L})$.

**Proposition 3.1.** (1) $\mathrm{Bs}(\overline{L}+\overline{M}) \subseteq \mathrm{Bs}(\overline{L}) \cup \mathrm{Bs}(\overline{M})$ *for any* $\overline{L},\overline{M} \in \widehat{\mathrm{Pic}}(X;C^0)$.
(2) *There is a positive integer* $m_0$ *such that* $\mathrm{SBs}(\overline{L}) = \mathrm{Bs}(mm_0\overline{L})$ *for all* $m \geq 1$.
(3) $\mathrm{SBs}(\overline{L}+\overline{M}) \subseteq \mathrm{SBs}(\overline{L}) \cup \mathrm{SBs}(\overline{M})$ *for any* $\overline{L},\overline{M} \in \widehat{\mathrm{Pic}}(X;C^0)$.
(4) $\mathrm{SBs}(\overline{L}) = \mathrm{SBs}(m\overline{L})$ *for all* $m \geq 1$.

*Proof.* (1) is obvious by its definition.

(2) By using (1), it is sufficient to find a positive integer $m_0$ with $\mathrm{SBs}(\overline{L}) = \mathrm{Bs}(m_0\overline{L})$. Thus it is enough to see that if $\mathrm{SBs}(\overline{L}) \subsetneq \mathrm{Bs}(a\overline{L})$, then there is $b$ with $\mathrm{Bs}(ab\overline{L}) \subsetneq \mathrm{Bs}(a\overline{L})$. Indeed, choose $x \in \mathrm{Bs}(a\overline{L}) \setminus \mathrm{SBs}(\overline{L})$. Then there is $b$ with $x \notin \mathrm{Bs}(b\overline{L})$, so that $x \notin \mathrm{Bs}(ab\overline{L})$ by (1).

(3) This is a consequence of (1) and (2).

(4) Clearly $\mathrm{SBs}(\overline{L}) \subseteq \mathrm{SBs}(m\overline{L})$. We choose $m_0$ with $\mathrm{SBs}(\overline{L}) = \mathrm{Bs}(m_0\overline{L})$. Then $\mathrm{SBs}(m\overline{L}) \subseteq \mathrm{Bs}(m_0m\overline{L}) = \mathrm{SBs}(\overline{L})$. ■

Let $\iota : \widehat{\mathrm{Pic}}(X;C^0) \to \widehat{\mathrm{Pic}}_{\mathbb{Q}}(X;C^0)(:= \widehat{\mathrm{Pic}}(X;C^0) \otimes \mathbb{Q})$ be the natural homomorphism. For $\overline{L} \in \widehat{\mathrm{Pic}}_{\mathbb{Q}}(X;C^0)$, there are a positive integer $n$ and $\overline{M} \in \widehat{\mathrm{Pic}}(X;C^0)$ such that $\overline{L} = (1/n)\iota(\overline{M})$. Then, by the above (4), we can see that $\mathrm{SBs}(\overline{M})$ does not depend on the choice of $n$ and $\overline{M}$, so that $\mathrm{SBs}(\overline{L})$ is defined by $\mathrm{SBs}(\overline{M})$. The *augmented base-locus* $\mathrm{SBs}_+(\overline{L})$ of $\overline{L}$ is defined to be

$$\mathrm{SBs}_+(\overline{L}) = \bigcap_{\substack{\overline{A}\in\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X;C^\infty)\\ \overline{A}\text{ : ample}}} \mathrm{SBs}(\overline{L}-\overline{A}).$$

**Proposition 3.2.** *Let* $\overline{B}_1,\ldots,\overline{B}_r$ *be ample* $C^\infty$*-hermitian* $\mathbb{Q}$*-invertible sheaves on* $X$. *Then there is a positive number* $\epsilon_0$ *such that*

$$\mathrm{SBs}_+(\overline{L}) = \mathrm{SBs}(\overline{L} - \epsilon_1\overline{B}_1 - \cdots - \epsilon_r\overline{B}_r)$$

*for all rational numbers* $\epsilon_1,\ldots,\epsilon_r$ *with* $0 < \epsilon_1 \leq \epsilon_0,\ldots,0 < \epsilon_r \leq \epsilon_0$.

*Proof.* Since $X$ is a noetherian space, there are ample $C^\infty$-hermitian $\mathbb{Q}$-invertible sheaves $\overline{A}_1,\ldots,\overline{A}_l$ on $X$ such that $\mathrm{SBs}_+(\overline{L}) = \bigcap_{i=1}^{l}\mathrm{SBs}(\overline{L}-\overline{A}_i)$. We choose a positive number $\epsilon_0$ such that, for all rational numbers $\epsilon_1,\ldots,\epsilon_r$ with $0 < \epsilon_1 \leq \epsilon_0,\ldots,0 < \epsilon_r \leq \epsilon_0$,

$$\overline{A}_i - \epsilon_1\overline{B}_1 - \cdots - \epsilon_r\overline{B}_r$$

is ample for every $i = 1, \dots, l$. Then, by (2) in Proposition 3.1,

$$\begin{aligned}\mathrm{SBs}(\overline{L} - \epsilon_1\overline{B}_1 - \cdots - \epsilon_r\overline{B}_r) &= \mathrm{SBs}(\overline{L} - \overline{A}_i + (\overline{A}_i - \epsilon_1\overline{B}_1 - \cdots - \epsilon_r\overline{B}_r)) \\ &\subseteq \mathrm{SBs}(\overline{L} - \overline{A}_i) \cup \mathrm{SBs}(\overline{A}_i - \epsilon_1\overline{B}_1 - \cdots - \epsilon_r\overline{B}_r) \\ &= \mathrm{SBs}(\overline{L} - \overline{A}_i),\end{aligned}$$

which implies

$$\mathrm{SBs}_+(\overline{L}) \subseteq \mathrm{SBs}(\overline{L} - \epsilon_1\overline{B}_1 - \cdots - \epsilon_r\overline{B}_r) \subseteq \bigcap_{i=1}^{l} \mathrm{SBs}(\overline{L} - \overline{A}_i) = \mathrm{SBs}_+(\overline{L}).$$

■

## 4. Arithmetic Picard group and cones

According as [7], we fix several notations. Let $X$ be a projective arithmetic variety. Let $C^0(X)$ be the set of real valued continuous functions $f$ on $X(\mathbb{C})$ with $F_\infty^*(f) = f$, where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the complex conjugation map on $X(\mathbb{C})$. Let $\overline{\mathcal{O}} : C^0(X) \to \widehat{\mathrm{Pic}}(X; C^0)$ be the homomorphism given by

$$\overline{\mathcal{O}}(f) = (\mathcal{O}_X, \exp(-f)|\cdot|_{can}).$$

We define $\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X; C^0)$ and $\widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0)$ to be

$$\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X; C^0) := \widehat{\mathrm{Pic}}(X; C^0) \otimes \mathbb{Q} \quad \text{and} \quad \widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0) := \widehat{\mathrm{Pic}}(X; C^0) \otimes \mathbb{R}.$$

We denote the natural homomorphism $\widehat{\mathrm{Pic}}(X; C^0) \to \widehat{\mathrm{Pic}}_{\mathbb{Q}}(X; C^0)$ by $\iota$. Let $N(X)$ be the subgroup of $\widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0)$ consisting of elements

$$\overline{\mathcal{O}}(f_1) \otimes x_1 + \cdots + \overline{\mathcal{O}}(f_r) \otimes x_r \quad (f_1, \dots, f_r \in C^0(X),\ x_1, \dots, x_r \in \mathbb{R})$$

with $x_1 f_1 + \cdots + x_r f_r = 0$. We define $\widehat{\mathrm{Pic}}_{\mathbb{R}}(X; C^0)$ to be

$$\widehat{\mathrm{Pic}}_{\mathbb{R}}(X; C^0) := \widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0)/N(X).$$

Let $\pi : \widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0) \to \widehat{\mathrm{Pic}}_{\mathbb{R}}(X; C^0)$ be the natural homomorphism. Here we give the strong topology to $\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X; C^0)$, $\widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0)$ and $\widehat{\mathrm{Pic}}_{\mathbb{R}}(X; C^0)$. Then the homomorphisms

$$\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X; C^0) \hookrightarrow \widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0) \quad \text{and} \quad \pi : \widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0) \to \widehat{\mathrm{Pic}}_{\mathbb{R}}(X; C^0)$$

are continuous. Moreover, $\pi : \widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0) \to \widehat{\mathrm{Pic}}_{\mathbb{R}}(X; C^0)$ is an open map (cf. Conventions and terminology 4). We denote the composition of homomorphisms

$$\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X; C^0) \hookrightarrow \widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0) \xrightarrow{\pi} \widehat{\mathrm{Pic}}_{\mathbb{R}}(X; C^0)$$

by $\rho$. Then $\rho$ is also continuous. Note that $\rho$ is not necessarily injective (cf. [7, Example 4.5]).

Let $\widehat{\mathrm{Amp}}(X)$ be the sub-semigroup of $\widehat{\mathrm{Pic}}(X; C^0)$ consisting of all ample $C^\infty$-hermitian invertible sheaves on $X$. Let us observe the following lemma.

**Lemma 4.1.** *Let $\overline{A}$ be an ample invertible sheaf on $X$. For any $\overline{L} \in \widehat{\mathrm{Pic}}(X; C^0)$, there are a positive integer $n_0$ and $f \in C^0(X)$ such that $f \geq 0$ and*

$$\overline{L} + n\overline{A} - \overline{\mathcal{O}}(f) \in \widehat{\mathrm{Amp}}(X)$$

*for all $n \geq n_0$.*

*Proof.* Let $|\cdot|$ be the hermitian metric of $\overline{L}$ and $|\cdot|_0$ a $C^\infty$-hermitian metric of $L$. We set $|\cdot| = \exp(-f_0)|\cdot|_0$ for some $f_0 \in C^0(X)$. We can take a constant $c$ with $f_0 + c \geq 0$, and put $f = f_0 + c$. Then $f \geq 0$ and $\exp(f)|\cdot|$ is $C^\infty$, which means that $\overline{L} - \overline{\mathcal{O}}(f)$ is $C^\infty$. Thus there is a positive integer $n_0$ such that

$$(\overline{L} - \overline{\mathcal{O}}(f)) + n\overline{A} \in \widehat{\operatorname{Amp}}(X)$$

for all $n \geq n_0$. ■

**Proposition 4.2.** *Let $\widehat{C}$ be a sub-monoid of $\widehat{\operatorname{Pic}}(X; C^0)$ such that*

$$\{\overline{\mathcal{O}}(f) \mid f \in C^0(X),\ f \geq 0\} \subseteq \widehat{C}.$$

*We set $\widehat{B} = \operatorname{Sat}(\widehat{\operatorname{Amp}}(X) + \widehat{C})$ (cf. see Conventions and terminology 1 for the saturation). Then we have the following.*

(1) *$\widehat{B}$ is open, that is, for any $\overline{L} \in \widehat{B}$ and $\overline{M} \in \widehat{\operatorname{Pic}}(X; C^0)$, there is a positive integer $n$ such that $n\overline{L} + \overline{M} \in \widehat{B}$.*

(2) *If we set*

$$\begin{cases} \widehat{B}_{\mathbb{Q}} := \operatorname{Cone}_{\mathbb{Q}}(\iota(\widehat{B})) & \text{in } \widehat{\operatorname{Pic}}_{\mathbb{Q}}(X; C^0), \\ \widehat{B}_{\otimes\mathbb{R}} := \operatorname{Cone}_{\mathbb{R}}(\widehat{B}_{\mathbb{Q}}) & \text{in } \widehat{\operatorname{Pic}}_{\otimes\mathbb{R}}(X; C^0), \\ \widehat{B}_{\mathbb{R}} := \operatorname{Cone}_{\mathbb{R}}(\rho(\widehat{B}_{\mathbb{Q}})) & \text{in } \widehat{\operatorname{Pic}}_{\mathbb{R}}(X; C^0), \end{cases}$$

*then $\widehat{B}_{\mathbb{Q}}$, $\widehat{B}_{\otimes\mathbb{R}}$ and $\widehat{B}_{\mathbb{R}}$ are open in $\widehat{\operatorname{Pic}}_{\mathbb{Q}}(X; C^0)$, $\widehat{\operatorname{Pic}}_{\otimes\mathbb{R}}(X; C^0)$ and $\widehat{\operatorname{Pic}}_{\mathbb{R}}(X; C^0)$ respectively.*

(3)

$$\begin{cases} \iota^{-1}\left(\widehat{B}_{\mathbb{Q}}\right) = \widehat{B}, \quad \widehat{B}_{\otimes\mathbb{R}} \cap \widehat{\operatorname{Pic}}_{\mathbb{Q}}(X; C^0) = \widehat{B}_{\mathbb{Q}}, \\ \pi^{-1}\left(\widehat{B}_{\mathbb{R}}\right) = \widehat{B}_{\otimes\mathbb{R}}, \quad \rho^{-1}\left(\widehat{B}_{\mathbb{R}}\right) = \widehat{B}_{\mathbb{Q}}. \end{cases}$$

*Proof.* (1) Let $\overline{L} \in \widehat{B}$ and $\overline{M} \in \widehat{\operatorname{Pic}}(X; C^0)$. Then there is a positive integer $n_0$ such that $n_0\overline{L} = \overline{A} + \overline{E}$ for some $\overline{A} \in \widehat{\operatorname{Amp}}(X)$ and $\overline{E} \in \widehat{C}$. By Lemma 4.1, there are a positive integer $n_1$ and $f \in C^0(X)$ such that $f \geq 0$ and $\overline{M} + n_1\overline{A} - \overline{\mathcal{O}}(f) = \overline{A}'$ for some $\overline{A}' \in \widehat{\operatorname{Amp}}(X)$. Then

$$n_1 n_0 \overline{L} + \overline{M} = n_1(\overline{A} + \overline{E}) + \overline{M} = \overline{A}' + (n_1\overline{E} + \overline{\mathcal{O}}(f)) \in \widehat{B}.$$

(2) follows from (3) in Proposition 1.1.5, (2) in Proposition 1.1.4 and (2.1) in Proposition 1.1.1.

(3) Let us consider the following claim:

**Claim 4.2.1.** $\widehat{B}_{\otimes\mathbb{R}} + N(X) \subseteq \widehat{B}_{\otimes\mathbb{R}}$.

First of all, let us see the following formula:

$$\widehat{B}_{\otimes\mathbb{R}} + \iota(\widehat{C}) \subseteq \widehat{B}_{\otimes\mathbb{R}}. \tag{4.2.2}$$

Indeed, as $\widehat{B} + \widehat{C} \subseteq \widehat{B}$, we have $\iota(\widehat{B} + \widehat{C}) \subseteq \iota(\widehat{B})$. Thus, by (3) in Proposition 1.1.4,

$$\widehat{B}_{\otimes\mathbb{R}} + \iota(\widehat{C}) \subseteq \operatorname{Cone}_{\mathbb{R}}(\iota(\widehat{B})) + \operatorname{Cone}_{\mathbb{R}}(\iota(\widehat{C})) = \operatorname{Cone}_{\mathbb{R}}(\iota(\widehat{B} + \widehat{C})) \subseteq \widehat{B}_{\otimes\mathbb{R}}.$$

Let $a \in \widehat{B}_{\otimes\mathbb{R}}$ and $x \in N(X)$. We set $x = \overline{\mathcal{O}}(f_1) \otimes a_1 + \cdots + \overline{\mathcal{O}}(f_r) \otimes a_r$ with $a_1 f_1 + \cdots + a_r f_r = 0$, where $f_1, \ldots, f_r \in C^0(X)$ and $a_1, \ldots, a_r \in \mathbb{R}$. Let us take a sequence $\{a_{in}\}_{n=1}^{\infty}$ in $\mathbb{Q}$ such that $a_i = \lim_{n\to\infty} a_{in}$. We set $\phi_n = a_{1n}f_1 + \cdots + a_{rn}f_r$. Then

$$\begin{aligned}\|\phi_n\|_{\sup} &= \|(a_{1n} - a_1)f_1 + \cdots + (a_{rn} - a_r)f_r\|_{\sup} \\ &\leq |a_{1n} - a_1|\|f_1\|_{\sup} + \cdots + |a_{rn} - a_r|\|f_r\|_{\sup}.\end{aligned}$$

Thus $\lim_{n\to\infty} \|\phi_n\|_{\sup} = 0$. We choose a sequence $\{b_n\}$ in $\mathbb{Q}$ such that $b_n \geq \|\phi_n\|_{\sup}$ and $\lim_{n\to\infty} b_n = 0$. Then $\phi_n + b_n \geq 0$. If we put

$$x_n = \overline{\mathcal{O}}(f_1) \otimes a_{1n} + \cdots + \overline{\mathcal{O}}(f_r) \otimes a_{rn} + \overline{\mathcal{O}}(1) \otimes b_n,$$

then $\lim_{n\to\infty} x_n = x$. On the other hand, as $x_n = \overline{\mathcal{O}}(\phi_n + b_n)$ in $\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X; C^0)$, $x_n \in \iota(\widehat{C})$. By (2), $\widehat{B}_{\otimes\mathbb{R}}$ is an open set in $\widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0)$. Thus, if $n \gg 1$, then $(x - x_n) + a \in \widehat{B}_{\otimes\mathbb{R}}$. Hence the claim follows because

$$x + a = ((x - x_n) + a) + x_n \in \widehat{B}_{\otimes\mathbb{R}} + \iota(\widehat{C}) \subseteq \widehat{B}_{\otimes\mathbb{R}}.$$

□

The first formula follows from (2) in Proposition 1.1.5. The second is derived from (1) in Proposition 1.1.4. We can see the third by using (2.2) in Proposition 1.1.1 and the above claim. The last formula follows from the second and the third. ■

## 5. BIG HERMITIAN INVERTIBLE SHEAVES WITH RESPECT TO AN ARITHMETIC SUBVARIETY

Let $X$ be a projective arithmetic variety and $Y$ an arithmetic subvariety of $X$, that is, $Y$ is an integral subscheme of $X$ such that $Y$ is flat over $\mathrm{Spec}(\mathbb{Z})$. A continuous hermitian invertible sheaf $\overline{L}$ is said to be *$Y$-effective* (or *effective with respect to $Y$*) if there is $s \in \hat{H}^0(X, \overline{L})$ with $s|_Y \neq 0$. For $\overline{L}_1, \overline{L}_2 \in \widehat{\mathrm{Pic}}(X)$, if $\overline{L}_1 - \overline{L}_2$ is $Y$-effective, then we denote it by $\overline{L}_1 \geq_Y \overline{L}_2$. We define $\widehat{\mathrm{Eff}}(X; Y)$ to be

$$\widehat{\mathrm{Eff}}(X; Y) := \left\{ \overline{L} \in \widehat{\mathrm{Pic}}(X; C^0) \mid \overline{L} \text{ is } Y\text{-effective} \right\}.$$

Then it is easy to see the following (cf. Proposition 4.2):

(a) $\widehat{\mathrm{Eff}}(X; Y)$ is a sub-monoid of $\widehat{\mathrm{Pic}}(X; C^0)$.
(b) $\{\overline{\mathcal{O}}(f) \mid f \in C^0(X),\ f \geq 0\} \subseteq \widehat{\mathrm{Eff}}(X; Y)$.

Here we define $\widehat{\mathrm{Big}}(X; Y)$, $\widehat{\mathrm{Big}}_{\mathbb{Q}}(X; Y)$, $\widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$ and $\widehat{\mathrm{Big}}_{\mathbb{R}}(X; Y)$ to be

$$\begin{cases} \widehat{\mathrm{Big}}(X; Y) := \mathrm{Sat}(\widehat{\mathrm{Amp}}(X) + \widehat{\mathrm{Eff}}(X; Y)), \\ \widehat{\mathrm{Big}}_{\mathbb{Q}}(X; Y) := \mathrm{Cone}_{\mathbb{Q}}(\iota(\widehat{\mathrm{Big}}(X; Y))), \\ \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y) := \mathrm{Cone}_{\mathbb{R}}(\widehat{\mathrm{Big}}_{\mathbb{Q}}(X; Y)), \\ \widehat{\mathrm{Big}}_{\mathbb{R}}(X; Y) := \mathrm{Cone}_{\mathbb{R}}(\rho(\widehat{\mathrm{Big}}_{\mathbb{Q}}(X; Y))), \end{cases}$$

where $\iota$, $\pi$ and $\rho$ are the natural homomorphisms as follows:

$$\begin{tikzcd} \widehat{\mathrm{Pic}}(X; C^0) \arrow[r, "\iota"] \arrow[drr, "\gamma"'] & \widehat{\mathrm{Pic}}_{\mathbb{Q}}(X; C^0) \arrow[r, hook] \arrow[dr, "\rho"] & \widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X; C^0) \arrow[d, "\pi"] \\ & & \widehat{\mathrm{Pic}}_{\mathbb{R}}(X; C^0) \end{tikzcd}$$

For the definition of the saturation, see Conventions and terminology 1. By Proposition 4.2, $\widehat{\mathrm{Big}}_{\mathbb{Q}}(X;Y)$, $\widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X;Y)$ and $\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y)$ are open in $\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X;C^0)$, $\widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X;C^0)$ and $\widehat{\mathrm{Pic}}_{\mathbb{R}}(X;C^0)$ respectively. Moreover,

$$\begin{cases} \iota^{-1}\left(\widehat{\mathrm{Big}}_{\mathbb{Q}}(X;Y)\right) = \widehat{\mathrm{Big}}(X;Y),\ \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X;Y) \cap \widehat{\mathrm{Pic}}_{\mathbb{Q}}(X;C^0) = \widehat{\mathrm{Big}}_{\mathbb{Q}}(X;Y), \\ \pi^{-1}\left(\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y)\right) = \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X;Y),\ \rho^{-1}\left(\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y)\right) = \widehat{\mathrm{Big}}_{\mathbb{Q}}(X;Y). \end{cases}$$

A continuous hermitian invertible sheaf $\overline{L}$ on $X$ is said to be $Y$*-big* (or *big with respect to* $Y$) if $\overline{L} \in \widehat{\mathrm{Big}}(X;Y)$. In the remaining of this section, we will observe several basic properties of $Y$-big continuous hermitian invertible sheaves. Let us begin with the following proposition.

**Proposition 5.1.** (1) *Let $\overline{L}$ be a continuous hermitian invertible sheaf on $X$. Then the following are equivalent:*

(1.1) *$\overline{L}$ is $Y$-big.*

(1.2) *For any $\overline{A} \in \widehat{\mathrm{Amp}}(X)$, there is a positive integer $n$ with $n\overline{L} \geq_Y \overline{A}$.*

(1.3) *$Y \not\subseteq \mathrm{SBs}_+(\overline{L})$.*

(2) *If $\overline{L}$ is $Y$-big, then there is a positive integer $m_0$ such that $m\overline{L}$ is $Y$-effective for all $m \geq m_0$.*

*Proof.* (1) (1.1) $\Longrightarrow$ (1.2) : There is a positive integer $n$ such that $n\overline{L} = \overline{B} + \overline{M}$ for some $\overline{B} \in \widehat{\mathrm{Amp}}(X)$ and $\overline{M} \in \widehat{\mathrm{Eff}}(X;Y)$. Let $\overline{A}$ be an ample $C^\infty$-hermitian invertible sheaf on $X$. We choose a positive number $n_1$ such that $n_1\overline{B} - \overline{A}$ is $Y$-effective. Then

$$n_1 n\overline{L} - \overline{A} = (n_1\overline{B} - \overline{A}) + n_1\overline{M}$$

is $Y$-effective.

(1.2) $\Longrightarrow$ (1.3) : For an ample $C^\infty$-hermitian invertible $\overline{A}$ sheaf, there is a positive integer $n$ such that $n\overline{L} \geq_Y \overline{A}$. Thus there is $s \in \hat{H}^0(X, n\overline{L} - \overline{A})$ with $s|_Y \neq 0$, which means that $Y \not\subseteq \mathrm{Bs}(n\overline{L} - \overline{A})$. Note that

$$\mathrm{Bs}(n\overline{L} - \overline{A}) \supseteq \mathrm{SBs}(n\overline{L} - \overline{A}) = \mathrm{SBs}(\overline{L} - (1/n)\overline{A}) \supseteq \mathrm{SBs}_+(\overline{L}).$$

Hence $Y \not\subseteq \mathrm{SBs}_+(\overline{L})$.

(1.3) $\Longrightarrow$ (1.1) : Let $\overline{A}$ be an ample $C^\infty$-hermitian invertible sheaf. Then, by Proposition 3.2, there is a positive number $n$ such that

$$\mathrm{SBs}_+(\overline{L}) = \mathrm{SBs}(\overline{L} - (1/n)\overline{A}) = \mathrm{SBs}(n\overline{L} - \overline{A}).$$

Thus, by (2) in Proposition 3.1, we can find a positive integer $m$ such that

$$\mathrm{SBs}_+(\overline{L}) = \mathrm{Bs}(m(n\overline{L} - \overline{A})),$$

so that there is $s \in \hat{H}^0(X, m(n\overline{L} - \overline{A}))$ with $s|_Y \neq 0$ because $Y \not\subseteq \mathrm{SBs}_+(\overline{L})$. This means that $mn\overline{L} \geq_Y m\overline{A}$, as required.

(2) We choose an ample $C^\infty$-hermitian invertible sheaf $\overline{A}$ such that $\overline{A}$ and $\overline{L} + \overline{A}$ is $Y$-effective. Moreover, we can take a positive integer $a$ such that $a\overline{L} - \overline{A}$ is $Y$-effective because $\overline{L}$ is $Y$-big. Note that $a\overline{L} = (a\overline{L} - \overline{A}) + \overline{A}$ and $(a+1)\overline{L} = (a\overline{L} - \overline{A}) + (\overline{L} + \overline{A})$. Thus $a\overline{L}$ and $(a+1)\overline{L}$ are $Y$-effective. Let $m$ be an integer with $m \geq a^2 + a$. We set $m = aq + r$ $(0 \leq r < a)$. Then $q \geq a$, so that there is an integer $b$ with $q = b + r$ and $b > 0$. Therefore, $m\overline{L}$ is $Y$-effective because $m\overline{L} = b(a\overline{L}) + r((a+1)\overline{L})$. ∎

**Proposition 5.2.** *Let $X$ be a projective arithmetic variety, $Y$ a $d'$-dimensional arithmetic subvariety of $X$ and $\overline{L}$ a continuous hermitian invertible sheaf on $X$. Let $Z_{\cdot} : Z_0 = Y \supset Z_1 \supset Z_2 \supset \cdots \supset Z_{d'}$ be a good flag over a prime $p$ on $Y$. If $\overline{L}$ is $Y$-big, then*

$$\left\{ \left(\nu_{Z_{\cdot}}\left(\left.s\right|_Y\right), m\right) \mid s \in \hat{H}^0(X, m\overline{L}) \text{ and } \left.s\right|_Y \neq 0 \right\}$$

*generates $\mathbb{Z}^{d'+1}$ as a $\mathbb{Z}$-module.*

To prove the above proposition, we need the following two lemmas.

**Lemma 5.3.** *Let $X$ be either a projective arithmetic variety or a projective variety over a field. Let $Z$ be a reduced and irreducible subvariety of codimension $1$ and $x$ a closed point of $Z$. Let $I$ be the defining ideal sheaf of $Z$. We assume that $I$ is principal at $x$ (it holds if $X$ is regular at $x$). Let $H$ be an ample invertible sheaf on $X$. Then there is a positive integer $n_0$ with the following property: for all $n \geq n_0$, we can find $s \in H^0(X, nH \otimes I)$ such that $s \neq 0$ in $nH \otimes I \otimes \kappa(x)$, where $\kappa(x)$ is the residue field at $x$.*

*Proof.* Let $m_x$ be the maximal ideal at $x$. Since $I$ is invertible around $x$, we have the exact sequence

$$0 \to nH \otimes I \otimes m_x \to nH \otimes I \to nH \otimes I \otimes \kappa(x) \to 0.$$

As $H$ is ample, there is a positive integer $n_0$ such that

$$H^1(X, nH \otimes I \otimes m_x) = 0$$

for all $n \geq n_0$, which means that $H^0(X, nH \otimes I) \to nH \otimes I \otimes \kappa(x)$ is surjective, as required. ■

**Lemma 5.4.** *Let $X$ be a projective arithmetic variety and $Y$ a $d'$-dimensional arithmetic subvariety of $X$. Let $Z_{\cdot} : Z_0 = Y \supset Z_1 \supset Z_2 \supset \cdots \supset Z_{d'}$ be a good flag over a prime $p$ on $Y$. Let $H$ be an ample invertible sheaf on $X$. Let $e_1, \ldots, e_{d'}$ be the standard basis of $\mathbb{Z}^{d'}$. Then there is a positive integer $n_0$ such that, for all $n \geq n_0$, we can find $s_1, \ldots, s_{d'} \in H^0(X, nH)$ with $\nu_{Z_{\cdot}}(\left.s_1\right|_Y) = e_1, \ldots, \nu_{Z_{\cdot}}(\left.s_{d'}\right|_Y) = e_{d'}$.*

*Proof.* First of all, we can find $n'_0$ such that, for all $n \geq n'_0$,

$$H^0(X, nH) \to H^0(Z_i, \left.nH\right|_{Z_i})$$

are surjective for all $i$. We set $Z_{d'} = \{z\}$. For $i = 1, \ldots, d'$, let $I_i$ be the defining ideal sheaf of $Z_i$ in $Z_{i-1}$. Then, by Lemma 5.3, there is a positive integer $n'_i$ such that, for all $n \geq n'_i$, we can find $s'_i \in H^0(Z_i, \left.nH\right|_{Z_{i-1}} \otimes I_i)$ such that $s'_i \neq 0$ in $\left.nH\right|_{Z_{i-1}} \otimes I_i \otimes \kappa(z)$. Thus, if $n \geq \max\{n'_0, n'_1, \ldots, n'_{d'}\}$, then there are $s_1, \ldots, s_{d'} \in H^0(X, nH)$ such that $\left.s_i\right|_{Y_{i-1}} = s'_i$ for $i = 1, \ldots, d'$. By our construction, it is easy to see that $\nu_{Z_{\cdot}}(\left.s_i\right|_Y) = e_i$. ■

*The proof of Proposition 5.2.* Let us begin with the following claim:

**Claim 5.5.1.** *There are an ample $C^\infty$-hermitian invertible sheaf $\overline{A}$ and $s_0, s_1, \ldots, s_{d'} \in \hat{H}^0(X, \overline{A}) \setminus \{0\}$ and $t \in \hat{H}^0(X, \overline{A} + \overline{L}) \setminus \{0\}$ such that*

$$\left.s_0\right|_Y \neq 0,\ \left.s_1\right|_Y \neq 0, \ldots,\ \left.s_{d'}\right|_Y \neq 0,\ \left.t\right|_Y \neq 0$$

*and*

$$\nu_{Z_{\cdot}}(\left.s_0\right|_Y) = 0,\ \nu_{Z_{\cdot}}(\left.s_1\right|_Y) = e_1,\ \ldots,\ \nu_{Z_{\cdot}}(\left.s_{d'}\right|_Y) = e_{d'} \text{ and } \nu_{Z_{\cdot}}(\left.t\right|_Y) = 0.$$

Let $B$ be an ample invertible sheaf on $X$. By Lemma 5.4, there are positive integer $n$, $s_0, s_1, \ldots, s_d \in H^0(X, nB) \setminus \{0\}$ and $t \in H^0(X, nB + L) \setminus \{0\}$ such that

$$\nu_{Z.}(s_0|_Y) = 0,\ \nu_{Z.}(s_1|_Y) = e_1,\ \ldots,\ \nu_{Z.}(s_d|_Y) = e_d \text{ and } \nu_{Z.}(t|_Y) = 0.$$

We choose a $C^\infty$-hermitian metric of $B$ such that $\overline{B}$ is ample, $s_0, s_1, \ldots, s_d \in \hat{H}^0(X, n\overline{B})$ and $t \in \hat{H}^0(X, n\overline{B} + \overline{L})$. □

Let $M$ be the $\mathbb{Z}$-submodule generated by

$$\{(\nu_{Z.}(s|_Y), m) \mid s \in \hat{H}^0(X, m\overline{L}) \text{ and } s|_Y \neq 0\}.$$

Since $\overline{L}$ is $Y$-big, there is a positive integer $a$ with $a\overline{L} \geq_Y \overline{A}$, that is, there is $e \in \hat{H}^0(X, a\overline{L} - \overline{A})$ with $e|_Y \neq 0$. Note that

$$t \otimes e \in \hat{H}^0(X, (a+1)\overline{L}) \quad \text{and} \quad s_0 \otimes e \in \hat{H}^0(X, a\overline{L}).$$

Moreover $\nu_{Z.}(t \otimes e|_Y) = \nu_{Z.}(e|_Y)$ and $\nu_{Z.}(s_0 \otimes e|_Y) = \nu_{Z.}(e|_Y)$. Thus

$$(\nu_{Z.}(t \otimes e|_Y), a+1) - (\nu_{Z.}(s_0 \otimes e|_Y), a) = (0, \ldots, 0, 1) \in M.$$

Further, as $s_i \otimes e, s_0 \otimes e \in \hat{H}^0(X, a\overline{L})$, we obtain

$$\begin{aligned}(\nu_{Z.}(s_i \otimes e|_Y), m) &- (\nu_{Z.}(s_0 \otimes e|_Y), m) \\ &= (e_i + \nu_{Z.}(e|_Y), m) - (\nu_{Z.}(e|_Y), m) = (e_i, 0) \in M.\end{aligned}$$

Hence $M = \mathbb{Z}^{d+1}$. ■

## 6. ARITHMETIC RESTRICTED VOLUME

Let $X$ be a projective arithmetic variety and $Y$ a $d'$-dimensional arithmetic subvariety of $X$. For an invertible sheaf $L$ on $X$, $\mathrm{Image}(H^0(X, L) \to H^0(Y, L|_Y))$ is denoted by $H^0(X|Y, L)$. We assign an arithmetic linear series $\hat{H}^0_\bullet(X|Y, \overline{L})$ of $\overline{L}\big|_Y$ to each continuous hermitian invertible sheaf $\overline{L}$ on $X$ with the following properties:

(1) $\mathrm{Image}(\hat{H}^0(X, \overline{L}) \to H^0(X|Y, L)) \subseteq \hat{H}^0_\bullet(X|Y, \overline{L})$.
(2) $s \otimes s' \in \hat{H}^0_\bullet(X|Y, \overline{L} + \overline{M})$ for all $s \in \hat{H}^0_\bullet(X|Y, \overline{L})$ and $s' \in \hat{H}^0_\bullet(X|Y, \overline{M})$.

This correspondence $\overline{L} \mapsto \hat{H}^0_\bullet(X|Y, \overline{L})$ is called an *assignment of arithmetic restricted linear series from $X$ to $Y$*. As examples, we have the following:

- $\hat{H}^0_{\mathrm{CL}}(X|Y, \overline{L})$ : $\hat{H}^0_{\mathrm{CL}}(X|Y, \overline{L})$ is the convex lattice hull of

$$\mathrm{Image}(\hat{H}^0(X, \overline{L}) \to H^0(X|Y, L))$$

in $H^0(X|Y, L)$. This is actually an assignment of arithmetic restricted linear series from $X$ to $Y$. The above property (1) is obvious. For (2), let $s_1, \ldots, s_r \in \mathrm{Image}(\hat{H}^0(X, \overline{L}) \to H^0(X|Y, L))$ and $s'_1, \ldots, s'_{r'} \in \mathrm{Image}(\hat{H}^0(X, \overline{M}) \to H^0(X|Y, M))$, and let

$$\lambda_1, \ldots, \lambda_r \quad \text{and} \quad \lambda'_1, \ldots, \lambda'_{r'}$$

be non-negative real numbers with $\lambda_1 + \cdots + \lambda_r = 1$ and $\lambda'_1 + \cdots + \lambda'_{r'} = 1$. Then

$$(\lambda_1 s_1 + \cdots + \lambda_r s_r) \otimes (\lambda'_1 s'_1 + \cdots + \lambda'_{r'} s'_{r'}) = \sum_{i,j} \lambda_i \lambda'_j (s_i \otimes s_j)$$

and

$$\sum_{i,j} \lambda_i \lambda'_j = (\lambda_1 + \cdots + \lambda_r)(\lambda'_1 + \cdots + \lambda'_{r'}) = 1,$$

as required.

• $\hat{H}^0_{\rm quot}(X|Y,\overline{L})$ : Let $\|\cdot\|^{X|Y}_{\rm sup,quot}$ be the quotient norm of $H^0(X|Y,L)$ induced by the norm $\|\cdot\|_{\rm sup}$ on $H^0(X,L)$ and the natural surjective homomorphism $H^0(X,L)\to H^0(X|Y,L)$. Then $\hat{H}^0_{\rm quot}(X|Y,\overline{L})$ is defined to be

$$\hat{H}^0_{\rm quot}(X|Y,\overline{L})=\{s\in H^0(X|Y,L)\mid \|s\|^{X|Y}_{\rm sup,quot}\leq 1\}.$$

This is obviously an assignment of arithmetic restricted linear series from $X$ to $Y$.

• $\hat{H}^0_{\rm sub}(X|Y,\overline{L})$ : Let $\|\cdot\|_{Y,\rm sup}$ be the norm on $H^0(Y,L|_Y)$ given by $\|s\|_{Y,\rm sup}=\sup_{y\in Y(\mathbb{C})}|s|(y)$. Let $\|\cdot\|^{X|Y}_{\rm sup,sub}$ be the sub-norm of $H^0(X|Y,L)$ induced by $\|\cdot\|_{Y,\rm sup}$ on $H^0(Y,L|_Y)$ and the natural injective homomorphism $H^0(X|Y,L)\hookrightarrow H^0(Y,L|_Y)$. Then $\hat{H}^0_{\rm sub}(X|Y,\overline{L})$ is defined to be

$$\hat{H}^0_{\rm sub}(X|Y,\overline{L})=\left\{s\in H^0(X|Y,L)\mid \|s\|^{X|Y}_{\rm sup,sub}\leq 1\right\}.$$

This is obviously an assignment of arithmetic restricted linear series from $X$ to $Y$.

Note that

$$\hat{H}^0_{\rm CL}(X|Y,\overline{L})\subseteq\hat{H}^0_{\rm quot}(X|Y,\overline{L})\subseteq\hat{H}^0_{\rm sub}(X|Y,\overline{L})$$

for any continuous hermitian invertible sheaf $\overline{L}$. An assignment $\overline{L}\mapsto\hat{H}^0_\bullet(X|Y,\overline{L})$ of arithmetic restricted linear series from $X$ to $Y$ is said to be *proper* if, for each $\overline{L}\in\widehat{\rm Pic}(X,C^0)$, there is a symmetric and bounded convex set $\Delta$ in $H^0(X|Y,L)\otimes\mathbb{R}$ such that $\hat{H}^0_\bullet(X|Y,\overline{L}+\overline{\mathcal{O}}(\lambda))=H^0(X|Y,L)\cap\exp(\lambda)\Delta$ for all $\lambda\in\mathbb{R}$. For example, the assignments $\overline{L}\mapsto\hat{H}^0_{\rm quot}(X|Y,\overline{L})$ and $\overline{L}\mapsto\hat{H}^0_{\rm sub}(X|Y,\overline{L})$ are proper.

Let us fix an assignment $\overline{L}\mapsto\hat{H}^0_\bullet(X|Y,\overline{L})$ of arithmetic restricted linear series from $X$ to $Y$. Then we define the restricted arithmetic volume with respect to the assignment to be

$$\widehat{\rm vol}_\bullet\left(X|Y,\overline{L}\right):=\limsup_{m\to\infty}\frac{\log\#\hat{H}^0_\bullet(X|Y,m\overline{L})}{m^{d'}/d'!}.$$

Let us begin with the following proposition.

**Proposition 6.1.** (1) *If $\overline{L}\leq_Y\overline{M}$, then $\#\hat{H}^0_\bullet(X|Y,\overline{L})\leq\#\hat{H}^0_\bullet(X|Y,\overline{M})$. In particular, $\widehat{\rm vol}_\bullet\left(X|Y,\overline{L}\right)\leq\widehat{\rm vol}_\bullet\left(X|Y,\overline{M}\right)$.*

(2) *We assume that the assignment $\overline{L}\mapsto\hat{H}^0_\bullet(X|Y,\overline{L})$ is proper. Then, for any $\overline{L}\in\widehat{\rm Pic}(X;C^0)$ and $f\in C^0(X)$,*

$$\left|\widehat{\rm vol}_\bullet\left(X|Y,\overline{L}+\overline{\mathcal{O}}(f)\right)-\widehat{\rm vol}_\bullet\left(X|Y,\overline{L}\right)\right|\leq d'\,{\rm vol}(X_{\mathbb{Q}}|Y_{\mathbb{Q}},L_{\mathbb{Q}})\|f\|_{\rm sup},$$

*where* ${\rm vol}(X_{\mathbb{Q}}|Y_{\mathbb{Q}},L_{\mathbb{Q}})$ *is the algebraic restricted volume* (*cf.* [3]).

*Proof.* (1) Let us choose $t\in\hat{H}^0(X,\overline{M}-\overline{L})$ with $t|_Y\neq 0$. Then $t|_Y\in\hat{H}^0_\bullet(X|Y,\overline{M}-\overline{L})$ and

$$s\otimes(t|_Y)\in\hat{H}^0_\bullet(X|Y,\overline{M})$$

for any $s\in\hat{H}^0_\bullet(X|Y,\overline{L})$, which means that we have the injective map

$$\hat{H}^0_\bullet(X|Y,\overline{L})\to\hat{H}^0_\bullet(X|Y,\overline{M})$$

given by $s\mapsto s\otimes(t|_Y)$. Thus (1) follows.

(2) First let us see that

$$\left|\widehat{\rm vol}_\bullet\left(X|Y,\overline{L}+\overline{\mathcal{O}}(\lambda)\right)-\widehat{\rm vol}_\bullet\left(X|Y,\overline{L}\right)\right|\leq d'\,{\rm vol}(X_{\mathbb{Q}}|Y_{\mathbb{Q}},L_{\mathbb{Q}})|\lambda|. \tag{6.1.1}$$

for any $\overline{L} \in \widehat{\mathrm{Pic}}(X; C^0)$ and $\lambda \in \mathbb{R}$. Without loss of generality, we may assume that $\lambda \geq 0$. As the assignment is proper, for each $m \geq 1$, there is a symmetric and bounded convex set $\Delta_m$ such that

$$\hat{H}^0_\bullet(X|Y, m\overline{L} + \overline{\mathcal{O}}(\mu)) = H^0(X|Y, mL) \cap \exp(\mu)\Delta_m$$

for all $\mu \in \mathbb{R}$. Thus, by using Lemma 1.2.2,

$$\begin{aligned} 0 &\leq \log \#\hat{H}^0_\bullet(X|Y, m(\overline{L} + \overline{\mathcal{O}}(\lambda))) - \log \#\hat{H}^0_\bullet(X|Y, m\overline{L}) \\ &= \log \#(H^0(X|Y, mL) \cap \exp(m\lambda)\Delta_m) - \log \#(H^0(X|Y, L) \cap \Delta_m) \\ &\leq \log(\lceil 2\exp(m\lambda)\rceil) \dim_{\mathbb{Q}} H^0(X_{\mathbb{Q}}|Y_{\mathbb{Q}}, mL_{\mathbb{Q}}), \end{aligned}$$

which implies (6.1.1).

For $f \in C^0(X)$, if we set $\lambda = \|f\|_{\sup}$, then $-\lambda \leq f \leq \lambda$. Thus the proposition follows from (6.1.1). ∎

The following theorem is the main result of this section.

**Theorem 6.2.** (1) *If $\overline{L}$ is $Y$-big, then*

$$\widehat{\mathrm{vol}}_\bullet\left(X|Y, \overline{L}\right) = \lim_{m\to\infty} \frac{\log \#\hat{H}^0_\bullet(X|Y, m\overline{L})}{m^{d'}/d'!}.$$

*In particular, if $\overline{L}$ is $Y$-big, then $\widehat{\mathrm{vol}}_\bullet\left(X|Y, n\overline{L}\right) = n^{d'}\widehat{\mathrm{vol}}_\bullet\left(X|Y, \overline{L}\right)$ for all non-negative integers $n$.*

(2) *If $\overline{L}$ and $\overline{M}$ are $Y$-big continuous hermitian invertible sheaves on $X$, then*

$$\widehat{\mathrm{vol}}_\bullet\left(X|Y, \overline{L} + \overline{M}\right)^{\frac{1}{d'}} \geq \widehat{\mathrm{vol}}_\bullet\left(X|Y, \overline{L}\right)^{\frac{1}{d'}} + \widehat{\mathrm{vol}}_\bullet\left(X|Y, \overline{M}\right)^{\frac{1}{d'}}.$$

(3) *If $\overline{L}$ is $Y$-big, then, for any positive $\epsilon$, there is a positive integer $n_0 = n_0(\epsilon)$ such that, for all $n \geq n_0$,*

$$\liminf_{k\to\infty} \frac{\log \#(K_{k,n})}{n^{d'}k^{d'}} \geq \frac{\widehat{\mathrm{vol}}_\bullet(X|Y, \overline{L})}{d'!} - \epsilon,$$

*where $K_{k,n}$ is the convex lattice hull of*

$$V_{k,n} = \{s_1 \otimes \cdots \otimes s_k \mid s_1, \ldots, s_k \in \hat{H}^0_\bullet(X|Y, n\overline{L})\}$$

*in $H^0(X|Y, knL)$.*

*Proof.* Let $Z_. : Z_0 = Y \supset Z_1 \supset Z_2 \supset \cdots \supset Z_{d'}$ be a good flag over a prime $p$ on $Y$.

(1) Let $\Delta$ be the closure of

$$\bigcup_{m=1}^{\infty} \frac{1}{m}\nu_{Z.}(\hat{H}^0_\bullet(X|Y, m\overline{L}) \setminus \{0\})$$

in $\mathbb{R}^{d'}$. Then, by Proposition 5.2, [11, Lemma 2.4] and [5, Proposition 2.1],

$$\mathrm{vol}(\Delta) = \lim_{m\to\infty} \frac{\#\nu_{Z.}(\hat{H}^0_\bullet(X|Y, m\overline{L}) \setminus \{0\})}{m^{d'}}.$$

By Corollary 2.3, there is a constant $c$ depending only on $\overline{L}$ such that

$$\begin{aligned} \nu_{Z.}(\hat{H}^0_\bullet(X|Y, m\overline{L}) \setminus \{0\}) \log p - \frac{cm^{d'}}{\log p} &\leq \log \#\hat{H}^0_\bullet(X|Y, m\overline{L}) \\ &\leq \nu_{Z.}(\hat{H}^0_\bullet(X|Y, m\overline{L}) \setminus \{0\}) \log p + \frac{cm^{d'}}{\log p}, \end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{vol}(\Delta)\log p - \frac{c}{\log p} &\leq \liminf_{m\to\infty} \frac{\log \#\hat{H}^0_{\bullet}(X|Y, m\overline{L})}{m^{d'}} \\ &\leq \limsup_{m\to\infty} \frac{\log \#\hat{H}^0_{\bullet}(X|Y, m\overline{L})}{m^{d'}} \leq \mathrm{vol}(\Delta)\log p + \frac{c}{\log p}.\end{aligned}$$

Hence

$$\limsup_{m\to\infty} \frac{\log \#\hat{H}^0_{\bullet}(X|Y, m\overline{L})}{m^{d'}} - \liminf_{m\to\infty} \frac{\log \#\hat{H}^0_{\bullet}(X|Y, m\overline{L})}{m^{d'}} \leq \frac{2c}{\log p}.$$

Thus, as $p$ goes to $\infty$, we have

$$\limsup_{m\to\infty} \frac{\log \#\hat{H}^0_{\bullet}(X|Y, m\overline{L})}{m^{d'}} = \liminf_{m\to\infty} \frac{\log \#\hat{H}^0_{\bullet}(X|Y, m\overline{L})}{m^{d'}}.$$

Moreover, we can see that

$$\left|\widehat{\mathrm{vol}}_{\bullet}\left(X|Y, \overline{L}\right) - \mathrm{vol}(\Delta) d'! \log p\right| \leq \frac{cd'!}{\log p}. \tag{6.2.1}$$

(2) Let $\Delta'$ and $\Delta''$ be the closure of

$$\bigcup_{m=1}^{\infty} \frac{1}{m}\nu_{Z.}(\hat{H}^0_{\bullet}(X|Y, m\overline{M}) \setminus \{0\}) \quad \text{and} \quad \bigcup_{m=1}^{\infty} \frac{1}{m}\nu_{Z.}(\hat{H}^0_{\bullet}(X|Y, m(\overline{L}+\overline{M})) \setminus \{0\})$$

in $\mathbb{R}^{d'}$. Since

$$\begin{aligned}&\nu_{Z.}(\hat{H}^0_{\bullet}(X|Y, m\overline{L}) \setminus \{0\}) + \nu_{Z.}(\hat{H}^0_{\bullet}(X|Y, m\overline{M}) \setminus \{0\}) \\ &\qquad = \{\nu_{Z.}(s \otimes s') \mid s \in \hat{H}^0_{\bullet}(X|Y, m\overline{L}) \setminus \{0\},\ s' \in \hat{H}^0_{\bullet}(X|Y, m\overline{M}) \setminus \{0\}\} \\ &\qquad\qquad \subseteq \nu_{Z.}(\hat{H}^0_{\bullet}(X|Y, m(\overline{L}+\overline{M})) \setminus \{0\}),\end{aligned}$$

we have $\Delta + \Delta' \subseteq \Delta''$. Thus, by Brunn-Minkowski's theorem,

$$\mathrm{vol}(\Delta'')^{\frac{1}{d'}} \geq \mathrm{vol}(\Delta + \Delta')^{\frac{1}{d'}} \geq \mathrm{vol}(\Delta)^{\frac{1}{d'}} + \mathrm{vol}(\Delta')^{\frac{1}{d'}}.$$

Note that (6.2.1) also holds for $\overline{L}$ and $\overline{L}+\overline{M}$ with another constants $c'$ and $c''$. Hence, for a small positive number $\epsilon$, if $p$ is a sufficiently large prime number, then

$$\begin{aligned}&\left|\widehat{\mathrm{vol}}_{\bullet}\left(X|Y, \overline{L}\right) - \mathrm{vol}(\Delta) d'! \log p\right| \leq \epsilon, \\ &\left|\widehat{\mathrm{vol}}_{\bullet}\left(X|Y, \overline{M}\right) - \mathrm{vol}(\Delta') d'! \log p\right| \leq \epsilon \quad \text{and} \\ &\left|\widehat{\mathrm{vol}}_{\bullet}\left(X|Y, \overline{L}+\overline{M}\right) - \mathrm{vol}(\Delta'') d'! \log p\right| \leq \epsilon\end{aligned}$$

hold. Therefore,

$$\left(\widehat{\mathrm{vol}}_{\bullet}\left(X|Y, \overline{L}+\overline{M}\right) + \epsilon\right)^{\frac{1}{d'}} \geq \left(\widehat{\mathrm{vol}}_{\bullet}\left(X|Y, \overline{L}\right) - \epsilon\right)^{\frac{1}{d'}} + \left(\widehat{\mathrm{vol}}_{\bullet}\left(X|Y, \overline{M}\right) - \epsilon\right)^{\frac{1}{d'}},$$

as required.

(3) Let $c$ be a constant for $Y$ and $\overline{L}\big|_Y$ as in Corollary 2.3. We choose a good flag $Z. : Z_0 = Y \supset Z_1 \supset Z_2 \supset \cdots \supset Z_{d'}$ over a prime $p$ with $c/(\log p) \leq \epsilon/3$. Let $\epsilon'$ be a positive number with $\epsilon' \log p \leq \epsilon/3$. By [5, Proposition 3.1], there is a positive integer $n_0$ such that

$$\lim_{k\to\infty} \frac{\#(k * \nu(\hat{H}^0_{\bullet}(X|Y, n\overline{L}) \setminus \{0\}))}{k^{d'} n^{d'}} \geq \mathrm{vol}(\Delta) - \epsilon'$$

for all $n \geq n_0$. Note that

$$\nu(K_{k,n} \setminus \{0\}) \supseteq k * \nu(\hat{H}^0_\bullet(X|Y, n\overline{L}) \setminus \{0\})$$

and

$$\log \#(K_{k,n}) \geq \#\nu(K_{k,n} \setminus \{0\}) \log p - (\epsilon/3) k^{d'} n^{d'}$$

by Corollary 2.3 for $k \gg 1$. Thus

$$\frac{\log \#(K_{k,n})}{k^{d'} n^{d'}} \geq \frac{\#(k * \nu(\hat{H}^0_\bullet(X|Y, n\overline{L}) \setminus \{0\})) \log p}{k^{d'} n^{d'}} - \epsilon/3,$$

which implies that

$$\begin{aligned} \liminf_{k \to \infty} \frac{\log \#(K_{k,n})}{k^{d'} n^{d'}} &\geq \lim_{k \to \infty} \frac{\#(k * \nu(\hat{H}^0_\bullet(X|Y, n\overline{L}) \setminus \{0\})) \log p}{k^{d'} n^{d'}} - \epsilon/3 \\ &\geq (\mathrm{vol}(\Delta) - \epsilon') \log p - \epsilon/3 \\ &\geq \mathrm{vol}(\Delta) \log p - 2\epsilon/3. \end{aligned}$$

Moreover, by (6.2.1),

$$\mathrm{vol}(\Delta) \log p \geq \frac{\widehat{\mathrm{vol}}_\bullet(X|Y, \overline{L})}{d'!} - \epsilon/3.$$

Thus we obtain (3). ■

In the remaining of this section, let us consider consequences of Theorem 6.2.

**Corollary 6.3.** *There is a unique continuous function*

$$\widehat{\mathrm{vol}}'_\bullet(X|Y, -) : \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y) \to \mathbb{R}$$

*with the following properties:*

(1) $\widehat{\mathrm{vol}}'_\bullet(X|Y, \iota(\overline{L})) = \widehat{\mathrm{vol}}_\bullet(X|Y, \overline{L})$ *holds for all* $\overline{L} \in \widehat{\mathrm{Big}}(X; Y)$.
(2) $\widehat{\mathrm{vol}}'_\bullet(X|Y, \lambda x) = \lambda^{d'} \widehat{\mathrm{vol}}'_\bullet(X|Y, x)$ *holds for all* $\lambda \in \mathbb{R}_{>0}$ *and* $x \in \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$.
(3) $\widehat{\mathrm{vol}}'_\bullet(X|Y, x + y)^{\frac{1}{d'}} \geq \widehat{\mathrm{vol}}'_\bullet(X|Y, x)^{\frac{1}{d'}} + \widehat{\mathrm{vol}}'_\bullet(X|Y, y)^{\frac{1}{d'}}$ *holds for all* $x, y \in \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$.

*Proof.* It follows from Theorem 6.2, Proposition 1.1.5 and Corollary 1.3.2. ■

**Corollary 6.4.** *If the assignment* $\overline{L} \mapsto \hat{H}^0_\bullet(X|Y, \overline{L})$ *is proper, then there is a unique continuous function*

$$\widehat{\mathrm{vol}}''_\bullet(X|Y, -) : \widehat{\mathrm{Big}}_{\mathbb{R}}(X; Y) \to \mathbb{R}$$

*with the following properties:*

(1) $\widehat{\mathrm{vol}}''_\bullet(X|Y, \pi(x')) = \widehat{\mathrm{vol}}'_\bullet(X|Y, x')$ *holds for all* $x' \in \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$.
(2) $\widehat{\mathrm{vol}}''_\bullet(X|Y, \lambda x) = \lambda^{d'} \widehat{\mathrm{vol}}''_\bullet(X|Y, x)$ *holds for all* $\lambda \in \mathbb{R}_{>0}$ *and* $x \in \widehat{\mathrm{Big}}_{\mathbb{R}}(X; Y)$.
(3) $\widehat{\mathrm{vol}}''_\bullet(X|Y, x + y)^{\frac{1}{d'}} \geq \widehat{\mathrm{vol}}''_\bullet(X|Y, x)^{\frac{1}{d'}} + \widehat{\mathrm{vol}}''_\bullet(X|Y, y)^{\frac{1}{d'}}$ *holds for all* $x, y \in \widehat{\mathrm{Big}}_{\mathbb{R}}(X; Y)$.

*Proof.* Let us begin with the following estimation.

$$\left| \widehat{\mathrm{vol}}'_\bullet\left(X|Y, \overline{L} + \overline{\mathcal{O}}(f)\right) - \widehat{\mathrm{vol}}'_\bullet\left(X|Y, \overline{L}\right) \right| \leq d' \,\mathrm{vol}(X_{\mathbb{Q}}|Y_{\mathbb{Q}}, L_{\mathbb{Q}}) \|f\|_{\sup}. \tag{6.4.1}$$

for any $\overline{L} \in \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$ and $f \in C^0(X)$ with $\overline{L} + \mathcal{O}(f) \in \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$. By using the openness of $\widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$ and the continuity of $\widehat{\mathrm{vol}}'_\bullet(X|Y, -)$ on $\widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$, it is sufficient to see (6.4.1) for $\overline{L} \in \widehat{\mathrm{Big}}_{\mathbb{Q}}(X;Y)$. Thus $\overline{L} = (1/n)\iota(\overline{M})$ for some $\overline{M} \in \widehat{\mathrm{Big}}(X;Y)$ and $n \in \mathbb{Z}_{>0}$, and hence, by Proposition 6.1,

$$
\begin{aligned}
&\left|\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,\overline{L}+\overline{\mathcal{O}}(f)\right)-\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,\overline{L}\right)\right| \\
&\qquad = \left|\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,(1/n)\iota(\overline{M}+\overline{\mathcal{O}}(nf))\right)-\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,(1/n)\iota(\overline{M})\right)\right| \\
&\qquad = (1/n)^{d'}\left|\widehat{\mathrm{vol}}_{\bullet}\left(X|Y,\overline{M}+\overline{\mathcal{O}}(nf)\right)-\widehat{\mathrm{vol}}_{\bullet}\left(X|Y,\overline{M}\right)\right| \\
&\qquad = (1/n)^{d'} d' \operatorname{vol}(X_{\mathbb{Q}}|Y_{\mathbb{Q}},M_{\mathbb{Q}})\|nf\|_{\sup} = d' \operatorname{vol}(X_{\mathbb{Q}}|Y_{\mathbb{Q}},L_{\mathbb{Q}})\|f\|_{\sup}.
\end{aligned}
$$

Let us observe that there is a function

$$
\widehat{\mathrm{vol}}''_{\bullet}(X|Y,-) : \widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y) \to \mathbb{R}
$$

such that the following diagram is commutative:

$$
\begin{tikzcd}
\widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X;Y) \arrow[r, "\widehat{\mathrm{vol}}'_{\bullet}(X|Y,-)"] \arrow[d, "\pi"'] & \mathbb{R} \\
\widehat{\mathrm{Big}}_{\mathbb{R}}(X;Y) \arrow[ur, "\widehat{\mathrm{vol}}''_{\bullet}(X|Y-)"'] &
\end{tikzcd}
$$

Namely, we need to show that if $\pi(x') = \pi(y')$ for $x', y' \in \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X;Y)$, then

$$
\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,x'\right) = \widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,y'\right).
$$

As $\pi(x') = \pi(y')$, there is $z \in N(X)$ such that $y' = x' + z$. We set $z = \overline{\mathcal{O}}(f_1) \otimes a_1 + \cdots + \overline{\mathcal{O}}(f_r) \otimes a_r$ with $a_1 f_1 + \cdots + a_r f_r = 0$, where $f_1, \ldots, f_r \in C^0(X)$ and $a_1, \ldots, a_r \in \mathbb{R}$. Let us take a sequence $\{a_{in}\}_{n=1}^{\infty}$ in $\mathbb{Q}$ such that $a_i = \lim_{n\to\infty} a_{in}$. We set $\phi_n = a_{1n}f_1 + \cdots + a_{rn}f_r$. Then

$$
\begin{aligned}
\|\phi_n\|_{\sup} &= \|(a_{1n}-a_1)f_1 + \cdots + (a_{rn}-a_r)f_r\|_{\sup} \\
&\qquad \leq |a_{1n}-a_1|\|f_1\|_{\sup} + \cdots + |a_{rn}-a_r|\|f_r\|_{\sup}.
\end{aligned}
$$

Thus $\lim_{n\to\infty}\|\phi_n\|_{\sup} = 0$. If we put $z_n = \overline{\mathcal{O}}(f_1)\otimes a_{1n} + \cdots + \overline{\mathcal{O}}(f_r)\otimes a_{rn}$, then $\lim_{n\to\infty} z_n = z$ in $\widehat{\mathrm{Pic}}_{\otimes\mathbb{R}}(X;C^0)$ and $z_n = \overline{\mathcal{O}}(\phi_n)$ in $\widehat{\mathrm{Pic}}_{\mathbb{Q}}(X;C^0)$. Thus, by (6.4.1),

$$
\begin{aligned}
&\left|\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,x'+z_n\right)-\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,x'\right)\right| \\
&\qquad = \left|\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,x'+\overline{\mathcal{O}}(\phi_n)\right)-\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,x'\right)\right| \\
&\qquad\qquad \leq d' \operatorname{vol}(X_{\mathbb{Q}}|Y_{\mathbb{Q}},x'_{\mathbb{Q}})\|\phi_n\|_{\sup}
\end{aligned}
$$

for $n \gg 1$. Therefore, as $n$ goes to $\infty$, $\widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,y'\right) = \widehat{\mathrm{vol}}'_{\bullet}\left(X|Y,x'\right)$.

The properties (2) and (3) are obvious. The continuity of $\widehat{\mathrm{vol}}''_{\bullet}(X|Y,-)$ follows from that $\pi$ is an open map. ∎

## 7. Restricted volume for ample $C^\infty$-hermitian invertible sheaf

In this section, let us consider the restricted volume for an ample $C^\infty$-hermitian invertible sheaf and observe several consequences.

**Theorem 7.1.** *Let $X$ be a projective arithmetic variety and $Y$ a $d'$-dimensional arithmetic subvariety of $X$. Let $\overline{A}$ be a $Y$-big continuous hermitian invertible sheaf on $X$. Then we have the following.*

(1) *We assume that there are a positive integer $a$ and strictly small sections $s_1, \ldots, s_l$ of $a\overline{A}$ with $\{x \in X_{\mathbb{Q}} \mid s_1(x) = \cdots = s_l(x) = 0\} = \emptyset$. Then*

$$\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y, \overline{A}) = \lim_{m\to\infty} \frac{\log \# \operatorname{Image}(\hat{H}^0(X, m\overline{A}) \to H^0(X|Y, mA))}{m^{d'}/d'!}.$$

*Moreover, if $A_{\mathbb{Q}}$ is ample on $X_{\mathbb{Q}}$, then*

$$\lim_{m\to\infty} \frac{\log \# \operatorname{Image}(\hat{H}^0(X, m\overline{A}) \to H^0(X|Y, mA))}{m^{d'}/d'!} > 0.$$

(2) *We assume that $X$ is generically smooth, $A_{\mathbb{Q}}$ is ample on $X_{\mathbb{Q}}$, the metric of $\overline{A}$ is $C^\infty$ and that $c_1(\overline{A})$ is semipositive on $X(\mathbb{C})$. Then*

$$\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y, \overline{A}) = \widehat{\mathrm{vol}}(Y, A|_Y).$$

*Proof.* (1) Let $\mathcal{I}$ be the defining ideal sheaf of $Y$. Let us begin with the following claim.

**Claim 7.1.1.** *We can find a positive integer $m_0$ and a positive number $\epsilon_0$ with the following property: for all $m \geq m_0$, there is a free basis $e_1, \ldots, e_N$ of $H^0(X, mA \otimes \mathcal{I})$ as a $\mathbb{Z}$-module such that $\|e_i\|_{\sup} \leq e^{-m\epsilon_0}$ for all $i$, where the norm $\|\cdot\|_{\sup}$ of $H^0(X, mA \otimes \mathcal{I})$ is the sub-norm induced by the inclusion map $H^0(X, mA \otimes \mathcal{I}) \hookrightarrow H^0(X, mA)$ and the sup norm of $H^0(X, mA)$.*

By [8, Corollary 3.3], there are positive constants $B$ and $c$ such that

$$\lambda_{\mathbb{Q}}(H^0(X, mA), \|\cdot\|_{\sup}) \leq B(m+1)^{\dim X(\dim X - 1)} \exp(-cm)$$

for all $m \geq 0$ (for the definition of $\lambda_{\mathbb{Q}}$, see [8]). We set $R = \bigoplus_{m\geq 0} H^0(X, mA)$ and $I = \bigoplus_{m\geq 0} H^0(X, mA \otimes \mathcal{I})$. Note that $R_{\mathbb{Q}}$ is noetherian by [8, Lemma 3.4]. Thus, as $I$ is a homogeneous ideal of $R$, $I_{\mathbb{Q}}$ is finitely generated as a $R_{\mathbb{Q}}$-module. Therefore, by [8, Lemma 2.2], there is a positive constant $B'$ such that

$$\lambda_{\mathbb{Q}}(H^0(X, mA \otimes \mathcal{I}), \|\cdot\|_{\sup}) \leq B'(m+1)^{\dim X(\dim X - 1)} \exp(-cm)$$

for all $m \geq 0$. Hence the claim follows by [8, Lemma 1.2]. $\square$

Let $\epsilon$ be an arbitrary positive number. Next let us see the following claim.

**Claim 7.1.2.** $\hat{H}^0_{\mathrm{quot}}(X|Y, m(\overline{A} - \overline{\mathcal{O}}(\epsilon))) \subseteq \operatorname{Image}(\hat{H}^0(X, m\overline{A}) \to H^0(X|Y, mA))$ *for* $m \gg 1$.

By Claim 7.1.1, if $m \gg 1$, then we can find a free basis $e_1, \ldots, e_N$ of $H^0(X, mA \otimes \mathcal{I})$ such that $\|e_i\|_{\sup} \leq e^{-\epsilon_0 m}$ for all $i$. We choose $e_{N+1}, \ldots, e_M \in H^0(X, mA)$ such that $e_{N+1}|_Y, \ldots, e_M|_Y$ form a free basis of $H^0(X|Y, mA)$. Then $e_1, \ldots, e_M$ form a free basis of $H^0(X, mA)$. Let $s \in \hat{H}^0_{\mathrm{quot}}(X|Y, m(\overline{A} - \overline{\mathcal{O}}(\epsilon)))$. Then there is $s' \in H^0(X, mA) \otimes \mathbb{R}$ such that $s'|_Y = s$ and $\|s'\|_{\sup} = \|s\|^{X|Y}_{\sup,\mathrm{quot}} \leq e^{-\epsilon m}$. We set $s' = \sum_{i=1}^{M} c_i e_i$ ($c_i \in \mathbb{R}$). Since

$$s'|_Y = \sum_{i=N+1}^{M} c_i e_i|_Y = s \in H^0(X|Y, mA),$$

we have $c_i \in \mathbb{Z}$ for all $i = N+1, \ldots, M$. Here we put

$$\tilde{s} = \sum_{i=1}^{N} \lceil c_i \rceil e_i + \sum_{i=N+1}^{M} c_i e_i.$$

Then $\tilde{s}|_Y = s$ and

$$\|\tilde{s}\|_{\sup} = \left\| s' + \sum_{i=1}^{N} (\lceil c_i \rceil - c_i) e_i \right\|_{\sup} \leq e^{-\epsilon m} + e^{-\epsilon_0 m} \operatorname{rk} H^0(X, mA),$$

which means that $\tilde{s} \in \hat{H}^0(X, m\overline{A})$ for $m \gg 1$. Therefore,

$$s \in \operatorname{Image}(\hat{H}^0(X, m\overline{A}) \to H^0(X|Y, mA)).$$

□

By the above claim, if we choose $\epsilon > 0$ such that $\overline{A} - \overline{\mathcal{O}}(\epsilon)$ is $Y$-big, then

$$\begin{aligned} \widehat{\operatorname{vol}}_{\mathrm{quot}}(X|Y, \overline{A} - \overline{\mathcal{O}}(\epsilon)) &\leq \liminf_{m\to\infty} \frac{\log \# \operatorname{Image}(\hat{H}^0(X, m\overline{A}) \to H^0(X|Y, mA))}{m^{d'}/d'!} \\ &\leq \limsup_{m\to\infty} \frac{\log \# \operatorname{Image}(\hat{H}^0(X, m\overline{A}) \to H^0(X|Y, mA))}{m^{d'}/d'!} \leq \widehat{\operatorname{vol}}_{\mathrm{quot}}(X|Y, \overline{A}). \end{aligned}$$

Hence the first assertion follows because, by Proposition 6.1,

$$\widehat{\operatorname{vol}}_{\mathrm{quot}}(X|Y, \overline{A} - \overline{\mathcal{O}}(\epsilon)) \geq \widehat{\operatorname{vol}}_{\mathrm{quot}}(X|Y, \overline{A}) - d'\epsilon \operatorname{vol}(X_{\mathbb{Q}}|Y_{\mathbb{Q}}, A_{\mathbb{Q}})$$

and $\epsilon$ can be taken as an arbitrary small number.

We further assume that $A_{\mathbb{Q}}$ is ample on $X_{\mathbb{Q}}$. Let us observe

$$\lim_{m\to\infty} \frac{\log \# \operatorname{Image}(\hat{H}^0(X, m\overline{A}) \to H^0(X|Y, mA))}{m^{d'}/d'!} > 0.$$

Let us choose a sufficiently large integer $n_0$ with the following properties:

(a) $H^0(X, n_0 A)$ has a free basis $\Sigma$ consisting of strictly small sections, which is possible by [8, Corollary 3.3 and Lemma 1.2].
(b) $\operatorname{Sym}^m(H^0(X, n_0 A) \otimes \mathbb{Q}) \to H^0(X, mn_0 A) \otimes \mathbb{Q}$ is surjective for all $m \geq 1$.
(c) $H^0(X, mn_0 A) \otimes \mathbb{Q} \to H^0(Y, mn_0 A|_Y) \otimes \mathbb{Q}$ is surjective for all $m \geq 1$.

We set $e^{-c} = \max\{\|s\|_{\sup} \mid s \in \Sigma\}$. Then $c > 0$. Moreover, we put

$$\Sigma_m = \{s_1 \otimes \cdots \otimes s_m \mid s_1, \ldots, s_m \in \Sigma\}.$$

Note that $\Sigma_m$ generates $H^0(X, mn_0 A) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space and that $\|s\|_{\sup} \leq e^{-mc}$ for all $s \in \Sigma_m$. Let $r_m$ be the rank of $H^0(Y, mn_0 A|_Y)$. Since $\{\, s|_Y \mid s \in \Sigma_m\}$ gives rise to a generator of $H^0(Y, mn_0 A|_Y) \otimes \mathbb{Q}$, we can find $s_1, \ldots, s_{r_m} \in \Sigma_m$ such that $\{\, s_1|_Y, \ldots, s_{r_m}|_Y\}$ forms a basis of $H^0(Y, mn_0 A|_Y) \otimes \mathbb{Q}$. We put

$$S_m = \{(a_1, \ldots, a_{r_m}) \in \mathbb{Z}^{r_m} \mid 0 \leq a_i \leq e^{cm}/r_m\}.$$

Then the map $S_m \to H^0(Y, mn_0 A|_Y)$ given by

$$(a_1, \ldots, a_{r_m}) \mapsto a_1 \ s_1|_Y + \cdots + a_{r_m} \ s_{r_m}|_Y$$

is injective. Moreover, for $(a_1, \ldots, a_{r_m}) \in S_m$,

$$\left\| \sum_{i=1}^{r_m} a_i s_i \right\|_{\sup} \leq \sum_{i=1}^{r_m} a_i \|s_i\|_{\sup} \leq \sum_{i=1}^{r_m} (e^{cm}/r_m) e^{-cm} = 1.$$

Hence

$$\#\,\mathrm{Image}(\hat{H}^0(X, mn_0\overline{A}) \to H^0(X|Y, mn_0 A)) \geq \#(S_m) \geq (e^{cm}/r_m)^{r_m}.$$

Thus the second assertion follows.

(3) Note that $\widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y, \overline{A}) \leq \widehat{\mathrm{vol}}(Y, \overline{A}\big|_Y)$. Thus, if $\widehat{\mathrm{vol}}(Y, \overline{A}\big|_Y) = 0$, then the assertion is obvious, so that we may assume that $\widehat{\mathrm{vol}}(Y, \overline{A}\big|_Y) > 0$. Let $\epsilon$ be an arbitrary positive number such that $(\overline{A} - \overline{\mathcal{O}}(\epsilon))\big|_Y$ is big. By (3) in Theorem 6.2, there is an integer $n_1 \geq 2$ such that if we set $\hat{H}^0(Y, n_1(\overline{A} - \overline{\mathcal{O}}(\epsilon))) = \{s_1, \ldots, s_l\}$, then

$$\liminf_{m\to\infty} \frac{\log \#\,\mathrm{CL}(\{s_1^{a_1} \otimes \cdots \otimes s_l^{a_l} \mid (a_1, \ldots, a_l) \in \Gamma_m\})}{m^{d'} n_1^{d'}/d'!} \geq \widehat{\mathrm{vol}}(Y, \overline{A} - \overline{\mathcal{O}}(\epsilon)\big|_Y) - \epsilon,$$

where $\Gamma_m = \{(a_1, \ldots, a_l) \in (\mathbb{Z}_{\geq 0})^l \mid a_1 + \cdots + a_l = m\}$. Note that $\|s_i\|_{Y,\sup} \leq e^{-n_1\epsilon}$ for all $i$. By [13, Theorem 3.3 and Theorem 3.5], if $m \gg 1$, then, for any $(a_1, \ldots, a_l) \in \Gamma_m$, there is $s(a_1, \ldots, a_l) \in H^0(X, mn_1 A)\otimes\mathbb{R}$ such that $s(a_1, \ldots, a_l)|_Y = s_1^{a_1}\otimes\cdots\otimes s_l^{a_l}$ and

$$\|s(a_1, \ldots, a_l)\|_{X,\sup} \leq e^{m\epsilon}\|s_1\|_{Y,\sup}^{a_1} \cdots \|s_l\|_{Y,\sup}^{a_l} \leq e^{-\epsilon m(n_1-1)} < 1,$$

which means that $s_1^{a_1} \otimes \cdots \otimes s_l^{a_l} \in \hat{H}^0_{\mathrm{quot}}(X|Y, mn_1\overline{A})$. Therefore,

$$\mathrm{CL}(\{s_1^{a_1} \otimes \cdots \otimes s_l^{a_l} \mid (a_1, \ldots, a_l) \in \Gamma_m\}) \subseteq \hat{H}^0_{\mathrm{quot}}(X|Y, mn_1\overline{A}).$$

Hence

$$\begin{aligned}
\widehat{\mathrm{vol}}(Y, \overline{A}\big|_Y) &\geq \widehat{\mathrm{vol}}_{\mathrm{quot}}(X|Y, \overline{A}) \\
&= \lim_{m\to\infty} \frac{\log \#\hat{H}^0_{\mathrm{quot}}(X|Y, mn_1\overline{A})}{(mn_1)^{d'}/d'!} \\
&\geq \liminf_{m\to\infty} \frac{\log \#\,\mathrm{CL}(\{s_1^{a_1} \otimes \cdots \otimes s_l^{a_l} \mid (a_1, \ldots, a_l) \in \Gamma_m\})}{m^{d'} n_1^{d'}/d'!} \\
&\geq \widehat{\mathrm{vol}}(Y, \overline{A} - \overline{\mathcal{O}}(\epsilon)\big|_Y) - \epsilon \geq \widehat{\mathrm{vol}}(Y, A|_Y) - \epsilon(d' \mathrm{vol}(Y_{\mathbb{Q}}, A_{\mathbb{Q}}) + 1),
\end{aligned}$$

as required. ■

**Corollary 7.2.** *Let $\overline{L} \mapsto \hat{H}^0_\bullet(X|Y, \overline{L})$ be an assignment of arithmetic restricted linear series from $X$ to $Y$. Then we have the following.*

(1) *If $X$ is generically smooth and $\overline{A}$ is an ample $C^\infty$-hermitian invertible sheaf on $X$, then*

$$\widehat{\mathrm{vol}}_\bullet(X|Y, \overline{A}) = \widehat{\mathrm{vol}}(Y, \overline{A}\big|_Y).$$

(2) *If $\overline{L}$ is a $Y$-big continuous hermitian invertible sheaf on $X$, then*

$$\widehat{\mathrm{vol}}_\bullet(X|Y, \overline{L}) > 0.$$

(3) *If $x \in \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X; Y)$, then $\widehat{\mathrm{vol}}_\bullet(X|Y, x) > 0$.*

*Proof.* (1) is a consequence of Theorem 7.1.

(2) As $\overline{L}$ is $Y$-big, there are a positive integer $n$ and an ample $C^\infty$-hermitian invertible sheaf $\overline{A}$ on $X$ such that $n\overline{L} \geq_Y \overline{A}$, so that, by (1) in Proposition 6.1 and (1) in Theorem 7.1,

$$n^{d'}\widehat{\mathrm{vol}}_\bullet(X|Y, \overline{L}) = \widehat{\mathrm{vol}}_\bullet(X|Y, n\overline{L}) \geq \widehat{\mathrm{vol}}_\bullet(X|Y, \overline{A}) > 0.$$

(3) If $x \in \widehat{\mathrm{Big}}_{\otimes\mathbb{R}}(X;Y)$, there are positive numbers $a_1, \ldots, a_r$ and $\overline{L}_1, \ldots, \overline{L}_r \in \widehat{\mathrm{Big}}(X;Y)$ such that $x = \overline{L}_1 \otimes a_1 + \cdots + \overline{L}_r \otimes a_r$. Hence, by (2) and Corollary 6.3,

$$\begin{aligned}\widehat{\mathrm{vol}}_{\bullet}(X|Y,x)^{\frac{1}{d'}} &\geq \widehat{\mathrm{vol}}_{\bullet}(X|Y,\overline{L}_1 \otimes a_1)^{\frac{1}{d'}} + \cdots + \widehat{\mathrm{vol}}_{\bullet}(X|Y,\overline{L}_r \otimes a_r)^{\frac{1}{d'}} \\ &= a_1\widehat{\mathrm{vol}}_{\bullet}(X|Y,\overline{L}_1)^{\frac{1}{d'}} + \cdots + a_r\widehat{\mathrm{vol}}_{\bullet}(X|Y,\overline{L}_r)^{\frac{1}{d'}} > 0.\end{aligned}$$

■

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