Congestion Games Viewed from M-convexity☆

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Abstract

Congestion games have extensively been studied till recently. It is shown by Fotakis (2010) that for every congestion game on an extension-parallel network, any best-response sequence reaches a pure Nash equilibrium of the game in n steps, where n is the number of players. We show that the fast convergence of best-response sequences results from M-convexity (of Murota (1996)) of the potential function associated with the game. We also give a characterization of M-convex functions in terms of greedy algorithms.

Keywords: Congestion games, discrete convexity, best-response dynamics, M-convex function, discrete convexity

1. Introduction

Congestion games (or finite potential games) have extensively been studied in the literature (e.g., [1, 8, 11, 12, 14]). An interesting special class of congestion games on networks, which is called *congestion games on extension-parallel networks*, is considered by Holzman and Law-yone [12] (see also [6, 14, 19]). Fotakis [8] showed a fast convergence behavior of the best-response dynamics for such games. That is, for every congestion game on an extension-parallel network, any best-response sequence converges to a pure Nash equilibrium of the game in n steps, where n is the number of the players. However, the underlying structure that brings the fast convergence has not yet fully been understood.

In the present paper we show that the fast convergence of best-response sequences results from M-convexity of the potential function associated with the game. M-convex functions were introduced by Murota [16, 17] based on Dress and Wenzel [4, 5]. The convex conjugate (or Legendre transform) of an M-convex function is called an L-convex function [17] (also see [7]). It has been recognized that L-/M-convex functions appear in various fields of operations research, economics, mathematics, and others (see, e.g., [2, 10, 18, 25]). We now find yet another interesting instance of M-convexity structure in congestion games.

The present paper is organized as follows. We briefly review congestion games in Section 2 and M-convex functions in Section 3. In Section 4 we examine the best-response dynamics for congestion games on extension-parallel networks and show that the essence of the fast convergence of the best-response dynamics is due to the M-convexity structure of the games. It is revealed in Section 5 that the best-response dynamics corresponds to a greedy algorithm for minimizing M-convex functions, which is a natural generalization of the greedy algorithm due to Dress and Wenzel [4] for valuated matroids. We also show a characterization of M-convex functions in terms of greedy algorithms.

2. Congestion Games

A *congestion game* [20] is a tuple $\Gamma = (N, A, (\mathcal{P}^{(i)} \mid i \in N), (c_a \mid a \in A))$, where

(a) N is a finite nonempty set of players,

(b) A is a set of resources,

(c) for each $i \in N$, $\mathcal{P}^{(i)}$ is a set of subsets of A, i.e., $\mathcal{P}^{(i)} \subseteq 2^A$,

(d) for each resource $a \in A$, $c_a : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a nondecreasing function satisfying $c_a(0) = 0$.

Each player $i \in N$ selects a set $P_i \in \mathcal{P}^{(i)}$, which forms a *strategy configuration* $\mathcal{P} = (P_i \mid i \in N)$. The *congestion* induced by $\mathcal{P}$ at resource $a \in A$ is

$$\nu_{\mathcal{P}}(a) = |\{i \in N \mid a \in P_i\}|. \tag{1}$$

Then, the incurred *individual cost* of player i is given by the sum of *congestion costs* over the resources that player i uses, i.e.,

$$\pi_i(\mathcal{P}) = \sum_{a \in P_i} c_a(\nu_{\mathcal{P}}(a)). \tag{2}$$

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For any player $i \in N$ and strategy $Q \in \mathcal{P}^{(i)}$ let $(\mathcal{P}_{-i}, Q)$ be the strategy configuration obtained from $\mathcal{P}$ by replacing P_i by Q.

A *potential function* associated with a strategy configuration $\mathcal{P} = (P_i \mid i \in N)$ of the congestion game Γ is introduced by Rosenthal [20] as follows:

$$\Phi(\mathcal{P}) = \sum_{a \in A} \hat{c}_a(\nu_{\mathcal{P}}(a)), \tag{3}$$

where $\hat{c}_a(\cdot)$ represents a function of accumulated congestion costs given by

$$\hat{c}_a(k) = \sum_{\ell=0}^{k} c_a(\ell) \qquad (\forall a \in A,\ \forall k \in \mathbb{Z}_{\geq 0}). \tag{4}$$

We then have the following fundamental relation

$$\Phi(\mathcal{P}_{-i}, Q) - \Phi(\mathcal{P}) = \pi_i(\mathcal{P}_{-i}, Q) - \pi_i(\mathcal{P}) \tag{5}$$

for any strategy configuration $\mathcal{P}$, $i \in N$, and $Q \in \mathcal{P}^{(i)}$. (A game having this property is called a *potential game*.) Hence local minima of potential Φ are exactly pure Nash equilibria of congestion game Γ, as shown by Rosenthal [20]. It is shown ([15]) that the class of congestion games coincides with the class of finite potential games.

3. M-convex Functions

The concept of M-convex function was introduced by Murota ([16, 17]). We briefly review some fundamental properties of M-convex functions.

Let W be a finite nonempty set and f be a function $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$. The *effective domain* $\mathrm{dom}(f)$ of f is defined by $\mathrm{dom}(f) = \{x \in \mathbb{Z}^W \mid f(x) < +\infty\}$. The *epigraph* $\mathrm{epi}(f)$ of f is defined by

$$\mathrm{epi}(f) = \{(x, \beta) \mid x \in \mathrm{dom}(f),\ f(x) \leq \beta \in \mathbb{R}\}. \tag{6}$$

We call f *convex-extensible* if the lower envelope of the convex hull of $\mathrm{epi}(f)$ gives a convex function $\bar{f} : \mathbb{R}^W \to \mathbb{R} \cup \{+\infty\}$ such that $\bar{f}(x) = f(x)$ for all $x \in \mathbb{Z}^W$. The function $\bar{f}$ is called the *convex extension* of f.

A function $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ is called an *M-convex function* if its effective domain is nonempty and it satisfies the exchange axiom:

(M-EXC) $\forall x, y \in \mathbb{Z}^W$, $\forall u \in W$ with $x(u) > y(u)$, $\exists v \in W$ with $x(v) < y(v)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v), \tag{7}$$

where χ_u for $u \in W$ is a unit vector in $\mathbb{Z}^W$ such that $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \neq u$, and we allow $+\infty \geq +\infty$.

The following is a well-known characterization of M-convex functions (see [16, 17] and also [9, 23]).

Proposition 1. *Let $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ be a convex-extensible function with a nonempty bounded effective domain. Let $\bar{f}$ be the convex extension of f. Then f is an M-convex function if and only if for every non-vertical edge L of the epigraph $\mathrm{epi}(\bar{f})$ the direction vector of the line segment obtained by the projection $((x, \beta) \mapsto x$ onto $\mathbb{R}^W)$ of L belongs to $\{\chi_u - \chi_v \mid u, v \in W,\ u \neq v\}$.* Here any two direction vectors are identified with each other if one is a non-zero scalar multiple of the other. (See Figure 1 for an example of M$^\natural$-concave function, a variant of M-concave function, which admits direction vectors of $\{\pm\chi_v \mid v \in W\}$ in addition to $\{\chi_u - \chi_v \mid u, v \in W,\ u \neq v\}$. Note that M-/M$^\natural$-concave functions are negative of M-/M$^\natural$-convex functions.)

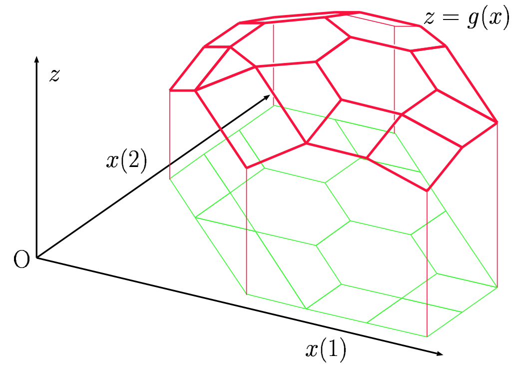

Figure 1: An M$^\natural$-concave function g ([9, Figure 17.4]).

An M-convex function f is a special case of M$^\natural$-convex function whose effective domain lies on a hyperplane $x(W)$ ($\equiv \sum_{v \in W} x(v)$) $= k$ for some integer k. We call the integer k the *rank* of f.

4. Congestion Games on Extension-parallel Networks

Suppose that we are given a directed graph $G = (V, A)$ with a vertex set V, an arc set A, and specified (distinct) source s and sink t in V. Arc set A is regarded as a set of resources. Consider a congestion game $\Gamma = (N, A, (\mathcal{P}^{(i)} \mid i \in N), (c_a \mid a \in A))$, where each $\mathcal{P}^{(i)}$ is a set of elementary directed paths from source s to sink t (st-paths) in graph $G = (V, A)$ and every st-path is regarded as a set of the arcs (resources) lying on the path.

An interesting special class of congestion games on networks is the class of congestion games on extension-parallel networks ([12]). An *extension-parallel network* with a source and a sink is constructed by finitely many repeated operations of *source/sink extension* and *parallel join* defined below (see Figure 2), starting from finitely many networks, each consisting of a single arc whose tail and head are, respectively, a source and a sink.

(1) (source extension): Given a network G, construct a new network G' by adding a new arc a to a source of G in such a way that the head of a is the source of G and the tail of a is the source of the new G'.

(2) (sink extension): Given G, construct a new network G' by adding a new arc a to a sink of G in such a way that the tail of a is the sink of G and the head of a is the sink of the new G'.

(3) (parallel join): Given two distinct networks G_1 and G_2, construct a new network G' by identifying their respective sources as a source of the new G' and by identifying their respective sinks as a sink of the new G'.

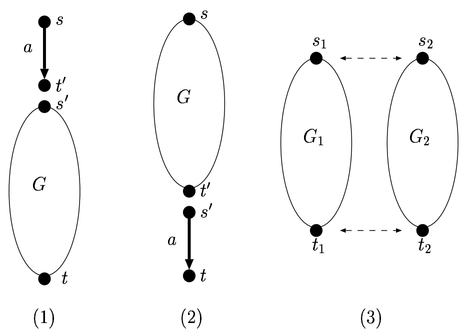

Figure 2: (1) Source extension, (2) Sink extension, and (3) Parallel join.

In the sequel we consider a symmetric game Γ such that every player's strategy set $\mathcal{P}^{(i)}$ is the set $\mathcal{P}^{\mathrm{all}}$ of all paths from source s to sink t in an extension-parallel network G.

Theorem 2 (Fotakis [8]). *For any symmetric congestion game $\Gamma = (N, A, (\mathcal{P}^{(i)}(= \mathcal{P}^{\mathrm{all}}) \mid i \in N), (c_a \mid a \in A))$ on an extension-parallel network, any sequence of strategies generated by the procedure,* Procedure(Best_Response)*, given below reaches a pure Nash equilibrium in $n(= |N|)$ steps.*

Procedure(Best_Response)

1. Start from any strategy configuration $\mathcal{P} = (P_i \mid i \in N)$.
   Let $(i_1, i_2, \cdots, i_n)$ be any permutation of N.
2. For each $i = i_1, i_2, \cdots, i_n$ do the following.
   Let $\hat{P} \in \mathcal{P}^{\mathrm{all}}$ be a minimizer of $\Phi((\mathcal{P}_{-i}, P))$ in $P \in \mathcal{P}^{\mathrm{all}}$.
   Put $\mathcal{P} \leftarrow (\mathcal{P}_{-i}, \hat{P})$.
3. Return $\mathcal{P}$.

It should be noted that minimizers of $\Phi((\mathcal{P}_{-i}, P))$ in $P \in \mathcal{P}^{\mathrm{all}}$ are exactly minimizers of the individual cost $\pi_i(\mathcal{P}_{-i}, Q)$ in $P \in \mathcal{P}^{\mathrm{all}}$, due to (5). Hence each choice of a minimizer $\hat{P} \in \mathcal{P}^{\mathrm{all}}$ for $i \in N$ in Step 2 is called a best response of player i.

More precisely, Fotakis [8] shows that, for extension-parallel networks, whenever the strategy of a player $i \in N$ is its best response with respect to the current state, then even after some other player changes its strategy to a best response strategy, the strategy of player i will remain a best response strategy. One way to rephrase Fotakis' result, is by stating that Procedure(Best_Response) always leads to a pure Nash equilibrium, no matter what permutation is chosen in the first step.

We will now show how the correctness of the procedure, Procedure(Best_Response), can be derived from results on M-convexity, thus implying Fotakis' result.

Each strategy configuration $\mathcal{P} = (P_i \mid i \in N)$ is made to correspond to a vector $x_{\mathcal{P}} \in \mathbb{Z}^{\mathcal{P}^{\mathrm{all}}}$ given by

$$x_{\mathcal{P}} = \sum_{i \in N} \chi_{P_i}, \tag{8}$$

where χ_{P_i} is a unit vector in $\mathbb{Z}^{\mathcal{P}^{\mathrm{all}}}$ such that $\chi_{P_i}(P) = 1$ if $P = P_i$ and $= 0$ otherwise. (Note that $\mathcal{P}$ is not uniquely determined by $x_{\mathcal{P}}$ in general.)

For each arc $a \in A$ denote by Q_a the set of paths $P \in \mathcal{P}^{\mathrm{all}}$ containing arc a. Then we have the following

Lemma 3. *The family of path sets Q_a $(a \in A)$ is laminar, i.e., for any distinct a, $a' \in A$ we have $Q_a \cap Q_{a'} = \emptyset$, $Q_a \subseteq Q_{a'}$, or $Q_a \supseteq Q_{a'}$.*

(Proof) Suppose that during the construction of the extension-parallel graph $G = (V, A)$ arc a belongs to a graph G_1 and arc a' to another graph G_2 and the operation of parallel join is made for G_1 and G_2, then there is no st-path that contains both a and a'. Hence $Q_a \cap Q_{a'} = \emptyset$. On the other hand, if arc a belongs to a graph G_1 and arc a' is used for a source/sink-extension of G_1, then we have $Q_a \subseteq Q_{a'}$. □

Remark 1. The laminarity structure has been recognized as a rooted tree structure ([11, 13]) such that the set of non-root vertices of the rooted tree is the resource set (arc set) A of G and the set of non-root vertices of every path in the tree from the root to a leaf is the arc set of a respective path in G from the source to the sink. The tree structure described in [11, 13] is closely related to the Tutte decomposition tree of a 2-connected graph into 3-connected components, cycles, and graphs of parallel arcs (see [26]), where only the latter two kinds of components appear (even for series-parallel graphs). Here, for a given extension-parallel (or, more generally, series-parallel) graph G with source s and sink t, we should define a graph G' obtained by adding to G a reference arc from s to t and consider the Tutte decomposition tree of G'.

For a given strategy configuration $\mathcal{P} = (P_i \mid i \in N)$, using the vector $x_{\mathcal{P}} \in \mathbb{Z}^{\mathcal{P}^{\mathrm{all}}}$ in (8), the potential function value $\Phi(\mathcal{P})$ in (3) is equal to

$$\Phi(\mathcal{P}) = \sum_{a \in A} \hat{c}_a(\nu_{\mathcal{P}}(a)) = \sum_{a \in A} \hat{c}_a(x_{\mathcal{P}}(Q_a)) \equiv \tilde{\Phi}(x_{\mathcal{P}}) \tag{9}$$

regarded as a function, denoted by $\tilde{\Phi}(\cdot)$, in $x_{\mathcal{P}}$, where $x_{\mathcal{P}}(Q_a) = \sum_{P \in Q_a} x_{\mathcal{P}}(P)$ for each $a \in A$. Note that $\tilde{\Phi}(x_{\mathcal{P}})$ is the sum over all $a \in A$ of scalar, discrete convex functions $\hat{c}_a(\cdot)$ on integers, and the sets Q_a $(\forall a \in A)$ form a laminar family. It follows that the function $\tilde{\Phi}(x)$ in $x \in \mathbb{Z}^{\mathcal{P}^{\mathrm{all}}}$ is what is called a *laminar convex* function with its effective domain $\Delta_n = \{x \in \mathbb{Z}^{\mathcal{P}^{\mathrm{all}}}_{\geq 0} \mid x(\mathcal{P}^{\mathrm{all}}) = n\}$, where $n = |N|$. This implies the following (see [3, 17]).

Lemma 4. *The function $\tilde{\Phi}(x)$ with its effective domain Δ_n is an M-convex function.*

As will be discussed in the next section, the M-convexity of $\tilde{\Phi}$ implies the validity of Theorem 2, the n step convergence of the best-response sequence, due to Fotakis [8]. Procedure(Best_Response) can be regarded as a specialized version of a greedy algorithm for M-convex functions.

5. A Greedy Algorithm for M-Convex Functions

We give a greedy algorithm for M-convex functions, which is a natural generalization of the greedy algorithm given by Dress and Wenzel [4] for valuated matroids. The greedy algorithm given in this section is slightly different from the existing algorithms given by Shioura [21, 22] and Tamura [24] (also see

[17]). We show the validity of our greedy algorithm by adapting Dress and Wenzel's proof in [4] for valuated matroids.

For a finite nonempty set W let us consider an M-convex function $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ having a nonempty bounded effective domain $\mathrm{dom}(f) = \{x \in \mathbb{Z}^W \mid f(x) < +\infty\}$. Suppose that the effective domain is included in the nonnegative orthant, i.e., $\mathrm{dom}(f) \subset \mathbb{Z}^W_{\geq 0}$ and that $x(W)$ $(= \sum_{v \in W} x(v)) = n \geq 1$ for all $x \in \mathrm{dom}(f)$, where we recall that n is equal to the rank of f. Also define $N = \{1, 2, \cdots, n\}$. It should be noted that W corresponds to the set $\mathcal{P}^{\mathrm{all}}$ of all st-paths and $\sigma(i)$ appearing in the algorithm given below corresponds to player i's initial strategy $P_i \in \mathcal{P}^{\mathrm{all}}$.

Greedy Algorithm

1. Start from any $x = x_0 \in \mathrm{dom}(f)$.
   Choose any mapping $\sigma : N \to W$ such that $x = \sum_{i \in N} \chi_{\sigma(i)}$.
2. For each $i = 1, 2, \cdots, n$ do the following.
   Find an element w^* of W such that
   (*) $f(x - \chi_{\sigma(i)} + \chi_{w^*}) = \min\{f(x - \chi_{\sigma(i)} + \chi_w) \mid w \in W\}$.
   Put $x \leftarrow x - \chi_{\sigma(i)} + \chi_{w^*}$.
3. Return x (a minimizer of f).

Remark 2. We have assumed that the effective domain $\mathrm{dom}(f)$ is included in the nonnegative orthant $\mathbb{Z}^W_{\geq 0}$, but this is equivalent to assuming that we are given a lower bound vector $b \in \mathbb{Z}^W$ of $\mathrm{dom}(f)$ such that $b \leq x$ for all $x \in \mathrm{dom}(f)$. Since the translation $x \mapsto x - b$ keeps M-convexity, we can apply the algorithm to the function $f(x + b)$ in x.

Theorem 5. *The greedy algorithm described above computes a minimizer of any M-convex function f with $\mathrm{dom}(f) \subseteq \mathbb{Z}^W_{\geq 0}$ in n steps, where n is the rank of f.*

(Proof) Let x_i be the x obtained after the ith execution of Step 2 for $i = 1, 2, \cdots, n$. Also denote by w_i^* the element $w^* \in W$ found at the ith execution of Step 2. It suffices to prove the following local optimality (see [17]):

$$\forall u,\ v \in W: \quad f(x_n - \chi_u + \chi_v) \geq f(x_n). \tag{10}$$

We show that for any M-convex function $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ of rank $n \geq 1$, the greedy algorithm obtains $x = x_n$ satisfying (10), by induction on the rank n of f, where recall that the effective domain of f lies on the hyperplane $x(W) = n$. Note that we fix W in the following arguments.

For any M-convex function of rank $n = 1$, (10) holds. Hence, let k be an integer with $k \geq 1$ and suppose that for any M-convex function of rank $n = k$ the greedy algorithm obtains $x = x_n$ satisfying (10), i.e., the greedy algorithm finds a minimizer of any M-convex function f when $x(W) = k$ for all $x \in \mathrm{dom}(f)$.

Now suppose $n = k + 1$. Since f remains to be M-convex by the restriction of its effective domain $B = \mathrm{dom}(f)$ to $B_1 = B \cap \{x \in \mathbb{Z}^W \mid x(\sigma(n)) \geq 1\}$, it follows from the induction hypothesis that

$$x_{n-1} \text{ is a minimizer of } f \text{ restricted on } B_1. \tag{11}$$

Let $y \in \mathrm{dom}(f)$ be a minimizer of f. Notice that if $y \in B_1$ then x_n is indeed a minimizer of f since $f(x_n) \leq f(x_{n-1}) = f(y)$, where the equality follows from (11). Thus, suppose $y(\sigma(n)) = 0$. Then $x_{n-1}(\sigma(n)) > y(\sigma(n))$. By the exchange axiom of M-convex functions there exists $j \in N \setminus \{\sigma(n)\}$ such that $y(j) > x_{n-1}(j)$ and

$$f(x_{n-1}) + f(y) \geq f(x_{n-1} - \chi_{\sigma(n)} + \chi_j) + f(y + \chi_{\sigma(n)} - \chi_j). \tag{12}$$

Since $y + \chi_{\sigma(n)} - \chi_j \in B_1$, it follows from (11) that

$$f(y + \chi_{\sigma(n)} - \chi_j) \geq f(x_{n-1}). \tag{13}$$

Also, since $x_{n-1} - \chi_{\sigma(n)} + \chi_j$ is a candidate for x_n, we get

$$f(x_{n-1} - \chi_{\sigma(n)} + \chi_j) \geq f(x_n). \tag{14}$$

It follows from (12)–(14) that $f(y) \geq f(x_n)$, i.e., x_n is a minimizer of f. $\square$

Remark 3. Consider N as the set of players and identify W with the set $\mathcal{P}^{\mathrm{all}}$ of all st-paths in an extension-parallel network G, and f with $\tilde{\Phi}$ in (9). Then the greedy algorithm for f becomes Procedure(Best_Response). It may also be worth mentioning that the number n of steps is independent of the size of W.

Remark 4. It follows from results in [22, 24] that as an algorithm for minimizing an M-convex function f we can skip those is which satisfy $\sigma(i) \in \{w_\ell^* \mid \ell = 1, \cdots, i-1\}$, which is also implicitly implied by the above proof of Theorem 5. This corresponds to the rule: keep the present strategy P_i if it is a best one even if there exist multiple (other) best strategies. It is a natural restriction of the behavior of the players in the case of congestion games.

Similarly as in [4] for valuated matroids, we have a converse of Theorem 5 and hence show the equivalence between the greediness and M-convexity as follows.

Theorem 6. *Let $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ be a function having a nonempty bounded effective domain $B \subset \mathbb{Z}^W_{\geq 0}$. For any $d \in \mathbb{R}^W$ define $f^d : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ by*

$$f^d(x) = f(x) + \langle d, x \rangle \qquad (\forall x \in \mathbb{Z}^W), \tag{15}$$

where $\langle d, x \rangle = \sum_{u \in W} d(u)x(u)$. Suppose that f is convex-extensible on $\mathbb{R}^W$. Then, f is an M-convex function if and only if for every $d \in \mathbb{R}^W$, the Greedy Algorithm minimizes the function f^d.

(Proof) Since adding a linear function keeps M-convexity, Theorem 5 implies the only-if part, and hence we show the if part.

Since f is convex-extensible, denoting by $\bar{f}$ the convex extension of f, it suffices to prove that every non-vertical edge vector of the epigraph of $\bar{f}$ projected on $\mathbb{R}^W$ belongs to $\{\chi_u - \chi_v \mid u, v \in W,\ u \neq v\}$, due to Proposition 1.

Let L be an arbitrary non-vertical edge of the epigraph of $\bar{f}$ and let $\hat{L}$ be the projection of L on $\mathbb{R}^W$. Also let $x_1, x_2 \in B$ be the end points of $\hat{L}$. Let $z \in B$ be the point in $(\hat{L} \setminus \{x_1\}) \cap B$ nearest to x_1. Then there exists a vector $d \in \mathbb{R}^W$ such that x_1 is the unique minimizer of f^d and $\{x \in B \mid f^d(x) \leq f^d(z)\} = \{z, x_1\}$. Hence, starting from z, the Greedy Algorithm for f^d must move to x_1 by the first improving step. By the definition of the Greedy Algorithm, the direction of the movement from z to x_1, which is a direction vector of $\hat{L}$, belongs to $\{\chi_u - \chi_v \mid u, v \in W,\ u \neq v\}$. $\square$

The proof of the if part of Theorem 6 can easily be adapted to that of the following fundamental fact on *generic* greedy algorithms. We call a transformation from $x \in \mathbb{Z}^W$ to $x - \chi_u + \chi_v$ for $u, v \in W$ a *basic local transformation*.

Theorem 7. *Let $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ be a convex-extensible function having a nonempty bounded* $\mathrm{dom}(f)$*. Suppose that there exists an algorithm* $\mathbf{P}$ *such that for every $d \in \mathbb{R}^W$ and every initial solution $x_0 \in \mathrm{dom}(f)$ Algorithm* $\mathbf{P}$ *finds a finite sequence of solutions $(x_0, x_1, \cdots, x_k)$ for some integer $k \geq 0$ satisfying*

(a) $f^d(x_0) \geq f^d(x_1) \geq \cdots \geq f^d(x_k)$.

(b) *For $i = 1, \cdots, k$, each x_i is obtained by a basic local transformation of x_{i-1}.*

(c) *x_k is a minimizer of f^d.*

Then, f is an M-convex function.

It follows from the present theorem that the validity of algorithms based on repeated basic local transformations such as those given by [21, 22, 24] implies M-convexity of the objective functions.

6. Concluding Remarks

We have revealed the M-convexity structure of congestion games on extension-parallel networks, which explains the fast convergence of the best-response dynamics shown by Fotakis [8]. We believe that there are phenomena in congestion games in general that can be viewed from M-convexity or other variants of discrete convexity (cf. [1, 27]). Finding and investigating such instances are left for future research.

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