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Unitary matrices and random permutations: conjecture and degenerated Laplacian

By

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Abstract

We propose a conjecture that a unitary matrix U of size n determines a probability p on a symmetric group S_n in such a manner that $|\det U(A,B)|^2 = \sum_{\sigma(A)=B} p(\sigma)$ for any subsets A and B of the index set. As a preliminary study we introduce a degenerated Laplacian on S_n and prove some of its properties, mainly for $n = 3$ and $n = 4$.

§ 1. Introduction

In [4] we studied the ergodic properties of translation invariant fermion(or determinantal) point processes on $\mathbb{Z}$. In particular, we gave upper and lower estimates of the metric entropy in term of the spectrum of the associated convolution operator. The method of the proof strongly suggested a conjecture stated below. It will presumably be a key to further study of fermion point processes and their Palm probabilities. Also, some people may speculate that it might be a clue to understand the probabilistic interpretation of the wave function in quantum mechanics in terms of random homeomorphisms(or, hopefully, random diffeomorphisms).

First of all, we introduce some notations. Let $U = (u_{jk})_{j,k=1}^{n}$ be a unitary matrix of size n. For subsets A and B of the index set $\Lambda = \{1, 2, \ldots, n\}$ we write

$$U(A,B) = (u_{a_j b_k})_{1\le j\le l,\ 1\le k\le m}$$

if $A = \{a_1 < \cdots < a_l\}$ and $B = \{b_1 < \cdots < b_m\}$. For simplicity of description we set $\det U(A,B) = 0$ unless $1 \le l = m \le n$ and $\det U(\emptyset,\emptyset) = 1$ for $m = 0$.

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Conjecture. *For a given unitary matrix* $U = (u_{jk})_{j,k=1}^n$ *there exists a probability* p *on the symmetric group* S_n *such that*

$$|\det U(A,B)|^2 = \sum_{\sigma(A)=B} p(\sigma)$$

for any subsets A *and* B *of* $\Lambda = \{1,2,\ldots,n\}$ *with same cardinality* $|A| = |B|$.

Remark. *The probability measure* p*, if exists, is not uniquely determined by nature except for* $n = 2$ *and for certain special cases such as* U *is the unit matrix. See Appendix.*

If the subsets A and B are restricted to singletons, then the conjecture is true. Indeed, the matrix $Q = (q_{jk})_{j,k=1}^n$ with $q_{jk} = |u_{jk}|^2$ is doubly stochastic, i.e., both Q and its transposed $Q^{\mathbf{T}}$ are stochastic matrices: $\sum_k q_{jk} = \sum_k q_{kj} = 1$ for all j. For the doubly stochastic matrix the following theorem is well-known :

Theorem 1.1. *If* $Q = (q_{jk})_{j,k=1}^n$ *is doubly stochastic, then there exists a probability* p *on the symmetric group* S_n *such that*

$$q_{jk} = \sum_{\sigma(j)=k} p(\sigma).$$

In the other words,

$$Q = \sum_{\sigma \in S_n} p(\sigma) E(\sigma)$$

where $E(\sigma)$ *stands for the doubly stochastic matrix associated with permutation* σ*:*

$$E(\sigma)_{jk} = 1(\sigma(j) = k) = \begin{cases} 1 & \text{if } \sigma(j) = k \\ 0 & \text{otherwise.} \end{cases}$$

Here notice that $E(\sigma)E(\tau) = E(\tau\sigma), \quad \sigma, \tau \in S_n$.

A probabilistic interpretation of Theorem 1.1 is that a symmetric finite Markov chain can always be lifted to an i.i.d.(independent identically distributed) sequence of random permutations. The idea is extended to more general cases. For instance, a diffusion process on a manifold can be lifted to a stochastic flow of diffeomorphisms under certain mild conditions. In the diffusion case the joint distribution of the sample paths starting at several points are known to be completely governed by the joint distribution at two points or by the two point correlation (cf. H.Kunita [1]).

Also the theorem above implies the following.

Corollary 1.2. *Write for $k = 1, 2, \ldots, n$*

$$\Lambda_k = \{A \subset \Lambda : \ |A| = k\}$$

and

$$q(A, B) = |\det U(A, B)|^2, \qquad A, B \in \Lambda_k. \tag{1.1}$$

Then for each $k = 1, 2, \ldots, n$ the matrix $(q(A, B))_{A,B\in\Lambda_k}$ is doubly stochastic and so there exists a probability measure $p^{(k)}$ on the symmetric group $S(\Lambda_k)$ on Λ_k such that

$$q(A, B) = \sum_{\Sigma\in S(\Lambda_k)} p^{(k)}(\Sigma).$$

Unfortunately, it seems (at least to the author) to be very difficult to reduce $p^{(k)}$ on $S(\Lambda_k)$ to p on $S_n = S(\Lambda)$. In below we would like to discuss some nature of the above conjecture.

In the subsequent sections we need some notations.

- $\Lambda = \{1, 2, \ldots, n\}$.
- $\mathcal{C}(S_n)$ stands for the space of functions on S_n with inner product
$$\langle f, g\rangle = \sum_{\sigma\in S_n} f(\sigma)\overline{g(\sigma)}, \quad f, g \in \mathcal{C}(S_n) \tag{1.2}$$
- $\mathcal{D}_n$ stands for the space of functions on the set $\{(A, B); A, B \subset \Lambda\, |A| = |B|\}$ which satisfy the condition that there exists a constant c_ϕ such that
$$\sum_{B\subset\Lambda} \phi(A, B) = \sum_{B\subset\Lambda} \phi(B, A) = c_\phi \ \text{ for any } A \subset \Lambda.$$
Obviously, the space $\mathcal{D}_n$ is spanned by doubly stochastic matrices $q(A, B)$ indexed by the subsets A and B of Λ. We define an inner product on $\mathcal{D}_n$ by
$$\langle \phi, \psi\rangle = \sum_{A,B\in\Lambda} \phi(A, B)\overline{\psi(A, B)}, \quad \phi,\ \psi \in \mathcal{D}_n \tag{1.3}$$

§ 2. Consistency conditions

If Conjecture is true, then, the function $q(A, B)$, $A, B \subset \Lambda$ should satisfy certain conditions required by the probability expression in p. For instance, $q(A, B)$ must be a doubly stochastic matrix as we already used in Corollary 1.2. The shortest proof may be as follows.

Consider the k-hold exterior product $\wedge^k U$. Then, it is a unitary operator on $\wedge^k \mathbb{C}^{S_n}$ since

$$(\wedge^k U)^*(\wedge^k U) = \wedge^k (U^* U) = I.$$

Moreover,

$$\det U(A,B) = \langle (\wedge^k U) e_B, e_A \rangle$$

where e_A's stand for the canonical basis, namely,

$$e_A = e_{a_1} \wedge e_{a_2} \wedge \cdots \wedge e_{a_k}, \qquad A = \{a_1 < \cdots < a_k\} \quad \text{etc.}$$

Hence, $q(A,B) = |\det U(A,B)|^2$ is a doubly stochastic matrix.

In this section we focus on the following:

Consistency conditions:

(i) For any subset A and $\tilde{B}$ of $\Lambda = \{1, 2, \ldots, n\}$ with $|A| \le |\tilde{B}|$,

$$\sum_{B:\ B \subset \tilde{B}} q(A,B) = \sum_{\tilde{A}:\ \tilde{A} \supset A} q(\tilde{A}, \tilde{B}).$$

(ii) For any subset $\tilde{A}$ and B of $\Lambda = \{1, 2, \ldots, n\}$ with $|\tilde{A}| \ge |B|$,

$$\sum_{A:\ A \subset \tilde{A}} \phi(A,B) = \sum_{\tilde{B}:\ \tilde{B} \supset B} \phi(\tilde{A}, \tilde{B}).$$

Indeed, Conjecture requires that

$$\begin{aligned}\sum_{B:\ B \subset \tilde{B}} q(A,B) &= \sum_{B:\ B \subset \tilde{B}} \sum_{\sigma:\ \sigma(A) = B} p(\sigma) = \sum_{\sigma:\ \sigma(A) \subset \tilde{B}} p(\sigma) \\ &= \sum_{\tilde{A}:\ \tilde{A} \supset A} \sum_{\sigma:\ \sigma(\tilde{A}) = \tilde{B}} p(\sigma) = \sum_{\tilde{A}:\ \tilde{A} \supset A} q(\tilde{A}, \tilde{B}).\end{aligned}$$

Hence, (i) must hold. Similarly, (ii) must hold.

It may not be so obvious that $q(A,B) = |\det U(A,B)|^2$ satisfies these conditions, such as

$$\sum_{l:\ l \ne k} q(\{i,j\}, \{k,l\}) = q(i,k) + q(j,k).$$

One can prove the condition in such a simple case, for instance, by using the relation $\|u \wedge v\|^2 = \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2$ among the norms of 2-vectors u and v, the norms of their exterior products $u \wedge v$ and their inner-product $\langle u, v \rangle$. A general proof in this direction may be possible but our verification will be done by introducing the following operator.

Definition 2.1. Define an operator $S : \mathcal{C}(S_n) \to \mathcal{D}_n$ by

$$Sf(A,B) = \sum_{\sigma \in S_n, \sigma(A)=B} f(\sigma), \qquad f \in \mathcal{C}(S_n). \tag{2.1}$$

The well-definedness of $S : \mathcal{C}(S_n) \to \mathcal{D}_n$ follows if we observe that

$$\sum_{B'} Sf(A,B') = \sum_{B'} \sum_{\sigma:\ \sigma(A)=B'} f(\sigma) = \sum_{\sigma} f(\sigma) = \sum_{A'} \sum_{\sigma:\ \sigma(A')=B} f(\sigma) = \sum_{A'} Sf(A',B)$$

for any $A, B \subset \Lambda$. In other words, $Sf \in \mathcal{D}_n$ with

$$c_{Sf} = \sum_{\sigma} f(\sigma).$$

The system of equations we want to solve can be written as

$$p \geq 0 \ \text{ and } \ Sp = q. \tag{2.2}$$

Here we can omit the normalization condition for p since it is automatic.

Firstly, our **consistency conditions** hold on the range of S.

Lemma 2.2 (consistency lemma). *Let $\phi \in \mathcal{D}_n$ and assume that $\phi = Sf$ for some $f \in \mathcal{C}(S_n)$. Then the* **consistency conditions** *hold for ϕ in place of q.*

Proof. We only show the condition (i) because the proof of (ii) is similar. Take $f \in \mathcal{C}(S_n)$ so that $\phi = Sf$. Then,

$$\begin{aligned}\sum_{B \subset \tilde{B}} \phi(A,B) &= \sum_{B \subset \tilde{B}} \sum_{\sigma \in S_n, \sigma(A)=B} f(\sigma) = \sum_{\sigma \in S_n, \sigma(A) \subset \tilde{B}} f(\sigma) \\ &= \sum_{\tilde{A} \supset A} \sum_{\sigma \in S_n, \sigma(\tilde{A})=\tilde{B}} f(\sigma) = \sum_{\tilde{A} \supset A} \phi(\tilde{A}, \tilde{B}).\end{aligned}$$

□

Next we show that $q(A,B) = |\det U(A,B)|^2$ lies in the range of S.

We need a few lemmas .

Lemma 2.3. *For a square matrix $X = (x_{ij})_{i,j=1}^n$ define a function $f_X \in \mathcal{C}(S_n)$ by*

$$f_X(\sigma) = sgn(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}. \tag{2.3}$$

Then, for any subsets A and B of the index set Λ

$$Sf_X(A,B) = \det X(A,B) \det X(A^c, B^c). \tag{2.4}$$

Proof. Firstly we prove the assertion when $A = B$. We identify $\sigma_1 \in S(A)$ with $\sigma_1 \otimes id_{A^c} \in S_n$.

$$
\begin{aligned}
& Sf_X(A,A) \\
&= \sum_{\sigma\in S_n,\sigma(A)=A} f_X(\sigma) \\
&= \sum_{\sigma\in S(A),\sigma(A)=A} sgn(\sigma)\left(\prod_{i\in A} x_{i,\sigma(i)}\right)\left(\prod_i x_{i\sigma(i)}\right) \\
&= \sum_{\sigma_1\in S(A),\sigma_2\in S(A^c)} sgn(\sigma_1\sigma_2)\left(\prod_{i\in A} x_{i,\sigma(i)}\right)\left(\prod_i x_{i\sigma(i)}\right) \\
&= \left(\sum_{\sigma_1\in S(A)} sgn(\sigma_1)\prod_{i\in A} x_{i,\sigma(i)}\right)\left(\sum_{\sigma_2\in S(A^c)} sgn(\sigma_2)\prod_i x_{i\sigma(i)}\right) \\
&= \det X_{AA}\det X_{A^cA^c}.
\end{aligned}
$$

Now we proceed to the proof of the general case. If $|A| \neq |B|$ then the both hand sides vanish so that the equality is trivial.

If $|A| = |B|$ we take a permutation τ such that $\tau(A) = B$ and write

$$X^\tau = (x_{i\tau(j)}) = UE(\tau),$$

where $E(\tau)$ is the matrix representing the action of τ, as before:

$$E(\tau)_{ij} = 1[\tau(i) = j].$$

Then,

$$Sf_X(A,B) = Sf_{X^\tau}(A,A) = \det X^\tau(A,A)\det X^\tau(A^c,A^c) = \det X(A,B)\det X(A^c,B^c).$$

$\square$

Recall generalized Cramer's cofactor formula for an invertible matrix .

Lemma 2.4. *If X is invertible, then for any subsets A and B*

$$\det X(B^c,A^c) = \varepsilon(A,B)(\det X)(\det X^{-1}(A,B)) \tag{2.5}$$

where $\varepsilon(A,B) = \varepsilon(A)\varepsilon(B)$ and $\epsilon(A) = \pm 1$ is defined by the relation

$$e_\Lambda = \varepsilon(A)e_A \wedge e_{A^c}.$$

Proof. The above formula does not seem to be found in standard textbooks so that we will give a proof here.

On one hand, recall the Laplace expansion of determinant by minor determinants :

$$\sum_{B:\ |B|=k} \varepsilon(A,B)\det X(A,B)\det X(C^c,B^c) = \delta(A,C)\det X \qquad \text{for any } A,$$
$$\sum_{A:\ |A|=k} \det X(A,B)\det X(A^c,C^c) = \delta(B,C)\det X \qquad \text{for any } B$$

Indeed,

$$\begin{aligned}\det X &= \langle(\wedge^n X)e_\Lambda, e_\Lambda\rangle = \langle\varepsilon(A)(\wedge^n X)(e_A\wedge e_{A^c}, e_\Lambda\rangle\\ &= \langle\varepsilon(A)(\wedge^k X)e_A\wedge(\wedge^{n-k}X)e_{A^c}, e_\Lambda\rangle\\ &= \varepsilon(A)\sum_{B:\ |B|=k}\langle(\wedge^k X)e_A, e_B\rangle\langle e_B\wedge(\wedge^{n-k}X)e_{A^c}, e_\Lambda\rangle\\ &= \varepsilon(A)\varepsilon(B)\sum_{B:\ |B|=k}\langle(\wedge^k X)e_A, e_B\rangle\langle(\wedge^{n-k}X)e_{A^c}, e_{B^c}\rangle\end{aligned}$$

On the other hand, the relation $(\wedge^k X)(\wedge^k(X^{-1})) = (\wedge^k(X^{-1}))(\wedge^k X) = I$ yields

$$\sum_{B:\ |B|=k} \varepsilon(A,B)\det X(A,B)\det X^{-1}(A^c,C^c) = \delta(A,C),$$
$$\sum_{A:\ |A|=k} \varepsilon(A,B)\det X^{-1}(C^c,A^c)\det X(A,B) = \delta(B,C).$$

Indeed,

$$\begin{aligned}\delta(A,C) &= \langle(\wedge^k X)(\wedge^k X^{-1})e_C, e_A\rangle\\ &= \sum_{B:\ |B|=k}\langle(\wedge^k X^{-1})e_C, e_B\rangle\langle(\wedge^k X)e_B, e_A\rangle\\ &= \sum_{B:\ |B|=k}\det(X^{-1})(B,C)\det X(A,B).\end{aligned}$$

Comparing these, we obtain the desired formula since $(\det X(A,B))_{A,B\in\Lambda_k}$ is an invertible matrix for each k.

□

Now we can prove

Theorem 2.5. *Let $U = (u_{ij})_{i,j=1}^n$ be a unitary matrix. Then, for any subsets A and B of the index set Λ*

$$q(A,B) = |\det U_{AB}|^2 = (\overline{\det U})Sf_U(A,B). \tag{2.6}$$

Consequently, $q(\cdot,\cdot)\in\mathcal{D}_n$ and so satisfies **consistency conditions**.

Proof. If U is a unitary matrix, then, it follows from above lemmas that

$$\begin{aligned} Sf_U(A,B) &= \det U(A,B) \det U(A^c,B^c) \\ &= \det U(A,B)(\det U) \det U^{-1}(B,A) = \det U(A,B)(\det U)\overline{\det U(A,B)} \\ &= (\det U)|\det U(A,B)|^2 \end{aligned}$$

□

§ 3. Degenerate Laplacian on S_n

Lemma 3.1. *The adjoint operator $S^* : \mathcal{D}_n \to \mathcal{C}(S_n)$ of S is given by*

$$S^*\phi(\sigma) = \sum_{A\subset\Lambda} \phi(A,\sigma(A)), \qquad \phi \in \mathcal{D}_n. \tag{3.1}$$

Proof. Let $f \in \mathcal{C}(S_n)$ and $\phi \in \mathcal{D}_n$. Then,

$$\begin{aligned} \langle Sf, \phi\rangle &= \sum_{A,B\in\Lambda} Sf(A,B)\overline{\phi(A,B)} = \sum_{A,B\in\Lambda}\ \sum_{\sigma\in S_n, \sigma(A)=B} f(\sigma)\overline{\phi(A,B)} \\ &= \sum_{A\in\Lambda} f(\sigma) \sum_{\sigma\in S_n} \overline{\phi(A,\sigma(A))}. \end{aligned}$$

□

Remark. *If we denote $\phi^c(A,B) = \phi(A^c,B^c)$, then $S^*\phi^c = S^*\phi$.*

Once one has a pair of adjoint operators, it might be a routine work to study the "Laplacian".

Definition 3.2. Let

$$T = S^*S. \tag{3.2}$$

We call T the degenerate Laplacian on S_n (although T is nondegenerate for $n = 2$).

To understand the nature of the operator T we consider the decomposition of a permutation σ into cyclic permutations γ_i's:

$$\sigma = \gamma_1\gamma_2\ldots\gamma_\nu, \quad \nu \geq 1.$$

Definition 3.3. Given a $\sigma \in S_n$, let ν_k be the number of k-cycles in the decomposition of σ into cyclic permutations for $k = 1, 2, \ldots, n$. Then we write

$$\sigma \in (1)^{\nu_1}(2)^{\nu_2}(3)^{\nu_3}\cdots(n)^{\nu_n}, \tag{3.3}$$

$$\nu_k(\sigma) = \nu_k, \qquad k = 1, 2, \ldots, n \tag{3.4}$$

$$\nu(\sigma) = \sum_{k=1}^{n} \nu_k(\sigma) \tag{3.5}$$

As usual, if n is fixed, we employ conventional notations such as

$$(1) = (1)^n,\ (2) = (1)^{n-2}(2),\ (2)^2 = (1)^{n-4}(2)^2,\ etc.$$

Lemma 3.4. *Each ν_k satisfies*

$$\nu_k(\sigma^{-1}) = \nu_k(\sigma) \text{ and } \nu_k(\tau^{-1}\sigma\tau) = \nu_k(\sigma) \tag{3.6}$$

for any $\sigma, \tau \in S_n$. As a consequence, ν satisfies

$$\nu(\sigma^{-1}) = \nu(\sigma) \text{ and } \nu(\tau^{-1}\sigma\tau) = \nu(\sigma) \tag{3.7}$$

Since the values of $\nu(\sigma)$ and $\nu_k(\sigma)$ are determined by the conjugate class (α), we sometimes write $\nu(\sigma)$ as $\nu(\alpha)$ and $\nu_k(\sigma)$ as $\nu_k(\alpha)$.

Proof. The assertions about σ^{-1} are obvious. The rest assertions are immediately proved by induction on $\nu(\sigma)$ if one observes that for each transposition (ij)
(a) if i and j are contained in a common cycle, then both of the right and left actions of (ij) cut the cycle into two cycles and the lengths of resultant cycles are same in those two cases.
(b) If i and j are contained in distinct cycles, then both of the right and left action of (ij) join the two cycles into one cycle. $\square$

Theorem 3.5. *Let $f \in \mathcal{C}(S_n)$. Then,*

$$Tf(\sigma) = \sum_{\tau \in S_n} 2^{\nu(\sigma\tau^{-1})} f(\tau), \tag{3.8}$$

Proof. Let $f \in \mathcal{C}(S_n)$. Then,

$$\begin{aligned}
S^*Sf(\sigma) &= \sum_{A\subset\Lambda} Sf(A, \sigma(A)) = \sum_{A\subset\Lambda}\left(\sum_{\tau\in S_n:\ \tau(A)=\sigma(A)} f(\tau)\right) \\
&= \sum_{\tau\in S_n}\left(\sum_{A\subset\Lambda:\ \tau(A)=\sigma(A)} f(\tau)\right) = \sum_{\rho\in S_n}\left(\sum_{A\subset\Lambda:\ \rho(A)=A} f(\sigma\rho)\right) \\
&= \sum_{\rho\in S_n} f(\sigma\rho)\left(\sum_{A\subset\Lambda:\ \rho(A)=A} 1\right).
\end{aligned}$$

Now let us compute the number $\sum_{A\subset\Lambda:\ \rho(A)=A} 1$. If $\nu(\rho) = k$, then, by definition, there is a partition $A_1 \sqcup \cdots \sqcup A_k$ of $\Lambda = \{1, 2, \ldots, n\}$ such that ρ is cyclic on each A_i. If A is invariant under ρ, then A must be a (posssibly, empty) union of these A_i's. Thus,

$$\sum_{A\subset\Lambda, \rho\in S(A)} 1 = 2^{\nu(\rho)}.$$

Consequently,

$$S^*Sf(\sigma) = \sum_{\rho\in S_n} 2^{\nu(\rho)} f(\sigma\rho) = \sum_{\tau\in S_n} 2^{\nu(\tau\sigma^{-1})} f(\tau).$$

$\square$

Remark . *In* [ST1] *, for a real number α(not a cycle class) we introduced the α-determinant* $\det_\alpha(X)$ *of a matrix* $X = (x_{ij})_{i,j=1}^n$ *by*

$$\det_\alpha(X) = \sum_{\sigma\in S_n} \alpha^{\nu(\sigma)} \prod_{i=1}^n x_{i\sigma(i)}.$$

If $\alpha = -1$*, then*

$$(-1)^{\nu(\sigma)} = sgn(\sigma)$$

and $\det_{-1}$ *is the usual determinant and describes the correlation functions of fermion(or determinantal) point processes. On the other hand, if* $\alpha = 1$*, then* $\det_1$ *is the permanent and describes the correlation functions of boson(or permanental) point processes.*

If $\alpha = 2$*, then* $\det_2$ *gives the correlation functions of "super-boson" point processes which are closely related with the (infinite dimensional) χ-square processes (which is nothing but the square of real-valued Gaussian processes).*

In [2,3] we proposed a conjecture on the sufficient condition for the existence of α-determinantal point processes which follows from(but is not equivalent to) the non-negative definiteness of the quadratic form:

$$Q_\alpha(f) = \langle T_\alpha f, f\rangle$$

where

$$T_\alpha f(\sigma) = \sum_{\sigma,\tau\in S_n} \alpha^{n-\nu(\sigma\tau^{-1})} f(\tau).$$

Our conjecture is related to the case where $\alpha = 1/2$*. In the above we showed that* $Q_{1/2}(f) = 2^n\langle S^*Sf, f\rangle \geq 0$*. It gives a proof of the nonnegative definiteness while we proved it by a probabilistic construction based on Gaussian processes in [3].*

Finally, we should notice that α-determinants are different from so-called q-determinants which are defined using the inversion number

$$\iota(\sigma) = |\{(i,j) :\ i < j,\ \sigma i > \sigma j\}|$$

in place of $\nu(\sigma)$ *except for* $\alpha = q = \pm 1$*. For q-determinants the positive definiteness is easily proved because* $d(\sigma,\tau) = \iota(\sigma\tau^{-1})$ *is a distance on* S_n*.*

The degenerate Laplacian T can be decomposed into the sum of mutually commuting nonnegative definite operators in two manners.

Definition 3.6. For a cycle class (α) write

$$T^{(\alpha)}f(\sigma) = \sum_{\rho\in(\alpha)} f(\sigma\rho). \tag{3.9}$$

Theorem 3.7. *The operators $T^{(\alpha)}$ are nonnegative definite and satisfy*

$$T^{(\alpha)}(sgn\ f) = sgn(\alpha)T^{(\alpha)}f \tag{3.10}$$

and

$$T = \sum_{(\alpha)} 2^{\nu(\alpha)}T^{(\alpha)}, \tag{3.11}$$

$$T^{(\alpha)}T^{(\beta)} = T^{(\beta)}T^{(\alpha)}, \qquad \text{for any } (\alpha), (\beta). \tag{3.12}$$

Proof. The nonnegative definiteness follows from $T = S^*S$. The first equation is obvious. The second equation follows from

$$\begin{aligned} T^{(\alpha)}T^{(\beta)}f(\sigma) &= \sum_{\rho\in(\alpha)} T^{(\beta)}f(\sigma\rho) = \sum_{\rho\in(\alpha)}\sum_{\pi\in(\beta)} f(\sigma\rho\pi) \\ &= \sum_{\rho\in(\alpha)}\sum_{\pi\in(\beta)} f(\sigma\rho\pi\rho^{-1}\rho) = \sum_{\rho\in(\alpha)}\sum_{\pi'\in\beta} f(\sigma\pi'\rho) = T^{(\beta)}T^{(\alpha)}f(\sigma) \end{aligned}$$

□

Now we proceed to a second decomposition.

Definition 3.8. For each $m = 1, 2, \ldots, n$, let

$$S_m f(A, B) = 1(|A| = |B| = m)Sf(A, B), \tag{3.13}$$

$$T_m = S_m^* S_m. \tag{3.14}$$

By definition, it is immediate to see

$$S_m^*\phi(\sigma) = \sum_{A:\ |A|=m} \phi(A, \sigma A). \tag{3.15}$$

Lemma 3.9. *Let $\kappa(\rho)$ be a number defined by*

$$\kappa_m(\rho) = \sum_{|A|=m} 1(\rho A = A).$$

Then, κ_m's are given by ν_k's as

$$\kappa_m = \sum_{\sum km_k = m} \prod_{k=1}^{n} \binom{\nu_k}{m_k} \tag{3.16}$$

In particular,

$$\kappa_m(\sigma^{-1}) = \kappa_m(\sigma),\ \ \kappa_m(\tau\sigma\tau^{-1}) = \kappa_m(\sigma)$$

for any $\sigma, \tau \in S_n$.

Moreover,

$$T\kappa_m = n! \ \ \text{for any } m = 0, 1, 2, \ldots, n \tag{3.17}$$

$$\sum_{m=0}^{n} \kappa_m(\sigma) = 2^{\nu(\sigma)}. \tag{3.18}$$

Proof. If $\rho A = A$, then A must consist of cycles in ρ. Let $|A| = m$ and A consist of m_k of k-cycles for each k. Then we obtain the desired expression. The identities might not seem so obvious but are immediately seen from our definitions by combinatorial argument. A key is the following:

$$\sum_{m\geq 0} \kappa_m z^m = \prod_{k=1}^{n} (1+z^k)^{\nu_k}. \tag{3.19}$$

□

Now we can proceed to the following:

Theorem 3.10.

$$T = \sum_{m=0}^{n} T_m, \tag{3.20}$$

$$T_m T_l = T_l T_m \ \ \text{if } m \neq l, \tag{3.21}$$

$$T_m f(\sigma) = \sum_{\rho} \kappa_m(\rho) f(\sigma\rho) = \sum_{(\alpha)} \kappa_m(\alpha) T^{(\alpha)} f(\sigma). \tag{3.22}$$

In particular,

$$T^{(\alpha)} T_m = T_m T^{(\alpha)} \ \ \text{for any } (\alpha) \text{ and } m.$$

Proof. The first two assertions are obvious from the definition. The expression of T_m is obtained in the following manner.

$$\begin{aligned} T_m f(\sigma) &= \sum_{|A|=m} Sf(A, \sigma A) = \sum_{|A|=m} \sum_{\tau:\ \tau A = \sigma A} f(\tau) \\ &= \sum_{|A|=m} \sum_{\rho:\ \rho A = A} f(\sigma\rho) = \sum_{\rho} f(\sigma\rho) \sum_{|A|=m} 1(\rho A = A) = \sum_{\rho} \kappa_m(\rho) f(\sigma\rho). \end{aligned}$$

□

Remark. *The operator S_1^* turns out to be injective on the space $\mathcal{D}_n$. In other words, if $\phi(i,j),\ i,j=1,2,\ldots,n$ satisfies*

$$\sum_{j=1}^{n}\phi(i,j)=\sum_{j=1}^{n}\phi(j,i)=c \ \text{ for any } i$$

for a constant c and if

$$S_1^*\phi(\sigma)=\sum_{i=1}^{n}\phi(i,\sigma i)=0 \ \text{ for any } \sigma,$$

then,

$$c=0 \quad \textit{and } \phi=0.$$

The injectivity problem of the operator S_m^* is left open for $m\geq 2$.

The following proof of the above remark might be interesting in itself.

Proof. Set

$$f_{ij}=\phi(i,j)-\phi(i,i)$$

Then, it follows from $S_1^*\phi(e)=\sum_{i=1}^n\phi(i,i)=0$ that

$$S_1^*\phi((ij))=f_{ij}+f_{ji}=0,\quad S_1^*\phi((ijk))=f_{ij}+f_{jk}+f_{ki}=0$$

Thus, $f_ij,\ i\neq j$ form a cocycle on the complete graph K_n: the summation of f_{ij} along each cyclic path vanishes. Hence, one can find g_i such that $f_{ij}=g_i-g_j$. Moreover, g_i can be chosen so that $\sum_i g_i=0$.

This means that

$$\phi(i,j)=\phi(i,i)+g_i-g_j.$$

Summing up these equations in j, one finds $\phi(i,i)+g_i=c$ and so

$$\phi(i,j)=c-g_j \ \text{ for any } i,j.$$

Now, summing them up in i, one finds

$$g_j=(1-\frac{1}{n})c \ \text{ for any } j.$$

Hence,

$$\phi(i,j)=\frac{c}{n}.$$

Consequently,

$$c=S_1^*\phi(e)=0$$

and so $g_j=0$ and

$$\phi(i,j)=0 \ \text{ for any } i,j.$$

□

Example 3.11. For $\rho \in S_n$, the solution to

$$S_1^* q(\sigma) = \nu_1(\sigma\rho), \qquad \sigma \in S_n$$

is given by $q(i,j) = 1(i = \rho j)$. In particular, $\tau = \rho$ if

$$\nu_1(\sigma\tau) = \nu_1(\sigma\rho), \qquad \sigma \in S_n.$$

Now our equation to be solved is written as

$$p \geq 0 \ \text{ and } \ Sp = S\tilde{f}_U, \qquad \tilde{f}_U = \overline{\det U} f_U. \tag{3.23}$$

In other words, our conjecture is reduced to the following:

Problem *Does there exist a nonnegative solution p such that*

$$p - \tilde{f}_U \in \ker S$$

for a given unitary matrix U.

The problem is still open. We only show a partial result on the dimension of $\ker S = \ker T$.

Proposition 3.12.

$$\dim \ker T = \begin{cases} 0 & \textit{for } \ n = 2, \\ 1 & \textit{for } \ n = 3. \end{cases} \tag{3.24}$$

and

$$\dim \ker T \geq n \ \textit{for} \ \ n \geq 4. \tag{3.25}$$

Proof.
For $n = 2$ we have $\nu(e) = 2$ and $\nu(12) = 1$ and so

$$(2^{\nu(\sigma\tau^{-1})})_{\sigma,\tau \in S_2} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

For $n = 3$ there appears a parameter t in Example 4.2 as will be stated in Appendix except for the cases where $\min_i q_{ii} = 0$ or $\min_{j \neq k} q_{jk} = 0$. It suggests that

$$\dim \ker S = \dim \ker S^* S = 1.$$

Indeed, if $f \in \ker S$, then,

$$f(e) + f(ij) = 0, \qquad f(ij) + f(ijk) = 0$$

Hence it follows

$$f(ij) = -f(e), \qquad f(ijk) = f(e).$$

In other words, $f(\sigma) = f(e)sgn(\sigma)$.

In general,

$$sgn \in \ker S.$$

For $n \geq 4$ consider the square matrix

$$X = (\mathbf{1}\ \mathbf{1}\ \cdots \mathbf{1}\ x)$$

for a given vector x where $\mathbf{1}$ stands for the vector all of whose entries are 1 and $\mathbf{1} \otimes \mathbf{1}$ means the square matrix all of whose components are 1. Then, it is obvious that $\det X(A, B) = 0$ if $|A| = |B| \geq 3$. If $|A| = |B| = 2$, then either $\det X(A, B) = 0$ or $\det X(A^c, B^c) = 0$ even when $n = 4$. Hence, $f_X \in \ker S$ for any x.

Here we omit the proof of the inequality $\dim \ker T \leq 4$ for $n = 4$.

□

We conclude the paper by the following :

For $n = 3$ the characteristic equation of the degenerated Laplacian T is given by

$$\det(\lambda I - T) = \lambda(\lambda - 6)^4(\lambda - 24).$$

For $n = 4$ the characteristic equation of the degenerated Laplacian T is given by

$$\det(\lambda I - T) = \lambda^{10}(\lambda - 12)^4(\lambda - 24)^9(\lambda - 120).$$

§ 4. Appendix

If $n = 2, 3$ our Conjecture is reduced to Theorem 0 (which is beautifully proved by using the Krein-Milman theorem on the extremal point representation for compact convex sets. Consult with Phelps' "Lecture on Choquet's theorem," Lecture Notes in Mathematics , Springer). Nevertheless, we give a direct proof here for the further consideration.

Example 4.1 (Case $n = 2$). The system of equations itself gives the (unique) solution probability p:

$$q_{11} = q_{22} = p(e), \qquad q_{12} = q_{21} = p(12). \tag{4.1}$$

Example 4.2 (Case $n=3$)**.** For any unitary matrix U there exists a solution probability p. The solution is not necessarily unique.

More precisely, there exists a solution probability p with $p(e)=t$ if and only if $t \geq 0$ and

$$\min_{i=1,2,3} q_{ii} - \min_{j,k=1,2,3;\ j\neq k} q_{jk} \leq t \leq \min_{i=1,2,3} q_{ii}. \tag{4.2}$$

Then the solution p is uniquely determined by t and is given by

$$\begin{aligned} &p(12)=q_{11}-t,\ p(23)=q_{22}-t,\ p(31)=q_{22}-t;\\ &p(123)=q_{12}-q_{33}+t=q_{23}-q_{11}+t=q_{31}-q_{22}+t,\\ &p(132)=q_{21}-q_{33}+t=q_{32}-q_{11}+t=q_{13}-q_{22}+t. \end{aligned} \tag{4.3}$$

In particular, the solution p is unique if U is diagonal (i.e., if $\min_{j\neq k} q_{jk}=0$) or if U is off-diagonal (i.e., $\min_i q_{ii}=0$).

Proof. Let $H=\{e,(123),(132)\}=(1)\cup(3)$ and

$$M_H=\frac{1}{3}(I+T^{(3)}).$$

Then the equation to be solved

$$3p+T^{(2)}p=S_1^*q$$

can be written as

$$3p-3M_Hp=S_1^*q-1.$$

Now it is immediate to see that the double stochasticity of $q_{i,j}$ implies

$$\begin{aligned} q_{12}-q_{33}=q_{23}-q_{11}=q_{31}-q_{22}=:d_1,\\ q_{21}-q_{33}=q_{32}-q_{11}=q_{13}-q_{22}=:d_2 \end{aligned}$$

It then follows

$$\begin{aligned} &S_1^*q(e)-1\ =\sum_{i=1}^{3} q_{ii}-1=-(d_1+d_2)\\ &S_1^*q(ij)-1\ =q_{ij}+q_{ji}+q_{kk}-1=(3q_{kk}-1)+(d_1+d_2)\ \text{ if } \{i,j,k\}=\{1,2,3\},\\ &S_1^*q(123)-1=q_{12}+q_{23}+q_{31}-1=-(d_1+d_2)+3d_1,\\ &S_1^*q(132)-1=q_{13}+q_{32}+q_{21}-1=-(d_1+d_2)+3d_2,\\ &M_HS_1^*q=1. \end{aligned}$$

Consequently, if $p(e) = t$, then,

$$p(123) = d_1 + t, \quad p(132) = d_2 + t,$$
$$p(12) = q_{33} - t, \quad p(23) = q_{11} - t, \quad p(31) = q_{22} - t$$

and the nonnegative constant t should satisfy the condition

$$-\min\{d_1, d_2\} \leq t \leq \min_i q_{ii}.$$

to guarantee $p \geq 0$.

Finally, a direct computation shows that

$$-\min\{d_1, d_2\} = \min_i q_{ii} - \min_{j \neq k} q_{jk}$$

and that the condition on t defines a nonempty closed interval, although the parameter t is uniquely determined if U is diagonal (i.e., if $\min_{j \neq k} q_{jk} = 0$) or if U is off-diagonal (i.e., $\min_i q_{ii} = 0$). $\square$

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Added in proof: Further discussions on the positivity of α-determinants can be found in

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