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Sampling the Lindelöf hypothesis with an ergodic transformation

Dedicated to Professor Eduard Wirsing at the occasion of his eightieth birthday

By

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Abstract

We study the distribution of values of the Riemann zeta-function $\zeta(s)$ on vertical lines $s = \sigma + i\mathbb{R}$ with respect to the ergodic transformation given by $Tx := \frac{1}{2}(x - \frac{1}{x})$ for $x \neq 0$. We show among other things that, for $\operatorname{Re} s > -\frac{1}{2}$, the mean-value of $\zeta(s + iT^n x)$ exists for almost all values $x \in \mathbb{R}$, as $n \to \infty$, and is independent of x; we determine its exact value and discuss our results with respect to the Lindelöf hypothesis on the growth of the zeta-function on the critical line. Moreover, we present an equivalent formulation for the Riemann hypothesis in terms of our ergodic transformation.

§ 1. Ergodic Cesàro Means of the Riemann Zeta-Function

The Riemann zeta-function $\zeta(s)$ is of great interest in number theory since relevant information on prime numbers is encoded in its value-distribution as a function of a complex variable. The zeta-function is initially defined by $\zeta(s) = \sum_{n\in\mathbb{N}} n^{-s}$ for $\operatorname{Re} s > 1$ and by analytic continuation elsewhere, except for a simple pole at $s = 1$. Whereas $\zeta(s)$ behaves rather regular in any compact subset of the half-plane of absolute convergence of the defining Dirichlet series, the behaviour inside the so-called critical strip $0 < \operatorname{Re} s < 1$ and on its boundary is not yet understood completely. The famous open Riemann hypothesis claims that there are no zeros of $\zeta(s)$ to the right of the critical line $s = \frac{1}{2} + i\mathbb{R}$ which is equivalent to find all nontrivial (non-real) zeros on this line. The functional equation implies a relation between the values of $\zeta(s)$ and of $\zeta(1 - s)$ with the critical line as symmetry axis.

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We shall study the value-distribution of $\zeta(s)$ on vertical lines $\sigma + i\mathbb{R}$ with respect to the ergodic transformation $T : \mathbb{R} \to \mathbb{R}$ given by $T0 := 0$ and $Tx := \frac{1}{2}(x - \frac{1}{x})$ for $x \neq 0$. Here and in the sequel we use the abbreviation Tx for $T(x)$ and $T^n x$ is defined by $T \circ T^{n-1}x$ and $T^0 x = x$. Given a complex number s, we shall show that for almost all real numbers x the mean-value of $\zeta(s + iT^n x)$ exists, as $n \to \infty$, and is independent of x (as follows from the ergodicity of T). Moreover, we shall determine the exact value of the limit, for example,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \zeta(2 + iT^n x) = \zeta(3) = 1.20205\ldots$$

for almost all $x \in \mathbb{R}$. Obvious exceptional values for x are the preimages of 0 (which include the points ± 1 and $\pm 1 \pm \sqrt{2}$ and so forth). On different lines we obtain different limits, e.g., in the most interesting case of the critical line, we have, for almost all x,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \zeta(\tfrac{1}{2} + iT^n x) = \zeta(\tfrac{3}{2}) - \tfrac{8}{3} = -0.05429\ldots \ . \tag{1.1}$$

Our first aim is the description of all appearing limits:

Theorem 1.1. *Let s be given with $\operatorname{Re} s > -\frac{1}{2}$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \zeta(s + iT^n x) = \frac{1}{\pi} \int_{\mathbb{R}} \zeta(s + i\tau) \frac{\mathrm{d}\tau}{1 + \tau^2}. \tag{1.2}$$

for almost all $x \in \mathbb{R}$.

Define

$$\ell(s) = \frac{1}{\pi} \int_{\mathbb{R}} \zeta(s + i\tau) \frac{\mathrm{d}\tau}{1 + \tau^2}.$$

If $\operatorname{Re} s < 1$, then

$$\ell(s) = \zeta(s + 1) - \frac{2}{s(2 - s)}; \tag{1.3}$$

here the case of $s = 0$ is included as $\ell(0) = \lim_{s\to 0} \ell(s) = \gamma - \frac{1}{2}$ where $\gamma := \lim_{M\to\infty}(\sum_{m=1}^{M} \frac{1}{m} - \log M) = 0.57721\ldots$ denotes the Euler-Mascheroni constant. If $s = 1 + it$ with some real number t, then

$$\ell(s) = \zeta(s + 1) - \frac{1}{s(2 - s)} = \zeta(2 + it) - \frac{1}{1 + t^2}. \tag{1.4}$$

Finally, if $\operatorname{Re} s > 1$, then

$$\ell(s) = \zeta(s + 1). \tag{1.5}$$

Interestingly, the function $\ell(s)$ is locally analytic apart from the line $s = 1 + i\mathbb{R}$. The proof will explain this irregularity.

Recently, Lifshits & Weber [28] published a paper entitled "*Sampling the Lindelöf Hypothesis with the Cauchy Random Walk*" which describes the content of their interesting paper very well. If (X_m) is an infinite sequence of independent Cauchy distributed random variables, the Cauchy random walk is defined by $\mathcal{C}_n = \sum_{m\leq n} X_m$. Lifshits & Weber proved among other things (in slightly different notation) that almost surely

$$\sum_{1\leq n\leq N} \zeta(\tfrac{1}{2} + i\mathcal{C}_n) = 1 + o(N^{-\frac{1}{2}}(\log N)^b) \tag{1.6}$$

for any $b > 2$. It should be noted that the expectations $\mathbf{E}X_m$ and $\mathbf{E}\mathcal{C}_n$ do not exist, and, indeed, the values of $\mathcal{C}_n$ provide a sampling of randomly distributed real numbers of unpredictable size. Hence, the almost sure convergence theorem of Lifshits & Weber shows that the expectation value of $\zeta(s)$ on the Cauchy random walk $s = \frac{1}{2} + i\mathcal{C}_n$ equals one, which indicates that the values of the zeta-function are small on average. The yet unproved Lindelöf hypothesis states that, for any $\epsilon > 0$,

$$\zeta(\tfrac{1}{2} + it) \ll t^\epsilon \tag{1.7}$$

as $t \to \infty$. The Riemann hypothesis implies the Lindelöf hypothesis (see [30], resp. [37], §14.2) and the Lindelöf hypothesis serves in some applications as valuable substitute. The presently best estimate in this direction is due to Huxley [16] who obtained the exponent $\frac{32}{205} + \epsilon$ in place of the tiny ϵ above. On the contrary, Soundararajan [34] proved the inequality

$$\max_{t\in[T,2T]} |\zeta(\tfrac{1}{2} + it)| \geq \exp\left((1 + o(1))\left(\frac{\log t}{\log\log t}\right)^{\frac{1}{2}}\right)$$

for sufficiently large T. Similar to the approach of Lifshits & Weber our Cesàro means $\frac{1}{N}\sum_{0\leq n<N} \zeta(s + iT^n x)$ provide ergodic samples for testing the Lindelöf hypothesis and their almost sure convergence indicates that most of the values of the zeta-function are not too big, which an optimist would interpret as evidence in favour for the truth of the Lindelöf hypothesis although hypothetical counterexamples may belong to a rather small set, as a pessimist would reply. It is remarkable that the ergodic mean of $\zeta(s + iT^n x)$, if existent, is essentially $\zeta(s + 1)$ which has in general a smaller absolute value than $\zeta(s + iT^n x)$ which indicates cancelation in the Cesàro means.

This paper is organized as follows. The next section contains the proof of Theorem 1.1 which is mainly the evaluation of $\ell(s)$; in Section 3 a slightly different approach is sketched. Section 4 contains equivalent formulations of the Riemann hypothesis and the Lindelöf hypothesis in terms of the ergodic transformation under consideration. Section 5 consists of a brief discussion of corresponding results for other zeta- and L-functions. We conclude with some numerical experiments and some remarks.

§ 2. Calculus of Residues

The proof of Theorem 1.1 relies on the pointwise ergodic theorem of Birkhoff and calculus of residues. It will turn out that our sampling is not unrelated to that of Lifshits & Weber [28].

For the sake of completeness, let us briefly recall some facts about transformation T. It is easy to see that T is measurable and, using the substitution $\tau = Tx$, $\mathrm{d}\tau = \frac{1}{2}(1 + \frac{1}{x^2})\,\mathrm{d}x$, it turns out that, for any Lebesgue integrable function f,

$$\int_{\mathbb{R}} f(Tx)\frac{\mathrm{d}x}{1+x^2} = \int_{\mathbb{R}} f(\tau)\frac{\mathrm{d}\tau}{1+\tau^2}.$$

Hence, T is measure preserving with respect to the probability measure $\mathbf{P}$ defined by

$$\mathbf{P}((\alpha,\beta)) = \frac{1}{\pi}\int_{(\alpha,\beta)} \frac{\mathrm{d}\tau}{1+\tau^2}; \tag{2.1}$$

alternatively, we could have used the addition theorem for the arctangent function. Finally, we observe that the only T-invariant sets A with respect to the related probability measure $\mathbf{P}$ are $A = \{0\}$ and $A = \mathbb{R}$ for which $\mathbf{P}(A) = 0$ or $= 1$. Hence, T is indeed ergodic and $(\mathbb{R}, \mathcal{B}, \mathbf{P}, T)$ is an ergodic system, where $\mathcal{B}$ denotes the Borel sigma-algebra associated with $\mathbb{R}$. This example of an ergodic transformation on the real line is due to Lind (cf. [10], Example 2.9); however, a related transformation can even be traced back to Boole [8] (cf. [1]). Actually, the mapping T originates from Newton's iteration applied to the function $f(x) := 1 + x^2$. If there would exist a real zero of f, the sequence of the iterations would converge, however, since $f(x)$ does not vanish for real x, the iteration diverges and thus provides an interesting transformation.

The pointwise ergodic theorem states that, given a measure preserving transformation T on a measurable space (X, μ) and an integrable function f, the limit of the Cesàro means

$$\frac{1}{N}\sum_{0\le n<N} f(T^n x)$$

exists as $N \to \infty$ for almost all $x \in X$; if the measure space is finite and T ergodic, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N} f(T^n x) = \frac{1}{\mu(X)}\int_X f\,\mathrm{d}\mu$$

almost everywhere (see [7, 23], resp. [10], §3.2).

In order to apply this identity between the 'time mean' and the 'space mean' in our situation we need to consider the growth rate of $\zeta(s)$ on vertical lines $\sigma + i\mathbb{R}$. By the Phragmén-Lindelöf principle and the functional equation for ζ, we have

$$\zeta(\sigma+it) \ll t^{\mu(\sigma)+\epsilon} \qquad \text{with} \quad \mu(\sigma) \le \begin{cases} 0 & \text{if } \sigma > 1, \\ \frac{1-\sigma}{2} & \text{if } 0 \le \sigma \le 1, \\ \frac{1}{2} - \sigma & \text{if } \sigma < 0, \end{cases} \tag{2.2}$$

as $t \to \infty$ (see [37], §5.1). Hence, the function $\tau \mapsto \frac{\zeta(s+i\tau)}{1+\tau^2}$ is Lebesgue integrable on $\mathbb{R}$ for fixed $\operatorname{Re} s > -\frac{1}{2}$ and we may apply the pointwise ergodic theorem which immediately yields (1.2) for almost all $x \in \mathbb{R}$.

For the evaluation of this ergodic limit in the case $s = \frac{1}{2}$ we can use another interpretation of the integral on the right-hand side of (1.2) which is implied by the work of Lifshits & Weber [28] mentioned above. Note that the density function of a Cauchy distributed random variable X is given by $\tau \mapsto \frac{1}{\pi(1+\tau^2)}$, hence, (2.1) is the associated probability measure and the integral in question is nothing but the expectation of $\zeta(\frac{1}{2} + iX)$,

$$\mathbf{E}\zeta(\tfrac{1}{2}+iX) = \frac{1}{\pi}\int_{\mathbb{R}} \zeta(\tfrac{1}{2}+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2}.$$

In their account to prove (1.6) Lifshits & Weber computed by elementary means several expectation values, in particular, this one by $\zeta(\frac{3}{2}) - \frac{8}{3}$, which yields (1.1), resp.(1.3) in the case $s = \frac{1}{2}$.

However, we want to give an independent analytic evaluation of $\ell(s)$ which is valid not only for $s = \frac{1}{2}$ or real values of s. For this aim we apply calculus of residues. (It seems we cannot miss Cauchy!) The integrand in (1.2) is a regular function of τ apart from the poles at $\tau = \pm i$ and $\tau = -i(1-s) = i(s-1)$ in the τ-plane (the latter one resulting from the simple pole of the zeta-function). We shall distinguish several cases according to the location of $i(s-1)$.

Firstly, suppose that $i(s-1)$ lies in the lower half of the τ-plane, i.e., $\operatorname{Re} s < 1$. Moreover, we assume that $i(s-1) \neq -i$, resp. $s \neq 0$. Then the integrand has two distinct simple poles in the lower half-plane. For a sufficiently large parameter $R > 1+|s|$ denote by Γ_R the counterclockwise oriented semicircle of radius R centered at the origin located in the lower half of the τ-plane. Then

$$\int_{[-R,+R]} \zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2} = \int_{\Gamma_R} \zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2} - 2\pi i\Sigma(s),$$

where $\Sigma(s)$ is the sum of residues inside $[-R,+R]+\Gamma_R$, that is here the sum of residues at the simple poles in $\tau = -i$ and $\tau = i(s-1)$. In view of (2.2) a short computation shows that

$$\int_{\Gamma_R} \zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2} \ll \frac{R}{1+R^2}\max_{\tau\in\Gamma_R}|\zeta(s+i\tau)| \ll R^{-\frac{1}{2}}.$$

Hence, this integral vanishes as $R \to +\infty$. On the other hand, the integral over $[-R, +R]$ tends under this limit to the integral we are interested in (up to the factor $\frac{1}{\pi}$). Thus, letting $R \to \infty$, we find

$$\frac{1}{\pi}\int_{\mathbb{R}} \zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2} = -2i\left\{\mathrm{Res}_{\tau=-i} + \mathrm{Res}_{\tau=i(s-1)}\right\}\frac{\zeta(s+i\tau)}{1+\tau^2}, \tag{2.3}$$

and it remains to compute the residues. A short calculation shows

$$\mathrm{Res}_{\tau=-i}\frac{\zeta(s+i\tau)}{1+\tau^2} = \lim_{\tau\to -i}(\tau+i)\frac{\zeta(s+i\tau)}{\tau^2+1} = -\frac{\zeta(s+1)}{2i}. \tag{2.4}$$

In order to compute the second residue we shall use the Laurent expansion of the zeta-function near its pole,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|) \tag{2.5}$$

(see [37], §2.1). We thus obtain

$$\begin{aligned}
&\mathrm{Res}_{\tau=i(s-1)}\frac{\zeta(s+i\tau)}{1+\tau^2}\\
&= \frac{1}{1+(i(s-1))^2}\lim_{\tau\to i(s-1)}(\tau - i(s-1))\left\{\frac{1}{s+i\tau-1} + O(1)\right\}\\
&= \frac{1}{i(1-(s-1)^2)} = -\frac{i}{s(2-s)}.
\end{aligned} \tag{2.6}$$

Inserting these values in (2.3) yields (1.3).

Now we assume that the integrand has a double pole in the lower half-plane, that is $-i = i(s-1)$, hence $s = 0$. In this case the sum $\Sigma(s)$ of residues reduces to a single residue which again can be calculated by use of the Laurent expansion (2.5). For this we note that, as $\tau \to -i$,

$$\zeta(i\tau)\frac{1}{1+\tau^2} = \left\{\frac{i}{\tau+i} + \gamma + O(|\tau+i|)\right\}\left\{\frac{i/2}{\tau+i} + \frac{1}{4} + O(|\tau+i|)\right\};$$

Hence,

$$\mathrm{Res}_{\tau=-i}\frac{\zeta(i\tau)}{1+\tau^2} = \frac{i}{2}(\gamma - \tfrac{1}{2}).$$

This implies $\ell(0) = \gamma - \frac{1}{2}$ which equals the limit $\lim_{s\to 0}\ell(s)$ as follows from (2.5).

Next we suppose that $i(s-1)$ lies in the upper half of the τ-plane, i.e., $\mathrm{Re}\, s > 1$. Then the integrand has just one pole in the lower half of the τ-plane and we proceed as in the corresponding case above. The corresponding residue is given by (2.4), hence we may deduce (1.5).

Finally, we consider the intermediate case when $i(s-1)$ lies on the real axis, i.e., $s = 1 + it$ for some real number t. Since the pole of the zeta-function is on the line of integration, we assign the Cauchy principal value to our improper integral,

$$\begin{aligned}&\int_{\mathbb{R}} \zeta(1+i(t+\tau))\frac{\mathrm{d}\tau}{1+\tau^2}\\&= \lim_{\substack{R\to+\infty\\ \epsilon\to 0+}} \left\{\int_{-t-R}^{-t-\epsilon} + \int_{-t+\epsilon}^{-t+R}\right\}\zeta(1+i(t+\tau))\frac{\mathrm{d}\tau}{1+\tau^2}.\end{aligned}$$

Now denote by γ_ϵ and γ_R the counterclockwise oriented semicircles of radius ϵ and R, respectively, both centered at $\tau = -t$ and located in the lower half of the τ-plane. Then, for sufficiently large R,

$$\begin{aligned}&\left\{\int_{-t-R}^{-t-\epsilon} + \int_{-t+\epsilon}^{-t+R}\right\}\zeta(1+i(t+\tau))\frac{\mathrm{d}\tau}{1+\tau^2}\\&=\left\{\int_{\gamma_R} - \int_{\gamma_\epsilon}\right\}\zeta(1+i(t+\tau))\frac{\mathrm{d}\tau}{1+\tau^2} + 2\pi i \mathrm{Res}_{\tau=-i}\frac{\zeta(1+i(t+\tau))}{1+\tau^2}.\end{aligned}\tag{2.7}$$

It is easily seen that

$$\lim_{R\to+\infty}\int_{\gamma_R}\zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2} = 0.$$

Hence, it remains to evaluate the integral over γ_ϵ as ϵ tends to zero. For this aim, using the parametrization $\tau = \epsilon\exp(i\varphi) - t$ and (2.5), we find

$$\begin{aligned}&\int_{\gamma_\epsilon}\zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2}\\&=\int_\pi^{2\pi}\zeta(1+i\epsilon\exp(i\varphi))\frac{i\epsilon\exp(i\varphi)}{1+(\epsilon\exp(i\varphi)-t)^2}\,\mathrm{d}\varphi\\&=\int_\pi^{2\pi}\left\{\frac{1}{i\epsilon\exp(i\varphi)} + \gamma + O(\epsilon)\right\}\frac{i\epsilon\exp(i\varphi)}{1+t^2+O(\epsilon)}\,\mathrm{d}\varphi.\end{aligned}$$

Hence,

$$\begin{aligned}&\lim_{\epsilon\to 0+}\int_{\gamma_1}\zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2}\\&=\int_\pi^{2\pi}\lim_{\epsilon\to 0+}\left\{\frac{\mathrm{d}\varphi}{1+t^2+O(\epsilon)} + O(\epsilon)\right\} = \frac{\pi}{1+t^2}.\end{aligned}$$

Inserting this and (2.4) into (2.7) proves (1.4). The proof of the theorem is complete.

It should be noted that including the pole at $\tau = -t$ in the interior of our contour of integration would have led to the same value for $\ell(s)$. In that case there would have been the additional residue (2.6) from which we would have to subtract the contribution of the integral over γ_ϵ, that is

$$-\frac{2}{s(2-s)}+\frac{1}{1+t^2}=-\frac{1}{1+t^2}$$

for $s=1+it$. It follows that the values of $\ell(s)$ on the boundary line $s=1+i\mathbb{R}$ of the critical strip are the arithmetic means of the limits $\lim_{\varepsilon\to 0\pm}\ell(s+\varepsilon)$.

By a similar reasoning as in the proof above we can recover parts of the results of Lifshits & Weber [28]. For example, since the Cauchy random walk $\mathcal{C}_n$ has distribution density $\tau\mapsto\frac{n}{\pi(n^2+\tau^2)}$, we find for the expectation of $\zeta(\frac12+i\mathcal{C}_n)$

$$\mathbf{E}\zeta(\tfrac12+i\mathcal{C}_n)=\frac{n}{\pi}\int_{\mathbb{R}}\zeta(\tfrac12+i\tau)\frac{\mathrm{d}\tau}{n^2+\tau^2}=\zeta(\tfrac12+n)-\frac{2n}{n^2-\frac14},$$

which is Formula (3.3) of [28]. To recover the theorem of Lifshits & Weber, however, one has to prove that the normalized random variables $X_n:=\zeta(\frac12+i\mathcal{C}_n)-\mathbf{E}\zeta(\frac12+i\mathcal{C}_n)$ are almost orthogonal which is a natural condition in almost sure convergence theorems of Rademacher-Menchoff type for dependent random variables. It seems a good piece of work to achieve this almost orthogonality by complex integration only.

We want to remark that, although there are quite many similarities in both samplings of the Lindelöf hypothesis, the one of Lifshits & Weber and the ergodic one presented here, both processes describe different phenomena which is already apparent by the different limits in (1.3) and (1.6). Moreover, whereas the Cauchy random walk is of random nature, given an initial value, the orbit of the ergodic transformation provides a completely deterministic sequence.

§ 3. The Dirichlet Problem for the Upper Half-Plane

There is a slightly different approach for the evaluation of the integrals $\ell(s)$ appearing in Theorem 1.1 which we want to discuss briefly. Given a domain D in the complex plane, the classical Dirichlet problem asks for finding a harmonic function H on D with given continuous boundary values $h=H\big|_{\partial D}$. The Poisson formula solves this problem for a disk (see [31], §3.9). The Möbius transformation $\omega\mapsto M\omega:=i\frac{1+\omega}{1-\omega}$, also called Cayley map, maps the unit disk to the real line which provides the solution of the Dirichlet problem for the upper half-plane $\mathbb{H}:=\{z=x+iy\,:\,y>0\}$: given a continuous real-valued function h on the real line, there exists a continuous function $H(x,y)$ defined on $\mathbb{R}\cup\mathbb{H}$ such that $H(x,0)=h(x)$ and which is harmonic in $\mathbb{H}$. The Poisson formula translates to

$$H(x,y)=\frac{y}{\pi}\int_{\mathbb{R}}\frac{h(t)}{y^2+(t-x)^2}\,\mathrm{d}t;$$

here $P_y(t) = \frac{1}{\pi}\frac{y}{y^2+t^2}$ is called the Poisson kernel which reminds us on the invariant measure $\mathbf{P}$ for our ergodic transformation T. We are interested in the boundary values given by $h(t) = \operatorname{Re}\zeta(s+it)$. The substitution $\tau = \frac{x-t}{y}$ leads to

$$H(x,y) = \operatorname{Re}\frac{1}{\pi}\int_{\mathbb{R}} \zeta(s+i(x-y\tau))\frac{\mathrm{d}\tau}{1+\tau^2}.$$

Since h is the real part of an analytic function, we may deduce $H(t,0) = h(t) = \operatorname{Re}\zeta(s+it)$. The same argument applies for the imaginary part as well, and combining both we end up with an analytic function

$$f(x+iy) = \frac{1}{\pi}\int_{\mathbb{R}} \zeta(s+i(x-y\tau))\frac{\mathrm{d}\tau}{1+\tau^2}$$

satisfying $f(x) = \zeta(s+ix)$ for all $x \in \mathbb{R}$. In view of Theorem 1.1, however, we are interested in the special value of f at $x+iy = -i$ since, for $x = 0, y = -1$,

$$f(-i) = \frac{1}{\pi}\int_{\mathbb{R}} \zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2}.$$

Unfortunately, $-i$ does not lie in the upper half-plane, however, analytic continuation allows to determine the value $f(-i)$. To conclude we notice that the analytic function f is uniquely determined by its values on the real axis. In the simplest case $\operatorname{Re} s > 1$ we thus find $f(x+iy) = \zeta(s-y+ix)$. The other cases can be treated similarly by taking the appearing singularities into account.

§ 4. Number-Theoretical Applications

We start with an equivalent formulation of the Lindelöf hypothesis in terms of our ergodic transformation.

Theorem 4.1. *The Lindelöf hypothesis is true if, and only if, for any $k \in \mathbb{N}$ and almost all $x \in \mathbb{R}$, the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N}|\zeta(\tfrac{1}{2}+iT^n x)|^{2k} \tag{4.1}$$

exists, which is also equivalent to the existence of the integrals

$$\int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+i\tau)|^{2k}\frac{\mathrm{d}\tau}{1+\tau^2}. \tag{4.2}$$

Proof. If the Lindelöf hypothesis (1.7) is true, $|\zeta(\frac{1}{2}+iT^n x)|^{2k}$ is $\mathbf{P}$-integrable for any k, hence the pointwise ergodic theorem is applicable and provides the almost sure existence of the limits (4.1) and their values are up to the factor $\frac{1}{\pi}$ given by (4.2).

It thus remains to show that the existence of the integrals (4.2) implies the Lindelöf hypothesis. For this purpose we suppose that the Lindelöf hypothesis is not true and derive a contradiction. We assume that there exist a positive real number δ, a sequence of real numbers $\tau_m \to \infty$ and a positive constant C_1 such that

$$|\zeta(\tfrac{1}{2} + i\tau_m)| > C_1 \tau_m^\delta.$$

Since $|\zeta'(\frac{1}{2} + it)| < C_2 t$ for any $t \geq 1$ with some positive constant C_2 (see [37], §13.2),

$$|\zeta(\tfrac{1}{2} + i\tau) - \zeta(\tfrac{1}{2} + i\tau_m)| = \left| \int_{\tau_m}^{\tau} \zeta'(\tfrac{1}{2} + it)\,\mathrm{d}t \right| < C_2 |\tau - \tau_m| \tau.$$

Thus, $|\zeta(\frac{1}{2} + i\tau)| > \frac{1}{2} C_1 \tau_m^\delta$ for any τ satisfying $|\tau - \tau_m| \leq \tau_m^{-1}$ with sufficiently large m. Choosing $T = \frac{2}{3}\tau_m$ the interval $\mathcal{I} := (\tau_m - \tau_m^{-1}, \tau_m + \tau_m^{-1})$ is contained in $(T, 2T)$ for large m; here T is a real quantity which should not be confused with the ergodic transformation. This leads to

$$\begin{aligned}
&\int_T^{2T} |\zeta(\tfrac{1}{2} + i\tau)|^{2k} \frac{\mathrm{d}\tau}{1 + \tau^2} \\
&\geq \left(\frac{C_1}{2}\right)^{2k} \int_{\mathcal{I}} \tau_m^{2k\delta - 2}\,\mathrm{d}\tau = 2\left(\frac{C_1}{2}\right)^{2k} \tau_m^{2k\delta - 3},
\end{aligned}$$

which is $\gg T^{2k\delta - 3}$, a contradiction for $k \to \infty$. This proves the theorem.

The proof of the latter implication is adopted from Hardy & Littlewood [15] (resp. [37], §13.2) where they showed that the Lindelöf hypothesis is true if, and only if, for any $k \in \mathbb{N}$,

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^{2k}\,\mathrm{d}t \ll T^{1+\epsilon}.$$

Note that Titchmarsh [37], §13.4, also proved the estimate

$$\int_{\frac{1}{2}T}^{T} \frac{|\zeta(\sigma + it)|^{2k}}{|\sigma + it|^2}\,\mathrm{d}t = O(1) \qquad \text{for} \quad \sigma > \tfrac{1}{2}$$

to be equivalent to the Lindelöf hypothesis by a similar reasoning. Actually, Titchmarsh [37], §12.5 and 12.6, showed that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta(\sigma + it)|^{2k}}{|\sigma + it|^2}\,\mathrm{d}t = \int_0^\infty \Delta_k^2(x) x^{-2\sigma - 1}\,\mathrm{d}x, \tag{4.3}$$

valid for $\sigma > \beta_k$, where $\Delta_k(x)$ is the error term in the divisor problem for $\zeta(s)^k$ and β_k is the average order of Δ_k (i.e., the least real number such that $\frac{1}{X}\int_0^X \Delta_k^2(x)\,\mathrm{d}x \ll X^{2\beta_k + \epsilon}$) and $\frac{1}{2} - \frac{1}{2k} \leq \beta_k \leq 1 - \frac{2}{k+1}$. His reasoning is based on the Perron-type representation

$$\Delta_k(x) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \zeta(s)^k \,\frac{x^s}{s}\,\mathrm{d}s$$

and an application of Parseval's formula for Mellin transforms. Obviously, we can interpret the improper integrals in (4.3) as either certain ergodic Cesàro limits, similar to those in Theorem 4.1, or as expectation value of the Cauchy random walk similar to the one introduced by Lifshits & Weber. There is also an interpretation as solution to a Dirichlet problem (as in Section 3) which implies that these integrals are essentially harmonic functions.

We should not forget to mention that Ivić [19] proved, among other identities,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\zeta(\sigma+it)|^2}{\sigma^2+t^2}\,\mathrm{d}t=\int_0^{\infty}(x-\lfloor x\rfloor)^2x^{-2\sigma-1}\,\mathrm{d}x$$

for $0<\sigma<1$, which covers case $k=1$ of (4.3).

Next we want to give an equivalent formulation of the Riemann hypothesis in terms of our ergodic transformation. It is widely expected that if the Riemann hypothesis is true, this should be related to the Euler product $\zeta(s)=\prod(1-p^{-s})^{-1}$, where the product is over all prime numbers p, although this representation is valid only for $\operatorname{Re} s>1$. This belief is grounded on counterexamples to the Riemann hypothesis which have a Dirichlet series expansion and satisfy a Riemann-type functional equation but lack an Euler product (see [37], §10.25). In many reformulations of the Riemann hypothesis one can find a multiplicative feature inside. For our purpose we replace the zeta-function by its logarithm which is, thanks to the Euler product, also a Dirichlet series in the half-plane of convergence $\operatorname{Re} s>1$. We denote the nontrivial (non-real) zeros of $\zeta(s)$ by ρ. Balazard, Saias & Yor [6] proved

$$\frac{1}{2\pi}\int_{\operatorname{Re} s=\frac12}\frac{\log|\zeta(s)|}{|s|^2}|\,\mathrm{d}s|=\sum_{\operatorname{Re}\rho>\frac12}\log\left|\frac{\rho}{1-\rho}\right|, \tag{4.4}$$

and deduced (the obvious consequence) that the Riemann hypothesis is true if, and only if, the integral vanishes; a slightly different proof has been given by Burnol [9]. Note that $\log|\zeta(s)|$ is integrable with respect to $\mathrm{d}s/|s|^2|$ for s on the critical line. Substituting $t=\frac{\tau}{2}$ the integral in (4.4) can be rewritten as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log|\zeta(\tfrac12+it)|\frac{\mathrm{d}t}{|\frac12+it|^2}=\frac{1}{\pi}\int_{\mathbb{R}}\log|\zeta(\tfrac12+\tfrac12 i\tau)|\frac{\mathrm{d}\tau}{1+\tau^2},$$

which Balazard, Saias & Yor also interpret in terms of a Brownian motion; if B_t denotes the complex Brownian motion starting at the origin and τ is the passage time to the critical line, then the imaginary part of B_τ has Cauchy distribution with scale 2 (or $\frac12$ depending on the literature). We may interpret this integral as limit of Cesàro means under application of the pointwise ergodic theorem; notice that $\log|\zeta(s)|/|s|^2$ is integrable on $s=\frac12+i\mathbb{R}$ (as a short computation shows). This leads to

Theorem 4.2. *For almost all $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \log|\zeta(\tfrac{1}{2} + \tfrac{1}{2} iT^n x)| = \sum_{\operatorname{Re}\rho>\frac{1}{2}} \log\left|\frac{\rho}{1-\rho}\right|;$$

in particular, the Riemann hypothesis is true if, and only if, one and thus either side vanishes, the left-hand side for almost all real x.

The set of exceptional values x is expected to be the set $\{T^{-n}0\}_{n\in\mathbb{N}_0}$ of preimages of 0. Although this equivalent does not provide any reasonable approach towards Riemann's hypothesis we could not resist to check the statement of Theorem 4.2 for various values of x numerically. For instance, with the initial value $x = 42$ we found

$$10^{-6} \sum_{0\le n<10^6} \log|\zeta(\tfrac{1}{2} + iT^n 42)| \;=\; -0.00004\,45327\ldots.$$

§ 5. Variations on Our Theme: Other Zeta-Functions

In the previous section we have already investigated the behaviour of other functions than $\zeta(s)$ under the ergodic transformation T, namely the real logarithm of the zeta-function and powers $|\zeta(s)|^{2k}$. It might be interesting to study with $\zeta(s)^k$ other powers as well or to consider L-functions. Exploiting the analyticity of $\ell(s)$ in Theorem 1.1, one may also use differentiation to obtain further results for derivatives. This leads to amusing identities as

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\zeta'}{\zeta}(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2} = \frac{\int_{\mathbb{R}} \zeta'(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2}}{\int_{\mathbb{R}} \zeta(s+i\tau)\frac{\mathrm{d}\tau}{1+\tau^2}},$$

valid for $\operatorname{Re} s > 1$. However, we want to study another type of zeta-function in this context which is related to work of Garunkštis and the author [11, 12].

It was first shown by Littlewood [30] that the Lindelöf hypothesis follows from the Riemann hypothesis. Later Backlund [3] proved that the Lindelöf hypothesis is equivalent to the much less drastic but yet unproved hypothesis that for every $\sigma > \frac{1}{2}$ the number of hypothetical exceptional zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta > \sigma$ and $T < \gamma < T + 1$ is $o(\log T)$ as $T \to \infty$. Furthermore, the Lindelöf hypothesis implies the classical density hypothesis which claims

$$N(\sigma, T) \ll T^{2(1-\sigma)+\epsilon}, \tag{5.1}$$

where $N(\sigma, T)$ counts the zeros $\rho = \beta + i\gamma$ with $\beta > \sigma$ and $0 < \gamma < T$ (see [18], Notes to Chapter 11). It should be noted that the Euler product is essential for this proof. For Lerch zeta-functions $L(\lambda, \alpha, s)$ with real parameters $\lambda, \alpha > 0$, initially defined by

$\sum_{m=0}^{\infty} \exp(2\pi i\lambda m)(m+\alpha)^{-s}$ for $\operatorname{Re} s > 1$ and by analytic continuation elswhere, except for a pole at $s = 1$ in case $\lambda \in \mathbb{Z}$, the analogue of the density hypothesis is not true in general. For instance, $L(\frac{1}{2}, 1, s) = (1 - 2^{1-s})\zeta(s)$ has infinitely many zeros off the critical line and violates the analogue of (5.1). However, $L(\frac{1}{2}, 1, s)$ obviously satisfies the analogue of the Lindelöf hypothesis provided $\zeta(s)$ satisfies the Lindelöf hypothesis. It is known that a generic Lerch zeta-function has many zeros off the critical line (see [11]) which indicates that if $L(\lambda, \alpha, s)$ has no Euler product representation, i.e., apart from $\lambda, \alpha \in \{\frac{1}{2}, 1\}$, the density hypothesis does not follow from the analogue of the Lindelöf hypothesis for Lerch zeta-functions, although the latter conjecture might be reasonable (see [12]).

A similar reasoning as in Section 2 yields, for $\operatorname{Re} s > -\frac{1}{2}$,

$$\frac{1}{\pi} \int_{\mathbb{R}} L(\lambda, \alpha, s + i\tau) \frac{\mathrm{d}\tau}{1 + \tau^2} = L(\lambda, \alpha, s + 1)$$

provided $\lambda \notin \mathbb{Z}$; otherwise the Lerch zeta-function reduces to the Hurwitz zeta-function $\zeta(s, \alpha) = L(1, \alpha, s)$ and we have to take the pole at $s = 1$ into account which, by calculus of residues, would add a further term on the right-hand side as in case of the Riemann zeta-function.

In view of Lerch and Hurwitz zeta-functions one may also think about ergodic transfomations of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ which we may identify with the unit interval $[0, 1)$. For instance, given a real number θ, the circle rotation R_θ is defined by $R_\theta : \mathbb{T} \to \mathbb{T},\ x \mapsto R_\theta x = x + \theta \bmod 1$. It is easily seen that R_θ is ergodic with respect to Lebesgue measure if, and only if, θ is irrational (see [10]). As another simple consequence of the pointwise ergodic theorem we note

Theorem 5.1. *Let θ be irrational and $s \neq 1$. Then, for almost all $x \in [0, 1)$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} |\zeta_*(s, R_\theta^n x)|^2 = \int_1^2 |\zeta(s, \alpha)|^2 \,\mathrm{d}\alpha.$$

where $\zeta_*(s, \alpha) := \zeta(s, \alpha) - \alpha^{-s}$.

There have been many investigations on the integral on the right-hand side. The most powerful method has been developed by Katsurada & Matsumoto in a series of papers starting with [22]. The particularly beautiful result

$$\begin{aligned} &\int_1^2 |\zeta(\tfrac{1}{2} + it, \alpha)|^2 \,\mathrm{d}\alpha \\ &= \gamma - \log 2\pi + \operatorname{Re} \frac{\Gamma'}{\Gamma}(\tfrac{1}{2} + it) - 2\operatorname{Re} \sum_{n=0}^{\infty} \frac{\zeta(\frac{1}{2} + n + it)}{\frac{1}{2} + n + it} \end{aligned}$$

belongs to Andersson [2]. Further mean-value results for the Hurwitz and Lerch zeta-functions can be found in Laurinčikas & Garunkštis [26], §3.4, which may be used to derive more results of this flavour.

§ 6. Numerical Experiments and Complex Iteration

We illustrate the dynamics of the transformation $x \mapsto Tx = \frac{1}{2}(x - \frac{1}{x})$ with two simple examples:

$$42 \to \tfrac{1763}{84} \to \tfrac{3\,101\,113}{296\,184} \to \tfrac{9\,529\,176\,876\,913}{1\,837\,000\,105\,584} \to \tfrac{87\,430\,642\,563\,577\,769\,204\,428\,513}{35\,010\,197\,858\,035\,584\,743\,964\,384} \to \ldots,$$

and

$$\sqrt{2} \to \tfrac{1}{4}\sqrt{2} \to -\tfrac{7}{8}\sqrt{2} \to -\tfrac{17}{112}\sqrt{2} \to \tfrac{5893}{3808}\sqrt{2} \to \tfrac{28\,545\,857}{45\,566\,528}\sqrt{2} \to \ldots .$$

If x is rational, so are the elements of the orbit $\{T^n x\}$. Moreover, algebraic irrationality, resp. transcendence is inherited by T. The first negative number in the orbit of 42 appears after seven iterations, the first negative in the orbit of $\sqrt{2}$ after only two iterations. Since T is ergodic, almost all orbits lie dense in $\mathbb{R}$ and thus contain arbitrarily large and arbitrarily small negative real numbers; more precisely, for almost all x the sojourn time of the orbit $\{T^n x\}$ in any interval (α, β) is given by the positive quantity

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbf{1}_{(\alpha,\beta)}(T^n x) &= \mathbf{P}((\alpha,\beta)) \\ = \frac{1}{\pi}\int_\alpha^\beta \frac{\mathrm{d}\tau}{1+\tau^2} &= \frac{1}{\pi}(\arctan\beta - \arctan\alpha),\end{aligned}$$

where $\mathbf{1}_{(\alpha,\beta)}$ is the indicator function of the interval (α, β). Poincaré's recurrence theorem and Kac's quantitative refinement [20] state that, given a measurable set A, almost all points $x \in A$ will return infinitely many times to A and the expectation for the first return equals $\mathbf{P}(A)^{-1}$. For an illustration we notice $\mathbf{P}((10^5, 10^6))^{-1} = 349\,065.85\ldots$. The first element of the orbit for $x = 42$ in the interval $(10^5, 10^6)$ appears after $n = 251\,524$ iterations; its first return to this interval is thus expected at time $n \approx 600\,589$. This shows that large values of $T^n x$ are rare events although they appear with a certain frequency. Therefore, the following numerical material can only provide a very modest impression.

The rapid growth of fractions is causing long computation time, hence we have used machine-size real numbers rather than the correct but inexpressable exact values in our numerical experiments. In particular, we have tested (1.1). For

$$c_k := \frac{1}{10^k} \sum_{0\le n<10^k} \zeta(\tfrac{1}{2} + iT^n 42),$$

we computed

$$c_4 = -0.04092\ldots + i\,0.00288\ldots,$$
$$c_5 = -0.05357\ldots + i\,0.00022\ldots,$$
$$c_6 = -0.05362\ldots - i\,0.00043\ldots.$$

We have also tried other values for x and observed a similar slow tendency towards $-0.05429\ldots$, the limit value of (1.1) (although this data is not indicating convergence). In general the convergence in ergodic theorems can be arbitrarily slow (cf. [24], §1.2.3) which is rather different from the Cauchy random walk (1.6) investigated by Lifshits & Weber. That the limit value is almost zero might be explained by the remarkable 'almost symmetry' of the graph of the curve $\mathbb{R} \ni t \mapsto \zeta(\frac{1}{2}+it)$ with respect to the real axis and the appearance of values $\zeta(\frac{1}{2}+it)$ in the left half-plane for small values of t.

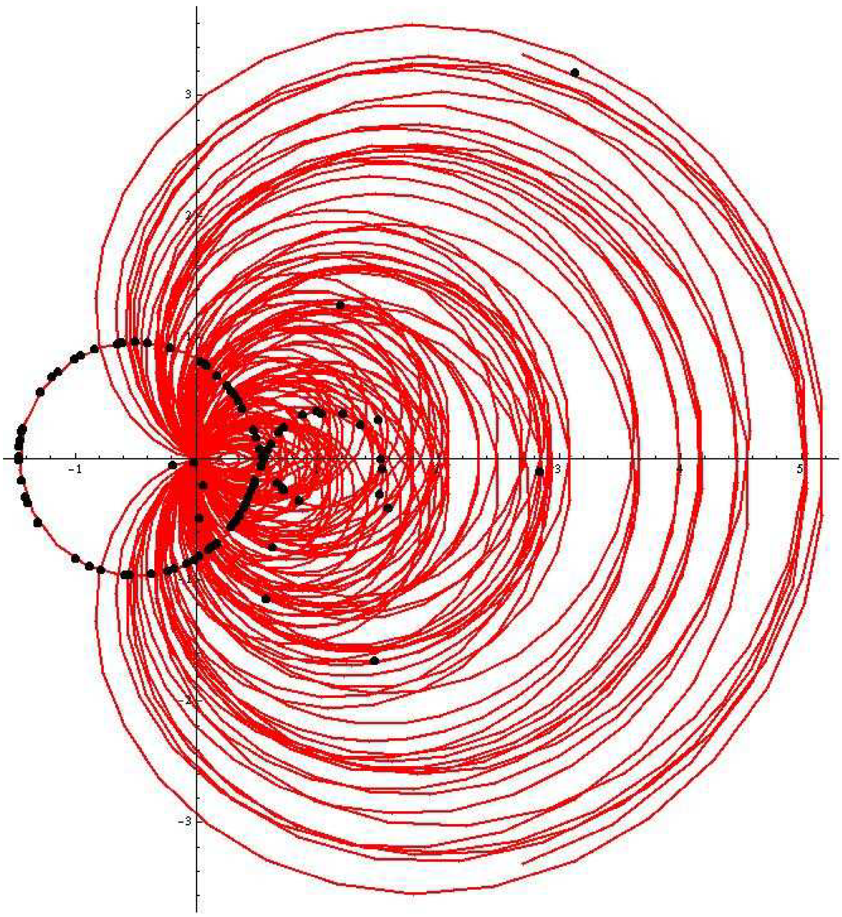

Figure 1. The values of $\zeta(\frac{1}{2}+it)$ as $-155 \leq t \leq 155$ in red and the values of $\zeta(\frac{1}{2}+iT^n x)$ with $x = 42$ for $0 \leq n < 100$ in black; the range for t is according to the values $T^n 42$.

Moreover, we have tested the behaviour on the vertical line $s = -1 + i\mathbb{R}$ which is outside the half-plane for which Theorem 1.1 is valid. Testing

$$d_k := \frac{1}{10^k} \sum_{0 \le n < 10^k} \zeta(-1 + iT^n 42),$$

we found a visible divergence:

$$\begin{aligned} d_4 &= -\ 7.8003\ldots + i\, 18.1683\ldots, \\ d_5 &= -\ 3.2030\ldots - i\ \ 6.2725\ldots, \\ d_6 &= -13.0623\ldots - i\, 18.6014\ldots. \end{aligned}$$

Notice that $\zeta(s)$ is not $\mathbf{P}$-integrable on vertical lines $s = \sigma + i\mathbb{R}$ with fixed $\sigma < -\frac{1}{2}$ (as follows from (2.2)). The following converse of the pointwise ergodic theorem is true: if the Cesàro means $\frac{1}{N}\sum_{0 \le n < N} f(T^n x)$ converge to a finite limit almost everywhere, where T is an ergodic transformation in some finite measure space and f is measurable and non-negative, then f is integrable (see [13], p. 32); however, as Gerstenhaber's counterexample (see [40], §4.1) shows, the assumption on f to be non-negative cannot be dropped. Nevertheless the above computations suggest the divergence of the Cesàro means for $\mathrm{Re}\, s < -\frac{1}{2}$.

It might be interesting to consider also complex x. Here computer experiments show a fast convergence of $T^n x$ to $\pm i$ which reminds us on the origin of T as Newton iteration of the quadratic polynomial $1 + \tau^2$. Following an idea of Schröder [32] (from 1871, so before the investigations of Julia and Fatou, and long before the Mandelbrot set became popular), we conjugate our map $x \mapsto Tx = \frac{1}{2}(x - \frac{1}{x})$ with the Möbius transform $\omega = M^{-1}\tau := \frac{\tau - i}{\tau + i}$, which maps the real line to the unit circle minus 1 (the image of infinity). Hence, with the inverse $M\omega = i\frac{1+\omega}{1-\omega}$ (which appeared already in Section 3) we obtain

$$M^{-1} \circ T \circ M\omega = M^{-1}\left(i\frac{1+\omega^2}{1-\omega^2}\right) = \omega^2.$$

Setting $\mathcal{T} = M^{-1} \circ T \circ M$, we have $\mathcal{T}^n = M^{-1} \circ T^n \circ M$ and thus we can deduce the behaviour of the iterations T^n from those of $\mathcal{T}$. The dynamics of the latter map are easy to understand and we may conclude that T has two fixed points $\pm i$ and the associated basins of attractions are the upper and the lower half-plane, respectively. The Julia set of T is their boundary, hence the real axis, which corresponds to our observations. Recall that the Julia set consists, roughly speaking, of those values for which an arbitrarily small perturbation can cause a drastic change in the sequence of iterated values. We conclude that $T^n x$ converges to $\pm i$ according to the imaginary part of x being positive or negative, and, consequently, the limit of the Cesàro mean equals $\zeta(s \mp 1)$.

§ 7. Concluding Remarks

The value-distribution of the zeta-function is a fascinating topic with beautiful results and a bunch of open questions. Selberg (unpublished) proved that, after a suitable normalization, the values of the zeta-function on the critical line are Gauss-normal distributed (cf. [37], §11.13): let $\mathcal{R}$ be an arbitrary fixed rectangle in the complex plane whose sides are parallel to the real and the imaginary axes, then

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{t\in(0,T]\,:\,\frac{\log\zeta\left(\frac{1}{2}+it\right)}{\sqrt{\frac{1}{2}\log\log T}}\in\mathcal{R}\right\}$$
$$=\frac{1}{2\pi}\iint_{\mathcal{R}}\exp\left(-\tfrac{1}{2}(x^2+y^2)\right)\,\mathrm{d}x\,\mathrm{d}y.$$

However, it is unknown whether the set of values of the zeta-function on the critical line is dense in the complex plane. It is even no complex number different from zero explicitly known which is assumed by $\zeta(s)$ on the critical line (infinitely often or just once). On the contrary, for vertical lines to the right there are deep results due to Bohr and his collaborators (see [37], Chapter XI) and the remarkable universality theorem (which will be briefly mentioned below). In studies of the value distribution of the zeta-function inside the critical strip the method of choice are often discrete and continuous moments.

Recently, other types of discrete first moments of the zeta-function than in our article have been studied. Kalpokas & Steuding [21] investigated the values of $\zeta(s)$ of the critical line which have a fixed argument φ modulo π and proved, among other things, the existence of their mean-value: for any $\phi\in[0,\pi)$, as $T\to\infty$,

$$\sum_{\substack{0<t\leq T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}}\zeta\left(\tfrac{1}{2}+it\right)=2e^{i\phi}\cos\phi\,\frac{T}{2\pi}\log\frac{T}{2\pi e}+O\left(T^{\frac{1}{2}+\epsilon}\right);$$

note that the number of values $0<t<T$ with $0\neq\zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}$ is asymptotically equal to $\frac{T}{2\pi}\log T$. Steuding & Wegert [36] considered the values of the zeta-function on certain arithmetic progressions on vertical lines in the crtitical strip; interestingly, the mean-value depends on the difference of the arithmetic progression in a rather irregular way (according to its diophantine nature). Both examples show the existence of the first moments indicating that the zeta-function takes not too many extremely large values on vertical lines inside the critical strip.

Besides these discrete moments for zeta on deterministic sequences also discrete moments associated with random sequences have been studied. The work of Lifshits & Weber has been extended by Shirai [33] to a subclass of Lévy processes for which a similar phenomenon was observed. His class consists of so-called symmetric α-stable processes $\mathcal{S}_n$ which includes besides the Cauchy random walk ($\alpha = 1$) also the Brownian motion ($\alpha = 2$). Here the characteristic function is given by

$$\mathbf{E}\exp(i\lambda\mathcal{S}_n) = \exp(-n|\lambda|^\alpha).$$

Shirai succeeds in proving an analogue of the theorem of Lifshits & Weber for $1 \leq \alpha \leq 2$. Surprisingly, also in his Rademacher-Menchoff type theorem the expectation of $\frac{1}{N}\sum_{1\leq n\leq N}\zeta(\frac{1}{2}+i\mathcal{S}_n)$ equals one, so is independent of α, and the only impact of α is visible in the remainder term. Shirai remarks that one should try to investigate besides these recurrent random walks also transient random walks, which appear for $\alpha \in (0,1)$, since they better fit to the assertion of the Lindelöf hypothesis which is a statement about the zeta-function at values on the critical line with increasing ordinates.

We conclude with a few historical remarks. Adler & Weiss [1] traced back the transformation $x \mapsto x - \frac{1}{x}$ to a paper of Boole [8] from the second half of the nineteenth century in which he observed the remarkable identity

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x - \tfrac{1}{x})\,\mathrm{d}x,$$

valid for all continuous functions f. Adler & Weiss proved that Boole's transformation is indeed ergodic as the dear reader probably already suspected. Other ergodic transformations of the real line are also of interest, in particular those considered by Li & Schweiger [27] and Ishitani & Ishitani [17]; it is the authors aim to study the zeta-function with respect to those in a sequel to this article. Another line of investigation concerns the Cesàro means (4.1) and the explicit evaluation of (4.2).

It should be noted that Birkhoff proved his famous pointwise ergodic theorem only for indicator functions. It was Khintchine [23] who extended this result to arbitrary integrable functions on a finite measure space. Therefore, it is appropriate to speak of the Birkhoff-Khintchine theorem or just the pointwise ergodic theorem. This ergodic theorem plays a central role in probabilistic proofs of universality results on approximation properties of $\zeta(s)$ and other L-functions due to Bagchi [4, 5] (see also [25, 35]). To get an impression recall that Voronin's universality theorem [38] states that, roughly speaking, any non-vanishing analytic function $f(s)$ on the right half of the critical strip can be uniformly approximated on arbitrary compact subsets K by certain imaginary shifts of the Riemann zeta-function: there exists a real number τ such that $|\zeta(s+i\tau) - f(s)| < \epsilon$ for all $s \in K$. In some sense ergodic theory replaces the corresponding part of Weyl's uniform distribution theorem in Voronin's original approach. Building on classical work of Bohr, Bagchi used universality to give an equivalent reformulation of the Riemann hypothesis in terms of a self-approximation property. However, the author could not find any other applications of ergodic theory to the zeta-function.

Finally, we would like to mention that Lindelöf [29] expressed his belief that $\zeta(s)$ is bounded on any vertical line $s = \sigma + i\mathbb{R}$ with fixed $\sigma \in (\frac{1}{2}, 1)$. This would imply (1.7), however, Lindelöf's boundedness conjecture is false as one can easily deduce, for example, from Voronin's universality theorem.

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