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On hybrid universality for L-functions

By

Łukasz Pańkowski*

Abstract

In this paper we present a recent survey on hybrid universality on L-functions, which is a connection of a universality theorem and the Kronecker approximation theorem.

§ 1. Introduction

Universality is one of the most interesting phenomenon in modern mathematics. K.-G. Grosse-Erdmann in [GE] proposed the following precise definition of universality in the language of topology. Let X and Y be two topological spaces and $T_\tau : X \to Y$ $(\tau \in I)$ be a family of mappings. Then an element $x \in X$ is called universal if each element in Y can be approximated by $T_\tau(x)$ for certain $\tau \in I$ or, equivalently, if the set $\{T_\tau(x) : \tau \in I\}$ is dense in Y. Following this we can give a definition of joint universality. Let X and $Y_1, \ldots, Y_n$ be topological spaces and $T_\tau^{(i)} : X \to Y_i$ $(\tau \in I,\ 1 \leqslant i \leqslant n)$ be a family of mappings. Then we say that elements $x_1, \ldots, x_n$ in X are jointly universal if the set $\{(T_\tau^{(1)}, \ldots, T_\tau^{(n)}) : \tau \in I\}$ is dense in the product space $Y_1 \times \cdots \times Y_n$.

For a historical overview of universality in functional spaces we refer to [GE] and [L1]. In our point of view, the most important result was shown by Voronin in [V1]. He indicated the first explicitly given universal object by proving that the Riemann zeta-function is universal with respect to imaginary shifts in the space of analytic functions.

Theorem A (Voronin [V2]). *Let $0 < r < \frac{1}{4}$ and a function $f(s)$ be non-vanishing and continuous on the disk $|s| \leqslant r$ and analytic in the interior. Then for every $\varepsilon > 0$*

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The author was partially supported by the grant no. N N201 6059 40 from National Science Centre.
*Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznan, POLAND.
e-mail: `lpan@amu.edu.pl`

there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s|\leqslant r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon. \tag{1.1}$$

This result was generalized in many directions. For example, Reich [R] and Bagchi [B1] showed that the disk can be replaced by a compact subset of $D := \{s \in \mathbb{C} : \frac{1}{2} < \mathcal{R} < 1\}$ with connected complement, and that the set of translations $\zeta(s+i\tau)$ has a positive lower density. Recall that for a Lebesgue measurable set $A \subset (0,\infty)$ we define the *natural density* of A as

$$\lim_{T\to\infty}\frac{\mu(A\cap(0,T])}{T},$$

if the limit exists, where μ denotes the Lebesgue measure on $\mathbb{R}$. Moreover, if the lower limit of the same quotient is positive, then we say that the set A has a positive lower density.

Moreover, for the last thirty years numerous analogues of the Voronin theorem for other L-functions important in number theory were discovered by many mathematicians. For instance, universality theorems were proved for Dirichlet L-functions, Hecke L-functions, L-functions associated to newforms and many others. Furthermore, it turns out that there exist also a lot of zeta-functions with strong universality property, where the attribute strong means that also functions having zeros can be approximated. The first example comes from Bagchi [B1] and Gonek [Go], who showed, independently, the strong universality theorem for Hurwitz zeta-functions $\zeta(s,\alpha)$, if α is transcendental or a rational number $\neq 1/2, 1$. Afterwards, strong universality was proved for many interesting generalizations of Hurwitz zeta-functions like Lerch zeta-functions (see [LM3], [N1]) or periodic Hurwitz zeta-functions (see [JL3], [LS]). For more examples and a comprehensive overview of the theory of universal zeta-functions we refer to the monograph [S].

On the other hand, let us consider the circle $S^1 = \{s \in \mathbb{C} : |s| = 1\}$ with topology induced from $\mathbb{C}$ and mappings $T_\tau : \mathbb{R} \to S^1$ ($\tau \in \mathbb{R}$) given by $T_\tau(x) = e(\tau x) := e^{2\pi i \tau x}$. In the above notation the classical Kronecker approximation theorem says that any real elements $\alpha_1, \dots, \alpha_n$ linearly independent over $\mathbb{Q}$ are joint universal in the product space $S^1 \times \cdots \times S^1$. Again it turns out that the set of $\tau \in \mathbb{R}$ such that $(e(\tau\alpha_1), \dots, e(\tau\alpha_n))$ approximates with error $\varepsilon > 0$ an arbitrary point $(e(\theta_1), \dots, e(\theta_n))$ has a positive density equal to $(2\varepsilon)^n$. If we denote by $||x||$ for $x \in \mathbb{R}$ the distance from x to the nearest integer, the Kronecker theorem could be formulated as follows (see [KV]).

Theorem B. *If $\alpha_1, \dots, \alpha_n$ are real numbers linearly independent over $\mathbb{Q}$ and $\theta_1, \dots, \theta_n$ are real numbers, then, for every positive $\varepsilon_1, \dots, \varepsilon_n < 1/2$, the set of τ such that*

$$\max_{1\leqslant i\leqslant n}||\tau\alpha_i - \theta_i|| < \varepsilon_i \tag{1.2}$$

has a positive density equal to $2^n \prod_{1\leqslant i\leqslant n} \varepsilon_i$.

The following question arises in a natural way: does the set of real numbers satisfying simultaneously (1.1) and (1.2) have a positive lower density? Using some ideas of Good [Goo], a partial answer was firstly given by Gonek for Dirichlet L-functions (for example see [SS, Theorem A]).

Theorem C (Gonek). *Let q be a positive integer and let $K \in D$ be a compact set with connected complement. Suppose that for each prime $p|q$ we have $0 \leqslant \theta_p < 1$ and that for each character χ mod q, the function $g(s)$ is continuous on K and analytic in the interior. Then, for every $\varepsilon > 0$, there is a real number τ such that*

$$\max_{p|q}\left\|\frac{\tau\log p}{2\pi}-\theta_p\right\|<\varepsilon \qquad \textit{and} \qquad \max_{\chi \bmod q}\max_{s\in K}|L(s+i\tau,\chi)-g(s)|<\varepsilon.$$

Recently, a slightly improvement was given by Kaczorowski and Kulas in [KK]. They showed that in Gonek's theorem it suffices to assume that Dirichlet characters are pairwise non-equivalent and that the set of all parameters τ satisfying the above inequalities has a positive lower density.

In this paper we present a recent survey of the author, which gives a complete answer to our question and significantly extend a class of functions having this property. For convenience, we use the following definition of hybrid universality.

Definition 1.1. We say that a set of functions $\{L_1,\ldots,L_n\}$ is *hybridly jointly universal* if for every compact set $K \subset D$ with connected complement, any functions $f_1,\ldots,f_n$ continuous and non-vanishing on K, which are analytic in the interior of K, any real numbers $(\alpha_i)_{(1\leqslant i\leqslant m)}$ linearly independent over $\mathbb{Q}$ and any real numbers $(\theta_i)_{1\leqslant i\leqslant m}$, the set of real numbers τ such that

$$\max_{1\leqslant j\leqslant n}\max_{s\in K}|L_j(s+i\tau)-f_j(s)|<\varepsilon \qquad \text{and} \qquad \max_{1\leqslant j\leqslant m}\|\tau\alpha_j-\theta_j\|<\varepsilon$$

has a positive lower density.

Moreover, we say that the set $\{L_1,\ldots,L_n\}$ is *hybridly strongly jointly universal* if functions having zeros can be approximated as well.

In Section 2 we define a wide class of functions having Euler product, which contains for instance Dirichlet L-functions. In this class we formulate the hybrid universality theorem and give the main steps of the proof. In Section 3 we shall deal with hybrid strong universality for an axiomatically defined class of zeta-functions without Euler product like Lerch zeta-functions, twists of Lerch zeta-functions and periodic Hurwitz zeta-functions. At the end, in Section 4, we present some application of hybrid universality. For example, we show how to apply this property to prove universality for new L-functions and obtain self-approximation property.

§ 2. Hybridly universal L-functions

§ 2.1. The class of L-functions

Throughout this section we will work with L-functions defined as the following Euler product (see [KV, Chapter 7]).

$$F(s) = \prod_{p \in \mathbb{P}} R_p(p^{-s}), \qquad (\sigma > 1) \tag{2.1}$$

where p runs over all primes and $R_p(z) = 1 + \sum_{m=1}^{\infty} a(p^m) z^m$ are rational functions, analytic and non-vanishing on the disk $|z| < 1$.

First of all, we assume the following Ramanujan conjecture on the size of the coefficients:

(E1) $\forall_{\varepsilon > 0}\ a(p^m) \ll_\varepsilon p^{\varepsilon m}$ uniformly in p.

One can easily show that (E1) implies that $F(s)$ is an absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \qquad (\sigma > 1)$$

where $a(n) = \prod_{p|n} a\left(p^{\nu_p(n)}\right)$ and $\nu_p(n)$ is the exponent of the prime p in the prime factorization of the integer n. Obviously, one can prove that $a(m) \ll m^\varepsilon$ for every ε.

Now, let us describe some analytic properties which should be satisfied by $F(s)$. Obviously, it is natural to ask for an analytic continuation, especially that, as was shown by Bohr [Bo], all absolutely convergent Dirichlet series are almost periodic function in the half-plane of absolute convergence and we can not expect the universality theorem for them.

(E2) F has meromorphic continuation to the half-plane $\sigma > \frac{1}{2}$. It can possess at most a finite number of poles and all of them must lie on the straight line $\sigma = 1$.

The following condition gives us a restriction on order of growth.

(E3) F is a function of finite order, which means that

$$F(s) \ll_\sigma |t|^{A(\sigma)} \qquad \text{for } \sigma > \frac{1}{2},\ |t| \to \infty.$$

The last axiom is required by Carlson's theorem (for example see [KV, Theorem A.2.10]) which plays a crucial role in the proof of universality.

(E4) For any fixed $\frac{1}{2} < \sigma < 1$ the square mean-value

$$\frac{1}{T} \int_{-T}^{T} |F(\sigma + it)|^2 dt$$

is bounded as $T \to \infty$.

By $\mathcal{E}$ we denote the class consisting of functions of the form (2.1) and satisfying (E1)–(E4).

§ 2.2. Approximation by a finite product and sum

In the proof of universality, the following property, called acceptability, plays a crucial role. Roughly speaking, it says that every analytic function can be approximated by a certain finite product $F_M(s,\Theta) = \prod_{p\in M} R_p(p^{-s}e(-\theta_p))$ associated to $F \in \mathcal{E}$, where $\Theta = (\theta_p)_{p\in\mathbb{P}} \subset \mathbb{R}$ and $M \subset \mathbb{P}$ is a finite set of primes. Following [KK] we call an open and bounded subset G of $\mathbb{C}$ *admissible*, when for every $\varepsilon > 0$ the set $G_\varepsilon = \{s \in \mathbb{C} : |s-w| < \varepsilon \text{ for certain } w \in G\}$ has connected complement.

Definition 2.1. We say that the set $\{F_1,\dots,F_n\} \subset \mathcal{E}$ is *acceptable* if it satisfies the following condition: for every finite set $A \subset \mathbb{P}$, arbitrary admissible domain G such that $\overline{G} \subset D$, every analytic and non-vanishing functions $f_1,\dots,f_n$ on the closure $\overline{G}$, there exists a sequence of finite sets $M_1 \subset M_2 \subset \dots \subset \mathbb{P}$ such that

$$\bigcup_{k=1}^{\infty} M_k = \{p \,:\, p \notin A\}$$

and for certain $\Theta_k = (\theta_{kp})_{p\in\mathbb{P}}$

$$F_{j,M_k}(s,\Theta_k) \to f_j(s) \qquad \text{uniformly for } s \in \overline{G},\ j=1,\dots,n$$

as $k \to \infty$.

Remark. The assumption that M_k's are disjoint to a given set A is crucial only to obtain hybrid type of universality.

By the continuity of logarithm and the following lemma, one can easily see that an approximation by a finite product $F_M(s,\Theta)$ in the above definition can be replaced by an approximation of functions $\log f_1(s),\dots,\log f_n(s)$ by finite sums of the form $\sum_{p\in M'} \frac{a(p)e(-\theta_p)}{p^s}$.

Lemma 2.2. *For any $F \in \mathcal{E}$ and any sequence $\Theta = (\theta_p)_p \in \Omega$ the series*

$$\sum_{p\in\mathbb{P}} \left(\log R_p(p^{-s}e(-\theta_p)) - \frac{a(p)e(-\theta_p)}{p^s} \right)$$

is absolutely convergent in the half-plane $\sigma > \frac{1}{2}$.

Unfortunately, in practice, it is difficult to check whether functions from $\mathcal{E}$ form an acceptable set. Usually (see for example the proof of Lemma 6 in [KK]), we need to work with certain Hilbert space of analytic functions and apply Phragmén–Lindelöf principle and Pechersky's generalization of the Riemann rearrangement theorem.

In order to easily verify the acceptability we use the following result, which combining with Lemma 2.2 gives some sufficient conditions for functions from an acceptable set.

Lemma 2.3. *Suppose that an increasing to infinity sequence $(\lambda_n) \subset \mathbb{R}$ and sequences $(a(n)), (b_1(n)), \dots, (b_m(n)) \subset \mathbb{C}$ are such that*

(i) *there exists an integer k such that* $\operatorname{rank}[b_j(n)]_{1\leqslant j\leqslant m}^{1\leqslant n\leqslant k} = m$

(ii) $\sum_{n=1}^{\infty} |a(n)b_j(n)|^2 e^{-2\lambda_n\sigma} < \infty$ *for $\sigma > 1/2$ and $j = 1, 2, \dots, m$*

(iii) *there exists $c > 0$ such that for all $B > 0$ and $n_0 \in \mathbb{N}$ we have*

$$\sum_{\substack{x<\lambda_n\leqslant x+B/x^2\\ (b_1(n),\dots,b_m(n))=(b_1(n_0),\dots,b_m(n_0))\\ |a(n)|>c}} 1 \gg e^{(1-\varepsilon)x}$$

for every sufficiently small $\varepsilon > 0$.

Moreover let G be an admissible set with $\overline{G} \subset D$, $f_1, \dots, f_m$ any analytic functions on $\overline{G}$, and $A \subset \mathbb{N}$ a finite set. Then, for every $y > 0$ and $\varepsilon > 0$, there exists a finite set $M \subset \mathbb{N}\setminus A$ containing all integers less than y, and there exist real numbers θ_n, ($n \in M$), such that

$$\max_{1\leqslant j\leqslant m} \max_{s\in\overline{G}} \left| \sum_{n\in M} \frac{a(n)b_j(n)e(-\theta_n)}{e^{\lambda_n s}} - f_j(s) \right| < \varepsilon.$$

Proof. The proof of the lemma can be found in [P1] and basically follows the proof of Lemma 6 in [KK]. □

Proposition 2.4. *Let $F_1, \dots, F_n \in \mathcal{E}$, where*

$$F_j(s) = \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{a^{(j)}(p^m)}{p^{sm}}\right).$$

Moreover, assume that $\lambda_m = \log p_m$, where p_m is the m-th prime number, and $b^{(j)}(m) := a^{(j)}(p_m)$ satisfy the condition (i), (iii) from Lemma 2.3. Then the set $\{F_1, \dots, F_n\}$ is acceptable.

§ 2.3. The classical Kronecker approximation theorem

The next step in the proof of universality is to show that the factors $e(-\theta_p)$ in $F_M(s, \Theta)$ can be replaced by $p^{i\tau} = e\left(\frac{\tau \log p}{2\pi}\right)$ for infinitely many real numbers τ.

To prove it, it suffices to apply Theorem B and the fact that the numbers $\log p$ are linearly independent over the field $\mathbb{Q}$ of rational numbers.

In this step, we encounter the main obstacle in proving a hybrid version of universality (recall Definition 1.1). Obviously, in the case when $\alpha_1, \ldots, \alpha_n$ and $\left\{\frac{\log p}{2\pi}\right\}_{p\in\mathbb{P}}$ are linearly independent over $\mathbb{Q}$, it suffices to follow the original prove of the Voronin theorem.

In general case, we need to apply the following lemma (for $a_n = \frac{\log p_n}{2\pi}$), which plays a crucial role in the proof of hybrid universality.

Lemma 2.5 (Corollary 2.7 in [P3]). *Suppose that $(a_n)_{n=1}^{\infty}$ are real numbers linearly independent over $\mathbb{Q}$. Moreover assume that $\alpha_1, \ldots, \alpha_m$ are real numbers linearly independent over $\mathbb{Q}$ and $\theta_1, \ldots, \theta_m$ arbitrary real numbers. Then there exist finite sets $J \subset \{1, 2, \ldots, m\}$ and $A = A(\alpha_1, \ldots, \alpha_m) \subset \mathbb{Z}_+$ such that the set*

$$\{a_i\}_{i\in A\cup M} \cup \{\alpha_i\}_{i\in J}$$

is linearly independent over $\mathbb{Q}$ for every finite set of non-negative integers M with $M \cap A = \emptyset$.
Moreover, there exist real numbers θ_i^*, $i \in A$ and a positive integer N such that

$$\max_{i\notin J} \|N\tau\alpha_i - \theta_i\| < \varepsilon,$$

whenever the following inequalities hold

$$\max_{i\in J} \|\tau\alpha_i - \frac{\theta_i}{N}\| < \frac{\varepsilon}{N},$$

$$\max_{i\in A} \|\tau a_i - \frac{\theta_i^*}{N}\| < \frac{\varepsilon}{N}.$$

§ 2.4. Main theorem

In the last step of the proof of universality, using the Kronecker theorem for the logarithms of primes and Carlson's theorem, one shows that the set of all τ, such that $F(s + i\tau)$ approximates in some sense (see for example p.18 of [S]) the finite product $F_M(s, \Theta)$ for any given $\Theta = (\theta_p)$, has a positive lower density.

In order to get hybrid universality we need to use Lemma 2.5 for $a_n = \frac{\log p_n}{2\pi}$ instead of the Kronecker theorem. Then, since $p_n^{i\tau}$, for $n \in A$, are close to certain $e(-\theta_n^*)$ and can not approximate an arbitrary $e(-\theta_n)$, we can only obtain that the set of all τ such that $F(s+i\tau)_{|A} := F(s+i\tau)\prod_{n\in A} R_{p_n}^{-1}(a(p_n)p_n^{-s})$ approximates $F_M(s, \Theta)$, where $M \cap A = \emptyset$.

Finally combining this idea and the acceptability of the set $\{F_1, \dots, F_n\}$, one can prove that the set of real numbers τ satisfying the following inequalities

$$\max_{1\leqslant j\leqslant n} \max_{s\in\overline{G}} |F_j(s+i\tau)_{|A} - f_j(s) \prod_{n\in A} R_{p_n}^{-1}(p_n^{-s}e(-\theta_{p_n}^*))| < \varepsilon$$

$$\max_{n\in A} ||\tau \frac{\log p_n}{2\pi} - \theta_{p_n}^*|| < \varepsilon \qquad \text{and} \qquad \max_{1\leqslant i\leqslant m} ||\tau\alpha_i - \theta_i|| < \varepsilon$$

has a positive lower density, where G is a certain admissible set, θ_i's are an arbitrary given and the set A and real numbers $\theta_{p_n}^*$ come from Lemma 2.5 applying to $a_n = \frac{\log p_n}{2\pi}$.

Hence, using the Margelyan theorem and the inequality $\max_{n\in A} ||\tau\frac{\log p_n}{2\pi} - \theta_{p_n}^*|| < \varepsilon$ we get the following main theorem (for the complete proof see [P1]).

Theorem 2.6 (hybrid joint universality theorem). *An acceptable set $\{F_1, \dots, F_n\} \subset \mathcal{E}$ is hybridly jointly universal.*

Example 2.7. It is well-known that Dirichlet L-functions $L(s,\chi) = \prod_p (1 - \frac{\chi(p)}{p^s})^{-1}$ are meromorphic on $\mathbb{C}$ with possible pole at $s = 1$. Moreover $|\chi(\cdot)| = 1$ and $L(s,\chi)$ satisfies (E3) and (E4) by Theorem 5.4, Chapter IV of [Pr] and Theorem 7.2, Chapter VII of [T], respectively. Hence for every Dirichlet character χ the L-function $L(s,\chi)$ belongs to the class $\mathcal{E}$.

Moreover, putting $a(n) = 1$, $b_j(n) = \chi_j(n)$ and $\lambda_n = \log p_n$ in Lemma 2.3 one can easily see that the set $\{L(s,\chi_1), \dots, L(s,\chi_n)\}$ is acceptable for every pairwise non-equivalent Dirichlet characters $\chi_1, \dots, \chi_n$. Hence the set $\{L(s,\chi_1), \dots, L(s,\chi_n)\}$ is hybridly jointly universal.

Example 2.8. Now, let K be a finite Galois-extension of $\mathbb{Q}$ and $\chi_1, \dots, \chi_n$ linearly independent characters of the group $G(K/\mathbb{Q})$. Then it is known that Artin L-functions associated to χ_j satisfy the conditions (E1)-(E2) and have Euler product of the form (2.1). Unfortunately, for now it is only proved that $(E4)$ holds only in the half-place $1 - \frac{1}{2k} < \sigma < 1$, where $k = \sharp G(K/\mathbb{Q})$. Hence, verifying the assumption of Lemma 2.3, we can show that $\{L(s,\chi_1), \dots, L(s,\chi_n)\}$ is hybridly jointly universal in the half-plane $1 - \frac{1}{2k} < \sigma < 1$.

§ 3. Hybridly strongly universal zeta-functions

§ 3.1. Main theorem for zeta-functions without Euler product

Let us define the class $\mathcal{L}$ to consist of functions

$$L(s;\beta,a) = \sum_{n=1}^{\infty} \frac{a(n)}{(n+\beta)^s}, \qquad \sigma > 1,$$

where $0 < \beta < 1$ and a is a sequence of complex numbers, which satisfy the following conditions:

(L1) $a(m) \ll_\varepsilon m^\varepsilon$ for every $\varepsilon > 0$,

(L2) L has meromorphic continuation to the half-plane $\sigma > 1/2$. It can possess at most a finite number of poles and all of them must lie on the straight line $\sigma = 1$.

(L3) for every $\varepsilon > 0$, compact set $K \subset D$ and sufficiently large y we have

$$\max_{s\in K} \frac{1}{T}\int_{-T}^{T} |L(s+it;\beta,a) - L_M(s+it;\beta,a)|^2 dt < \varepsilon \qquad (T\to\infty),$$

where

$$M = \{n \in \mathbb{N} : n \leqslant y\} \qquad \text{and} \qquad L_M(s;\beta,a) = \sum_{n\in M} \frac{a(n)}{(n+\beta)^s}.$$

Essentially, the proof of hybrid strong universality for functions from $\mathcal{L}$ is very similar to the proof of Theorem 2.6. The main difference is caused by the fact that, in general, a function from $\mathcal{L}$ does not have Euler product, so we can not use an approximation by finite product over primes and the linearly independence of the logarithms of primes. In order to overcome this difficulties we need to use a finite sum $\sum_{n\in M} \frac{a(n)e(-\theta_n)}{(n+\beta)^s}$ instead of a finite product over primes. Moreover, to approximate arbitrary given analytic functions $f_1,\ldots,f_n$ it is sufficient to take n functions from $\mathcal{L}$ associated to algebraically independent real numbers $\beta_1,\ldots,\beta_n$. This algebraic independence provides that the numbers $\log(m+\beta_j)$, $m\in\mathbb{N}$, $1\leqslant j\leqslant n$ are linearly independent over $\mathbb{Q}$, which replaced the linear independence of the logarithms of primes in the proof of Theorem 2.6.

Similarly to Section 2, we use the following definition of acceptability in $\mathcal{L}$.

Definition 3.1. We say that a set $\{L_1(s;\beta,a_1),\ldots,L_m(s;\beta,a_m)\} \subset \mathcal{L}$ is *acceptable* if it satisfies the following condition: for every finite set $M\subset\mathbb{N}$, every admissible domain G with $\overline{G}\subset D$, and any analytic functions $f_1,\ldots,f_m$ on $\overline{G}$, there exists a sequence of finite sets $M_1\subset M_2\subset\ldots\subset\mathbb{N}$ such that

$$\bigcup_{k=1}^{\infty} M_k = \mathbb{N}\setminus M$$

and for suitable sequence $\Theta_k = (\theta_n^{(k)}) \in \prod_n \mathbb{R}$, as $k\to\infty$,

$$L_{j,M_k}(s;\beta,a_j\cdot\Theta_k) = \sum_{n\in M_k} \frac{a_j(n)e(-\theta_n^{(k)})}{(n+\beta)^s} \longrightarrow f_j(s)$$

uniformly for $s\in\overline{G}$, $j=1,2,\ldots,m$.

Thus, combining the idea described above and verifying the acceptability property by Lemma 2.3 gives the following main theorem.

Theorem 3.2 (hybrid strong joint universality theorem)**.** *Let us assume that for every fixed $1 \leqslant j \leqslant n$ the set $\{L_k^{(j)}(s;\beta_j,a_k^{(j)})\}_{1\leqslant k\leqslant m_j} \subset \mathcal{L}$ is acceptable and the numbers $\beta_1,\dots,\beta_n$ are algebraically independent. Then the set of all functions $L_k^{(j)}(s;\beta_j,a_k^{(j)})$, $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m_j$, is hybridly strongly jointly universal.*

For the complete proof of the above theorem see [P2].

§ 3.2. Examples

The first example of the strongly universal object was given by S. M. Gonek [Go] and B. Bagchi [B1], independently. They showed that the Hurwitz zeta-function $\zeta(s,\beta)$ is strongly universal when $\beta \neq 1/2, 1$ is a rational or transcendental number. Next, A. Laurinčikas and K. Matsumoto (see [L2], [LM1], [LM2], [LM3]) considered the joint value-distribution of Lerch zeta-function defined for $0 < \lambda, \beta \leqslant 1$ as the Dirichlet series

$$L(s;\beta,\lambda) = \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n+\beta)^s}, \qquad \Re s > 1.$$

For example, they proved the joint universality theorem for Lerch zeta-functions, which was improved by T. Nakamura in [N1]. For a great overview of the theory of Lerch zeta-functions we refer to [GL].

Obviously, the Lerch zeta-functions satisfy condition (L1) and (L2) from the definition of class $\mathcal{L}$. Moreover to prove condition (L3) it is sufficient to use the Montgomery-Vaughan theorem and follow the prove of Theorem 3.3.1 in [GL]. We skip the proof of this fact as the modifications needed are straightforward and can be left to the reader. Hence, using Lemma 2.3, it is clear that for $0 < \beta < 1$ and $0 \leqslant \lambda < 1$ the set $\{L(s;\beta;\lambda)\} \subset \mathcal{L}$ is acceptable. Therefore, Theorem 3.2 implies that for every algebraically independent $\beta_1,\dots,\beta_n$ the set $\{L(s;\beta_1;\lambda_1),\dots,L(s;\beta_n;\lambda_n)$ is hybridly strongly jointly universal.

Another generalization of the Hurwitz zeta-function is a periodic Hurwitz zeta-function, introduced by A. Javtokas and A. Laurinčikas in [JL1] and defined for a periodic sequence of complex numbers $\mathfrak{A} = (a_n)_{n=0}^{\infty}$ and real number $0 < \beta \leqslant 1$ as

$$\zeta(s;\beta,\mathfrak{A}) = \sum_{n=0}^{\infty} \frac{a_n}{(n+\beta)^s}, \qquad \Re s > 1.$$

More details on value-distribution and universality of this function could be found in [JL2], [JL3], however the strongest result was proved recently in [LS].

Note that also periodic Hurwitz zeta-functions belong to the class $\mathcal{L}$, since can be written as a linear combination of some Hurwitz zeta-functions associated to suitable parameters β. Moreover the set $\{L(s;\beta,\mathfrak{U}_1),\ldots,L(s;\beta,\mathfrak{U}_m)\}$ is acceptable, whenever periodic sequences $\mathfrak{U}_j=(a_{jn})$, $j=1,2,\ldots,m$, with period k_j satisfy the equation

$$\operatorname{rank}\begin{pmatrix} a_{11} & a_{21} & \ldots & a_{m1} \\ a_{12} & a_{22} & \ldots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \ldots & a_{mk}\end{pmatrix}=m, \tag{3.1}$$

where k denotes the least common multiple of all k_j.

Therefore, one can see that for every algebraically independent $0<\beta_j<1$, $(1\leqslant j\leqslant n)$, and every periodic sequences $\mathfrak{A}_k^{(j)}=(a_{nk}^{(j)})$, $(1\leqslant j\leqslant n,\ 1\leqslant k\leqslant m_j)$, such that $\operatorname{rank}[a_{nk}^{(j)}]_{1\leqslant n\leqslant N_j}^{1\leqslant k\leqslant m_j}=m_j$ for every $1\leqslant j\leqslant n$, where N_j is the least common multiply of periods of all $\mathfrak{A}_k^{(j)}$, $(1\leqslant k\leqslant m_j)$, we have that the set $\{L(s+i\tau;\beta_j,\mathfrak{A}_k^{(j)}):1\leqslant j\leqslant n,\ 1\leqslant k\leqslant m_j\}$ is hybridly strongly jointly universal.

The last example of hybridly strongly universal zeta-functions is twist of Lerch zeta-function, investigated by R. Garunkštis and J. Steuding in [G] and defined for a Dirichlet character χ as

$$L(s;\beta,\lambda,\chi)=\sum_{n=0}^{\infty}\frac{e(\lambda n)\chi(n)}{(n+\beta)^s},\qquad \Re s>1.$$

For some analytic properties (like analytic continuation and functional equation) of twisted Lerch zeta-function we refer to [G]. For instance, Theorem 1 and equation (3) from [G] implies that the class $\mathcal{L}$ contains twisted Lerch zeta-functions. Furthermore, it is easy to see that for pairwise non-equivalent Dirichlet's characters $\chi_1,\ldots,\chi_m$ the set $\{L(s;\beta,\lambda,\chi_1),\ldots,L(s;\beta,\lambda,\chi_m)\}$ is acceptable, since the hypotheses of Lemma 2.3 are fulfilled for $\lambda_n=\log(n+\beta)$, $a(n)=e(\lambda n)$, and $b_j\equiv\chi_j$. Hence the set $\{L(s;\beta_j,\lambda_k^{(j)},\chi_k^{(j)}):1\leqslant j\leqslant r,\ 1\leqslant k\leqslant m_j\}$ is hybridly strongly jointly universal, whenever we suppose that $0<\beta_j<1$ are algebraically independent numbers, $0<\lambda_k^{(j)}\leqslant 1$ are real for $1\leqslant j\leqslant r$, $1\leqslant k\leqslant m_j$, and for every fixed $1\leqslant j\leqslant r$ Dirichlet's characters $\chi_1^{(j)},\ldots,\chi_{m_j}^{(j)}$ are pairwise non-equivalent.

§ 4. Applications of hybrid universality

§ 4.1. On combination of hybridly universal L-functions

Firstly, let us recall that the Kronecker theorem on diophantine approximations was used by H. Bohr in [Bo] to prove that every Dirichlet series is almost-periodic in its half-plane of absolute convergence. More precisely, he showed that for any Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ and every $\varepsilon > 0$ we have for an arbitrary s lying in the half-plane of absolute convergence that

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \left|\sum_{n=1}^{\infty} \frac{a_n}{n^{s+i\tau}} - \sum_{n=1}^{\infty} \frac{a_n}{n^s}\right| < \varepsilon\right\} > 0.$$

In fact, Bohr applied the Kronecker theorem only for the sequence $\log p$, where p runs over all primes. Replacing $\log p$ by any sequence linearly independent over the field of rational numbers gives us easily an analogous result for general Dirichlet series defined as

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \qquad \text{where } a_n \in \mathbb{C},\ \lambda_n \in \mathbb{R}\setminus\{0\}.$$

Let us denote by $\mathcal{S}$ the ring of all general Dirichlet series which are absolutely convergent in the half-plane $\sigma > 1/2$.

Since hybrid universality combines diophantine approximations and the universality property, the following theorem is straightforward consequence of presented above Bohr's idea (see [NP2]).

Theorem 4.1 (Nakamura, Pańkowski [NP2]). *Suppose that a set $\{L_1, \dots, L_n\}$ is hybridly jointly universal and $P \in \mathcal{S}[X_1, \dots, X_n]$ be polynomial with degree greater than zero. Then, for any $\varepsilon > 0$ and any function $g(s)$ from an image $P\big(H(K), \dots, H(K)\big)$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mu\{\tau \in [0,T] : \|P(L_1(s+i\tau), \dots, L_n(s+i\tau)) - g(s)\|_\infty < \varepsilon\} > 0.$$

Moreover, if the set $\{L_1, \dots, L_n\}$ is hybridly strongly universal then the set $\{L_1, \dots, L_n\}$ can approximate $g \in P\left(H_0(K), \dots, H_0(K)\right)$.

Remark. The above theorem was inspired by the work of Sander and Steuding [SS], where they showed how to apply Theorem C to obtain the strong universality for the Hurwitz zeta-function $\zeta(s;\beta)$, which for rational $\beta = a/q$, where a, q are co-prime, can be written as

$$\zeta(s; a/q) = q^s \varphi(q)^{-1} \sum_{\chi \bmod q} \overline{\chi}(a) L(s,\chi),$$

where $\varphi(\cdot)$ is Euler's totient function.

There are a lot of examples of well-known zeta- and L-functions which can be written as polynomial type combination of hybridly universal functions. For example:

- Estermann zeta-functions (see Section 3 of [NP1]);
- zeta functions associated to symmetric matrices (see Section 4 of [NP1])

- Euler products of Igusa type (see Section 3 of [NP2])
- Euler-Zagier multiple zeta-functions (see Section 4 of [NP2])
- Tornheim-Hurwitz type of double zeta-functions (see Section 5 of [NP2])

For a comprehensive list of such functions and the precise definition we refer to the work of T. Nakamura and the author [NP1, NP2].

§ 4.2. Generalized strong recurrence

The second application of hybrid universality is related to Bagchi's observation, which connect the Riemann hypothesis and universality.

Theorem D (Bagchi [B2])**.** *The Riemann Hypothesis holds if and only if, for every compact set $K \subset D := \{s \in \mathbb{C} : \frac{1}{2} < \Re s < 1\}$ with connected complement and for every $\varepsilon > 0$, the set of real numbers τ satisfying the following inequality*

$$\max_{s\in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon,$$

has a positive lower density.

In the language of topological dynamics the above theorem says that the Riemann Hypothesis is equivalent to the strong recurrence of the Riemann zeta-function (see [GHK]).

Nakamura [N2] showed the following result which is a kind of generalization of the strong recurrence for the Dirichlet L-functions $L(s,\chi)$.

Theorem E (Nakamura)**.** *For almost all $\delta \in \mathbb{R}$, every compact set $K \subset D$ with connected complement and every $\varepsilon > 0$, the set of real numbers τ satisfying the following inequality*

$$\max_{s\in K} |L(s+i\delta\tau,\chi) - L(s+i\tau,\chi)| < \varepsilon, \tag{4.1}$$

has a positive lower density.

Let us mention that the set of real numbers δ, for which the above inequality holds, was not characterized in the proof of Nakamura's theorem, but analyzing his proof one can find out that (4.1) holds for every δ which can not be represented as a quotient $\log a/\log b$, where $a, b \in \mathbb{Q}$. It is due to the fact that the linear independence of the numbers $\log p$, $\log p^{\delta}$, where p runs through all primes, plays a crucial role in the proof of Nakamura's theorem. Fortunately, it turns out that only a finite number of primes p can possibly be involved in the linear dependence of these numbers. More precisely, using the Six Exponentials Theorem gives the following lemma.

Lemma 4.2 (Pańkowski [P3, Lemma 2.4]). *For an arbitrary irrational number δ there exists a finite set of primes $A = A(\delta)$ containing at most two elements such that the following set*

$$\{\log p\}_{p\in\mathbb{P}\setminus A} \cup \{\delta \log p\}_{p\in\mathbb{P}},$$

is linearly independent over $\mathbb{Q}$.

Then using the above lemma and slightly modifying the proof of hybrid universality one can get the following theorem, which trivially implies the improvement of Nakamura's result.

Theorem 4.3 (Pańkowski [P3, Theorem 1.1]). *Let $K \subset D$ be any compact set with connected complement, χ a Dirichlet character and f, g any functions which are non-vanishing and continuous on K and analytic in the interior. Then, for every $\delta \notin \mathbb{Q}$ and $\varepsilon > 0$, the set of real numbers τ satisfying the following inequalities:*

$$\max_{s\in K} |L(s + i\alpha\tau, \chi) - f(s)| < \varepsilon,$$

$$\max_{s\in K} |L(s + i\beta\tau, \chi) - g(s)| < \varepsilon,$$

has a positive lower density.

Remark. Noteworthy is the fact that a similar, but more subtle, method was used by P. Drungilas, R. Garunkštis and A. Kačėnas in [DGK] to prove universality of the Selberg zeta-function for the modular group.

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