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Riemann's zeta function and T-positivity (3): Kummer function and inner product representation

By

Yasunori OKABE*

Abstract

We consider Riemann's zeta function from the viewpoint of the theory of stationary Gaussian processes. In the previous two papers ([11, 12]), we proved that Riemann's zeta function satisfies an ordinary differential equation with time delay and then obtained a new representation of the $\mathrm{KM_2O}$-Langevin system which is the characteristics for the non-negative definite function associated with Riemann's zeta function. As a continuation of the previous papers, first, we introduce in this paper a derived Kummer function and prove a new representation theorem for an analytic continuation for Riemann's zeta function, by obtaining an analytic continuation of the derived Kummer function. Second, we prove an inner product representation theorem for the analytic continuation of Riemann's zeta function and the derived Kummer function, by constructing a Hamiltonian operator associated with a stationary Gaussian process with T-positivity.

§ 1. Introduction

Riemann's hypothesis for the zeta function

$$\zeta = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathrm{Re}(s) > 1) \tag{1.1}$$

has remained unsolved for 151 years ([14], [16]). By using the gamma function, Riemann obtained the following representation for an analytic continuation of the zeta function

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*Graduate School of Advanced Mathematical Sciences, Meiji University, Higashi-mita, Tama-ku, Kawasaki, 214-8571, Japan.

$\zeta = \zeta(s)$:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \frac{\theta(t)-1}{2}(t^{-\frac{1+s}{2}} + t^{-\frac{2-s}{2}})dt, \tag{1.2}$$

where $\Gamma = \Gamma(s)$ and $\theta = \theta(t)$ are the gamma function and the theta function, respectively, defined by

$$\Gamma(s) \equiv \int_0^\infty e^{-t}t^{s-1}dt \qquad (\mathrm{Re}(s) > 0), \tag{1.3}$$

$$\theta(t) \equiv \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \qquad (t > 0). \tag{1.4}$$

We note that

$$\frac{\theta(t)-1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 t} \qquad (t > 0). \tag{1.5}$$

Since the second term of the right-hand side in (1.2) is regular with respect to $s \in \mathbf{C}$, we see from the properties of the gamma function that Riemann's zeta function $\zeta = \zeta(s)$ can be analytically continued so that it is regular except at the point $s = 1$, where it has a pole of order 1 with residue 1 and vanishes on the set $\{-2n; n \in \mathbf{N}\}$, to be called the set of the trivial zero points. Riemann's hypothesis conjectures that real parts of all non-trivial zero points of the zeta function $\zeta = \zeta(s)$ lie on the vertical line $\{s \in \mathbf{C}; \mathrm{Re}(s) = \frac{1}{2}\}$([16],[5]).

The purpose of this paper consists of the following two: one is to introduce a derived Kummer function associated with the Kummer function and the theta function and to prove a new representation theorem for the analytic continuation of Riemann's zeta function, by obtainig an analytic continuation of the derived Kummer function; second is to prove an another representation theorem in terms of the innner product for the analytic continuation of Riemann's zeta function, by constructing a Hamiltonian operator associated with a stationary Gaussian process with T-positivity.

The main point is that when we rewrite the second term in the right-hand side of (1.2) into

$$\int_1^\infty \frac{\theta(t)-1}{2}(t^{-\frac{1+s}{2}} + t^{-\frac{2-s}{2}})dt = \int_0^\infty \frac{\theta(t+1)-1}{2}((t+1)^{-\frac{1+s}{2}} + (t+1)^{-\frac{2-s}{2}})dt,$$

we note that for each $s(0 < \mathrm{Re}(s) < 1)$, the terms $(t+1)^{-\frac{1+s}{2}}$ and $(t+1)^{-\frac{2-s}{2}}$ on the right-hand side of the above equation can represented as the Laplace transform of bounded complex valued Borel measures. In fact, we have

$$(t+1)^{-\frac{1+s}{2}} = \int_0^\infty e^{-t\lambda}\Gamma_{\frac{1+s}{2}}(d\lambda), \tag{1.6}$$

$$(t+1)^{-\frac{2-s}{2}} = \int_0^\infty e^{-t\lambda}\Gamma_{\frac{2-s}{2}}(d\lambda), \tag{1.7}$$

where for each $s(\mathrm{Re}(s) > 0)$, $\Gamma_s = \Gamma_s(d\lambda)$ is a bounded complex valued Borel measure on $[0, \infty)$ defined by

$$\Gamma_s(d\lambda) \equiv \frac{1}{\Gamma(s)} e^{-\lambda} \lambda^{s-1} d\lambda. \tag{1.8}$$

We note that if s is a positive real number, then Γ_s is the gamma distribution with mean s and variance s.

The detailed content of this paper is as follows.

In Section 2, we recall the proof the analytic continuation for Riemann's zeta function (1.2) due to Riemann, because the method used there is used in the sequel in this paper

In Section 3, we introduce a derived Kummer function defined by combining the Kummer function and the theta function, and obtain its analytic continuation, by noting that the principal part of the integrand in the analytic continuation of Riemann's zeta function can be regarded as the Laplace transform of a bounded complex valued Borel measure defined on $[0,\infty)$. Furthermore, we prove a new representation theorem of the analytic cointinuation for Riemann's zeta function, by using the analytic continuation for the derived Kummer function.

In Section 4, we introduce other two kinds of functions derived from the Kummer function and the theta function and obtaining their analytic continuation and prove a recurrence formula among them, according to the recurrence formula with respect to parameters of the Kummer function.

In order to clarify a mathematical structure of the notion of T-positivity coming from the axiomatic field theory([6],[13],[3]) from the view-point of the theory of stochastic processes, we constructed in [7] the Hamiltonian operator acting on the real splitting space associated with a stationary Gaussian process with T-positivity and derived an infinite-dimensional Langevin equation describing the time evolutiojn of the above process. In Section 5, by taking the same procedure as in [7], we construct a Hamiltonian operator acting on the complex splitting space associated with a stationary Gaussian process with T-positivity. Futhermore, we give a note concerning the Shwinger function of order 2 and the Wightmann function of order 2 in the axiomatic field theory.

In Section 6, we prove an inner product representation theorem for the analytic continuation of Riemann's zeta function and the derived Kummer function, by transforming the bounded complex valued Borel measure used in Section 3 to the gamma distribution $\Gamma_{\frac{3}{4}}$ and using the Hamiltonian operator associated with the stationary Gaussian process with T-positivity whose covarinace function is given by the Laplace transform of the gamma distribution $\Gamma_{\frac{3}{4}}$.

We shall investigate Riemann's hypothesis, by using the inner product representation theorem for the analytic continuation of Riemann's zeta function obtained in

Section 6 in the forthcoming paper. We would like to dedicate our hearty thanks for Prof. M.Klimek, Uppsala University, and the referee for giving valuable advices.

§ 2. The analytic continuation for Riemann's zeta function due to Riemann

First, we put together the fundamental properties concerning the gamma function which are used in this paper, together with the beta function $B = B(x,y)$ ($\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0$) defined by

$$B(x,y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{2.1}$$

Theorem 2.1. (i) *The gamma function $\Gamma = \Gamma(x)$ can be analytically continued so that it has no zero points and is regular at except the set $\{-n; n \in \mathbf{N}^*\}$ of poles with order 1 with residue 1.*

(ii) $\Gamma(x+1) = x\Gamma(x)$.

(iii) $\dfrac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1)$.

(iv) $\Gamma(1-x)\Gamma(x) = \dfrac{\pi}{\sin(\pi x)}$.

(v) $\Gamma(x)\Gamma(-x) = -\dfrac{\pi}{x\sin(\pi x)}$.

(vi) $\Gamma(\dfrac{x}{2})\Gamma(\dfrac{1+x}{2}) = 2^{1-x}\pi\Gamma(x)$.

(vii) $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad (\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0)$.

By (1.4), we note that the theta function $\theta = \theta(t)$ satisfies the following functional equation.

$$\theta(\frac{1}{t}) = \sqrt{t}\theta(t) \qquad (t > 0), \tag{2.2}$$

which is proved by applying Poisson's addition formula to the function $f(x) \equiv e^{-\pi n^2 x}$ ($x \in \mathbf{R}$).

Theorem 2.2. ([14]) *For any $s \in \{s \in \mathbf{C}; \mathrm{Re}(s) > 1\}$,*

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \frac{\theta(t)-1}{2}(t^{-\frac{1+s}{2}} + t^{-\frac{2-s}{2}})dt.$$

Proof. Fix any $n \in \mathbf{N}$ and $s \in \{s \in \mathbf{C}; \mathrm{Re}(s) > 1\}$. By the change of variables $t = \pi n^2 \lambda$ in (1.3), we have

$$\Gamma(\frac{s}{2}) = \pi^{\frac{s}{2}} n^s \int_0^\infty e^{-\pi n^2 \lambda} \lambda^{\frac{s}{2}-1} d\lambda$$

and so

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \frac{1}{n^s} = \int_0^\infty e^{-\pi n^2 \lambda} \lambda^{\frac{s}{2}-1} d\lambda.$$

By adding the above with respect to $n \in \mathbf{N}$, we see from (1.1) that

$$\begin{aligned} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) &= \int_0^\infty (\sum_{n=1}^\infty e^{-\pi n^2 \lambda}) \lambda^{\frac{s}{2}-1} d\lambda \\ &= \int_0^\infty \frac{\theta(\lambda)-1}{2} \lambda^{\frac{s}{2}-1} d\lambda \\ &= \int_0^1 \frac{\theta(\lambda)-1}{2} \lambda^{\frac{s}{2}-1} d\lambda + \int_1^\infty \frac{\theta(\lambda)-1}{2} \lambda^{\frac{s}{2}-1} d\lambda. \qquad (2.3) \end{aligned}$$

By the change of variables $\lambda = t^{-1}$ and by applying (2.2) to the first term of the above, we have

$$\begin{aligned} \int_0^1 \frac{\theta(\lambda)-1}{2} \lambda^{\frac{s}{2}-1} d\lambda &= \int_1^\infty \frac{\theta(t^{-1})-1}{2} t^{-(\frac{s}{2}-1)} t^{-2} dt \\ &= \int_1^\infty \frac{\sqrt{t}(\theta(t)-1) + (\sqrt{t}-1)}{2} t^{-(\frac{s}{2}+1)} dt \\ &= \int_1^\infty \frac{\theta(t)-1}{2} t^{-\frac{s+1}{2}} dt + \frac{1}{2} \int_1^\infty (t^{-\frac{s+1}{2}} - t^{-\frac{s+2}{2}}) dt. \qquad (2.4) \end{aligned}$$

On the other hand, by direct calculation, we have

$$\frac{1}{2} \int_1^\infty (t^{-\frac{s+1}{2}} - t^{-\frac{s+2}{2}}) dt = \frac{1}{s(s-1)}. \qquad (2.5)$$

Therefore, by substituting (2.4) and (2.5) to (2.3), we see that Theorem 2.2 holds. (Q.E.D.)

Lemma 2.3. ([14]) *The theta function $\theta = \theta(t)$ satisfies the following inequalities*:

$$e^{-\pi t} \leq \frac{\theta(t)-1}{2} \leq e^{-\pi t}(1-e^{-\pi})^{-1} \qquad (t > 1).$$

Proof. It follows from (1.5) that the inequality $e^{-\pi t} \leq \frac{\theta(t)-1}{2}$ holds. On the other hand, by applying the equality

$$\begin{aligned} n^2 t - (t+n-1) &= t(n^2-1) - (n-1) \qquad (2.6) \\ &= (n-1)(t(n+1)-1) \\ &= (n-1)n \geq 0 \end{aligned}$$

to (1.5), we see that

$$\begin{aligned}\frac{\theta(t)-1}{2} &\leq \sum_{n=1}^{\infty} e^{-\pi(t+n-1)}\\ &\leq e^{-\pi t}\sum_{n=1}^{\infty} e^{-\pi(n-1)}\\ &= e^{-\pi t}(1-e^{-\pi})^{-1},\end{aligned}$$

which proves Lemma 2.3. (Q.E.D.)

By using Lemma 2.3, we prove

Lemma 2.4. ([14]) *The function $\int_1^\infty \frac{\theta(t)-1}{2}t^{-s}dt$ is regular with respect to $s \in \mathbf{C}$.*

Proof. Fix any $s_0 \in \mathbf{C}$ such that $|s_0| < N$ with a natural number N. Since $\frac{\partial}{\partial s}t^{-s} = -(\log t)t^{-s}$, we see from Lemma 2.3 that for any $s \in \mathbf{C}$ such that $|s| < N$,

$$\begin{aligned}|\frac{\theta(t)-1}{2}\frac{\partial}{\partial s}t^{-s}| &\leq (1-e^{-\pi})^{-1}e^{-\pi t}|\log t|t^N\\ &= (1-e^{-\pi})^{-1}e^{-(\pi-2)t}|\frac{\log t}{e^t}|\frac{t^N}{e^t}\\ &\leq (1-e^{-\pi})^{-1}e^{-(\pi-2)t}|\frac{\log t}{t}|\frac{t^N}{t^N/N!}\\ &\leq N!(1-e^{-\pi})^{-1}e^{-(\pi-2)t} \in L^1((1,\infty),\mathcal{B}(\mathbf{R}),dt).\end{aligned}$$

Hence, by Lebesgue's convergence theorem, we see that Lemma 2.4 holds. (Q.E.D.)

By applying Lemma 2.4 to Theorem 2.2, we have

Theorem 2.5. ([14]) *The function $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ $(\mathrm{Re}(s) > 1)$ can be analytically continued on $\mathbf{C}$ so that it is regular except at two points $s = 0, 1$, where it has poles of order 1 with residue 1.*

By using Theorem 2.1 and Theorem 2.5, we see that

Theorem 2.6. ([14]) *Riemann's zeta function $\zeta = \zeta(s)$ can be analytically continued on $\mathbf{C}$ through the following representaion*

$$\zeta(s) = (\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}))^{-1}(\frac{1}{s(s-1)} + \int_1^\infty \frac{\theta(t)-1}{2}(t^{-\frac{1+s}{2}} + t^{-\frac{2-s}{2}})dt) \qquad (s \in \mathbf{C})$$

and it is regular except at the point $s = 1$, where it has a pole of order 1 with residue 1. It also vanishes on the set $\{-2n; n \in \mathbf{N}\}$.

Moreover, we see from Theorem 2.6 that

Theorem 2.7. ([14]) *The function* $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ *satisfies the following functional equation*:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \qquad (s \in \mathbf{C}).$$

§ 3. The derived Kummer function associated with the Kummer function and the theta function

As noted in Section 1, we can decompose the right-hand side in Theorem 2.2 as follows.

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + F_1(s) + F_2(s), \tag{3.1}$$

$$F_1(s) \equiv \int_0^\infty \frac{\theta(t+1)-1}{2}(t+1)^{-\frac{1+s}{2}}dt, \tag{3.2}$$

$$F_2(s) \equiv \int_0^\infty \frac{\theta(t+1)-1}{2}(t+1)^{-\frac{2-s}{2}}dt. \tag{3.3}$$

We note that the following functional equation holds.

$$F_1(s) = F_2(1-s) \qquad (s \in \mathbf{C}). \tag{3.4}$$

By Lemma 2.4, we have

Lemma 3.1. ([14]) *The functions* $F_1 = F_1(s)$ *and* $F_2 = F_2(s)$ *are regular on the complex plane* $\mathbf{C}$.

By applying (1.6) and (1.7) to (3.2) and (3.3), we have

Lemma 3.2. *For any* $s \in \{s \in \mathbf{C}; -1 < \mathrm{Re}(s) < 2\}$,

(i) $$F_1(s) = \Gamma(\frac{1+s}{2})^{-1}\sum_{n=1}^{\infty} e^{-\pi n^2}\int_0^\infty \frac{1}{\pi n^2+\lambda}e^{-\lambda}\lambda^{\frac{1+s}{2}-1}d\lambda,$$

(ii) $$F_2(s) = \Gamma(\frac{2-s}{2})^{-1}\sum_{n=1}^{\infty} e^{-\pi n^2}\int_0^\infty \frac{1}{\pi n^2+\lambda}e^{-\lambda}\lambda^{\frac{2-s}{2}-1}d\lambda.$$

Proof. By (1.5), (1.6), (1.8) and (3.2), we have

$$\begin{aligned}
F_1(s) &= \Gamma(\frac{1+s}{2})^{-1} \int_0^\infty \frac{\theta(t+1)-1}{2}(\int_0^\infty e^{-t\lambda} e^{-\lambda} \lambda^{\frac{1+s}{2}-1} d\lambda)dt \\
&= \Gamma(\frac{1+s}{2})^{-1} \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2(t+1)}(\int_0^\infty e^{-t\lambda} e^{-\lambda} \lambda^{\frac{1+s}{2}-1} d\lambda)dt \\
&= \Gamma(\frac{1+s}{2})^{-1} \sum_{n=1}^\infty e^{-\pi n^2} \int_0^\infty \sum_{n=1}^\infty (\int_0^\infty e^{-(\pi n^2+\lambda)t} e^{-\lambda} \lambda^{\frac{1+s}{2}-1} d\lambda)dt \\
&= \Gamma(\frac{1+s}{2})^{-1} \sum_{n=1}^\infty e^{-\pi n^2} \int_0^\infty \frac{1}{\pi n^2+\lambda} e^{-\lambda} \lambda^{\frac{1+s}{2}-1} d\lambda,
\end{aligned}$$

which proves (i). Property (ii) follows from (i) and (3.4). (Q.E.D.)

By using the bounded complex valued Borel measure $\Gamma_{\frac{1+s}{2}}$ in (1.6)(resp. $\Gamma_{\frac{2-s}{2}}$ in (1.7)), we find from Lemma 3.2 that the functions F_1 and F_2 can be rewritten in the following form:

$$F_1(s) = \sum_{n=1}^\infty e^{-\pi n^2} \int_0^\infty \frac{1}{\pi n^2+\lambda} \lambda^{\frac{2s-1}{4}} \Gamma_{\frac{1+s}{2}-1}(d\lambda), \tag{3.5}$$

$$F_2(s) = \sum_{n=1}^\infty e^{-\pi n^2} \int_0^\infty \frac{1}{\pi n^2+\lambda} \lambda^{\frac{1-2s}{4}} \Gamma_{\frac{2-s}{2}-1}(d\lambda). \tag{3.6}$$

In this Section 3, we use the representation for the functions F_1 and F_2 in Lemma 3.2. The representation (3.5) and (3.6) will be further studied in Section 6.

For each $s \in \{s \in \mathbf{C}; \mathrm{Re}(s) > 0\}$, we define a function $f = f(x;s)$ on $[0,\infty)$ by

$$f(x;s) \equiv \int_0^\infty e^{-xt} \frac{t^{s-1}}{1+t} dt. \tag{3.7}$$

Lemma 3.3. *For any* $s \in \{s \in \mathbf{C}; -1 < \mathrm{Re}(s) < 2\}$,

(i) $F_1(s) = \Gamma(\frac{1+s}{2})^{-1} \sum_{n=1}^\infty e^{-\pi n^2} (\pi n^2)^{\frac{1+s}{2}-1} f(\pi n^2; \frac{1+s}{2}),$

(ii) $F_2(s) = \Gamma(\frac{2-s}{2})^{-1} \sum_{n=1}^\infty e^{-\pi n^2} (\pi n^2)^{\frac{2-s}{2}-1} f(\pi n^2; \frac{2-s}{2}).$

Proof. By using the change of variables $\lambda = \pi n^2 t$ in Lemma 3.2(i), we see that (i) holds. Property (ii) follows from (i) and (3.4). (Q.E.D.)

Lemma 3.4. *For any* $s \in \{s \in \mathbf{C}; 0 < \mathrm{Re}(s) < 1\}$,

(i) $f(x;s) = e^x f(0;s) - \frac{\Gamma(s)}{x^{s-1}} \int_0^1 e^{x(1-t)} t^{-s} dt \qquad (x > 0),$

(ii) $f(0;s) = \dfrac{\pi}{\sin(\pi s)}$.

Proof. Fix any $s \in \{s \in \mathbf{C}; 0 < \mathrm{Re}(s) < 1\}$. Since the function $f = f(x;s)$ satisfies the following differential equation

$$\frac{d}{dx}f(x;s) - f(x;s) = -\frac{\Gamma(s)}{x^s} \qquad (x > 0),$$

we find that

$$f(x;s) = e^x f(0;s) - \int_0^x e^{x-y}\frac{\Gamma(s)}{y^s}dy.$$

By using the change of variables $y = xt$ in the above integral, we see that (i) holds. On the other hand, we have

$$\begin{aligned} f(0;s) &= \int_0^\infty t^{s-1}\frac{1}{1+t}dt \\ &= \int_0^\infty t^{s-1}(\int_0^\infty e^{-(1+t)\lambda}d\lambda)dt \\ &= \int_0^\infty e^{-\lambda}(\int_0^\infty t^{s-1}e^{-t\lambda}dt)d\lambda. \end{aligned}$$

By using the change of variables $t\lambda = u$ in the above integral, we have

$$\begin{aligned} f(0;s) &= \int_0^\infty e^{-\lambda}(\int_0^\infty (\frac{u}{\lambda})^{s-1}e^{-u}\frac{1}{\lambda}du)d\lambda \\ &= (\int_0^\infty e^{-\lambda}\lambda^{-s}d\lambda)(\int_0^\infty e^{-u}u^{s-1}du) \\ &= \Gamma(1-s)\Gamma(s). \end{aligned}$$

Hence, by Theorem 2.1(iv), we see that (ii) holds. (Q.E.D.)

Lemma 3.5. *For any* $s \in \{s \in \mathbf{C}; 0 < \mathrm{Re}(s) < 1\}$ *and any* $x > 0$,

$$\begin{aligned} \int_0^1 e^{x(1-t)}t^{-s}dt &= \sum_{n=0}^\infty \frac{x^n}{(1-s)(2-s)\cdots(n+1-s)} \\ &= (1-s)^{-1}\sum_{n=0}^\infty \frac{\Gamma(2-s)}{\Gamma(2-s+n)}x^n. \end{aligned}$$

Proof. Integration by parts gives us

$$\begin{aligned} \int_0^1 e^{x(1-t)}t^{-s}dt &= [e^{x(1-t)}\frac{t^{1-s}}{1-s}]_{t=0}^{t=1} + \frac{x}{1-s}\int_0^1 e^{x(1-t)}t^{1-s}dt \\ &= \frac{1}{1-s} + \frac{x}{1-s}\int_0^1 e^{x(1-t)}t^{1-s}dt. \end{aligned}$$

For any $N \in \mathbf{N}$, integrating by parts N times, we obtain

$$\text{(3.8)} \int_0^1 e^{x(1-t)}t^{-s}dt = \sum_{n=0}^{N} \frac{x^n}{(1-s)(2-s)\cdots(n+1-s)} + \frac{x^{N+1}}{(1-s)(2-s)\cdots(N+1-s)} \int_0^1 e^{x(1-t)}t^{N+1-s}dt.$$

Since

$$\left|\frac{x^{N+1}}{(1-s)(2-s)\cdots(N+1-s)}\right| \leq \frac{1}{1-\mathrm{Re}(s)} \frac{x^{N+1}}{N!}$$

and

$$|e^{x(1-t)}t^{N+1-s}| \leq e^x t^N \qquad (0 < t < 1)$$

for any $s \in \{s \in \mathbf{C}; 0 < \mathrm{Re}(s) < 1\}$ and any $x > 0$, we can let N go to ∞ in (3.8) to see that

$$\text{(3.9)} \qquad \int_0^1 e^{x(1-t)}t^{-s}dt = \sum_{n=0}^{\infty} \frac{x^n}{(1-s)(2-s)\cdots(n+1-s)}.$$

By replacing s by $2-s$ in Theorem 2.1(iii), we see that $\frac{\Gamma(2-s+n)}{\Gamma(2-s)} = (2-s)(3-s)\cdots(n+1-s)$. Hence, we conclude from (3.9) that Lemma 3.5 holds. (Q.E.D.)

Here, we shall recall the Kummer function ${}_1F_1(a;c;z)$ which is also called the hypergeometric function of confluent type with two parameters a and c ($a, c \in \mathbf{C}$)([1],[4]).

$$\text{(3.10)} \qquad {}_1F_1(a;c;z) \equiv \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{z^n}{n!} \qquad (z \in \mathbf{C}).$$

The function $u(z) \equiv {}_1F_1(a;c;z)$ satisfies the following hypergeometric differential equation of confluent type.

$$\text{(3.11)} \qquad z\frac{d^2}{dz}u(z) + (c-z)\frac{d}{dz}u(z) - au(z) = 0 \qquad (z \in \mathbf{C}),$$

which is also called Kummer's differential equation. We know that when $c \neq 0, -1, -2, \cdots$, the fundamental system $\{u_1, u_2\}$ of solutions to Kummer's differential equation is given by

$$\text{(3.12)} \quad u_1(z) \equiv {}_1F_1(a;c;z) \quad \text{and} \quad u_2(z) \equiv z^{1-c}\,{}_1F_1(a-c+1;2-c;z) \qquad (z \in \mathbf{C}).$$

We have the following integral formula for the Kummer function.

(3.13)

$${}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt}(1-t)^{c-a-1}t^{a-1}dt \qquad (0 < \mathrm{Re}(a) < \mathrm{Re}(c), z \in \mathbf{C}).$$

It follows from (3.10) that

$$ {}_1F_1(1;c;z) = \sum_{n=0}^{\infty} \frac{\Gamma(c)}{\Gamma(c+n)} z^n \qquad (z \in \mathbf{C}). \tag{3.14} $$

Hence, we see from Lemma 3.5 that

Lemma 3.6. *For any $s \in \{s \in \mathbf{C}; 0 < \mathrm{Re}(s) < 1\}$ and any $x > 0$,*

$$ \int_0^1 e^{x(1-t)} t^{-s} dt = (1-s)^{-1} \, {}_1F_1(1; 2-s; x). $$

By Lemmas 3.4 and 3.6, we have

Lemma 3.7. *For any $s \in \{s \in \mathbf{C}; 0 < \mathrm{Re}(s) < 1\}$ and any $x > 0$,*

$$ f(x;s) = e^x \frac{\pi}{\sin(\pi s)} - \frac{\Gamma(s)}{x^{s-1}} (1-s)^{-1} \, {}_1F_1(1; 2-s; x) \qquad (x > 0). $$

Lemma 3.8. *For any $s \in \{s \in \mathbf{C}; -1 < \mathrm{Re}(s) < 2\}$,*

(i) $$ F_1(s) = \sum_{n=1}^{\infty} (\pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \frac{1}{n^{1-s}} - \frac{2}{1-s} e^{-\pi n^2} \, {}_1F_1(1; \frac{3-s}{2}; \pi n^2)), $$

(ii) $$ F_2(s) = \sum_{n=1}^{\infty} (\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \frac{1}{n^s} - \frac{2}{s} e^{-\pi n^2} \, {}_1F_1(1; \frac{2+s}{2}; \pi n^2)). $$

Proof. By Lemmas 3.3, 3.5 and 3.7, we have

$$ \begin{aligned} F_1(s) = {} & \Gamma(\frac{1+s}{2})^{-1} \sum_{n=1}^{\infty} e^{-\pi n^2} (\pi n^2)^{\frac{1+s}{2}-1} \times \\ & \times (e^{\pi n^2} \frac{\pi}{\sin(\pi(1+s)/2)} - \frac{\Gamma((1+s)/2)}{1-(1+s)/2} (\pi n^2)^{1-\frac{1+s}{2}} \, {}_1F_1(1; 2 - \frac{1+s}{2}; \pi n^2)) \\ = {} & \sum_{n=1}^{\infty} (\pi n^2)^{-\frac{1-s}{2}} \times \\ & \times (\frac{\pi}{\Gamma((1+s)/2) \sin(\pi(1+s)/2)} - \frac{2}{1-s} e^{-\pi n^2} (\pi n^2)^{\frac{1-s}{2}} \, {}_1F_1(1; \frac{3-s}{2}; \pi n^2)) \\ = {} & \sum_{n=1}^{\infty} (\frac{\pi^{(1+s)/2}}{\Gamma((1+s)/2) \sin(\pi(1+s)/2)} \frac{1}{n^{1-s}} - \frac{2}{1-s} e^{-\pi n^2} \, {}_1F_1(1; \frac{3-s}{2}; \pi n^2)). \end{aligned} $$

Hence, by using the following formula for the gamma function

$$\frac{\pi^{(1+s)/2}}{\Gamma((1+s)/2)\sin(\pi(1+s)/2)} = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2}) \qquad (0 < \mathrm{Re}(s) < 1)$$

following from Theorem 2.1(iii), we see that (i) holds. Property (ii) follows from (i) and (3.4). (Q.E.D.)

Lemma 3.9. *For any fixed* $s \in \{s \in \mathbf{C}; \mathrm{Re}(s) > \frac{3}{2}\}$, *the following series is absolutely convergent*:

$$\sum_{n=1}^{\infty} e^{-\pi n^2}(\sum_{m=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s+m)}(\pi n^2)^m).$$

Proof. Put $s = \sigma + i\tau$ $(\sigma, \tau \in \mathbf{R})$. By Lemma 3.5 and Theorem 2.1(iii), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} e^{-\pi n^2}(\sum_{m=0}^{\infty} |\frac{\Gamma(s)}{\Gamma(s+m)}|(\pi n^2)^m) \\
&= |s-1| \sum_{n=1}^{\infty} e^{-\pi n^2}(\sum_{m=0}^{\infty} |\frac{1}{(s-1)s(s+1)\cdots(s+m-1)}|(\pi n^2)^m) \\
&\le |s-1| \sum_{n=1}^{\infty} e^{-\pi n^2}(\sum_{m=0}^{\infty} \frac{1}{(\sigma-1)\sigma(\sigma+1)\cdots(\sigma+m-1)}(\pi n^2)^m) \\
&= |s-1| \sum_{n=1}^{\infty} e^{-\pi n^2}(\int_0^1 e^{\pi n^2(1-t)} t^{\sigma-2} dt) \\
&= |s-1| \int_0^1 (\sum_{n=1}^{\infty} e^{-\pi n^2 t}) t^{\sigma-2} dt.
\end{aligned}$$

Hence, by the change of variables $t = \lambda^{-1}$, we obtain

$$\sum_{n=1}^{\infty} e^{-\pi n^2}(\sum_{m=0}^{\infty} |\frac{\Gamma(s)}{\Gamma(s+m)}|(\pi n^2)^m) \le |s-1| \int_1^{\infty} (\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{\lambda}})\lambda^{-\sigma} d\lambda. \tag{3.15}$$

On the other hand, it follows from (2.2) that

$$\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{\lambda}} = \sqrt{\lambda}\sum_{n=1}^{\infty} e^{-\pi n^2 \lambda} + \frac{\sqrt{\lambda}-1}{2}. \tag{3.16}$$

Therefore, by (3.15) and (3.16), we have

$$\sum_{n=1}^{\infty} e^{-\pi n^2}(\sum_{m=0}^{\infty} |\frac{\Gamma(s)}{\Gamma(s+m)}|(\pi n^2)^m)$$
$$\le |s-1|(\int_1^{\infty}(\sum_{n=1}^{\infty} e^{-\pi n^2\lambda})\lambda^{\frac{1}{2}-\sigma}d\lambda + \int_1^{\infty}\frac{\sqrt{\lambda}-1}{2}\lambda^{-\sigma}d\lambda)$$
$$= |s-1|(\int_1^{\infty}(\sum_{n=1}^{\infty} e^{-\pi n^2\lambda})\lambda^{\frac{1}{2}-\sigma}d\lambda + \frac{1}{4(\sigma-3/2)(\sigma-1)}).$$

Since it follows from Lemma 2.4 that the first term of the bottom part in the above equation is finite, we conclude that Lemma 3.9 holds. (Q.E.D.)

By virtue of Lemma 3.9, we can introduce a function $K_\theta = K_\theta(s)$ on $\{s \in \mathbf{C}; \mathrm{Re}(s) > \frac{3}{2}\}$ defined by

$$K_\theta(s) \equiv \sum_{n=1}^{\infty} e^{-\pi n^2}(\sum_{m=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s+m)}(\pi n^2)^m). \tag{3.17}$$

We note from (3.13) that

$$K_\theta(s) = \sum_{n=1}^{\infty} e^{-\pi n^2}\,{}_1F_1(1;s;\pi n^2). \tag{3.18}$$

We call the function $K_\theta = K_\theta(s)$ the derived Kummer function associated with the Kummer function and the theta function.

Lemma 3.10. (i) *The derived Kummer function $K_\theta = K_\theta(s)$ can be analytically continued on $\mathbf{C}$ so that it is regular except at the point $s = \frac{2}{3}$, where it has a pole of order 1 with resdue 1*

(ii) $F_1(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) - \frac{2}{1-s}K_\theta(\frac{3-s}{2}) \qquad (s \neq 0,1).$

(iii) $F_2(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) - \frac{2}{s}K_\theta(\frac{2+s}{2}) \qquad (s \neq 0,1).$

Proof. By Lemma 3.8, we see that for any $s \in \{s \in \mathbf{C}; -1 < \mathrm{Re}(s) < 0\}$,

$$F_1(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) - \frac{2}{1-s}K_\theta(\frac{3-s}{2}).$$

By multiplying the above by $\frac{1-s}{2}$, we have

$$K_\theta(\frac{3-s}{2}) = \frac{1-s}{2}F_1(s) - \frac{1-s}{2}\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s).$$

Therefore, it follows from Theorems 2.5, 2.6 and 3.12 that (i) holds. Property (ii) follows from (i) and the functional equation in (3.4). (Q.E.D.)

Furthermore, by using Lemma 3.10, we have

Lemma 3.11.

(i) $F_1(s) = \dfrac{2}{s}K_\theta(\dfrac{2+s}{2}) - \dfrac{1}{s(s-1)} \qquad (s \neq 0,1).$

(ii) $F_2(s) = \dfrac{2}{1-s}K_\theta(\dfrac{3-s}{2}) - \dfrac{1}{s(s-1)} \qquad (s \neq 0,1).$

Proof. By (iii) in Lemma 3.10, we have

$$\frac{2}{s}K_\theta(\frac{2+s}{2}) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) - F_2(s).$$

By substituting (3.1) to the right-hand side of the above, we see that $\frac{2}{s}K_\theta(\frac{2+s}{2}) = \frac{1}{s(s-1)} + F_1(s)$, which proves (i). Property (ii) follows from (i) and (3.4). (Q.E.D.)

We define a function $\xi = \xi(s)$ on $\mathbf{C}$ by

$$\xi(s) \equiv \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s). \tag{3.19}$$

It follows from Theorem 2.7 that the function $\xi = \xi(s)$ satisfies the following functional equation.

$$\xi(s) = \xi(1-s) \qquad (s \in \mathbf{C}). \tag{3.20}$$

We are now going to prove one of the main theorems of this paper.

Theorem 3.12. *The function ξ has the following representation*:

$$\xi(s) = \frac{1}{2} - (sK_\theta(\frac{3-s}{2}) + (1-s)K_\theta(\frac{2+s}{2})).$$

Proof. By Lemma 3.10, we have

$$\begin{aligned} F_1(s) + F_2(s) = {} & \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) + \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \\ & -(\frac{2}{1-s}K_\theta(\frac{3-s}{2}) + \frac{2}{s}K_\theta(\frac{2+s}{2})). \end{aligned}$$

By combining this with (3.1), we have

$$\begin{aligned} \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) + \frac{1}{s(s-1)} = {} & \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) + \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \\ & -(\frac{2}{1-s}K_\theta(\frac{3-s}{2}) + \frac{2}{s}K_\theta(\frac{2+s}{2})). \end{aligned}$$

Therefore, from Theorem 2.7, we have

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \frac{2}{1-s}K_\theta(\frac{3-s}{2}) + \frac{2}{s}K_\theta(\frac{2+s}{2}).$$

By multiplying the above by $\frac{s(s-1)}{2}$, we conclude that Theorem 3.12 holds. (Q.E.D.)

§ 4. A recurrence formula for the derived Kummer function

In this section, we introduce other two kinds of functions derived from the Kummer function and the theta function and obtain their analytic continuation. Furthermore, we prove some recurrence formulae among them, by using the recurrence formulas with respect to parameters of the Kummer function.

[4.1] We recall several recurrence formulae for the Kummer function ${}_1F_1 = {}_1F_1(a;c;z)$ with two parameters a and c([1],[4]).

Theorem 4.1. ([1],[4])

(i) $\dfrac{\partial}{\partial z}\,{}_1F_1(a;c;z) = \dfrac{a}{c}\,{}_1F_1(a+1;c+1;z).$

(ii) ${}_1F_1(a;c;z) = e^z\,{}_1F_1(c-a;c;-z).$

(iii) $a\,{}_1F_1(a+1;c+1;z) = (a-c)\,{}_1F_1(a;c+1;z) + c\,{}_1F_1(a;c;z).$

(iv) $z\,{}_1F_1(a+1;c+1;z) = c(\,{}_1F_1(a+1;c;z) - {}_1F_1(a;c;z)).$

(v) $a\,{}_1F_1(a+1;c;z) = (z+2a-c)\,{}_1F_1(a;c;z) + (c-a)\,{}_1F_1(a-1;c;z).$

(vi) $(c-a)z\,{}_1F_1(a;c+1;z) = c(z+c-1)\,{}_1F_1(a;c;z) + c(1-c)\,{}_1F_1(a;c-1;z).$

(vii) $\displaystyle\lim_{c\to -n}\frac{1}{\Gamma(c)}\,{}_1F_1(a;c;z) = \frac{z^{n+1}(a)_{n+1}}{(n+1)!}\,{}_1F_1(a+n+1;n+2;z) \qquad (n=0,1,\ldots),$

where $(a)_n$ *is defined by*

$$(a)_n \equiv a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(n)}.$$

[4.2] In (3.17), we introduced the derived Kummer function $K_\zeta = K_\zeta(s)$ associated with the Kummer function and the theta function. As a refinement, by using the Kummer function ${}_1F_1(a;s;z)$ with a parameter $a(>0)$, we introduce a derived Kummer function $K_\theta(a) = K_\theta(a;s)$ with a parameter $a(>0)$ by

$$K_\theta(a;s) \equiv \sum_{n=1}^{\infty} e^{-\pi n^2}(\sum_{m=0}^{\infty}\frac{\Gamma(a+m)}{\Gamma(a)}\frac{\Gamma(s)}{\Gamma(s+m)}\frac{(\pi n^2)^m}{m!}). \tag{4.1}$$

We note that

$$K_\theta(a;s) = \sum_{n=1}^{\infty} e^{-\pi n^2}\,{}_1F_1(a;s;\pi n^2), \tag{4.2}$$

$$K_\theta(1;s) = K_\theta(s). \tag{4.3}$$

If $s > a + \frac{1}{2}$, then the following Lemma 4.2, which is a refinement of Lemma 3.10, assures that

Lemma 4.2. *For any fixed $a > 0$ and $s \in \{s \in \mathbf{C}; \mathrm{Re}(s) > a + \frac{1}{2}\}$, the following series is absolutely convergent and its convergence is uniform in the set $\{s \in \mathbf{C}; \mathrm{Re}(s) \geq \sigma_0\}$ for any $\sigma_0 > a + \frac{1}{2}$:*

$$\sum_{n=1}^{\infty} e^{-\pi n^2} \Big(\sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(a)} \frac{\Gamma(s)}{\Gamma(s+m)} \frac{(\pi n^2)^m}{m!}\Big).$$

Proof. By using the integral formula (3.13), we can prove Lemma 4.2, using the same procedure as in Lemma 3.10. We give here a proof of completeness.

Put $s = \sigma + i\tau$ $(\sigma, \tau \in \mathbf{R})$. Noting that $\sigma \geq \sigma_0 > a + \frac{1}{2} > 0$, we see from Theorem 2.1(iii) that

$$\begin{aligned}
\Big|\frac{\Gamma(s)}{\Gamma(s+m)}\Big| &\leq \frac{1}{|s(s+1)\cdots(s+m-1)|} \\
&\leq \frac{1}{\sigma_0(\sigma_0+1)\cdots(\sigma_0+m-1)} \\
&= \frac{\Gamma(\sigma_0)}{\Gamma(\sigma_0+m)}.
\end{aligned}$$

Therefore, we see from Theorem 2.1(iii) and (3.13) that for any $z > 0$

$$\begin{aligned}
\sum_{m=0}^{\infty} \Big|\frac{\Gamma(a+m)}{\Gamma(a)} \frac{\Gamma(s)}{\Gamma(s+m)} \frac{z^m}{m!}\Big| &\leq \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(a)} \frac{\Gamma(\sigma_0)}{\Gamma(\sigma_0+m)} \frac{z^m}{m!} \\
&= {}_1F_1(a;\sigma_0;z) \\
&= \frac{\Gamma(\sigma_0)}{\Gamma(a)\Gamma(\sigma_0-a)} \int_0^1 e^{z(1-t)} t^{\sigma_0-(a+1)} (1-t)^{a-1} dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} e^{-\pi n^2} \Big(\sum_{m=0}^{\infty} \Big|\frac{\Gamma(a+m)}{\Gamma(a)} \frac{\Gamma(s)}{\Gamma(s+m)} \frac{(\pi n^2)^m}{m!}\Big|\Big) \\
&\leq \frac{\Gamma(\sigma_0)}{\Gamma(a)\Gamma(\sigma_0-a)} \int_0^1 \Big(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\Big) t^{\sigma_0-(a+1)} (1-t)^{a-1} dt.
\end{aligned}$$

By the change of variables $t=\lambda^{-1}$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}e^{-\pi n^2}(\sum_{m=0}^{\infty}|\frac{\Gamma(a+m)}{\Gamma(a)}\frac{\Gamma(s)}{\Gamma(s+m)}\frac{(\pi n^2)^m}{m!}|)\\ &\le\frac{\Gamma(\sigma_0)}{\Gamma(a)\Gamma(\sigma_0-a)}\int_1^\infty(\sum_{n=1}^{\infty}e^{-\frac{\pi n^2}{\lambda}})\lambda^{-\sigma_0-(a+1)}(1-\lambda^{-1})^{a-1}\lambda^{-2}d\lambda.\end{aligned}$$

By using (3.16), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}e^{-\pi n^2}(\sum_{m=0}^{\infty}|\frac{\Gamma(a+m)}{\Gamma(a)}\frac{\Gamma(s)}{\Gamma(s+m)}\frac{(\pi n^2)^m}{m!}|)\\ &\le\frac{\Gamma(\sigma_0)}{\Gamma(a)\Gamma(\sigma_0-a)}\int_1^\infty(\sqrt{\lambda}\sum_{n=1}^{\infty}e^{-\pi n^2\lambda}+\frac{\sqrt{\lambda}-1}{2})\lambda^{-\sigma_0-(a+1)}(1-\lambda^{-1})^{a-1}\lambda^{-2}d\lambda\end{aligned}$$

$$(4.4)=\frac{\Gamma(\sigma_0)}{\Gamma(a)\Gamma(\sigma_0-a)}\{\int_1^\infty(\sum_{n=1}^{\infty}e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{\sigma_0-(a-1/2)}}d\lambda+\frac12\int_1^\infty(\frac{\lambda-1}{\lambda})^{a-1}(\frac{1}{\lambda^{\sigma_0-(a-1/2)}}+\frac{1}{\lambda^{\sigma_0-(a-1)}})d\lambda\}.$$

We calculate the second term in (4.4). By the change of variables $\frac{\lambda-1}{\lambda}=t$, we obtain

$$\begin{aligned}\int_1^\infty(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a-\frac12)}}dt&=\int_0^1t^{a-1}(1-t)^{s-(a-\frac12)}(1-t)^{-2}dt\\ &=\int_0^1t^{a-1}(1-t)^{s-(a+\frac32)}dt\\ &=B(a,s-a-\frac12).\end{aligned}\tag{4.5}$$

Similarly, we have

$$\int_1^\infty(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a+1)}}dt=B(a,s-a).$$

Hence, we have

$$\int_1^\infty(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a-1/2)}}dt=\frac{\Gamma(a)\Gamma(s-(a+1/2))}{\Gamma(s-1/2)}-\frac{\Gamma(a)\Gamma(s-a)}{\Gamma(s)}.\tag{4.6}$$

We consider the first term in (4.4). First, we consider the case where $a\ge1$. Since $(\frac{\lambda-1}{\lambda})^{a-1}$ is bounded in $[1,\infty)$, we see from Lemma 2.3 that the integrand of the first term in (4.4) is integrable. Hence, the first term of the right-hand side in (4.4) is finite.

Next, we also consider the case where $0<a<1$. We decompose the integral part of the first term in (4.4) into

$$\begin{aligned}(4.7)\qquad &\int_1^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2\lambda}\right)\left(\frac{\lambda-1}{\lambda}\right)^{a-1}\frac{1}{\lambda^{\sigma_0-(a-1/2)}}d\lambda\\ &=\int_1^2 \left(\sum_{n=1}^\infty e^{-\pi n^2\lambda}\right)\left(\frac{\lambda-1}{\lambda}\right)^{a-1}\frac{1}{\lambda^{\sigma_0-(a-1/2)}}d\lambda\\ &\quad+\int_2^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2\lambda}\right)\left(\frac{\lambda-1}{\lambda}\right)^{a-1}\frac{1}{\lambda^{\sigma_0-(a-1/2)}}d\lambda.\end{aligned}$$

We have the decomposition $(\frac{\lambda-1}{\lambda})^{a-1}=\frac{1}{(\lambda-1)^{1-a}}\lambda^{1-a}$. Since $\frac{1}{(\lambda-1)^{1-a}}$ is integrable in $[1,2]$, we see from Lemma 2.3 that the integrand of the first term in (4.8) is integrable in $[1,2]$. Furthermore, since $(\frac{\lambda-1}{\lambda})^{a-1}$ is bounded in $[2,\infty)$, we see from Lemma 2.3 that the integrand of the second term in (4.8) is integrable in $[2,\infty)$. Hence, the first term of the right-hand side in (4.4) is finite.

Thus, we have proved Lemma 4.2. (Q.E.D.)

By noting (3.10) and (3.12), we can arrange the proof of Lemma 4.2 to see that

Lemma 4.3. *For any fixed $a>0$ and $s\in\{s\in\mathbf{C};\mathrm{Re}(s)>a+\frac{1}{2}\}$, the following series*

$$\sum_{n=1}^\infty e^{-\pi n^2}\frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)}\int_0^1 e^{\pi n^2 t}(1-t)^{s-a-1}t^{a-1}dt$$

is absolutely convergent and the following relation holds:

$$\begin{aligned}&\sum_{n=1}^\infty e^{-\pi n^2}\frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)}\int_0^1 e^{\pi n^2 t}(1-t)^{s-a-1}t^{a-1}dt\\ &=\frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)}\{\int_1^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2\lambda}\right)\left(\frac{\lambda-1}{\lambda}\right)^{a-1}\frac{1}{\lambda^{s-(a-1/2)}}d\lambda\\ &\qquad+\frac{1}{2}\left(\frac{\Gamma(a)\Gamma(s-(a+1/2))}{\Gamma(s-1/2)}-\frac{\Gamma(a)\Gamma(s-a)}{\Gamma(s)}\right).\end{aligned}$$

Concerning the analytic continuation for the derived Kummer function $K_\theta(a)=K_\theta(a;s)$ with a parameter a, we have

Theorem 4.4. *Let a be any fixed positive number.*

(i) $K_\theta(a;s)=\dfrac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)}\displaystyle\int_0^1\left(\sum_{n=1}^\infty e^{-\pi n^2 t}\right)(1-t)^{s-a-1}t^{a-1}dt.$

(ii) $K_\theta(a;s)=\dfrac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)}\{\displaystyle\int_1^\infty\left(\sum_{n=1}^\infty e^{-\pi n^2\lambda}\right)\left(\frac{\lambda-1}{\lambda}\right)^{a-1}\frac{1}{\lambda^{s-(a-1/2)}}d\lambda$

$$+\frac{1}{2}(\frac{\Gamma(a)\Gamma(s-(a+1/2))}{\Gamma(s-1/2)}-\frac{\Gamma(a)\Gamma(s-a)}{\Gamma(s)})\}.$$

(iii) *The following function*

$$(\frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)})^{-1}K_\theta(a;s)$$

can be analytically continued on $\mathbf{C}$ *so that it is regular except at the set* $\{a-\frac{2n-1}{2},$ $a-n; n\in\mathbf{N}^*\}\cap\mathbf{Z}$, *where it has poles of order 1.*

(iv) *The function* $K_\theta(a;s)$ *can be analytically continued on* $\mathbf{C}$ *so that it is regular except on the set* $\{a-\frac{2n-1}{2}, -n; n\in\mathbf{N}^*\}\cap\mathbf{Z}$, *where it has poles of order 1.*

Proof. We see from Lemmas 2.3 and 2.4 that (i) and (ii) hold.

Next, we prove (iii). For that purpose, we have only to prove that

$$\int_1^\infty(\sum_{n=1}^\infty e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a-1/2)}}d\lambda \text{ is regular in } \mathbf{C}. \tag{4.8}$$

First, we consider the case where $a\geq 1$. By noting that $(\frac{\lambda-1}{\lambda})^{a-1}$ is bounded in $[1,\infty)$, we can use the same estimate as in the proof of Lemma 2.4 to see that (4.8) is proved.

Next, we also consider the case where $0<a<1$. We docompose the integral part in (4.8) into

$$\begin{aligned}(4.9)\quad &\int_1^\infty(\sum_{n=1}^\infty e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a-1/2)}}d\lambda\\ &=\int_1^2(\sum_{n=1}^\infty e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a-1/2)}}d\lambda+\int_2^\infty(\sum_{n=1}^\infty e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a-1/2)}}d\lambda.\end{aligned}$$

By noting that $(\frac{\lambda-1}{\lambda})^{a-1}=\frac{1}{(\lambda-1)^{1-a}}\lambda^{1-a}$, we decompose the integrand of the first term in the right-hand side in (4.9) into

$$(\sum_{n=1}^\infty e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a-1/2)}}=\frac{1}{(\lambda-1)^{1-a}}(\sum_{n=1}^\infty e^{-\pi n^2\lambda})\frac{1}{\lambda^{s-(a+1/2)}}.$$

Since $\frac{1}{(\lambda-1)^{a-1}}$ is integrable in [1,2], we can use the same estimate as in the proof of Lemma 2.4 to see that the first term of the right-hand side in (4.9) is regular in $\mathbf{C}$. Furthermore, since $(\frac{\lambda-1}{\lambda})^{a-1}$ is bounded in $[2,\infty)$, we can use the same estimate as in the proof of Lemma 2.4 to see that the second term of the right-hand side in (4.9) is regular in $\mathbf{C}$. This proves (4.8).

Property (iv) comes from (iii) and Theorem 2.1(i). (Q.E.D.)

Theorem 4.5. *The following relation holds: for any $a>0$,*

$$aK_\theta(a+1;s+1)=(a-s)K_\theta(a;s+1)+sK_\theta(a;s) \quad (s\neq a+\frac{1}{2},a-\frac{1}{2},0,-1,-2,\ldots).$$

[4.3] As in Lemma 4.2, by taking into account Theorem 4.1(iii), we prove

Lemma 4.6. *For any fixed $a>0$ and $s\in\{s\in\mathbf{C};\mathrm{Re}(s)>a+\frac{3}{2}\}$, the following series is absolutely convergent:*

$$\sum_{n=1}^{\infty}\pi n^2 e^{-\pi n^2}(\sum_{m=0}^{\infty}\frac{\Gamma(a+m)}{\Gamma(a)}\frac{\Gamma(s)}{\Gamma(s+m)}\frac{(\pi n^2)^m}{m!}).$$

Proof. From the proof of Lemma 4.2

$$\sum_{n=1}^{\infty}\pi n^2 e^{-\pi n^2}(\sum_{m=0}^{\infty}|\frac{\Gamma(a+m)}{\Gamma(a)}\frac{\Gamma(s)}{\Gamma(s+m)}\frac{(\pi n^2)^m}{m!}|)$$
$$\leq\frac{\Gamma(\sigma)}{\Gamma(a)\Gamma(\sigma-a)}\int_0^1(\sum_{n=1}^{\infty}\pi n^2 e^{-\pi n^2 t})t^{\sigma-(a+1)}(1-t)^{a-1}dt.$$

By the change of variables $t=\lambda^{-1}$, we have

$$\sum_{n=1}^{\infty}\pi n^2 e^{-\pi n^2}(\sum_{m=0}^{\infty}|\frac{\Gamma(a+m)}{\Gamma(a)}\frac{\Gamma(s)}{\Gamma(s+m)}\frac{(\pi n^2)^m}{m!}|)$$
$$\leq\frac{\Gamma(\sigma)}{\Gamma(a)\Gamma(\sigma-a)}\int_1^{\infty}(\sum_{n=1}^{\infty}\pi n^2 e^{-\frac{\pi n^2}{\lambda}})\lambda^{-\sigma+a+1}(1-\lambda^{-1})^{a-1}\lambda^{-2}d\lambda. \tag{4.10}$$

By differentiating (3.16) with respect to λ, we have

$$-\lambda^{-2}\sum_{n=1}^{\infty}\pi n^2 e^{-\frac{\pi n^2}{\lambda}}=\frac{1}{2}\lambda^{-\frac{1}{2}}\sum_{n=1}^{\infty}e^{-\pi n^2\lambda}-\sqrt{\lambda}\sum_{n=1}^{\infty}\pi n^2 e^{-\pi n^2\lambda}+\frac{1}{4}\lambda^{-\frac{1}{2}}$$

and so

$$\sum_{n=1}^{\infty}\pi n^2 e^{-\frac{\pi n^2}{\lambda}}=-\frac{1}{2}\lambda^{\frac{3}{2}}\sum_{n=1}^{\infty}e^{-\pi n^2\lambda}+\lambda^{\frac{5}{2}}\sum_{n=1}^{\infty}\pi n^2 e^{-\pi n^2\lambda}-\frac{1}{4}\lambda^{\frac{3}{2}}. \tag{4.11}$$

By substituting this into (4.10), we have

$$\sum_{n=1}^{\infty}\pi n^2 e^{-\pi n^2}(\sum_{m=0}^{\infty}|\frac{\Gamma(a+m)}{\Gamma(a)}\frac{\Gamma(s)}{\Gamma(s+m)}\frac{(\pi n^2)^m}{m!}|) \tag{4.12}$$
$$\leq\frac{\Gamma(\sigma)}{\Gamma(a)\Gamma(\sigma-a)}\{-\frac{1}{2}\int_1^{\infty}(\sum_{n=1}^{\infty}e^{-\pi n^2\lambda})(1-\lambda^{-1})^{a-1}\lambda^{-\sigma+\frac{1}{2}}d\lambda$$
$$+\int_1^{\infty}(\sum_{n=1}^{\infty}\pi n^2 e^{-\pi n^2\lambda})(1-\lambda^{-1})^{a-1}\lambda^{-\sigma+\frac{3}{2}}d\lambda-\frac{1}{4}\int_1^{\infty}(1-\lambda^{-1})^{a-1}\lambda^{-\sigma+a+\frac{1}{2}}d\lambda\}.$$

Similarly as in the proof of Lemma 4.2, we find that the first term of the bottom part in (4.12) is finite.

By using (2.6), we have

$$\sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 \lambda} \leq (\pi n^2 e^{-\pi(n-1)}) e^{-\pi\lambda} \in L^1([0,\infty), \mathcal{B}(\mathbf{R}), d\lambda).$$

Hence, similarly as in the proof of Lemma 4.2, we find that the second term of the bottom in (4.12) is finite.

Since it follows from (4.5) and Theorem 4.1(v) that

$$\int_1^{\infty} (\frac{\lambda-1}{\lambda})^{a-1} \frac{1}{\lambda^{\sigma-(a+1/2)}} dt = B(a, \sigma - a - \frac{3}{2}) = \frac{\Gamma(a)\Gamma(\sigma-(a+3/2))}{\Gamma(\sigma-3/2)},$$

we find that the third term of the bottom part in (4.12) is finite. Hence, we have proved Lemma 4.6. (Q.E.D.)

By virtue of Lemma 4.6, we can introduce another derived Kummer function $K_\theta^\bullet(a) = K_\theta^\bullet(a;s)$ with a parameter $a(>0)$ by

$$K_\theta^\bullet(a;s) \equiv \sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2} (\sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(a)} \frac{\Gamma(s)}{\Gamma(s+m)} \frac{(\pi n^2)^m}{m!}). \tag{4.13}$$

Concerning the analytic continuation for the function $K_\theta^\bullet(a) = K_\theta^\bullet(a;s)$, we have

Theorem 4.7. *Let a be any fixed positive number.*

(i) $$K_\theta^\bullet(a;s) = \frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)} \{ -\frac{1}{2} \int_1^{\infty} (\sum_{n=1}^{\infty} e^{-\pi n^2 \lambda})(\frac{\lambda-1}{\lambda})^{a-1} \frac{1}{\lambda^{s-(a+1/2)}} d\lambda$$
$$+ \int_1^{\infty} (\sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 \lambda})(\frac{\lambda-1}{\lambda})^{a-1} \frac{1}{\lambda^{s-(a+3/2)}} d\lambda - \frac{1}{4} \frac{\Gamma(a)\Gamma(s-(a+3/2))}{\Gamma(s-3/2)} \}.$$

(ii) *The following function*

$$\frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)})^{-1} K_\theta^\bullet(a;s)$$

can be analytically continued on $\mathbf{C}$ *so that it is regular except on the set of* $\{a - \frac{2n+1}{2}; n \in \{-2,-1,0,1,2,\ldots\}\} \cap \mathbf{Z}$*, where it has poles of order* 1.

(iii) *The function* $K_\theta^\bullet(a;s)$ *can be analytically continued on* $\mathbf{C}$ *so that it is regular except on the set* $\{a - \frac{2n+1}{2}, -n; n \in \{-2,-1,0,1,2,\ldots\}\} \cap \mathbf{Z}$*, where it has poles of order* 1.

Proof. We see from Lemma 4.6 that

$$K_\zeta^\bullet(a;s) = \frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)} \{ -\frac{1}{2} \int_1^{\infty} (\sum_{n=1}^{\infty} e^{-\pi n^2 \lambda})(\frac{\lambda-1}{\lambda})^{a-1} \frac{1}{\lambda^{s-(a+1/2)}} d\lambda \tag{4.14}$$
$$+ \int_1^{\infty} (\sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 \lambda})(\frac{\lambda-1}{\lambda})^{a-1} \frac{1}{\lambda^{s-(a+3/2)}} d\lambda - \frac{1}{4} \int_1^{\infty} (\frac{\lambda-1}{\lambda})^{a-1} \lambda^{-s+a+\frac{1}{2}} d\lambda \}.$$

On the other hand, it follows from (4.5) and Theorem 2.1(v) that

$$\int_1^\infty (\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a+1/2)}}dt = \frac{\Gamma(a)\Gamma(s-(a-1/2))}{\Gamma(s+1/2)}. \tag{4.15}$$

Hence, it follows from (4.14) and (4.15) that (i) holds. Similarly as in the proof of Lemma 4.6(ii), we can use the same estimate as in Lemma 2.4 to see that both the first term and the second term of the bottom part in (i) are regular on **C**. Hence, we see from Theorem 2.1(i) that (ii) holds. Property (iii) follows from (ii) and Theorem 4.1(i). (Q.E.D.)

We prove another representation for the function $K_\theta^\bullet(a;s)$.

Theorem 4.8. *For any $a>0$,*

$$K_\theta^\bullet(a;s) = \frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)}\{-\int_0^1 (\sum_{n=1}^\infty \pi n^2 e^{-\pi n^2 t})(1-t)^{a-1}t^{s-(a+1)}dt$$
$$-\int_0^1 (\sum_{n=1}^\infty e^{-\pi n^2 t})(1-t)^{a-1}t^{s-(a+2)}dt - \frac{1}{2}\frac{\Gamma(a)\Gamma(s-(a+1))}{\Gamma(s-1)}\}.$$

Proof. First, by the change of variables $\frac{\lambda-1}{\lambda}=t$ in Theorem 4.7(i), we see that the first essential term of the right-hand side of Theorem 4.7(i) is rewritten into

$$\begin{aligned}\int_1^\infty (\sum_{n=1}^\infty e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a+1/2)}}d\lambda &= \int_0^1 (\sum_{n=1}^\infty e^{-\frac{\pi n^2}{1-t}})t^{a-1}(1-t)^{s-(a+\frac{1}{2})}(1-t)^{-2}dt \\ &= \int_0^1 (\sum_{n=1}^\infty e^{-\frac{\pi n^2}{1-t}})t^{a-1}(1-t)^{s-(a+\frac{5}{2})}dt.\end{aligned}$$

By using the functional equation in (3.16) and Theorem 2.1(v), we have

$$\begin{aligned}&\int_1^\infty (\sum_{n=1}^\infty e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a+1/2)}}d\lambda \\ &= \int_0^1 (\sum_{n=1}^\infty e^{-\pi n^2(1-t)})t^{a-1}(1-t)^{s-(a+2)}dt \\ &\qquad + \frac{1}{2}\int_0^1 (t^{a-1}(1-t)^{s-(a+2)} - t^{a-1}(1-t)^{s-(a+\frac{5}{2})})dt\end{aligned}$$

$$= \int_0^1 (\sum_{n=1}^{\infty} e^{-\pi n^2(1-t)}) t^{a-1}(1-t)^{s-(a+2)} dt$$
$$+ \frac{1}{2}(B(a, s-(a+1)) - B(a, s-(a+\frac{3}{2})))$$
$$= \int_0^1 (\sum_{n=1}^{\infty} e^{-\pi n^2(1-t)}) t^{a-1}(1-t)^{s-(a+2)} dt$$
$$+ \frac{1}{2}(\frac{\Gamma(a)\Gamma(s-(a+1))}{\Gamma(s-1)} - \frac{\Gamma(a)\Gamma(s-(a+3/2))}{\Gamma(s-3/2)}).$$

Furthermore, by using the change of variables $\frac{\lambda-1}{\lambda} = t$ in Theorem 4.7(i), we also see that the second essential term of the right-hand side of Theorem 4.7(i) is rewritten into

$$\int_1^{\infty} (\sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 \lambda})(\frac{\lambda-1}{\lambda})^{a-1} \frac{1}{\lambda^{s-(a+3/2)}} d\lambda$$
$$= \int_0^1 (\sum_{n=1}^{\infty} \pi n^2 e^{-\frac{\pi n^2}{1-t}}) t^{a-1}(1-t)^{s-(a+\frac{3}{2})}(1-t)^{-2} dt$$
$$= \int_0^1 (\sum_{n=1}^{\infty} \pi n^2 e^{-\frac{\pi n^2}{1-t}}) t^{a-1}(1-t)^{s-(a+\frac{7}{2})} dt.$$

By using the functional equation in (4.11), we have

$$\int_1^{\infty} (\sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2 \lambda})(\frac{\lambda-1}{\lambda})^{a-1} \frac{1}{\lambda^{s-(a+3/2)}} d\lambda$$
$$= -\frac{1}{2}\int_0^1 (\sum_{n=1}^{\infty} e^{-\pi n^2(1-t)}) t^{a-1}(1-t)^{s-(a+2)} dt$$
$$+ \int_0^1 (\sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2(1-t)}) t^{a-1}(1-t)^{s-(a+1)} dt - \frac{1}{4}\int_0^1 t^{a-1}(1-t)^{s-(a+2)} dt$$
$$= -\frac{1}{2}\int_0^1 (\sum_{n=1}^{\infty} e^{-\pi n^2(1-t)}) t^{a-1}(1-t)^{s-(a+2)} dt$$
$$+ \int_0^1 (\sum_{n=1}^{\infty} \pi n^2 e^{-\pi n^2(1-t)}) t^{a-1}(1-t)^{s-(a+1)} dt - \frac{1}{4}B(a, s-(a+1)).$$

Hence, by using Theorem 2.1(vii), we have

$$(4.16)\ \int_1^\infty (\sum_{n=1}^\infty \pi n^2 e^{-\pi n^2\lambda})(\frac{\lambda-1}{\lambda})^{a-1}\frac{1}{\lambda^{s-(a+3/2)}}d\lambda$$
$$= -\frac{1}{2}\int_0^1 (\sum_{n=1}^\infty e^{-\pi n^2(1-t)})t^{a-1}(1-t)^{s-(a+2)}dt$$
$$+\int_0^1 (\sum_{n=1}^\infty \pi n^2 e^{-\pi n^2(1-t)})t^{a-1}(1-t)^{s-(a+1)}dt - \frac{1}{4}\frac{\Gamma(a)\Gamma(s-(a+1))}{\Gamma(s-1)}.$$

Hence, by substituting (4.16) to Theorem 4.7(i), we have

$$K_\theta^\bullet(a;s) = \frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)}\{-\frac{1}{2}\int_0^1 (\sum_{n=1}^\infty e^{-\pi n^2(1-t)})t^{a-1}(1-t)^{s-(a+2)}dt$$
$$-\frac{1}{4}(\frac{\Gamma(a)\Gamma(s-(a+1))}{\Gamma(s-1)} - \frac{\Gamma(a)\Gamma(s-(a+3/2))}{\Gamma(s-3/2)})$$
$$-\frac{1}{2}\int_0^1 (\sum_{n=1}^\infty e^{-\pi n^2(1-t)})t^{a-1}(1-t)^{s-(a+2)}dt$$
$$+\int_0^1 (\sum_{n=1}^\infty \pi n^2 e^{-\pi n^2(1-t)})t^{a-1}(1-t)^{s-(a+1)}dt - \frac{1}{4}\frac{\Gamma(a)\Gamma(s-(a+1))}{\Gamma(s-1)}$$
$$-\frac{1}{4}\frac{\Gamma(a)\Gamma(s-(a+3/2))}{\Gamma(s-3/2)}\}$$
$$= \frac{\Gamma(s)}{\Gamma(a)\Gamma(s-a)}\{(-\int_0^1 (\sum_{n=1}^\infty e^{-\pi n^2(1-t)})t^{a-1}(1-t)^{s-(a+2)}$$
$$+\int_0^1 (\sum_{n=1}^\infty \pi n^2 e^{-\pi n^2(1-t)})t^{a-1}(1-t)^{s-(a+1)}dt - \frac{1}{2}\frac{\Gamma(a)\Gamma(s-(a+1)}{\Gamma(s-1)}\},$$

which proves Theorem 4.8. (Q.E.D.)

Corresponding to Theorem 4.7, by using the recurrence formulae (iii),(iv) and (v) in Theorem 4.1, we see from Theorem 4.7 that

Theorem 4.9. *The following relations hold: for any $a > 0$,*

(i) $K_\theta^\bullet(a+1;s+1) = s(K_\theta(a+1;s) - K_\theta(a;s))$,

(ii) $aK_\theta(a+1;s) = K_\theta^\bullet(a;s) + (2a-s)K_\theta(a;s) + (s-a)K_\theta(a-1;s)$,

(iii) $(s-a)K_\zeta^\bullet(a;s+1) = sK_\zeta^\bullet(a;s) + s(s-1)K_\zeta(a;s) + s(1-s)K_\zeta(a;s-1)$,

where $s \notin \{a+\frac{5}{2}, a+\frac{3}{2}, a+\frac{1}{2}, 1, 0, -1, -2, \ldots\} \cap \mathbf{Z}$.

§ 5. A Hamiltonian operator associated with a stationary Gaussian process having T-positivity

Let $\mathbf{X} = (X(t); t \in \mathbf{R})$ be any real valued stationary Gaussian process on a probability space $(\Omega, \mathcal{B}, P)$ with mean 0 and the covariance function $R = R(t) : \mathbf{R} \to \mathbf{R}$:

$$E(X(t)) = 0 \qquad (t \in \mathbf{R}), \tag{5.1}$$

$$E(X(t)X(s)) = R(t-s) \qquad (t, s \in \mathbf{R}). \tag{5.2}$$

Furthermore, we consider the case where the process $\mathbf{X}$ satisfies T-positivity, that is, there exists a bounded Borel measure $\sigma = \sigma(d\lambda)$ on $[0,\infty)$ such that the covariance function $R = R(t)$ can be represented as the Laplace transform of the Borel measure σ:

$$R(t) = \int_0^\infty e^{-|t|\lambda}\sigma(d\lambda) \qquad (t \in \mathbf{R}). \tag{5.3}$$

[5.1] We define a complex Hilbert space $\mathbf{M}(\mathbf{X})$ as the closed subspace of the complex Hilbert space $L^2(\Omega, \mathcal{B}, P)$ by

$$\mathbf{M}(\mathbf{X}) \equiv \text{the closed linear hull of } \{X(t); t \in \mathbf{R}\}. \tag{5.4}$$

We denote by $(*, \star)_{\mathbf{M}(\mathbf{X})}$ and $\| * \|_{\mathbf{M}(\mathbf{X})}$ the inner product and the norm in the complex Hilbert space $\mathbf{M}(\mathbf{X})$, respectively:

$$(Y, Z)_{\mathbf{M}(\mathbf{X})} \equiv E(Y\bar{Z}) \qquad (Y, Z \in \mathbf{M}(\mathbf{X})), \tag{5.5}$$

$$\|Y\|_{\mathbf{M}(\mathbf{X})} \equiv \sqrt{(Y, Y)_{\mathbf{M}(\mathbf{X})}} \qquad (Y \in \mathbf{M}(\mathbf{X})). \tag{5.6}$$

Furthermore, we define three complex closed subspaces $\mathbf{M}^+(\mathbf{X})$, $\mathbf{M}^-(\mathbf{X})$ and $\mathbf{M}^{-/+}(\mathbf{X})$ of the complex Hilbert space $\mathbf{M}(\mathbf{X})$ by

$$\mathbf{M}^+(\mathbf{X}) \equiv \text{the closed linear hull of } \{X(t); t \geq 0\}, \tag{5.7}$$

$$\mathbf{M}^-(\mathbf{X}) \equiv \text{the closed linear hull of } \{X(t); t \leq 0\}, \tag{5.8}$$

$$\mathbf{M}^{-/+}(\mathbf{X}) \equiv \text{the closed linear hull of } \{P_{\mathbf{M}^+(\mathbf{X})}Y; Y \in \mathbf{M}^-(\mathbf{X})\}, \tag{5.9}$$

where $P_{\mathbf{M}^+(\mathbf{X})}$ stands for the projection operator on the closed space $\mathbf{M}^+(\mathbf{X})$. We call these complex closed subspaces $\mathbf{M}^+(\mathbf{X}), \mathbf{M}^-(\mathbf{X})$ and $\mathbf{M}^{-/+}(\mathbf{X})$ the future space, the past space and the splitting space associated with the process $\mathbf{X}$, respectively.

In [7], we treated the real Hilbert space $\mathbf{M}_{\text{real}}(\mathbf{X})$ defined by

$$\mathbf{M}_{\text{real}}(\mathbf{X}) \equiv \text{the closure of } \{\textstyle\sum_{n=1}^N c_n X(t_n); c_n, t_n \in \mathbf{R} (1 \leq n \leq N), N \in \mathbf{N}\} \tag{5.10}$$

and constructed a Hamiltonian operator acting on the real splitting subspace $\mathbf{M}_{\text{real}}^{-/+}(\mathbf{X})$ of the space $\mathbf{M}_{\text{real}}(\mathbf{X})$. Since we can obtain the same results as in [7] by taking the same procedure as in [7], we state only the results of [7] which will be needed in this paper.

Since $\mathbf{X}$ is a real valued and stationary process, we can construct a strongly continuous one-parameter group $U = (U(t); t \in \mathbf{R})$ of unitary operators and a unitary and self-adjoint operator T acting on the complex Hilbert space $\mathbf{M}(\mathbf{X})$ such that

$$U(t)(X(s)) = X(s+t) \qquad (t, s \in \mathbf{R}), \tag{5.11}$$

$$T(X(s)) = X(-s) \qquad (s \in \mathbf{R}). \tag{5.12}$$

The operator T is called a time reflection operator.

Furthermore, we define an operator S acting on the complex Hilbert space $\mathbf{M}(\mathbf{X})$ by

$$S \equiv P_{\mathbf{M}^+(\mathbf{X})} T P_{\mathbf{M}^+(\mathbf{X})}. \tag{5.13}$$

It then follows that

Lemma 5.1. ([7])

(i) *The operator S is a bounded self-adjoint operator.*

(ii) $S(\mathbf{M}(\mathbf{X})) \subset \mathbf{M}^{-/+}(\mathbf{X})$.

(iii) $SX(0) = X(0)$.

(iv) $S^{\frac{1}{2}}X(0) = X(0)$.

Since $R(t+s) = (X(t), TX(s))_{\mathbf{M}(\mathbf{X})}$ $(t, s \in \mathbf{R})$, we find from (5.3) that

Lemma 5.2. ([7]) *The self-adjoint operator S is non-negative, that is,*

$$(S(Y), Y)_{\mathbf{M}(\mathbf{X})} \geq 0 \qquad (Y \in \mathbf{M}(\mathbf{X})).$$

Remark. Since the non-negative property of the operator S is equivalent to the T-positivity of the process $\mathbf{X}$([2]), the notion of T-positivity is also called a reflection positivity.

Lemma 5.3. ([7]) $U(t)(\mathbf{M}^+(\mathbf{X}) \ominus \mathbf{M}^{-/+}(\mathbf{X})) \subset \mathbf{M}^+(\mathbf{X}) \ominus \mathbf{M}^{-/+}(\mathbf{X}) \qquad (t \geq 0)$.

By using the strongly continuous one-parameter group $\{U(t); t \in \mathbf{R}\}$ of unitary operators in (5.11), we define a one-parameter semi-group $\{V(t); t \geq 0\}$ of bounded operators acting on the splitting space $\mathbf{M}^{-/+}(\mathbf{X})$ by

$$V(t)Y \equiv P_{\mathbf{M}^{-/+}(\mathbf{X})}U(t)Y \qquad (Y \in \mathbf{M}^{-/+}(\mathbf{X})). \tag{5.14}$$

Lemma 5.4. ([7]) $(V(t); t \geq 0)$ *is the strongly continuous one-parameter semi-group of contraction operator.*

Concerning the kernel space of the operator S in (5.13), we have

Lemma 5.5. ([7]) $\{Y \in \mathbf{M}^{+}(\mathbf{X});\ SY = 0\} = \mathbf{M}^{+}(\mathbf{X}) \ominus \mathbf{M}^{-/+}(\mathbf{X})$.

It follows from Lemmas 5.1, 5.2 and 5.5 that

Lemma 5.6. ([7]) (i) *The bounded self-adjoint operator* $S : \mathbf{M}^{-/+}(\mathbf{X}) \to \mathbf{M}^{-/+}(\mathbf{X})$ *is positive and contractive.*

(ii) $S^{\frac{1}{2}}(\mathbf{M}^{-/+}(\mathbf{X}))$ *is dense in* $\mathbf{M}^{-/+}(\mathbf{X})$.

After the above preparations, we have

Theorem 5.7. ([7]) *There exists uniquely a strongly continuous one-parameter semi-group* $\{T(t); t \geq 0\}$ *of bounded symmetric operators acting on the splitting space* $\mathbf{M}^{-/+}(\mathbf{X})$ *such that*

$$T(t)S^{\frac{1}{2}} = S^{\frac{1}{2}}V(t) \qquad (t \geq 0).$$

As a characteristic difference property between the one-parameter group $\{U(t); t \in \mathbf{R}\}$ of unitary operators and the one-parameter semi-group $\{T(t); t \geq 0\}$ of bounded symmetric operators, we show

Proposition 5.8.

(i) $(U(t)X(0), U(s)X(0))_{\mathbf{M}(\mathbf{X})} = R(t-s) \qquad (t, s \in \mathbf{R})$.

(ii) $(T(t)X(0), T(s)X(0))_{\mathbf{M}(\mathbf{X})} = R(t+s) \qquad (t, s \geq 0)$.

Proof. (i) comes from (5.2) and (5.11). Take any $t, s \geq 0$ and fix them. By

Lemmas 5.1(i), 5.1(iv) and Theorem 5.7, we have

$$\begin{aligned}(T(t)X(0), T(s)X(0))_{\mathbf{M}(\mathbf{X})} &= (T(s)T(t)X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (T(s+t)X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (T(s+t)S^{\frac{1}{2}}X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (S^{\frac{1}{2}}V(s+t)X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (V(s+t)X(0), S^{\frac{1}{2}}X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (V(s+t)X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (P_{\mathbf{M}^{-/+}(\mathbf{X})}U(s+t)X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (U(s+t)X(0), P_{\mathbf{M}^{-/+}(\mathbf{X})}X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (U(s+t)X(0), X(0))_{\mathbf{M}(\mathbf{X})},\end{aligned}$$

which, together with (i), proves (ii). (Q.E.D.)

We denote by $-H_{\mathbf{X}}$ the infinitesimal generator of the strongly continuous one-parameter semi-group $\{T(t); t \geq 0\}$:

$$T(t) = e^{-tH_{\mathbf{X}}} \qquad (t \geq 0). \tag{5.15}$$

Sine the operator $H_{\mathbf{X}}$ is a self-adjoint and non-negative operator acting on the splitting space $\mathbf{M}^{-/+}(\mathbf{X})$, we have a decomposition $\{E(\lambda); \lambda \geq 0\}$ of identity associated with the operator $H_{\mathbf{X}}$([17]):

$$H_{\mathbf{X}} = \int_0^\infty \lambda dE(\lambda). \tag{5.16}$$

We note that each operator $E(\lambda)$ is a projection operator acting on the space $\mathbf{M}^{-/+}(\mathbf{X})$. We call the self-adjoint operator $H_{\mathbf{X}}$ the Hamiltonian operator associated with the stationary Gaussian process $\mathbf{X}$.

As an application of Proposition 5.8(ii), we find that the Borel measure σ in (5.3) can be represented by the spectral resolution $(E(\lambda); \lambda \geq 0)$.

Proposition 5.9. ([7]) $\sigma(d\lambda) = d(E(\lambda)X(0), X(0))_{\mathbf{M}(\mathbf{X})}$.

Proof. It follows from (5.15), (5.16) and Proposition 5.8(ii) that for any $t \geq 0$,

$$\begin{aligned}R(t) &= (T(t)X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (e^{-tH_{\mathbf{X}}}X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= (\int_0^\infty e^{-t\lambda}dE(\lambda)X(0), X(0))_{\mathbf{M}(\mathbf{X})}\\ &= \int_0^\infty e^{-t\lambda}d(E(\lambda)X(0), X(0))_{\mathbf{M}(\mathbf{X})}.\end{aligned}$$

Therefore, by noting (5.3), we see from the uniqueness of the Laplace transform that Proposition 5.9 holds.. (Q.E.D.)

As a special property of the vector $X(0)$ of the splitting space $\mathbf{M}^{-/+}(\mathbf{X})$, we have

Lemma 5.10. ([7]) *The Hamiltonian operator $H_{\mathbf{X}}$ has a simple spectrum and the vector $X(0)$ is a generating vector of the splitting space $\mathbf{M}^{-/+}(\mathbf{X})$ with respect to the operator $H_{\mathbf{X}}$, that is,*

$$\{T(t)X(0); t \geq 0\} \textit{ is dense in } \mathbf{M}^{-/+}(\mathbf{X}).$$

By virtue of Lemma 5.10, we can apply Theorem 7.10 in [15] to the operator $H_{\mathbf{X}}$ to obtain

Theorem 5.11. ([7]) (i) *There exists a unitary operator V from $\mathbf{M}^{-/+}(\mathbf{X})$ onto $L^2([0,\infty), \mathcal{B}([0,\infty)), \sigma)$ such that*

$$Y = \int_0^\infty (VY)(\lambda) dE(\lambda) X(0) \qquad (Y \in \mathbf{M}^{-/+}(\mathbf{X})).$$

(ii) *Setting $H_\sigma \equiv V H_{\mathbf{X}} V^{-1}$, we obtain*

$$\mathcal{D}(H_\sigma) = \{f \in L^2([0,\infty), \mathcal{B}([0,\infty)), \sigma); \lambda f(\lambda) \in L^2([0,\infty), \mathcal{B}([0,\infty)), \sigma)\}$$

and

$$(H_\sigma f)(\lambda) = \lambda f(\lambda) \qquad (f \in \mathcal{D}(H_\sigma)).$$

From Lemma 5.10 and Proposition 5.9, we obtain

Proposition 5.12. ([7]) *The following three conditions* (*i*),(*ii*) *and* (*iii*) *are equivalent*:

(i) $\sigma(\{0\}) = 0$.

(ii) $s\text{-}\lim_{t\to\infty} T(t) = 0$.

(iii) $H_{\mathbf{X}}$ *is injective.*

[5.2] By taking account of (5.15), we can do an analytic continuation of the one-parameter semigroup $\{T(t); t \geq 0\}$ constructed in Theorem 5.7 from $[0,\infty)$ to $\{z \in \mathbf{C}; \mathrm{Re}(z) \geq 0\}$ to define a family $\{T(z); z \in \mathbf{C}, \mathrm{Re}(z) \geq 0\}$ of bounded operators acting on the splitting space $\mathbf{M}^{-/+}(\mathbf{X})$ by

$$T(z)Y \equiv \int_0^\infty e^{-z\lambda} dE(\lambda) Y \quad (Y \in \mathbf{M}^{-/+}(\mathbf{X})). \tag{5.17}$$

We show

Theorem 5.13.

(i) $T(z_1)T(z_2) = T(z_1 + z_2) \quad (z_j \in \mathbf{C}, \mathrm{Re}(z_j) \geq 0 (1 \leq j \leq 2))$.

(ii) *For each* $Y \in \mathbf{M}^{-/+}(\mathbf{X})$,

$$(T(z_1)Y, T(z_2)Y)_{\mathbf{M}(\mathbf{X})} = \int_0^\infty e^{-(z_1 + \overline{z_2})\lambda}\sigma(\lambda).$$

(iii) *For each* $Y \in \mathbf{M}^{-/+}(\mathbf{X})$, *the following function*

$$\{z \in \mathbf{C}; \mathrm{Re}(z) > 0\} \ni z \to T(z)Y \in \mathbf{M}^{-/+}(\mathbf{X})$$

is differentiable and $\frac{T(z)Y}{dz} = \int_0^\infty (-\lambda e^{-z\lambda}) dE(\lambda)Y$.

Proof. By noting the property of sepctral resolution, we find from (5.17) that $dE(\lambda)T(z_2)Y = e^{-z_2\lambda}dE(\lambda)Y$ and so $T(z_1)T(z_2)Y = \int_0^\infty e^{-z_1\lambda}e^{-z_2\lambda}dE(\lambda)Y = T(z_1 + z_2)Y$, which proves (i). (ii) comes from Proposition 5.9. Fix any $z \in \{z \in \mathbf{C}; \mathrm{Re}(z) > 0\}$. For any $w \in \{w \in \mathbf{C}; 0 < |w| < 1\}$,

$$\frac{T(z+w)Y - T(z)Y}{w} - \int_0^\infty (-\lambda e^{-z\lambda}) dE(\lambda)Y = \int_0^\infty e^{-z\lambda}\left(\frac{e^{-w\lambda} - 1}{w} + \lambda\right) dE(\lambda)Y.$$

Since $|\frac{e^{-w\lambda}-1}{w} + \lambda| \leq \frac{|w|\lambda^2}{2}$, $\lim_{w\to 0} \frac{e^{-w\lambda}-1}{w} + \lambda = 0$ and $\lambda e^{-z\lambda}$ is bounded, we can use the Lebegue's convergence theorem to see that

$$\lim_{w\to 0}\left\|\frac{T(z+w)Y - T(z)Y}{w} - \int_0^\infty (-\lambda e^{-z\lambda}) dE(\lambda)Y\right\|_{\mathbf{M}(\mathbf{X})} = 0,$$

which implies that (iii) holds. (Q.E.D.)

We define a complex valued stochastic process $\mathbf{X_W} = (X_{\mathbf{W}}(\tau); \tau \in \mathbf{R})$ by

$$X_{\mathbf{W}}(\tau) \equiv T(i\tau)X(0). \tag{5.18}$$

Theorem 5.14. *The complex valued stochastic process* $\mathbf{X_W}$ *is stationary whose mean is* 0 *and covariance function* $R_{\mathbf{X_W}} = (R_{\mathbf{X_W}}(\tau); \tau \in \mathbf{R})$ *is given by*

$$R_{\mathbf{X_W}}(\tau) \equiv (X_{\mathbf{W}}(\tau + t), X_{\mathbf{W}}(t))_{\mathbf{M}(\mathbf{X})} = \int_0^\infty e^{-i\tau\lambda}\sigma(d\lambda) \qquad (\tau, t \in \mathbf{R}).$$

Proof. By (5.17) and (5.18), we have

$$(X_{\mathbf{W}}(\tau + t), X_{\mathbf{W}(t)})_{\mathbf{M}(\mathbf{X})} = \int_0^\infty e^{-i(\tau+t)\lambda}e^{it\lambda}d(E(\lambda)X(0), X(0)),$$

which, together with Proposition 5.9, implies that Theorem 5.14 holds. (Q.E.D.)

We explain a relation between the real valued stationary Gaussian process $\mathbf{X}$ and the complex valued stationary process $\mathbf{X_W}$ from the view-point of the axiomatic field theory. We find from (5.2) and Theorem 5.14 that the covarince function R and $R_{\mathbf{X_W}}$ of the stationary Gaussian process $\mathbf{X}$ and the stationary process $\mathbf{X_W}$ is given by the Laplace transform and the one-sided Fourier transform of the bounded Borel measure $\sigma = \sigma(d\lambda)$ in $[0, \infty)$, respectively. For this reason, the covarinace function R and $R_{\mathbf{X_W}}$ is qualified to be called the Schwinger function of order 2 and the Wightmann function of order 2, respectively([6],[13],[3]).

§ 6. An inner product representation for the analytic continuation for Riemann's zeta function and the derived Kummer function

[6.1] By transforming the complex valued Borel measure $\Gamma_{\frac{1+s}{2}}$ in (3.5) and $\Gamma_{\frac{2-s}{2}}$ in (3.6) to the real valued gamma distribution $\Gamma_{\frac{3}{4}}$, we find that the functions F_1 in (3.1) and F_2 in (3.2) can be rewritten in the following form.

Lemma 6.1.

(i) $$F_1(s) = \frac{\Gamma(3/4)}{\Gamma((1+s)/2)} \sum_{n=1}^{\infty} e^{-\pi n^2} \int_0^{\infty} \frac{1}{\pi n^2 + \lambda} \lambda^{\frac{2s-1}{4}} \Gamma_{\frac{3}{4}}(d\lambda).$$

(i) $$F_2(s) = \frac{\Gamma(3/4)}{\Gamma((2-s)/2)} \sum_{n=1}^{\infty} e^{-\pi n^2} \int_0^{\infty} \frac{1}{\pi n^2 + \lambda} \lambda^{\frac{1-2s}{4}} \Gamma_{\frac{3}{4}}(d\lambda).$$

Let $\mathbf{X}_{\Gamma_{3/4}} = (X_{\Gamma_{3/4}}(t); t \in \mathbf{R})$ be the one-dimensional stationary Gaussian process with mean 0 whose covariance function $R_{\mathbf{X}_{\Gamma_{3/4}}} = (R_{\mathbf{X}_{\Gamma_{3/4}}}(t); t \in \mathbf{R})$ is given by

$$R_{\mathbf{X}_{\Gamma_{3/4}}}(t) \equiv \int_0^{\infty} e^{-|t|\lambda} \Gamma_{\frac{3}{4}}(d\lambda) \qquad (t \in \mathbf{R}). \tag{6.1}$$

By applying Theorem 5.7 to the above stationary Gaussian process $\mathbf{X}_{\Gamma_{3/4}}$ with T-positivity, we have the strongly continuous one-parameter group $\{T_{\mathbf{X}_{\Gamma_{3/4}}}(t); t \geq 0\}$ and the Hamiltonian operator $H_{\mathbf{X}_{\Gamma_{3/4}}}$ associated with the stationary Gaussian process $\mathbf{X}_{\Gamma_{3/4}}$:

$$T_{\mathbf{X}_{\Gamma_{3/4}}}(t) = e^{-tH_{\mathbf{X}_{\Gamma_{3/4}}}} \qquad (t \geq 0). \tag{6.2}$$

Since $\Gamma_{\frac{3}{4}}(\{0\}) = 0$, it follows from Proposition 5.12 that

Proposition 6.2. *The Hamiltonian operator $H_{\mathbf{X}_{\Gamma_{3/4}}}$ is injective.*

For each $\alpha > 0$, we define the resolvent operator G_α for the semi-group $\{T_{\mathbf{X}_{\Gamma_{3/4}}}(t); t \geq 0\}$ by

$$G_\alpha \equiv \int_0^\infty e^{-\alpha t} T_{\mathbf{X}_{\Gamma_{3/4}}}(t) dt = (\alpha + H_{\mathbf{X}_{\Gamma_{3/4}}})^{-1}. \tag{6.3}$$

In particular, we see from (5.16) and (6.3) that

$$G_\alpha = \int_0^\infty \frac{1}{\alpha + \lambda} dE(\lambda). \tag{6.4}$$

Corresponding to the derived Kummer function K_θ defined by (3.18), we introduce an operator G_θ acting on the splitting space $\mathbf{M}^{-/+}(\mathbf{X}_{\Gamma_{3/4}})$ by

$$G_\theta \equiv \sum_{n=1}^\infty e^{-\pi n^2} G_{\pi n^2}. \tag{6.5}$$

Furthermore, we define a function $G = G(\lambda) : [0, \infty) \to \mathbf{R}$ by

$$G(\lambda) \equiv \sum_{n=1}^\infty e^{-\pi n^2} \frac{1}{\pi n^2 + \lambda}. \tag{6.6}$$

Lemma 6.3.

(i) $\frac{e^{-\pi}}{\pi + \lambda} \leq G(\lambda) \leq \frac{1}{1 - e^{-\pi}} \frac{e^{-\lambda}}{\pi + \lambda}$.

(ii) *G_θ is a bounded operator and*

$$G_\theta Y = \int_0^\infty G(\lambda) dE(\lambda) Y \qquad (Y \in \mathbf{M}^{-/+}(\mathbf{X}_{\Gamma_{3/4}})).$$

Proof. By noting (1.5), we have

$$\begin{aligned}
G(\lambda) &= \sum_{n=1}^\infty e^{-\pi n^2} \int_0^\infty e^{-(\pi n^2 + \lambda)t} dt \\
&= \int_0^\infty (\sum_{n=1}^\infty e^{-\pi n^2 (t+1)}) e^{-\lambda t} dt \\
&= \int_0^\infty \frac{\theta(t+1) - 1}{2} e^{-\lambda t} dt \\
&= \int_1^\infty \frac{\theta(t) - 1}{2} e^{-\lambda(t-1)} dt \\
&= e^\lambda \int_1^\infty \frac{\theta(t) - 1}{2} e^{-\lambda t} dt.
\end{aligned}$$

Therefore, by the change of variables $x = \lambda t$, we have

$$G(\lambda) = \frac{e^{\lambda}}{\lambda}\int_{\lambda}^{\infty} \frac{\theta(x/\lambda) - 1}{2} e^{-x} dx. \tag{6.7}$$

By applying the left-hand side of the inequality in Lemma 2.3 to (6.7), we have

$$G(\lambda) \geq \frac{e^{\lambda}}{\lambda}\int_{\lambda}^{\infty} e^{-\pi\frac{x}{\lambda}} e^{-x} dx = \frac{e^{\lambda}}{\lambda}\int_{\lambda}^{\infty} e^{-(\frac{\pi}{\lambda}+1)x} dx = \frac{e^{\lambda}}{\lambda}\frac{e^{-(\pi/\lambda+1)\lambda}}{\pi/\lambda + 1} = \frac{e^{-\pi}}{\pi + \lambda},$$

which gives the left-hand side in the inequality of (i).

On the other hand, by applying the right-hand side of the inequality in Lemma 2.3 to (6.7), we have

$$G(\lambda) \leq \frac{e^{\lambda}}{\lambda}\int_{\lambda}^{\infty} \frac{e^{-\pi(x/\lambda)}}{1 - e^{-\pi}} e^{-x} dx = \frac{1}{1 - e^{-\pi}}\frac{e^{-\lambda}}{\pi + \lambda},$$

which gives the right-hand side in the inequality of (i).

Furthermore, we see from (6.4) and (6.5) that

$$G_{\theta} = \sum_{n=1}^{\infty} e^{-\pi n^2}(\int_0^{\infty} \frac{1}{\pi n^2 + \lambda} dE(\lambda)).$$

By virtue of (i), we can use Fubini's theorem to change the order between the summation and the integral in the above equality and find from (6.6) that

$$G_{\theta} = \int_0^{\infty} (\sum_{n=1}^{\infty} e^{-\pi n^2}\frac{1}{\pi n^2 + \lambda}) dE(\lambda) = \int_0^{\infty} G(\lambda) dE(\lambda),$$

which proves (ii).

Thus, we conclude that Lemma 6.3 holds. (Q.E.D.)

[6.2] By virtue of Proposition 6.2, for each $\sigma \in \mathbf{R}$, we can define a self-adjoint operator H_{σ} acting on the splitting space $\mathbf{M}^{-/+}(\mathbf{X}_{\Gamma_{3/4}})$ by

$$H_{\sigma} \equiv (H_{\mathbf{X}_{\Gamma_{3/4}}})^{\frac{2\sigma-1}{4}} = \int_0^{\infty} \lambda^{\frac{2\sigma-1}{4}} dE(\lambda), \tag{6.8}$$

$$\mathcal{D}(H_{\sigma}) \equiv \{Y \in \mathbf{M}^{-/+}(\mathbf{X}_{\Gamma_{3/4}}); VY(\lambda)\lambda^{\frac{2\sigma-1}{4}} \in L^2([0,\infty), \mathcal{B}([0,\infty)), \Gamma_{\frac{3}{4}})\}. \tag{6.9}$$

Lemma 6.4. *For any $\sigma \in \mathbf{R}$, $G_{\theta}H_{\sigma}$ is a symmetric operator and $G_{\theta}H_{\sigma} = H_{\sigma}G_{\theta}$.*

Proof. Since it follows from Lemma 6.3(ii) that G_{θ} is a bounded operator, $\mathcal{D}(G_{\theta}H_{\sigma}) = \mathcal{D}(H_{\sigma})$. Take any $Y \in \mathcal{D}(H_{\sigma})$ and fix it. Since it follows from Lemma 6.3(ii) that $G(\lambda)$ is a bounded function, we note that

$$\lambda^{\frac{2\sigma-1}{4}} G(\lambda) VY(\lambda) = G(\lambda)(\lambda^{\frac{2\sigma-1}{4}} VY(\lambda)) \in L^2([0,\infty), \mathcal{B}([0,\infty)), \Gamma_{\frac{3}{4}}). \tag{6.10}$$

Furthermore, by Theorem 5.11(ii) and Lemma 6.3, we have

$$V(G_\theta H_\sigma Y)(\lambda) = \lambda^{\frac{2\sigma-1}{4}} G(\lambda) VY(\lambda). \tag{6.11}$$

Hence, we see from (6.8) and (6.9) that $G_\theta Y \in \mathcal{D}(H_\sigma)$ and $G_\theta H_\sigma Y = H_\sigma G_\theta Y$. On the other hand, take any $Y \in \mathcal{D}(H_\sigma)$ and $Z \in \mathbf{M}^{-/+}(\mathbf{X}_{\Gamma_{3/4}})$. By (6.9), we see from Theorem 5.11(ii) that

$$\begin{aligned}(G_\theta H_\sigma Y, Z)_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} &= \int_0^\infty G(\lambda)\lambda^{\frac{2\sigma-1}{4}} VY(\lambda)\overline{VZ(\lambda)}\Gamma_{\frac{3}{4}}(d\lambda) \\ &= \int_0^\infty VY(\lambda)\overline{G(\lambda)\lambda^{\frac{2\sigma-1}{4}} VZ(\lambda)}\Gamma_{\frac{3}{4}}(d\lambda) \\ &= (Y, G_\theta H_\sigma Z)_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})},\end{aligned}$$

which proves that $G_\theta H_\sigma$ is a symmetric operator.

Therefore, we conclude that Lemma 6.4 holds. (Q.E.D.)

[6.3] After the above preparations, we shall obtain an inner product representation for the analytic continuation of Riemann's zeta function.

For that purpose, contact to (5.18), we define a complex valued stochastic process $\mathbf{W} = (W(\tau); \tau \in \mathbf{R})$ by

$$W(\tau) \equiv \int_0^\infty \lambda^{i\tau} dE(\lambda)X(0). \tag{6.12}$$

We note that

$$W(0) = X(0). \tag{6.13}$$

Lemma 6.5. *The complex valued stochastic process* $\mathbf{W}$ *is stationary whose mean is* 0 *and covariance function* $R_{\mathbf{W}} = (R_{\mathbf{W}}(\tau); \tau \in \mathbf{R})$ *is given by*

$$R_{\mathbf{W}}(\tau) \equiv (W(\tau+t), W(t))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} = \int_0^\infty \lambda^{i\tau\lambda}\sigma(d\lambda) \qquad (\tau, t \in \mathbf{R}).$$

Proof. By (6.12), we have

$$(W(\tau+t), W(t))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} = \int_0^\infty \lambda^{i(\tau+t)}\lambda^{-it} d(E(\lambda)X(0), X(0)),$$

which, together with Proposition 5.9, implies that Lemma 6.5 holds. (Q.E.D.)

Lemma 6.6. *For any* $s = \sigma + i\tau (0 < \sigma < 1, \tau \in \mathbf{R})$,

(i) $V(X(0)) = 1$,

(ii) $X(0) \in \mathcal{D}(H_\sigma) \cap \mathcal{D}(H_{1-\sigma})$,

(iii) $V(W(\tau)) = 1$,

(iv) $W(\tau) \in \mathcal{D}(H_\sigma) \cap \mathcal{D}(H_{1-\sigma})$.

Proof. Since $\lim_{\lambda\to\infty} E(\lambda) = I$ and $\lim_{\lambda\to 0} E(\lambda) = 0$, (i) holds. By virtue of (i), in order to prove (ii), we have only to show

$$\lambda^{\frac{2\sigma-1}{4}}, \lambda^{\frac{1-2\sigma}{4}} \in L^2([0,\infty), \mathcal{B}([0,\infty)), \Gamma_{3/4}). \tag{6.14}$$

By noting (1.8), we have

$$\int_0^\infty |\lambda^{\frac{2\sigma-1}{4}}|^2 \Gamma_{\frac{3}{4}}(d\lambda) = \int_0^\infty \lambda^{\frac{2\sigma-1}{2}} e^{-\lambda}\lambda^{\frac{3}{4}-1} d\lambda = \int_0^\infty e^{-\lambda}\frac{1}{\lambda^{(3-4\sigma)/4}} d\lambda.$$

Since $0<\sigma<1$, we see that $-\frac{1}{4} < \frac{3-4\sigma}{4} < \frac{3}{4}$ and so that $\lambda^{\frac{2\sigma-1}{4}} \in L^2([0,\infty), \mathcal{B}([0,\infty)), \Gamma_{\frac{3}{4}})$. By replacing σ to $1-\sigma$, we find that $\lambda^{\frac{1-2\sigma}{4}} \in L^2([0,\infty), \mathcal{B}([0,\infty)), \Gamma_{\frac{3}{4}})$, which proves (6.14). Since $V(W(\tau)) = \lambda^{i\tau}$, (iii) holds. By virtue of (iii), (iv) can be proved similarly as in (ii).

Therefore, we conclude that Lemma 6.6 holds. (Q.E.D.)

Lemma 6.7. *For any* $s = \sigma + i\tau (0 < \sigma < 1, \tau \in \mathbf{R})$,

(i) $F_1(s) = \dfrac{\Gamma(3/4)}{\Gamma((1+s)/2)}(G_\theta H_\sigma(W(\dfrac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}$,

(ii) $F_2(s) = \dfrac{\Gamma(3/4)}{\Gamma((2-s)/2)}(G_\theta H_{1-\sigma}(W(-\dfrac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}$.

Proof. By taking the same consideration as in Lemma 6.3(ii) and using Lemma 6.6(iii), we can change the order of the summation and the integral in Lemma 6.1 to see from Theorem 5.7 that

$$\begin{aligned} F_1(s) &= \frac{\Gamma(3/4)}{\Gamma((1+s)/2)} \int_0^\infty (\sum_{n=1}^\infty e^{-\pi n^2}\frac{1}{\pi n^2+\lambda})\lambda^{\frac{2s-1}{4}}\Gamma_{\frac{3}{4}}(d\lambda) \\ &= \frac{\Gamma(3/4)}{\Gamma((1+s)/2)} \int_0^\infty G(\lambda)\lambda^{\frac{2s-1}{4}}\Gamma_{\frac{3}{4}}(d\lambda) \\ &= \frac{\Gamma(3/4)}{\Gamma((1+s)/2)} \int_0^\infty G(\lambda)\lambda^{\frac{2\sigma-1}{4}}\lambda^{i\frac{\tau}{2}}\Gamma_{\frac{3}{4}}(d\lambda) \\ &= \frac{\Gamma(3/4)}{\Gamma((1+s)/2)} \int_0^\infty V(G_\theta H_\sigma(W(\frac{\tau}{2})))\overline{VW(0)}\Gamma_{\frac{3}{4}}(d\lambda) \\ &= \frac{\Gamma(3/4)}{\Gamma((1+s)/2)}(G_\theta H_\sigma(W(\frac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}, \end{aligned}$$

which proves (i). Similarly, we find that (ii) holds. (Q.E.D.)

We shall prove one of the main results in this paper.

Theorem 6.8. *For any* $s = \sigma + i\tau (0 < \sigma < 1, \tau \in \mathbf{R})$,

$$\begin{aligned}
&\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \\
&= \frac{1}{s(s-1)}(W(0), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} + \frac{\Gamma(3/4)}{\Gamma((1+s)/2)}(G_\theta H_\sigma(W(\frac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} \\
&\qquad + \frac{\Gamma(3/4)}{\Gamma((2-s)/2)}(G_\theta H_{1-\sigma}(W(-\frac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}.
\end{aligned}$$

Proof. By Lemma 6.6(i), we have

$$\frac{1}{s(s-1)} = \frac{1}{s(s-1)}(W(0), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}. \tag{6.15}$$

Therefore, by combining (6.15) with Lemma 6.7, we find that Theorem 6.8 holds. (Q.E.D.)

In particular, we see from (3.19) and Theorem 6.8 that

Theorem 6.9. *For any* $s = \sigma + i\tau (0 < \sigma < 1, \tau \in \mathbf{R})$,

$$\begin{aligned}
\xi(s) &= \frac{1}{2}(W(0), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} + \frac{s(s-1)\Gamma(3/4)}{2\Gamma((1+s)/2)}(G_\theta H_\sigma(W(\frac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} \\
&\qquad + \frac{s(s-1)\Gamma(3/4)}{2\Gamma((2-s)/2)}(G_\theta H_{1-\sigma}(W(-\frac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}.
\end{aligned}$$

[6.4] Next, we prove an inner product representation theorem for the analytic continuation of the derived Kummer function associated with the Kummer function and the theta function.

Theorem 6.10. *For any* $s = \sigma + i\tau (0 < \sigma < 1, \tau \in \mathbf{R})$,

(i) $K_\theta(\frac{2+s}{2}) = (\frac{2}{s-1}W(0) + \frac{s\Gamma(3/4)}{2\Gamma((1+s)/2)}G_\theta H_\sigma(W(\frac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}$,

(ii) $K_\theta(\frac{3-s}{2}) = (-\frac{2}{s}W(0) + \frac{(1-s)\Gamma(3/4)}{2\Gamma((2-s)/2)}G_\theta H_{1-\sigma}(W(-\frac{\tau}{2})), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}$.

Proof. By Lemmas 3.11(i) and 6.7(i), we have

$$\begin{aligned}
\frac{2}{s}K_\theta(\frac{2+s}{2}) &= \frac{1}{s(s-1)} + F_1(s) \\
&= \frac{1}{s(s-1)} + \frac{\Gamma(3/4)}{\Gamma((1+s)/2)}(G_\theta H_\sigma(W(\frac{\tau}{2})), X(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})},
\end{aligned}$$

which, together with (6.15), implies that (i) holds. Similarly, we see from Lemmas 3.11(ii) and 6.7(ii) that (ii) holds. Thus we conclude that Theorem 6.10 holds. (Q.E.D.)

By transforming the variable $\frac{s+1}{2}$ to the variable s in Theorem 6.10, we have the inner product representation theorem for the derived Kummer function associated with the Kummer function and the theta function.

Theorem 6.11. *For any* $s = \sigma + i\tau (1 < \sigma < \frac{3}{2}, \tau \in \mathbf{R})$,

$$K_\theta(s) = (\frac{2}{2s-3}W(0) + \frac{(s-1)\Gamma(3/4)}{\Gamma((2s-1)/2)}G_\theta H_{\sigma-1}(W(\tau)), W(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}.$$

[6.5] Finally, we introduce a complex valued random field and obtain another inner product representation theorem for the analytic continuation of Riemann's zeta function and the derived Kummer function associated with the Kummer function and the theta function.

As an generalization of (6.10), we show

Lemma 6.12. *For any* $s \in \mathbf{C}$ *such that* $\mathrm{Re}(s) > -\frac{3}{8}$,

(i) $\lambda^s \in L^2([0,\infty), \mathcal{B}([0,\infty)), \Gamma_{3/4})$,

(ii) $(\log\lambda)\lambda^s \in L^2([0,\infty), \mathcal{B}([0,\infty)), \Gamma_{3/4})$.

Proof. By (1.8), we have

$$\int_0^\infty |\lambda^s|^2 \Gamma_{\frac{3}{4}}(d\lambda) = \frac{1}{\Gamma(3/4)}\int_0^\infty \lambda^{2\sigma}e^{-\lambda}\lambda^{\frac{3}{4}-1}d\lambda = \frac{1}{\Gamma(3/4)}\int_0^\infty e^{-\lambda}\frac{1}{\lambda^{1/4-2\sigma}}d\lambda.$$

Since $\frac{1}{4} - 2\sigma < 1$, we see that the above integral is finite, which proves (i). Similarly, we have

$$\int_0^\infty |(\log\lambda)\lambda^s|^2 \Gamma_{\frac{3}{4}}(d\lambda) = \frac{1}{\Gamma(3/4)}\int_0^\infty e^{-\lambda}(\log\lambda)^2\lambda^{2\sigma}\lambda^{\frac{3}{4}-1}d\lambda.$$

Take any σ_0 such that $\frac{1}{4} - 2\sigma < \sigma_0 < 1$. Then

$$\int_0^\infty |(\log\lambda)\lambda^s|^2 \Gamma_{\frac{3}{4}}(d\lambda) = \frac{1}{\Gamma(3/4)}\int_0^\infty e^{-\lambda}(\lambda^{\sigma_0-(1/4-\sigma)}(\log\lambda)^2))\frac{1}{\lambda_0^\sigma}d\lambda$$

Since $\sigma_0 - (1/4-\sigma) > 0$ and $0 < \sigma_0 < 1$, we see that the above integral is finite, which proves (ii). (Q.E.D.)

By virtue of Lemma 6.12, we can enlarge the time domain of the complex valued stochastic process $\mathbf{W}$ in (6.12) to define the complex valued random field $\mathbf{Y} = (Y(s); s \in \mathbf{C}, \mathrm{Re}(s) > -\frac{3}{8})$ by

$$Y(s) \equiv \int_0^\infty \lambda^s dE(\lambda)X(0). \tag{6.16}$$

We note that the random variable of the field $\mathbf{Y}$ on the imaginary axis corresponds to the stochastic process $\mathbf{W}$ in (6.12) and

$$Y(0) = X(0) \tag{6.17}$$

Theorem 6.13.

(i) $(Y(s_1), Y(s_2))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} = \frac{\Gamma(s_1+\overline{s_2}+3/4)}{\Gamma(3/4)} \qquad (\mathrm{Re}(s_j) > -\frac{3}{8}(j=1,2)).$

(ii) *The following function*

$$\{s \in \mathbf{C}; \mathrm{Re}(s) > -\frac{3}{8}\} \ni s \to Y(s) \in \mathbf{M}^{-/+}(\mathbf{X}_{\Gamma_{3/4}})$$

is differentiable and $\frac{Y(s)}{ds} = \int_0^\infty (\log\lambda)\lambda^s dE(\lambda)X(0)$.

Proof. By (1.8), (6.16) and Proposition 5.9, we have

$$\begin{aligned}
(Y(s_1), Y(s_2))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} &= \int_0^\infty \lambda^{s_1+\overline{s_2}}\Gamma_{3/4}(d\lambda) \\
&= \frac{1}{\Gamma(3/4)}\int_0^\infty e^{-\lambda}\lambda^{s_1+\overline{s_2}}\lambda^{3/4-1}d\lambda \\
&= \frac{1}{\Gamma(3/4)}\int_0^\infty e^{-\lambda}\lambda^{(s_1+\overline{s_2}+3/4)-1}d\lambda \\
&= \frac{\Gamma(s_1+\overline{s_2}+3/4)}{\Gamma(3/4)},
\end{aligned}$$

which proves (i). Fix any $s_0 = \sigma_0 + i\tau_0 \in \{s \in \mathbf{C}; \mathrm{Re}(s) > -\frac{3}{8}\}$. Take $\delta > 0$ such that $\frac{1}{4} - 2\sigma_0 + 2\delta < 1$. For any $s \in \{w \in \mathbf{C}; 0 < |s| < \frac{\delta}{2}\}$,

$$\frac{Y(s_0+s)-Y(s_0)}{s} - \int_0^\infty (\log\lambda)\lambda^{s_0}dE(\lambda)X(0) = \int_0^\infty \lambda^{s_0}(\frac{\lambda^s-1}{s} - \log\lambda)dE(\lambda)X(0)$$

and so

$$\begin{aligned}
&\|\frac{Y(s_0+s)-Y(s_0)}{s} - \int_0^\infty (\log\lambda)\lambda^{s_0}dE(\lambda)X(0)\|^2_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} \\
&= \int_0^\infty \lambda^{2\sigma_0}|\frac{\lambda^s-1}{s} - \log\lambda|^2\Gamma_{3/4}(d\lambda) \\
&= \frac{1}{\Gamma(3/4)}\int_0^\infty e^{-\lambda}|\frac{\lambda^s-1}{s} - \log\lambda|^2\frac{1}{\lambda^{1/4-2\sigma_0}}d\lambda
\end{aligned}$$

$$= \int_0^1 e^{-\lambda}|\frac{\lambda^s-1}{s} - \log\lambda|^2\frac{1}{\lambda^{1/4-2\sigma_0}}d\lambda + \int_1^\infty e^{-\lambda}|\frac{\lambda^s-1}{s} - \log\lambda|^2\frac{1}{\lambda^{1/4-2\sigma_0}}d\lambda. \tag{6.18}$$

On the other hand, we note that

$$|\frac{\lambda^s-1}{s} - \log\lambda|^2 \le \frac{s^2(\log\lambda)^4}{4}e^{2|s||\log\lambda|}. \tag{6.19}$$

First, we consider the first term in the right-hand side of (6.18). Since it follows from (6.19) that

$$\begin{aligned}|\frac{\lambda^s-1}{s}-\log\lambda|^2\frac{1}{\lambda^{1/4-2\sigma_0}} &\leq \frac{s^2(\log\lambda)^4}{4}e^{-\delta\log\lambda}\frac{1}{\lambda^{1/4-2\sigma_0}}\\ &= \frac{s^2(\log\lambda)^4}{4}\frac{\lambda^{-\delta}}{\lambda^{1/4-2\sigma_0}}\\ &= \frac{s^2(\log\lambda)^4}{4}\frac{\lambda^{\delta}}{\lambda^{1/4-2\sigma_0+2\delta}}\end{aligned}$$

Hence, since $\lim_{s\to 0}\frac{\lambda^s-1}{s}-\log\lambda=0, 1/4-2\sigma_0+2\delta<1$ and $\frac{s^2(\log\lambda)^4}{4}\lambda^\delta$ is bounded in $(0,1)$, we can apply Lebegue's convergence theorm to find from (6.18) and (6.19) that the first term in the right-hand side of (6.18) tends to zero as s tends to zero.

Next, we consider the second term in the right-hand side of (6.18). Since it follows from (6.19) that

$$\begin{aligned}|\frac{\lambda^s-1}{s}-\log\lambda|^2\frac{1}{\lambda^{1/4-2\sigma_0}} &\leq \frac{s^2(\log\lambda)^4}{4}e^{\delta\log\lambda}\frac{1}{\lambda^{1/4-2\sigma_0}}\\ &= \frac{s^2(\log\lambda)^4}{4}\frac{\lambda^{\delta}}{\lambda^{1/4-2\sigma_0}},\end{aligned}$$

we can apply Lebegue's convergence theorm to find from (6.18) and (6.19) that the second term in the right-hand side of (6.18) tends to zero as s tends to zero.

Thus, we conclude that Theorem 6.13. (Q.E.D.)

By using the random field $\mathbf{Y}$ introduced in (6.16) and noting (6.8) and (6.17), we can rewrite Lemma 6.7 into

Lemma 6.14. *For any* $s=\sigma+i\tau(0<\sigma<1,\tau\in\mathbf{R})$,

(i) $F_1(s)=\dfrac{\Gamma(3/4)}{\Gamma((1+s)/2)}(G_\theta Y(\dfrac{2s-1}{4}),Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}$,

(ii) $F_2(s)=\dfrac{\Gamma(3/4)}{\Gamma((2-s)/2)}(G_\theta Y(\dfrac{1-2s}{4}),Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}$.

Similarly, we can rewrite Theorem 6.8 which is one of the main results in this paper into

Theorem 6.15. *For any* $s=\sigma+i\tau(0<\sigma<1,\tau\in\mathbf{R})$,

$$\begin{aligned}&\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)\\ &=\frac{1}{s(s-1)}(Y(0),Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}+\frac{\Gamma(3/4)}{\Gamma((1+s)/2)}(G_\theta Y(\frac{2s-1}{4}),Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}\\ &\quad+\frac{\Gamma(3/4)}{\Gamma((2-s)/2)}(G_\theta Y(\frac{1-2s}{4}),Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}.\end{aligned}$$

As its corollary, we can rewrite Theorem 6.9 into

Theorem 6.16. *For any* $s = \sigma + i\tau (0 < \sigma < 1, \tau \in \mathbf{R})$,

$$\xi(s) = \frac{1}{2}(Y(0), Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})} + \frac{s(s-1)\Gamma(3/4)}{2\Gamma((1+s)/2)}(G_\theta Y(\frac{2s-1}{4}), Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}$$
$$+\frac{s(s-1)\Gamma(3/4)}{2\Gamma((2-s)/2)}(G_\theta Y(\frac{1-2s}{4}), Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}.$$

Finally, concerning the derived Kummer function, we can rewrite Theorem 6.11 into

Theorem 6.17. *For any* $s = \sigma + i\tau (1 < \sigma < \frac{3}{2}, \tau \in \mathbf{R})$,

$$K_\theta(s) = (\frac{2}{2s-3}Y(0) + \frac{(s-1)\Gamma(3/4)}{\Gamma((2s-1)/2)}G_\theta Y(\frac{\sigma - 1 + i2\tau}{2}), Y(0))_{\mathbf{M}(\mathbf{X}_{\Gamma_{3/4}})}.$$

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