On joint universality for the zeta-functions of newforms and periodic Hurwitz zeta-functions

By

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Abstract

In the paper, a short survey on universality for zeta-functions both having and not having Euler’s product is given. Also, a joint universality theorem for the zeta-function of newforms and periodic Hurwitz zeta-functions is proved.

§1. Introduction

Let Δ be a vertical strip on the complex plane $\mathbb{C}$. Denote by $\mathcal{K}(\Delta)$ the class of compact subsets of the strip $\Delta$ with connected complements, for a compact set $K$, denote by $\mathcal{H}(K)$ the class of continuous functions on $K$ which are analytic in the interior of $K$, and by $\mathcal{H}_0(K)$ the subclass of $\mathcal{H}(K)$ consisting of functions which are non-vanishing on $K$.

It is well known, see [1], [5], [12], [14], [29], [36], [37], [38], that the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, which is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

and is analytically continued to the whole complex plane, except for a simple pole at $s = 1$, is universal in the sense that if $K \in \mathcal{K}(D)$, $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, and $f \in \mathcal{H}_0(K)$, then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s+i\tau) - f(s)| < \epsilon \right\} > 0.$$
Here and in the sequel, $p$ denotes a prime number, and $\text{meas}\{A\}$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Now let $\alpha$, $0 < \alpha \leq 1$, be a fixed parameter. Then the Hurwitz zeta-function $\zeta(s, \alpha)$ which is defined, for $\sigma > 1$, by

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{s}},$$

and is analytically continued to the whole complex plane, except for a simple pole at $s = 1$, is also in a similar sense universal. Namely, [1], [5], [21], [35] if $\alpha$ is a transcendental or rational number $\neq 1, \frac{1}{2}$, $K \in \mathcal{K}(D)$ and $f \in \mathcal{H}(K)$, then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s+i\tau, \alpha) - f(s)| < \epsilon \right\} > 0.$$

Since the cases $\alpha = 1$ ($\zeta(s, 1) = \zeta(s)$) and $\alpha = \frac{1}{2}$ ($\zeta(s, \frac{1}{2}) = (2^{s} - 1)\zeta(s)$) in the later statement are excluded, the function $\zeta(s, \alpha)$ has no Euler’s product over primes, and this is reflected in its universality: the shifts $\zeta(s + i\tau, \alpha)$ approximate every function $f \in \mathcal{H}(K)$, the restriction of the class $\mathcal{H}_{0}(K)$ is removed. Thus the universality of $\zeta(s, \alpha)$ is more general than that of $\zeta(s)$, and is called a strong universality.

Note that the universality of $\zeta(s, \alpha)$ with algebraic irrational parameter $\alpha$ remains an open problem.

Mishou in [31] obtained a very interesting joint universality theorem for the functions $\zeta(s)$ and $\zeta(s, \alpha)$.

**Theorem 1.1** ([31]). Suppose that $\alpha$ is a transcendental number, $K_{1}, K_{2} \in \mathcal{K}(D)$, $f_{1} \in \mathcal{H}_{0}(K)$ and $f_{2} \in \mathcal{H}(D)$. Then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K_{1}} |\zeta(s+i\tau) - f_{1}(s)| < \epsilon, \right.$$

$$\left. \sup_{s \in K_{2}} |\zeta(s+i\tau, \alpha) - f_{2}(s)| < \epsilon \right\} > 0.$$

Theorem 1.1 joins the universality and strong universality. We will call this type of the joint universality a mixed universality.

The functions $\zeta(s)$ and $\zeta(s, \alpha)$ have their generalizations. Suppose that $a = \{a_{m} : m \in \mathbb{N}\}$ and $b = \{b_{m} : m \in \mathbb{N}_{0} = \mathbb{N} \cup \{0\}\}$ are two periodic sequences of complex numbers with minimal periods $k_{1} \in \mathbb{N}$ and $k_{2} \in \mathbb{N}$, respectively. Then the functions

$$\zeta(s; a) = \sum_{m=1}^{\infty} \frac{a_{m}}{m^{s}} \quad \text{and} \quad \zeta(s, \alpha; b) = \sum_{m=0}^{\infty} \frac{b_{m}}{(m+\alpha)^{s}}, \quad \sigma > 1,$$
are called the periodic zeta and periodic Hurwitz zeta-function, respectively. The equalities
\[ \zeta(s; a) = \frac{1}{k_1^s} \sum_{k=1}^{k_1} a_k \zeta(s, \frac{k}{k_1}), \quad \zeta(s, \alpha; b) = \frac{1}{k_2^s} \sum_{k=0}^{k_2-1} b_k \zeta(s, \frac{k+\alpha}{k_2}) \]
give for the functions \( \zeta(s; a) \) and \( \zeta(s, \alpha; b) \) meromorphic continuation to the whole complex plane with possible simple pole at \( s = 1 \).

The universality of the function \( \zeta(s; a) \) with multiplicative sequence \( a \), has been studied by Steuding [36], and Laurinčikas and Šiaučiūnas [28]. In a general case, the problem was solved by Kaczorowski [10].

The strong universality of the function \( \zeta(s, \alpha; b) \) with transcendental parameter \( \alpha \) has been obtained by Javtokas and Laurinčikas [7], [8]. Nakamura [34] studied \( \zeta(s, \alpha; b) \) with a special bounded sequence.

A generalization of Theorem 1.1 for the functions \( \zeta(s; a) \) and \( \zeta(s, \alpha; b) \) with multiplicative sequence \( a \) has been obtained in [11]. A joint universality theorem for periodic zeta-functions with multiplicative coefficients satisfying a certain “independence” condition has been proved in [22]. The joint universality of Hurwitz zeta-functions by different methods has been considered in [33] and [19]. A series of works [15]-[18] and [9], [26], [27] are devoted to joint universality of periodic Hurwitz zeta-functions. A mixed universality theorem for zeta-functions with periodic coefficients can be found in [20].

In [24], Laurinčikas and Matsumoto observed that, for Lerch zeta-functions
\[ L(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m}}{(m + \alpha_j)^s}, \quad \sigma > 1, \quad j = 1, \ldots, r, \]
a more general setting of joint universality is possible. To each parameter \( \alpha_j \), they attached a collection of the parameters \( \lambda_j \). For periodic Hurwitz zeta-functions, the latter idea was applied by Laurinčikas [18], and Laurinčikas and Skerstonaite [27]. We will state the latter result. For \( j = 1, \ldots, r \), let \( l_j \in \mathbb{N} \), and, for \( j = 1, \ldots, r \) and \( l = 1, \ldots, l_j \), let \( b_{jl} = \{b_{mj} : m \in \mathbb{N}_0\} \) be a periodic sequence of complex numbers with minimal period \( k_{jl} \in \mathbb{N} \), and \( \zeta(s, \alpha_j; b_{jl}) \) denotes the corresponding periodic Hurwitz zeta-function. Moreover, let
\[ L(\alpha_1, \ldots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \ldots, r\}, \]
\( k_j \) be the least common multiple of the periods \( k_{j1}, \ldots, k_{jl_j} \), and
\[ B_j = \begin{pmatrix} b_{1j1} & b_{1j2} & \cdots & b_{1jl_j} \\ b_{2j1} & b_{2j2} & \cdots & b_{2jl_j} \\ \cdots & \cdots & \cdots & \cdots \\ b_{kj_1} & b_{kj_2} & \cdots & b_{kjl_j} \end{pmatrix}, \quad j = 1, \ldots, r. \]
Theorem 1.2 ([27]). Suppose that the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and that $\text{rank}(B_j) = l_j, j = 1, \ldots, r$. For $j = 1, \ldots, r$ and $l = 1, \ldots, l_j$, let $K_{jl} \in \mathcal{K}(D)$, and let $f_{jl}(s) \in \mathcal{H}(K_{jl})$. Then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \inf \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} | \zeta(s + i\tau, \alpha_j; b_{jl}) - f_{jl}(s)| < \epsilon \right\} > 0.$$  

Note that in the latter theorem the information on the values of $b_{mjl}$ related only to $\alpha_j$ is used.

In [4], a mixed universality theorem for the Riemann zeta-function and periodic Hurwitz zeta-functions in the frame of Theorem 1.2 has been proved.

Theorem 1.3 ([4]). Suppose that $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that all hypotheses of Theorem 1.2 for $b_{jl}, K_{jl}$ and $f_{jl}$ are satisfied. Moreover, let $K \in \mathcal{K}(D)$ and $f(s) \in \mathcal{H}_0(K)$. Then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \inf \left\{ \tau \in [0, T] : \sup_{s \in K} | \zeta(s + i\tau) - f(s)| < \epsilon, \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} | \zeta(s + i\tau, \alpha_j; b_{jl}) - f_{jl}(s)| < \epsilon \right\} > 0.$$  

The aim of this paper is to replace the function $\zeta(s)$ in Theorem 1.3 by zeta-functions of newforms. To state our result, we need some definitions and notation. Let

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

be the full modular group. For $N \in \mathbb{N}$, the subgroup of $SL_2(\mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0(\text{mod} \ N) \right\}$$

is called the Hecke subgroup or the congruence subgroup mod $N$. Suppose that $F(z)$ is a holomorphic function in the upper half-plane $\Im z > 0$, and $\kappa \in 2\mathbb{N}$. The function $F(z)$ is called a cusp form of weight $\kappa$ and level $N$ if

$$F \left( \begin{pmatrix} az + b \\ cz + d \end{pmatrix} \right) = (cz + d)^\kappa F(z) \quad \text{for all} \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \Gamma_0(N),$$

and $F(z)$ is holomorphic and vanishing at the cusps. In this case, $F(z)$ has the following Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}.$$
Denote by $S_{\kappa}(\Gamma_0(N))$ the space of all cusp forms of weight $\kappa$ and level $N$. For every $d \mid N$, the elements of $S_{\kappa}(\Gamma_0(d))$ also belong to $S_{\kappa}(\Gamma_0(N))$. $F \in S_{\kappa}(\Gamma_0(N))$ is called a newform if $F$ is not a cusp form of level less than $N$, and $F$ is an eigenfunction of all Hecke operators. Then we have that $c(1) \neq 0$, so we may assume that $c(1) = 1$, i.e., $F$ is a normalized newform. To each cusp form we can attach the zeta-function

$$
\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.
$$

In view of the Deligne estimate [3]

$$
|c(m)| \leq m^{\frac{\kappa-1}{2}} d(m),
$$

where $d(m)$ denotes the divisor function, the series for $\zeta(s, F)$ converges absolutely for $\sigma > \frac{\kappa+1}{2}$. In this region, $\zeta(s, F)$ also has a representation by Euler’s product. If $F$ is a newform, then this representation is of the form

$$
\zeta(s, F) = \prod_{p|N} \left(1 - \frac{c(p)}{p^s}\right)^{-1} \prod_{p \mid N} \left(1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s+1-\kappa}}\right)^{-1}.
$$

Moreover, the function $\zeta(s, F)$ can by analytically continued to an entire function. These and other facts of the theory of modular forms can be found, for example, in [6] and [32].

The universality for zeta-functions of normalized Hecke eigen cusp forms was obtained by Laurinčikas and Matsumoto [23], and for zeta-functions of newforms by Laurinčikas, Matsumoto and Steuding [25].

Denote $D_{\kappa} = \{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$.

**Theorem 1.4** ([25]). Suppose that $F$ is a normalized newform of weight $\kappa$ and level $N$, $K \in \mathcal{K}(D_{\kappa})$ and $f(s) \in \mathcal{H}_0(K)$. Then, for every $\epsilon > 0$,

$$
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s+i\tau, F) - f(s)| < \epsilon \right\} > 0.
$$

Our main result is the following theorem. It joins Theorem 1.2 with algebraically independent numbers $\alpha_1, ..., \alpha_r$ over $\mathbb{Q}$ with Theorem 1.4.

**Theorem 1.5.** Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $\text{rank}(B_j) = l_j$, $j = 1, ..., r$. Let $K$ and $f(s)$ be the same as in Theorem 1.4, and $K_{jl}$ and $f_{jl}$ be the same as in Theorem 1.2. Then, for every $\epsilon > 0$,

$$
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s+i\tau, \alpha_j; b_{jl}) - f_{jl}(s)| < \epsilon, \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} |\zeta(s+i\tau, \alpha_j; b_{jl}) - f_{jl}(s)| < \epsilon \right\} > 0.
$$
For the proof of Theorem 1.5, we will apply a modification of the probabilistic method used in [4] which is based on a joint limit theorem in the space of analytic functions. The case of Theorem 1.5 is more complicated because we deal with two strips $D_\kappa$ and $D$.

§2. Joint functional limit theorems

Let $G$ be a region on $\mathbb{C}$. Denote by $H(G)$ the space of analytic functions on $G$ equipped with the topology of uniform convergence on compacta. For $u = l_1 + \cdots + l_r$ and $v = u + 1$, let

$$H^v(D_\kappa, D) = H(D_\kappa) \times H(D) \times \cdots \times H(D).$$

Moreover, we set $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $\underline{b} = (b_{11}, \ldots, b_{1l_1}, \ldots, b_{r1}, \ldots, b_{rl_r})$. This section is devoted to a limit theorem in the space $H^v(D_\kappa, D)$ for the vector

$$\zeta(\hat{s}, s, \underline{\alpha}; \underline{\omega}; \underline{b}, F) = (\zeta(\hat{s}, F), \zeta(s, \alpha_1; b_{11}, \ldots, \zeta(s, \alpha_r; b_{r1}), \ldots, \zeta(s, \alpha_r; b_{rl_r})).$$

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space $S$, let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\hat{\Omega} = \prod_p \gamma_p \text{ and } \Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes $p$, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, the tori $\hat{\Omega}$ and $\Omega$ with the product topology and pointwise multiplication are compact topological Abelian groups. Therefore, on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ and $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measures $\hat{m}_H$ and $m_H$, respectively, exist, and we have two probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$ and $(\Omega, \mathcal{B}(\Omega), m_H)$. Moreover, let

$$\Omega = \hat{\Omega} \times \prod_{j=1}^{r} \Omega_j,$$

where $\Omega_j = \Omega$ for all $j = 1, \ldots, r$. Similarly as above, we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where $m_H$ is the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \hat{\Omega}$ to $\gamma_p$, and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_j$ to $\gamma_m$. Let $\underline{\omega} = (\hat{\omega}, \omega_1, \ldots, \omega_r)$ be the elements of $\Omega$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^v(D_\kappa, D)$-valued random element $\zeta(\hat{s}, s, \underline{\alpha}; \underline{\omega}; \underline{b}, F)$ by the formula

$$\zeta(\hat{s}, s, \underline{\alpha}; \underline{\omega}; \underline{b}, F) = (\zeta(\hat{s}, \hat{\omega}, F), \zeta(s, \alpha_1, \omega_1; b_{11}, \ldots, \zeta(s, \alpha_1, \omega_1; b_{1l_1}), \ldots, \zeta(s, \alpha_r, \omega_r; b_{r1}), \ldots, \zeta(s, \alpha_r, \omega_r; b_{rl_r})).$$
where
\[
\zeta(\hat{s}, \hat{\omega}, F) = \prod_{p|N} \left(1 - \frac{c(p)\hat{\omega}(p)}{p^s}\right)^{-1} \prod_{p\nmid N} \left(1 - \frac{c(p)\hat{\omega}(p)}{p^s} + \frac{\hat{\omega}^2(p)}{p^{2s-1+\kappa}}\right)^{-1}
\]
and
\[
\zeta(s, \alpha_j, \omega_j; \beta_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j.
\]

Denote by $P_\zeta$ the distribution of the random element $\zeta(\hat{s}, s, \alpha, \omega; \beta, F)$, i.e., for $A \in \mathcal{B}(H^v(D_\kappa, D))$,
\[
P_\zeta(A) = m_H(\omega \in \Omega : \zeta(\hat{s}, s, \alpha, \omega; \beta, F) \in A).
\]

**Theorem 2.1.** Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then
\[
P_T(A) \defeq \frac{1}{T} \text{meas}\{\tau \in [0, T] : \zeta(\hat{s}+i\tau, s+i\tau, \alpha; \beta, F) \in A\}, \quad A \in \mathcal{B}(H^v(D_\kappa, D)),
\]
converges weakly to $P_\zeta$ as $T \to \infty$.

The proof of Theorem 2.1 is similar to that Theorem 4 of [4], therefore, we will give only its sketch.

Let $\sigma_1 > \frac{1}{2}$ be a fixed number, and
\[
v_n(m) = \exp \left\{ -\left( \frac{m}{n} \right)^{\sigma_1} \right\}, \quad m, n \in \mathbb{N},
\]
\[
v_n(m, \alpha_j) = \exp \left\{ -\left( \frac{m+\alpha_j}{n+\alpha_j} \right)^{\sigma_1} \right\}, \quad m, n \in \mathbb{N}_0, \quad j = 1, \ldots, r.
\]

Define
\[
\zeta_n(\hat{s}, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s}
\]
and
\[
\zeta_n(s, \alpha_j; \beta_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j.
\]

By a standard method involving an application of the Mellin formula can be proved that the series for $\zeta_n(\hat{s}, F)$ and $\zeta_n(s, \alpha_j; \beta_{jl})$ are absolutely convergent for $\Re \hat{s} > \frac{\kappa}{2}$ and $\sigma > \frac{1}{2}$, respectively.

The formula
\[
\hat{\omega}(m) = \prod_{p|m|} \omega^l(p), \quad m \in \mathbb{N},
\]
where $p^l \parallel m$ means that a power $p^l$ occurs precisely in the canonical representation of $m$, extends the functions $\hat{\omega}(p)$ to the set $\mathbb{N}$. Define
\[\zeta_n(s, \hat{\omega}, F) = \sum_{m=1}^{\infty} \frac{c(m)\hat{\omega}(m)v_n(m)}{m^s},\]
and
\[\zeta_n(s, \alpha_j, \omega_j; b_j) = \sum_{m=0}^{\infty} \frac{b_{mj}\omega_j(m)v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, ..., r, \quad l = 1, ..., l_j,\]
the series being absolutely convergent for $\Re s > \frac{\kappa}{2}$ and $\sigma > \frac{1}{2}$, respectively. Moreover, we set
\[\underline{\zeta}_n(s, \alpha; \underline{b}, F) = (\zeta_n(s, \hat{\omega}, F), \zeta_n(s, \alpha_1; b_{11}), ..., \zeta_n(s, \alpha_r; b_{r1}), ..., \zeta_n(s, \alpha_{l_r}; b_{rl_r})),\]
and
\[\underline{\zeta}_n(s, \alpha; \omega; \underline{b}, F) = (\zeta_n(s, \hat{\omega}, F), \zeta_n(s, \alpha_1, \omega_1; b_{11}), ..., \zeta_n(s, \alpha_r, \omega_r; b_{r1}), ..., \zeta_n(s, \alpha_{l_r}, \omega_{l_r}; b_{rl_r})).\]
The first step in the proof of Theorem 2.1 is the following statement.

**Lemma 2.2.** Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures
\[P_{T,n}(A) \overset{\text{def}}{=} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \zeta_n(\hat{s} + i\tau, s + i\tau, \alpha; \underline{b}, F) \in A \right\},\]
and
\[P_{T,n,\omega}(A) \overset{\text{def}}{=} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \underline{\zeta}_n(\hat{s} + i\tau, s + i\tau, \alpha; \omega; \underline{b}, F) \in A \right\},\]
for any fixed $\omega \in \Omega$, to the same probability measure $P_n$ on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ as $T \to \infty$.

Lemma 2.2 is a result of the application of Theorem 5.1 from [2] and a limit theorem on the torus $\Omega$ which is contained in the next lemma obtained in [4], Lemma 1. Let $\mathcal{P}$ be the set of all prime numbers.

**Lemma 2.3.** Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then
\[\frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : ((p^{-i\tau} : p \in \mathcal{P}), ((m + \alpha_j)^{-i\tau} : m \in \mathbb{N}_0, j = 1, ..., r)) \in A \right\},\]
converges weakly to the Haar measure $m_H$ as $T \to \infty$.\]
The next step of the proof of Theorem 2.1 contains the results which allow to pass from the vector $\zeta_n(\hat{s}, s, \underline{\alpha}; \underline{b}, F)$ to $\zeta(\hat{s}, s, \underline{\alpha}; \underline{b}, F)$. For this, we need a metric on $H^v(D_\kappa, D)$.

It is well known that there exist a sequence $\{K_m : m \in \mathbb{N}\}$ of compact subsets of $D_\kappa$, and a sequence $\{K_m : m \in \mathbb{N}\}$ of $D$ such that

$$D_\kappa = \bigcup_{m=1}^\infty \hat{K}_m \quad \text{and} \quad D = \bigcup_{m=1}^\infty K_m.$$ 

Moreover, the sets $\hat{K}_m$ and $K_m$ can be chosen to satisfy $\hat{K}_m \subset \hat{K}_{m+1}, K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and, for every compact subsets $\hat{K} \subset D_\kappa$ and $K \subset D$, there exists $\hat{m}, m \in \mathbb{N}$ such that $\hat{K} \subset \hat{K}_{\hat{m}}$ and $K \subset K_m$. For $\hat{g}_1, \hat{g}_2 \in H(D_\kappa)$ and $g_1, g_2 \in H(D)$, define

$$\hat{\rho}(\hat{g}_1, \hat{g}_2) = \sum_{m=1}^\infty 2^{-m} \frac{\sup_{s \in \hat{K}_m} |\hat{g}_1(s) - \hat{g}_2(s)|}{1 + \sup_{s \in \hat{K}_m} |\hat{g}_1(s) - \hat{g}_2(s)|}$$

and

$$\rho(g_1, g_2) = \sum_{m=1}^\infty 2^{-m} \frac{\sup_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|}.$$ 

Then $\hat{\rho}$ and $\rho$ are the metrics on $H(D_\kappa)$ and $H(D)$, respectively, inducing the topology of uniform convergence on compacta. For $f = (\hat{f}, f_{11}, ..., f_{l_1}, ..., f_{r_1}, ..., f_{r_l}), \ g = (\hat{g}, g_{11}, ..., g_{l_1}, ..., g_{r_1}, ..., g_{r_l}) \in H^v(D_\kappa, D)$, let

$$\rho_v(f, g) = \max \left( \hat{\rho}(\hat{f}, \hat{g}), \max_{1 \leq j \leq r} \max_{1 \leq l \leq l_j} \rho(f_{jl}, g_{jl}) \right).$$

Then $\rho_v$ is a metric on the space $H^v(D_\kappa, D)$ which induces its topology.

Now we are able to approximate $\zeta(\hat{s}, s, \underline{\alpha}; \underline{b}, F)$ by $\zeta_n(\hat{s}, s, \underline{\alpha}; \underline{b}, F)$ in the mean.

**Lemma 2.4.** We have

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_v \left( \zeta(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{b}, F), \zeta_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{b}, F) \right) d\tau = 0.$$ 

As it was observed in [25], the zeta-functions associated to newforms constitute a subclass of Matsumoto zeta-functions. Therefore, the lemma follows from the relation

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \hat{\rho} \left( \zeta(\hat{s} + i\tau, F), \zeta_n(\hat{s} + i\tau, F) \right) d\tau = 0.$$
which is a corollary of Lemma 8 from [13], and from the equalities
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(\zeta(s + i\tau, \alpha_j; b_{jl}), \zeta_n(s + i\tau, \alpha_j; b_{jl})) \, d\tau = 0,
\]
\[j = 1, \ldots, r, \quad l = 1, \ldots, l_j,
\]
which are deduced from formula (3) of [18].

An analogue of Lemma 2.4 is also true for \(\zeta(\hat{s}, s, \underline{\alpha}; \underline{\omega}; \underline{\mathrm{b}}, F)\) and \(\zeta_n(\hat{s}, s, \underline{\alpha}; \underline{\omega}; \underline{\mathrm{b}}, F)\), where
\[
\zeta(\hat{s}, s, \underline{\alpha}; \underline{\omega}; \underline{\mathrm{b}}, F) = \left(\zeta(\hat{s}, \hat{\omega}, F), \zeta(s, \alpha_1, \omega_1; b_{11}), \ldots, \zeta(s, \alpha_1, \omega_1; b_{1l_1}), \ldots, \right.
\]
\[\left.\zeta(s, \alpha_r, \omega_r; b_{r1}), \ldots, \zeta(s, \alpha_r, \omega_r; b_{rl_r})\right).
\]

**Lemma 2.5.** Suppose that the numbers \(\alpha_1, \ldots, \alpha_r\) are algebraically independent over \(\mathbb{Q}\). Then, for almost all \(\omega \in \Omega\), we have
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_v(\zeta(\hat{s} + s + i\tau, \alpha_1, \omega; b), \zeta(\hat{s} + s + i\tau, \alpha_1, \omega; b)) \, d\tau = 0.
\]

**Proof.** Lemma 11 of [13], for almost all \(\hat{\omega} \in \hat{\Omega}\), implies the relation
\[
(2.1) \quad \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(\zeta(\hat{s} + i\tau, \hat{\omega}), \zeta_n(\hat{s} + i\tau, \hat{\omega})) \, d\tau = 0.
\]

Let
\[
\rho_u(\tilde{s}, \tilde{\omega}) = \max_{1 \leq j \leq r} \max_{1 \leq l \leq l_j} \rho(f_{jl}, g_{jl}).
\]

Denote by \(m_H\) the Haar measure on \((\Omega, \mathcal{B}(\Omega))\), where \(\Omega = \Omega_1 \times \cdots \times \Omega_r\). Then, for almost all \(\omega \in \Omega\),
\[
(2.2) \quad \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_u(\zeta(\tilde{s} + s + i\tau, \tilde{\omega}; b), \zeta(\tilde{s} + s + i\tau, \tilde{\omega}; b)) \, d\tau = 0,
\]
see formula (2.5) of [4]. Here \(\zeta(s, \alpha; \omega; b, F)\) and \(\zeta_n(s, \alpha; \omega; b, F)\) are obtained from \(\zeta(\hat{s}, \hat{\omega}, F)\) and \(\zeta_n(\hat{s}, \hat{\omega}, F)\), respectively. Since the measure \(m_H\) is the product of the measures \(\tilde{m}_H\) and \(\tilde{m}_H\), the lemma follows from (2.1), (2.2), and the definition of \(\rho_u\). \(\square\)

We can deduce from Lemmas 2.2 and 2.4 the weak convergence for the measure \(P_T\), as \(T \to \infty\). However, the identification of the limit measure requires the next lemma.
Lemma 2.6. Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures $P_T$ and

$$\frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \zeta(\hat{s} + i\tau, s + i\tau, \omega; b, F) \in A \right\}, \quad A \in \mathcal{B}(H^v(D_\kappa, D)),$$

both converge weakly, for almost all $\omega \in \Omega$, to the same probability measure $P$ on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ as $T \to \infty$.

Proof. We omit the details which are similar to those of [4]. Let $\theta$ be a random variable defined on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ and uniformly distributed on $[0, 1]$. On the later probability space, define the $H^v(D_\kappa, D)$-valued random element $X_{T,n}$ by the formula

$$X_{T,n}(\hat{s}, s) = \left( X_{T,n}(\hat{s}), X_{T,n,1,1}(s), ..., X_{T,n,1,l_1}(s), ..., X_{T,n,r,1}(s), ..., X_{T,n,r,l_r}(s) \right) \text{ def } \zeta_n(\hat{s} + i\theta T, s + i\theta T, \alpha; b, F).$$

Then, denoting by $\overset{D}{\to}$ the convergence in distribution, we have, by Lemma 2.2, that

$$(2.3) \quad X_{T,n}(\hat{s}, s) \overset{D}{\to} X_n(\hat{s}, s),$$

where $X_n(\hat{s}, s)$ is the $H^v(D_\kappa, D)$-valued random element with the distribution $P_n$ ($P_n$ is the limit measure in Lemma 2.2). Our first task is to prove the tightness of the family $\{P_n : n \in \mathbb{N}\}$.

In view of the Deligne estimate (1.1), the well-known properties of the mean square of Dirichlet series and Cauchy integral formula show that, for all $n \in \mathbb{N}$,

$$(2.4) \quad \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_m} |\zeta_n(s + i\tau, F)| d\tau \leq \hat{C}_m \left( \sum_{k=1}^{\infty} \frac{c^2(k)}{k^{2\sigma_m}} \right)^{\frac{1}{2}}, \quad m \in \mathbb{N},$$

with some $\hat{C}_m > 0$ and $\hat{\sigma}_m > \frac{\kappa}{2}$. Similarly, for all $n \in \mathbb{N}$,

$$(2.5) \quad \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in \hat{K}_m} |\zeta_n(s + i\tau, \alpha_j; b_{jl})| d\tau \leq C_m \left( \sum_{k=0}^{\infty} \frac{|b_{kjl}|^2}{(k + \alpha_j)^{2\sigma_m}} \right)^{\frac{1}{2}}, \quad m \in \mathbb{N}, j = 1, ..., r, l = 1, ..., l_j.$$

with some $C_m > 0$ and $\sigma_m > \frac{1}{2}$, $m \in \mathbb{N}$, $j = 1, ..., r, l = 1, ..., l_j$. The compact sets $\hat{K}_m$ and $K_m$ come from the definition of the metric $\rho_v$.

Now let

$$\hat{R}_m = \hat{C}_m \left( \sum_{k=1}^{\infty} \frac{c^2(k)}{k^{2\sigma_m}} \right)^{\frac{1}{2}}, \quad R_{jlm} = C_m \left( \sum_{k=0}^{\infty} \frac{|b_{kjl}|^2}{(k + \alpha_j)^{2\sigma_m}} \right)^{\frac{1}{2}}.$$
Taking $\hat{M}_m = \hat{R}_m 2^{m+1} \epsilon^{-1}$ and $M_{jlm} = R_{jlm} 2^{u+m+1} \epsilon^{-1}$, where $m \in \mathbb{N}$ and $\epsilon > 0$ is an arbitrary number, we find by (2.4) and (2.5) that
\[
\limsup_{T \to \infty} \mathbb{P} \left( \left( \sup_{\hat{s} \in \hat{K}_m} |X_{T,n}(\hat{s})| > \hat{M}_m \right) \lor \left( \exists j, l : \sup_{s \in K_m} |X_{T,n,j,l}(s)| > M_{jlm} \right) \right) \leq \frac{\epsilon}{2^m}.
\]
This together with (2.3) implies
\[
\mathbb{P} \left( \left( \sup_{\hat{s} \in \hat{K}_m} |X_n(\hat{s})| > \hat{M}_m \right) \lor \left( \exists j, l : \sup_{s \in K_m} |X_{n,j,l}(s)| > M_{jlm} \right) \right) \leq \frac{\epsilon}{2^m},
\]
where $X_n(\hat{s}), X_{n,j,l}(s), j = 1, ..., r, l = 1, ..., l_j$, are the elements of the random vector $X_n(\hat{s}, s)$. From this, we obtain that
\[
P_n(H^v_\epsilon) \geq 1 - \epsilon,
\]
where
\[
H^v_\epsilon = \left\{ f \in H^v(D_\kappa, D) : \sup_{\hat{s} \in \hat{K}_m} |\hat{f}(\hat{s})| \leq \hat{M}_m, \sup_{s \in K_m} |f_{j}(s)| \leq M_{jlm}, \right. \\
\left. j = 1, ..., r, l = 1, ..., l_j, m \in \mathbb{N} \right\}
\]
is a compact subset of the space $H^v(D_\kappa, D)$. This proves the tightness of the family $\{P_n : n \in \mathbb{N}\}$.

By the Prokhorov theorem, the family $\{P_n : n \in \mathbb{N}\}$ is relatively compact. Hence, there exists a sequence $n_k \to \infty$ and a probability measure $P$ on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ such that
\[
(2.6) \quad X_{n_k}(\hat{s}, s) \overset{D}{\to} P_{k \to \infty} P.
\]

On the probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$, define the $H^v(D_\kappa, D)$-valued random element $X_T(\hat{s}, s)$ by the formula
\[
X_T(\hat{s}, s) = \zeta(\hat{s} + i\theta T, s + i\theta T, \alpha; \omega; \mathbf{b}, F).
\]
Then Lemma 2.4 yields, for every $\epsilon > 0$, the relation
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P} \left( \rho_v \left( X_T(\hat{s}, s), X_{T,n}(\hat{s}, s) \right) \geq \epsilon \right) = 0.
\]
This, (2.3), (2.6) and Theorem 4.2 of [2] show that $X_T(\hat{s}, s) \overset{D}{\to} P_T$, or $P_T$ converges weakly to $P$ as $T \to \infty$.

Using the random elements
\[
\zeta_n(\hat{s} + i\theta T, s + i\theta T, \alpha; \omega; \mathbf{b}, F)
\]
and

$$\zeta(s + i\theta T, s + i\theta T, \alpha, \omega; b, F),$$

as well as Lemma 2.5, we obtain in a similar way that the second measure of Lemma 2.6 also converges weakly to $P$ as $T \to \infty$.

The end of the proof of Theorem 2.1 is standard. We apply Lemma 2.6, the ergodicity of the one-parameter group $\{\Phi_t : t \in \mathbb{R}\}$ of measurable and measure preserving transformations on $\Omega$, where, for $\omega \in \Omega$ and $\tau \in \mathbb{R},$

$$\Phi_\tau(\omega) = ((p^{-i\tau} : p \in \mathcal{P}), ((m + \alpha_j)^{-i\tau} : m \in \mathbb{N}_0, j = 1, \ldots, r)) \omega,$$

see Lemma 7 of [20], as well as the classical Birkhoff-Khintchine theorem.

§ 3. Support of the measure $P_\zeta$

The space $H^u(D_\kappa, D)$ is separable, therefore the support of $P_\zeta$ is the minimal closed set $S_{P_\zeta} \subset H^u(D_\kappa, D)$ such that $P_\zeta(S_{P_\zeta}) = 1$. The set $S_{P_\zeta}$ consists of all points $g \in H^u(D_\kappa, D)$ such that, for every open neighbourhood $G$ of $g$, the inequality $P_\zeta(G) > 0$ holds.

Define

$$S_\kappa = \{g \in H(D_\kappa) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 3.1. Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $\text{rank}(B_j) = l_j, j = 1, \ldots, r$. Then the support of the measure $P_\zeta$ is the set $S_\kappa \times H^u(D)$.

Proof. We have that

$$H^u(D_\kappa, D) = H(D_\kappa) \times H^u(D).$$

In view of separability of the above spaces, the equality

$$\mathcal{B}(H^u(D_\kappa, D)) = \mathcal{B}(H(D_\kappa)) \times \mathcal{B}(H^u(D))$$

is true [2]. Therefore, it suffices to investigate $P_\zeta(A)$ for $A = B \times C$, where $B \in \mathcal{B}(H(D_\kappa))$ and $C \in \mathcal{B}(H^u(D))$. We already have mentioned that the measure $m_H$ is the product of the measures $\hat{m}_H$ and $m_H$. Therefore, we have that

$$P_\zeta(A) = m_H \left( \omega \in \Omega : \zeta(s, \alpha, \omega; b, F) \in A \right)$$

$$= m_H \left( \omega \in \Omega : \zeta(s, \omega, F) \in B, \zeta(s, \alpha, \omega; b) \in C \right)$$

$$= \hat{m}_H \left( \hat{\omega} \in \hat{\Omega} : \zeta(\hat{s}, \hat{\omega}, F) \in B \right) m_H \left( \omega \in \Omega : \zeta(s, \alpha, \omega; b) \in C \right).$$

(3.1)
In [25], Lemma 9, it was obtained that the support of the random element \( \zeta(\hat{s}, \hat{\omega}, F) \) is the set \( S_\kappa \), i.e., \( S_\kappa \) is a minimal closed subset of \( H(D_\kappa) \) such that

(3.2) \[ \hat{m}_H \left( \hat{\omega} \in \hat{\Omega} : \zeta(\hat{s}, \hat{\omega}, F) \in S_\kappa \right) = 1. \]

To be precise, in [25] the space \( H(D_{\kappa,M}) \), where \( D_{\kappa,M} = \{ s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}, |t| < M \} \), is considered, however, all arguments remain valid for the space \( H(D_\kappa) \). Also, in [27], Theorem 3.1, it was proved that the support of the random element \( \zeta(s, \alpha, \omega; \underline{b}) \) is the whole of \( H^u(D) \), i.e., \( H^u(D) \) is a minimal closed set of \( H^u(D) \) such that

\[ \hat{m}_H \left( \omega \in \hat{\Omega} : \zeta(s, \alpha, \omega; \underline{b}) \in H^u(D) \right) = 1. \]

From this and (3.1), (3.2), the theorem follows. \( \square \)

§ 4. Proof of Theorem 1.5

We first recall the Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma 4.1. Suppose that \( K \) is a compact subset on the complex plane with connected complement, and that \( f(s) \) is a continuous function on \( K \) which is analytic in the interior of \( K \). Then, for every \( \epsilon > 0 \), there exists a polynomial \( p(s) \) such that

\[ \sup_{s \in K} |f(s) - p(s)| < \epsilon. \]

Proof of the lemma is given in [30], see also [39].

Proof. of Theorem 1.5. In view of Lemma 4.1, there exist polynomials \( p(s) \) and \( p_{jl}(s) \) such that

(4.1) \[ \sup_{s \in K} |f(s) - p(s)| < \frac{\epsilon}{4} \]

and

(4.2) \[ \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |f_{jl}(s) - P_{jl}(s)| < \frac{\epsilon}{2}. \]

Since \( f(s) \neq 0 \) on \( K \), we have that \( p(s) \neq 0 \) on \( K \) as well if \( \epsilon \) is small enough. Therefore, we can define a continuous branch of \( \log p(s) \) on \( K \) which will be analytic in the interior of \( K \). By Lemma 4.1 again, there exists a polynomial \( q(s) \) such that

\[ \sup_{s \in K} \left| p(s) - e^{q(s)} \right| < \frac{\epsilon}{4}. \]
From this and (4.1), we have that

\[(4.3) \quad \sup_{s \in K} |f(s) - e^{q(s)}| < \frac{\epsilon}{2}.\]

Define

\[G = \left\{ g \in H^v(D_\kappa, D) : \sup_{s \in K} |\hat{g}(s) - e^{q(s)}| < \frac{\epsilon}{2}, \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |g_{jl}(s) - P_{jl}(s)| < \frac{\epsilon}{2} \right\}.\]

In view of Theorem 3.1, the vector \((e^{q(s)}, p_{jl}, j = 1, ..., r, l = 1, ..., l_j)\), is an element of the support of the measure \(P_\zeta\). Since \(G\) is an open set, this shows that \(P_\zeta(G) > 0\). Therefore, Theorem 2.1 together with an equivalent of the weak convergence in terms of open sets yields the inequality

\[\lim_{T \to \infty} \inf \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - e^{q(s)}| < \frac{\epsilon}{2}, \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathbf{b}_{jl}) - p_{jl}(s)| < \frac{\epsilon}{2} \right\} > 0.\]

From this, (4.2) and (4.3), the assertion of the theorem follows.

\[\square\]

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**References**


