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Zeros of zeta functions and zeta distributions on $\mathbb{R}^d$

Dedicated to Professor Makoto Maejima for his retirement

By

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Abstract

The class of zeta distributions is one of the classical classes of probability distributions on $\mathbb{R}$. For $0 < u \leq 1$, let $\zeta(z, u), z \in \mathbb{C}$, be the Hurwitz zeta function. Zeros of zeta functions are one of the major subjects in number theory due to the Riemann Hypothesis. The Hurwitz zeta function $\zeta(z, u)$ has many zeros except in some conditions. In particular, $\zeta(z, 1)$ is the Riemann zeta function. Let $z = \sigma + \mathrm{i}t$, where $\sigma > 1$ and $t \in \mathbb{R}$, then $\zeta(z, 1)$ does not have zeros. In [8], it is shown that a normalized function $f_{\sigma,1}(t) := \zeta(\sigma + \mathrm{i}t, 1)/\zeta(\sigma, 1)$ is an infinitely divisible characteristic function. The corresponding distributions are called the Riemann zeta distributions on $\mathbb{R}$ and studied in [9]. A distribution with $f_{\sigma,u}(t) := \zeta(\sigma + \mathrm{i}t, u)/\zeta(\sigma, u)$, $0 < u \leq 1$, as its characteristic function is called the Hurwitz zeta distribution on $\mathbb{R}$. In [7], it is proved that $f_{\sigma,1/2}(t) = \zeta(\sigma + \mathrm{i}t, 1/2)/\zeta(\sigma, 1/2)$ is also an infinitely divisible characteristic function and $u = 1/2$ or 1 are the only cases to be so. Usually, $\zeta(z, u)$ is given by the Dirichlet series or, when $\zeta(z, u)$ does not have zeros, by the Euler products. In [1], a new multidimensional Euler product and corresponding zeta distributions on $\mathbb{R}^d$ are introduced and the infinite divisibility of them is studied. As one of the generalizations of the Dirichlet series, the Shintani zeta functions are well-known. In this paper, we define multidimensional Shintani zeta functions as a further generalization of the Shintani zeta functions. We show that the distributions on $\mathbb{R}^d$ produced by them are not infinitely divisible and also give some examples.

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§ 1. Introduction

§ 1.1. Infinitely divisible distributions

Infinitely divisible distributions are known as one of the most important class of distributions in probability theory. They are the marginal distributions of stochastic processes having independent and stationary increments such as Brownian motion and Poisson processes. In 1930's, such stochastic processes were well-studied by P. Lévy and now we usually call them Lévy processes. We can find the detail of Lévy processes in [13] but they are a bit far from our story, so we omit to comment about them any more in this paper.

In this section, we mention some known properties of infinitely divisible distributions.

Definition 1.1 (Infinitely divisible distribution). A probability measure μ on $\mathbb{R}^d$ is infinitely divisible if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that

$$\mu = \mu_n^{n*},$$

where μ_n^{n*} is the n-fold convolution of μ_n.

Example 1.2. Normal, degenerate, Poisson and compound Poisson distributions are infinitely divisible. (See, also Example 1.4.)

Denote by $I(\mathbb{R}^d)$ the class of all infinitely divisible distributions on $\mathbb{R}^d$. Let $\widehat{\mu}(z) := \int_{\mathbb{R}^d} e^{\mathrm{i}\langle z,x\rangle}\mu(dx)$, $z \in \mathbb{R}^d$, be the characteristic function of a distribution μ, where $\langle \cdot\,,\,\cdot \rangle$ is the inner product. Write $a \wedge b := \min\{a, b\}$.

The following is well-known.

Proposition 1.3 (Lévy–Khintchine representation (see, e.g. [13])). *(i) If $\mu \in I(\mathbb{R}^d)$, then*

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \frac{\mathrm{i}\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right],\ z \in \mathbb{R}^d, \tag{1.1}$$

where A is a symmetric nonnegative-definite $d\times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty, \tag{1.2}$$

and $\gamma \in \mathbb{R}^d$.

(ii) The representation of $\widehat{\mu}$ in (i) by A, ν, and γ is unique.

(iii) Conversely, if A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure satisfying (1.2), and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by (1.1).

The measure ν and (A, ν, γ) are called the Lévy measure and the Lévy–Khintchine triplet of $\mu \in I(\mathbb{R}^d)$, respectively.

Example 1.4. The characteristic functions of distributions given in Example 1.2 are the following.
(*i*) Let μ_{Nor} be a normal distribution on $\mathbb{R}^d$ with covariance matrix A, which is a symmetric positive-definite $d \times d$ matrix, and mean vector $\gamma \in \mathbb{R}^d$. Then,

$$\widehat{\mu}_{\mathrm{Nor}}(z) = \exp\left(-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle\right), \ z \in \mathbb{R}^d.$$

(*ii*) Let $\mu_\gamma = \delta_\gamma$, where δ_γ is a degenerate distribution at $\gamma \in \mathbb{R}^d$. Then,

$$\widehat{\mu}_\gamma(z) = \exp\left(\mathrm{i}\langle \gamma, z\rangle\right), \ z \in \mathbb{R}^d.$$

(*iii*) Let μ_{Po} be a Poisson distribution with mean $c > 0$. Then,

$$\widehat{\mu}_{\mathrm{Po}}(z) = \exp\left(c\left(e^{\mathrm{i}z} - 1\right)\right), \ z \in \mathbb{R}.$$

(*iv*) Let μ_{CPo} be a compound Poisson distribution. Then, for some $c > 0$ and for some distribution ρ on $\mathbb{R}^d$ with $\rho(\{0\}) = 0$,

$$\widehat{\mu}_{\mathrm{CPo}}(z) = \exp\left(c\left(\widehat{\rho}(z) - 1\right)\right), \ z \in \mathbb{R}^d.$$

The Poisson distribution is a special case when $d = 1$ and $\rho = \delta_1$.

§ 1.2. Riemann and Hurwitz zeta distributions

Zeta functions play one of the key roles in number theory. The Riemann zeta function is regarded as the prototype. First results about this function were obtained by L. Euler in the eighteenth century. It is named after B. Riemann, who in the memoir “On the Number of Primes Less Than a Given Magnitude”, published in 1859, established a relation between its zeros and the distribution of prime numbers.

In this section, we introduce the Riemann and Hurwitz zeta distributions and some known properties.

Definition 1.5 (Riemann zeta function (see, e.g. [2])). A function $\zeta(z)$ is called the Riemann zeta function if

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \ \text{where} \ z = \sigma + \mathrm{i}t, \ \sigma > 1, \ t \in \mathbb{R}.$$

The Dirichlet series of $\zeta(z)$ converges absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. It is known that the Riemann zeta function is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at $z = 1$ with residue 1.

Put

$$f_\sigma(t) := \frac{\zeta(\sigma + \mathrm{i}t)}{\zeta(\sigma)}, \ t \in \mathbb{R},$$

then $f_\sigma(t)$ is known to be a characteristic function. (See, e.g. [5].)

Definition 1.6 (Riemann zeta distribution on $\mathbb{R}$)**.** A distribution μ_σ on $\mathbb{R}$ is said to be a Riemann zeta distribution with parameter σ if it has $f_\sigma(t)$ as its characteristic function.

The Riemann zeta distribution is known to be infinitely divisible. The characteristic functions and the Lévy measures of them can be given of the form as in the following.

Proposition 1.7 (See, e.g. [5])**.** *Let $\mathbb{P}$ be the set of all prime numbers and μ_σ be a Riemann zeta distribution on $\mathbb{R}$ with characteristic function $f_\sigma(t)$. Then, μ_σ is compound Poisson on $\mathbb{R}$ and*

$$\begin{aligned}\log f_\sigma(t) &= \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \left(e^{-\mathrm{i}rt \log p} - 1 \right) \\ &= \int_0^\infty \left(e^{-\mathrm{i}tx} - 1 \right) N_\sigma(dx),\end{aligned}$$

where N_σ is given by

$$N_\sigma(dx) = \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \delta_{r \log p}(dx),$$

where δ_x is the delta measure at x.

As a generalization of the Riemann zeta function, the following function is well-known.

Definition 1.8 (Hurwitz zeta function (see, e.g. [2]))**.** For $0 < u \leq 1$, a function $\zeta(z, u)$ is called the Hurwitz zeta function if

$$\zeta(z, u) = \sum_{n=0}^{\infty} \frac{1}{(n+u)^z}, \ \text{where} \ z = \sigma + \mathrm{i}t, \ \sigma > 1, \ t \in \mathbb{R}.$$

In number theory, the Hurwitz zeta function, named after A. Hurwitz, is one of the well-known zeta functions. The Dirichlet series of $\zeta(z, u)$ converges absolutely in the half-plane $\sigma > 1$. The Hurwitz zeta function is analytically continuable to a meromorphic function, which has a simple pole at $z = 1$ with residue 1 as same as the property of the Riemann zeta case. Let $\Gamma(u)$, $u > 0$, be the gamma function. Lerch showed that

$$\left[\frac{d}{dz}\zeta(z,u)\right]_{z=0} = \log\Gamma(u) - \frac{1}{2}\log(2\pi).$$

Hence $\Gamma(u)$ can be written by the Hurwitz zeta function.

We put the corresponding normalized function and a discrete one-sided random variable $X_{\sigma,u}$ as follows.

$$f_{\sigma,u}(t) := \frac{\zeta(\sigma + \mathrm{i}t, u)}{\zeta(\sigma, u)}, \ t \in \mathbb{R},$$

and

$$\Pr\left(X_{\sigma,u} = \log(n+u)\right) = \frac{(n+u)^{-\sigma}}{\zeta(\sigma,u)} \quad \text{for } n \in \mathbb{N} \cup \{0\}.$$

Then $f_{\sigma,u}$ is known to be a characteristic function of $-X_{\sigma,u}$.

Proposition 1.9 ([7]). *(i) The Laplace-Stieltjes transform of $X_{\sigma,u}$ is $\Psi_{\sigma,u}(s) = \zeta(\sigma+s,u)/\zeta(\sigma,u)$, $s > 1-\sigma$.*
(ii) The characteristic function of $-X_{\sigma,u}$ is $f_{\sigma,u}$.

Therefore, we can define the following distribution.

Definition 1.10 (Hurwitz zeta distribution on $\mathbb{R}$). A distribution $\mu_{\sigma,u}$ on $\mathbb{R}$ is said to be a Hurwitz zeta distribution with parameter (σ, u) if it has $f_{\sigma,u}$ as its characteristic function.

The infinite divisibility of $\mu_{\sigma,u}$ is studied in [7].

Proposition 1.11 ([7]). *The Hurwitz zeta distribution $\mu_{\sigma,u}$ is infinitely divisible if and only if*

$$u = \frac{1}{2} \quad \text{or} \quad u = 1.$$

The Lévy measure of $\mu_{\sigma,\frac{1}{2}}$ is also given as follows.

Proposition 1.12 ([7]). *The Hurwitz zeta distribution $\mu_{\sigma,\frac{1}{2}}$ is compound Poisson (infinitely divisible) with its Lévy measure $N_{\sigma,\frac{1}{2}}$ given by*

$$N_{\sigma,\frac{1}{2}}(dx) = \sum_{p>2}\sum_{r=1}^{\infty}\frac{p^{-r\sigma}}{r}\delta_{r\log p}(dx),$$

where the first sum is taken over all odd primes p.

§ 2. Shintani zeta functions and zeta distributions on $\mathbb{R}^d$

§ 2.1. Multidimensional Shintani zeta function

The Riemann zeta function $\zeta(z,1)$ does not vanish when $\sigma > 1$ by the Euler product. According to $\zeta(s,1/2) = (2^s-1)\zeta(s,1)$, the Hurwitz zeta function $\zeta(z,1/2)$ also does not vanish in the region of absolute convergence. On the other hand, Davenport and Heilbronn [4] showed that if u is rational or transcendental except the case $u = 1/2$ or 1, then $\zeta(z,u)$ has infinitely many zeros in the region $\sigma > 1$. Moreover, Cassels [3] showed that $\zeta(z,u)$ has the same property when u is algebraic irrational. Hence we have the following.

Proposition 2.1. *The Hurwitz zeta function $\zeta(z,u)$ does not vanish in the region $1 < \sigma$ if and only if*

$$u = \frac{1}{2} \quad \textit{or} \quad u = 1.$$

This result should be compared with Proposition 1.11. Actually, we can see that the Hurwitz zeta distribution $\mu_{\sigma,u}$, $u \neq 1$ or $1/2$, is not infinitely divisible by Proposition 2.1 and the following well-known fact.

Proposition 2.2 (See, e.g. [13]). *If μ is infinitely divisible, then $\widehat{\mu}(z)$ has no zero, that is, $\widehat{\mu}(z) \neq 0$ for any $z \in \mathbb{R}^d$.*

Thus we obtain another proof of the one side of Proposition 2.1 proved by Hu et al. [7]. We consider this method for other zeta functions. So that we define the following multidimensional Shintani zeta function as a generalization of the Hurwitz zeta function.

Definition 2.3 (Multidimensional Shintani zeta function, $Z_S(\vec{s})$). Let $d,m,r \in \mathbb{N}$, $\xi > 0$, $\vec{s} \in \mathbb{C}^d$ and $(n_1,\ldots,n_r) \in \mathbb{Z}^r_{\geq 0}$. For $\lambda_{lj}, u_j > 0$, $\vec{c_l} \in \mathbb{R}^d_{\geq 0}$, where $1 \leq j \leq r$ and $1 \leq l \leq m$, and a function $\theta(n_1,\ldots,n_r) \in \mathbb{C}$ with $\theta(n_1,\ldots,n_r) \not\equiv 0$ and $|\theta(n_1,\ldots,n_r)| = O((n_1+\cdots+n_r)^\xi)$, we define a multidimensional Shintani zeta function given by

$$Z_S(\vec{s}) := \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\theta(n_1,\ldots,n_r)}{\prod_{l=1}^{m}(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r))^{\langle \vec{c_l},\vec{s}\rangle}}, \tag{2.1}$$

where $\Re\langle \vec{c_l},\vec{s}\rangle > (r+\xi)/m$.

This is a multidimensional case of the Shintani multiple zeta function, when $\theta(n_1,\ldots,n_r)$ in (2.1) is a product of Dirichlet characters, considered by Hida [6].

Put

$$\vec{s} := \vec{\sigma} + \mathrm{i}\vec{t}, \ \ \vec{\sigma}, \vec{t} \in \mathbb{R}^d.$$

It is important to note the following.

Lemma 2.4. *The series defined by* (2.1) *converges absolutely in the region* $\Re\langle \vec{c}_l, \vec{s}\rangle > 0$ *and* $-\xi + \sum_{l=1}^m \langle \vec{c}_l, \vec{\sigma}\rangle > r$.

Proof. Put $\lambda := \min\{\lambda_{lj}\}$ and $u := \min\{u_j\}$. Obviously, we have

$$(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r))^{-1} \le \lambda^{-1}(n_1+\cdots+n_r+ru))^{-1}.$$

Then,

$$\begin{aligned}
|Z_S(\vec{s})| \le& \sum_{n_1,\dots,n_r=0}^{\infty} \frac{\lambda^{-\sum_{l=1}^m \langle \vec{c}_l,\vec{\sigma}\rangle}(n_1+\cdots+n_r+ru)^{\xi}}{\prod_{l=1}^m (n_1+\cdots+n_r+ru)^{\langle \vec{c}_l,\vec{\sigma}\rangle}} \\
\le& \sum_{n_1,\dots,n_r=0}^{\infty} \frac{\lambda^{-\sum_{l=1}^m \langle \vec{c}_l,\vec{\sigma}\rangle}}{(n_1+\cdots+n_r+ru)^{-\xi+\sum_{l=1}^m \langle \vec{c}_l,\vec{\sigma}\rangle}} \\
\le& \lambda^{-\sum_{l=1}^m \langle \vec{c}_l,\vec{\sigma}\rangle}\left((ru)^{\xi-\sum_{l=1}^m \langle \vec{c}_l,\vec{\sigma}\rangle} + \int_0^\infty \!\!\cdots\! \int_0^\infty \frac{dx_1\cdots dx_r}{(x_1+\cdots+x_r+ru)^{-\xi+\sum_{l=1}^m \langle \vec{c}_l,\vec{\sigma}\rangle}}\right) \\
<& \infty
\end{aligned}$$

since $-\xi + \sum_{l=1}^m \langle \vec{c}_l, \vec{\sigma}\rangle > r$. □

Remark 1. This type of generalization of the Euler product is given in [1]. The multidimensional zeta distribution on $\mathbb{R}^d$ is also introduced and the infinite divisibility of them is studied.

By Lemma 2.4.1 of [6], we can show that $Z_S(\vec{s})$ converges absolutely and uniformly on any compact subset in the region $\Re\langle \vec{c}_l, \vec{s}\rangle > (r+\xi)/m$ for all $1 \le l \le m$. Moreover, when $d = m$, $c_1 = (1,0,\dots,0),\dots,c_m = (0,\dots,0,1)$ and $\theta(n_1,\dots,n_r)$ is a product of Dirichlet characters, it is proved that $Z_S(\vec{s})$ can be continued to the whole space $\mathbb{C}^d$ as a meromorphic function in [6].

Example 2.5. (i) When $d = m = r = \lambda_{11} = u_1 = c_1 = 1$, $\theta(n) = -\log(n+1)$, we have

$$Z_S(\vec{s}) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s} = \zeta'(s), \tag{2.2}$$

the derivative of the Riemann zeta function.

(*ii*) When $d=m=r=\lambda_{11}=c_1=1$ and $\theta(n)=e^{2\pi i v n}$, where $v\in\mathbb{R}$, we have

$$Z_S(\vec{s})=\sum_{n=0}^{\infty}\frac{e^{2\pi i v n}}{(n+u)^s}, \tag{2.3}$$

the Lerch zeta function which is a generalization of the Hurwitz zeta functions.

(*iii*) When $d=m=r$, $\lambda_{11}=\cdots=\lambda_{mr}=1$, $\vec{c}_1=(1,0,\ldots,0),\ldots,\vec{c}_m=(0,\ldots,0,1)$, and $\theta(n_1,\ldots,n_m)=1$ if $n_1>\cdots>n_r>0$ otherwise $\theta(n_1,\ldots,n_m)=0$, then one has

$$\begin{aligned}Z_S(\vec{s})&=\sum_{n_1>\cdots>n_r>0}^{\infty}\frac{1}{(n_1+u_1)^{s_1}(n_2+u_2)^{s_2}\cdots(n_r+u_r)^{s_r}}\\&=\sum_{n_1,\ldots,n_r=1}^{\infty}\frac{1}{(n_1+\cdots+n_r+u_1)^{s_1}(n_2+\cdots+n_r+u_2)^{s_2}\cdots(n_r+u_r)^{s_r}},\end{aligned} \tag{2.4}$$

the Euler-Zagier-Hurwitz type of multiple zeta function.

§ 2.2. Shintani zeta distributions on $\mathbb{R}^d$

In this section, we introduce a new probability distribution on $\mathbb{R}^d$ produced by Z_S and consider their infinite divisibility. Let $\theta(n_1,\ldots,n_r)$ be a nonnegative or nonpositive definite function and write $\vec{c}_l=(c_{l1},\ldots,c_{ld})\in\mathbb{R}^d_{\geq 0}$ in Definition 2.3. Again, we use the notation $\vec{s}:=\vec{\sigma}+i\vec{t}$, $\vec{\sigma},\vec{t}\in\mathbb{R}^d$.

Definition 2.6 (Multidimensional Shintani zeta distribution). For $(n_1,\ldots,n_r)\in\mathbb{Z}^r_{\geq 0}$ and $\vec{\sigma}$ satisfying $\Re\langle\vec{c}_l,\vec{s}\rangle>0$ and $-\xi+\sum_{l=1}^m\langle\vec{c}_l,\vec{\sigma}\rangle>r$ as in Lemma 2.4, we define a multidimensional Shintani zeta random variable $X_{\vec{\sigma}}$ with probability distribution on $\mathbb{R}^d$ given by

$$\begin{aligned}\Pr\Bigg(X_{\vec{\sigma}}=\Bigg(&-\sum_{l=1}^m c_{l1}\log\big(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r)\big),\\&\ldots,-\sum_{l=1}^m c_{ld}\log\big(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r)\big)\Bigg)\Bigg)\\=\;&\frac{\theta(n_1,\ldots,n_r)}{Z_S(\vec{\sigma})}\prod_{l=1}^m\big(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r)\big)^{-\langle\vec{c}_l,\vec{\sigma}\rangle}.\end{aligned}$$

It is easy to see these distributions are probability distributions since

$$\frac{\theta(n_1,\ldots,n_r)}{Z_S(\vec{\sigma})}\prod_{l=1}^m\big(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r)\big)^{-\langle\vec{c}_l,\vec{\sigma}\rangle}\geq 0$$

for each $(n_1,\ldots,n_r) \in \mathbb{Z}^r_{\geq 0}$ when $\theta(n_1,\ldots,n_r)$ is nonnegative or nonpositive definite, and

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\theta(n_1,\ldots,n_r)}{Z_S(\vec{\sigma})} \prod_{l=1}^{m} \big(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r)\big)^{-\langle \vec{c}_l,\vec{\sigma}\rangle} = \frac{Z_S(\vec{\sigma})}{Z_S(\vec{\sigma})} = 1$$

by Lemma 2.4.

The characteristic function of $X_{\vec{\sigma}}$ is as follows.

Theorem 2.7. *Let $X_{\vec{\sigma}}$ be a multidimensional Shintani zeta random variable. Then its characteristic function $f_{\vec{\sigma}}$ is given by*

$$f_{\vec{\sigma}}(\vec{t}) = \frac{Z_S(\vec{\sigma}+\mathrm{i}\vec{t})}{Z_S(\vec{\sigma})}, \ \vec{t} \in \mathbb{R}^d.$$

Proof. By the definition, we have

$$\begin{aligned} f_{\vec{\sigma}}(\vec{t}) &= \sum_{n_1,\ldots,n_r=0}^{\infty} e^{\mathrm{i}\langle \vec{t},X_{\vec{\sigma}}\rangle} \frac{\theta(n_1,\ldots,n_r)}{Z_S(\vec{\sigma})} \prod_{l=1}^{m} \big(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r)\big)^{-\langle \vec{c}_l,\vec{\sigma}\rangle} \\ &= \frac{Z_S(\vec{\sigma}+\mathrm{i}\vec{t})}{Z_S(\vec{\sigma})}. \end{aligned}$$

This completes the proof. □

This theorem shows that our definitions of multidimensional Shintani zeta function and distribution give a new generalization of zeta distributions on $\mathbb{R}$ mentioned in Section 1.2 to $\mathbb{R}^d$-valued.

For multidimensional Shintani zeta distributions, we have the following theorem by Proposition 2.2.

Theorem 2.8. *Multidimensional Shintani zeta distributions with $f_{\vec{\sigma}}$ having zeros in the region $\Re\langle \vec{c}_l,\vec{s}\rangle > 0$ and $-\xi+\sum_{l=1}^m\langle \vec{c}_l,\vec{\sigma}\rangle > r$ are not infinitely divisible.*

By the theorem above, we can see that zeta distributions produced by following functions are not infinitely divisible:

1. Partial zeta functions $\sum_{n\leq N} n^{-s}$ for some suitable integer N,
2. The derivative of the Riemann zeta function (2.2),
3. For some Dirichlet series with periodic coefficients, which contains the Hurwitz zeta functions with $u \neq 1/2$ and v are rational, treated by Saias and Weingartner [12],
4. Euler-Zagier-Hurwitz type of multiple zeta functions (2.4) when $u_1,\ldots,u_r$ are algebraically independent over $\mathbb{Q}$ proved in Proposition 3.2 of [10].

It should be noted that Saito and Tanaka [11] showed that the distributions defined by Euler-Zagier multiple zeta star functions are not infinitely divisible by reformulating the method used in [7].

Throughout this paper, we have treated zeta distributions on $\mathbb{R}^d$ and multiple zeta functions having zeros by defining the multidimensional Shintani zeta functions. Though, in one dimensional case, there exist some zeta functions which appear not to have zeros. In such cases, zeta functions may have Euler products. So that it seems to be quite natural to consider zeta functions not having zeros in multidimensional cases. As mentioned in Remark 1, we have also introduced a new multidimensional Euler product for them and defined new multidimensional zeta distributions on $\mathbb{R}^d$ produced by them. We have also studied the infinite divisibility of them in [1].

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