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Stability analysis for a planar traveling wave solution in an excitable system

By

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§ 1. Introduction

Reaction-diffusion systems have been used to describe various phenomena with spatial or spatio-temporal pattern formation. Propagating waves are the most important spatial patterns appearing in reaction-diffusion models with excitability, called excitable systems. For example, the FitzHugh-Nagumo system can generate impulses propagating in nerve axon ([8] and [16]). The Oregonator model realizes target patterns and spiral patterns in the Belouzov-Zhabotinsky reaction, which is simply called BZ reaction ([14]). A combustion experiment was modeled by a reaction-diffusion system ([10], [11]). In this experiment, various spatial patterns occurs (see Figure 1), and a planar propagating wave can be observed. These propagating waves can be characterized by traveling wave solutions with a pulse shape, which are simply called traveling pulses. It is well-known that the FitzHugh-Nagumo system and the Oregonator model have a traveling pulse. Similarly, it is shown that the combustion model has a traveling pulse ([10]). Hence reaction-diffusion systems with excitability exhibit propagating waves in common. Thus the question in this paper is to investigate what kind of similarity traveling pulses in excitable systems possess.

The FitzHugh-Nagumo system

$$\text{(FHN)} \qquad \begin{cases} \varepsilon\tau \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u + f(u) - v, \\ \dfrac{\partial v}{\partial t} = d\Delta v + u - \gamma v \end{cases}$$

was originally introduced in [8] and [16], which is also called the Bonhoeffer-van der Pol model. This system with $d = 0$ is a simplification of a class of excitable oscillatory including the Hodgkin-Huxley equations which were proposed in 1952 to unravel the

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dynamics ionic conductances that generate the nerve action potential. As described in [13], the small positive parameter ε does not mean anything about the magnitude of the physical diffusion coefficient. This is simply a scaling of the space variable to make the wave front appear steep, which is a procedure that facilitates the study of the wave as whole. The variable u diffuses spatially while v is not, because u represents the membrane potential and v does a slow ionic current or gating variable. The nonlinear term $f(u)$ is often given as a cubic function $u(1-u)(u-\alpha)$ for $0<\alpha<1/2$.

It is well-known [1] and [19] that (FHN) can generate a traveling pulse under a suitable condition. These results are obtained in the case of $d=0$ for (FHN). Recently, (FHN) with $d>0$ has been studied from a mathematical point of view, and it was proved in [9] that (FHN) with $d>0$ also generates a traveling pulse via a bifurcation from a standing pulse. Here a standing pulse is a stationary solution of (FHN) with two transition layers. Actually, the global bifurcation diagram of a traveling pulse can be much complicated by a suitable choice of nonlinearity as shown in Figure 10 in [9].

Recently, it was reported in [15] that a reaction-diffusion mechanism can explain that sutures are straight in the newborn human skull, but adult sutures are interdigitated. In this article the authors classified the key molecules that are directly involved in osteogenic differentiation in sutural tissue, and stated that there are systems to promote and inhibit osteogenesis in suture development. This mechanism is simply formulated as a modified model of a FitzHugh-Nagumo type reaction-diffusion system with a bistable nonlinearity. Since suture system is understood by interactions between diffusible signaling molecule and cells with random movement, the reaction-diffusion system has two diffusible components and exhibits the destabilization of a planar pattern in spite that initial states have almost planar pattern. This article suggests that it is important to study (FHN) not only mathematically but also practically.

The BZ reaction is one of the most widely studied oscillating reaction both theoretically and experimentally, which indicates the possibility that chemical reactions generate oscillatory behavior ([23]). In the meanwhile this reaction generates various spatial patterns, which include target patterns and spiral waves. For these phenomena, a simple reaction-diffusion system

$$\text{(BZ)}\qquad \begin{cases} \varepsilon\tau\dfrac{\partial u}{\partial t}=\varepsilon^2\Delta u+u(1-u)-2fv\dfrac{u-q}{u+q}, \\ \dfrac{\partial v}{\partial t}=d\Delta v+u-v \end{cases}$$

has been studied ([14]). This model is based on the Field-Körös-Noyes reaction mechanism, a prototype that suggests the general kinetic scheme of reactions which can be referred as the Oregonator. The small parameter ε is determined by rate constants and the concentration of hydrogen ions in [5]. Since u and v represent concentrations of some chemical products, this model has two diffusion terms for both u and v. Although

(BZ) generates a solution with an oscillatory behavior, it becomes an excitable system for a suitable parameter.

Finally we would like to introduce a combustion model given in [10], which was proposed to theoretically answer what is the reason why diverse patterns appear in the cinder paper in a combustion experiment when the Peclet number Pe is suitably varied. The spatial patterns like Figure 1 were first observed at an experiment in space shuttle that microgravity smoldering combustion expanding from radiative ignition exhibits complex, unexpected patterns. Moses and his group set up the experiment to understand this complexity more qualitatively under microgravity environment, and found that when the value of Pe is varied, various cinder patterns of the paper are observed in spite that the gas is supplied in a uniform flow, opposite to the direction of the front propagation and the paper undergoes uniform ignition. These patterns are qualitatively classified into four types of patterns; a planar pattern, a smooth but uneven front pattern, a fingering pattern with tip-splitting and a fingering pattern without tip-splitting, respectively. In order to theoretically answer the question above, an exothermic reaction-diffusion system was proposed in [4], [10]. A simple (non-dimensionalized) version of the model is

$$\begin{cases} \dfrac{\partial T}{\partial t} = Le\Delta T - \lambda_1 \dfrac{\partial T}{\partial x} + \gamma k(T)pw - aT, \\ \dfrac{\partial p}{\partial t} = -k(T)pw, \\ \dfrac{\partial w}{\partial t} = d\Delta w - \lambda_2 \dfrac{\partial w}{\partial x} - k(T)pw, \end{cases}$$

where T, p and w are respectively the temperature in Kelvin, the density of paper and the gas concentration of flow. In order to simplify this equation, we formally let the function p be a constant. Under a suitable condition, we rewrite the system above into

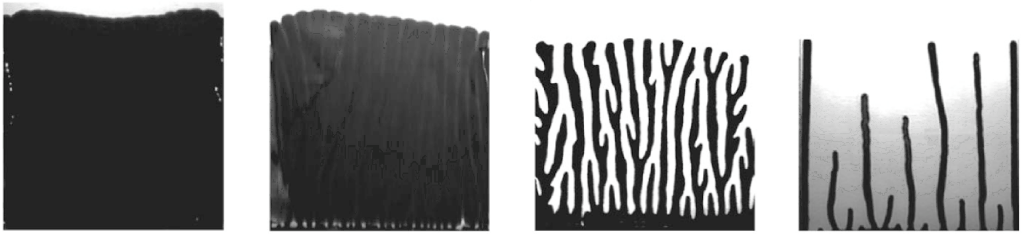
$$\begin{cases} \varepsilon \dfrac{\partial T}{\partial t} = \varepsilon^2 \Delta T - \varepsilon\lambda_1 \dfrac{\partial T}{\partial x} + \gamma k(T)w - aT, \\ \dfrac{\partial w}{\partial t} = d\Delta w - \lambda_2 \dfrac{\partial w}{\partial x} - k(T)w, \end{cases} \tag{1.1}$$

where $k(T)$ is the only nonlinear term included in the model, which is called the Arrhenius kinetics in chemical reactions. As an example, take

$$k(T) = \begin{cases} A\exp(-B/(T-\theta)), & T > \theta, \\ 0, & 0 \le T \le \theta \end{cases}$$

for some constants $A, B > 0$ and $\theta \ge 0$.

Various theoretical and numerical results show appearance of target patterns and spiral patterns in (FHN) and (BZ) with $d = 0$ or small $d > 0$ dependently on ε. A wave



(a) (b) (c) (d)

Figure 1. Cinder patterns of the paper where (a) $Pe = 18$, $(b)Pe = 15$, $(c)Pe = 5 \pm 0.5$, (d) $Pe = 0.45 \pm 0.2$. ([18])

with pulse shape first arises around a core, a central part of the pattern, and propagate with an almost constant speed. In the end the wave converges to a planar traveling pulse as the time goes to infinity, which implies that the planar traveling pulse must be stable. This has been already proved in [2] by a singular perturbation analysis in the case of $d = 0$. On the other hand, a traveling pulse which uniformly propagates exhibits destabilization in (1.1), and a fingering pattern emerges, which is observed numerically. In spite that these systems have excitability in common, the difference of magnitude of the diffusive coefficient strongly influences the stability of a planar pattern, and the stability results are completely different. Thus our aim in this article is to study an excitable system with a general nonlinearity and advection terms such as

$$
(1.2) \qquad \begin{cases} \varepsilon\tau \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u - \varepsilon\lambda_1 \dfrac{\partial u}{\partial x} + f(u,v), & (x,y) \in (-\infty,\infty) \times D, \quad t > 0, \\ \dfrac{\partial v}{\partial t} = d\Delta v - \lambda_2 \dfrac{\partial v}{\partial x} + g(u,v), & (x,y) \in (-\infty,\infty) \times D, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & (x,y) \in (-\infty,\infty) \times \partial D, \quad t > 0, \end{cases}
$$

and see how the diffusive coefficient influences the stability of a traveling pulse, where $D \subset \mathbb{R}^N$ is a domain with a smooth boundary, n is the outward normal unit vector on $(-\infty,\infty) \times \partial D$, ε, d, τ are positive, and λ_1, λ_2 are nonnegative constants. Since we take (1.1) into account, the advection terms only have the derivative of u and v with respect to the spatial variable x. In this paper we formally reduce (1.2) to a free boundary problem (2.3) and prove the existence and stability of a traveling pulse solution in (2.3). Since we have already studied the stability of a planar traveling pulse in the case of $d = 0$ as described above, we focus on the case of $d > 0$ in this article. Our main results will be stated in Section 2.

The stability of a traveling pulse in one dimension was studied in [12] and [22], where the FitzHugh-Nagumo system without the diffusive coefficient d was the object. The linearized eigenvalue problem in these papers induces a complex function which is called Evans function ([3]), and is defined as the Wronskian. The zero points of the Evans function completely correspond to the eigenvalues in the complex plane out of the essential spectrums. In order to look for the zero points, it is vital to bring out various properties of the traveling pulse, which can be verified by many calculations. Hence the analysis of the Evans function may be complicated and difficult. So we reduce our system to a free boundary problem and simplify the eigenvalue problem.

This paper is organized as follows; In Section 2, we state our results (Theorems 2.1, 2.2). In addition we describe several assumptions and notation used throughout this paper. In Section 3, we construct a traveling pulse and prove Theorem 2.1. Finally, we derive the linearized eigenvalue problem associated with (2.3) and prove Theorem 2.2.

§ 2. Assumptions and Main Results

In this section we describe several assumptions and notation used throughout this paper. At first, the nonlinear terms f, g are supposed to have suitable conditions, which give us a sufficient condition to guarantee the existence of a traveling pulse. We point out that the specific systems, (FHN), (BZ), and (1.1) satisfy the following assumptions under suitable conditions of parameters.

Suppose that the nonlinear terms f, g satisfy the following conditions;

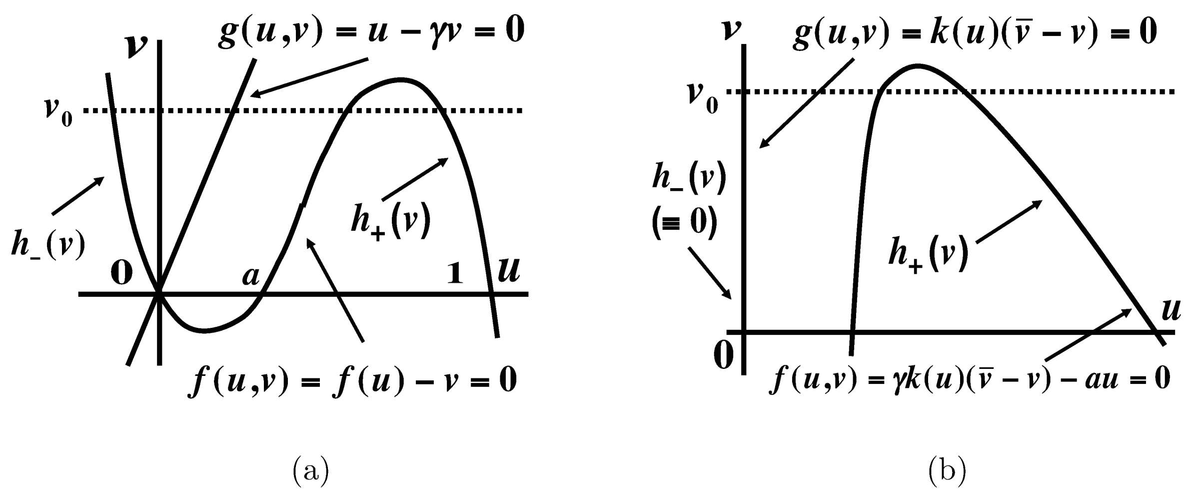


Figure 2. Nullclines of f and g. The figure (a) shows the nullcline of the nonlinearity in the FitzHugh-Nagumo system and (b) does that in the combustion model. The functions T, w in (1.1) are replaced into $u, \bar{v} - v$ in the right figure. The constant $\bar{v}$ is a given parameter, which originally appears in a boundary condition (see [11]).

(A1) $f(u,v), g(u,v)$ are defined on an open subset $\mathcal{O} \subset \mathbb{R}^2$ and smooth with respect to u, v.

(A2) There are $v_0 > 0$ and smooth functions $h_\pm(v), h_0(v)$ on $[0, v_0]$ $(h_-(v) < h_0(v) < h_+(v))$ such that the nullcline $\{(u,v) \in \mathcal{O} \mid f(u,v) = 0\}$ consists of three curves C_i $(i = -, 0, +)$ defined by $C_i = \{(u,v) \in \mathcal{O} \mid u = h_i(v), v \in [0, v_0]\}$ (see Figure 2).

(A3) $(0,0)$ is a unique solution of $f(u,v) = g(u,v) = 0$ in $\mathcal{O}$.

(A4) It holds true that $f_u(h_\pm(v), v) < 0$ on $v \in [0, v_0]$.

(A5) Let $J(v) = \int_{h_-(v)}^{h_+(v)} f(s,v)ds$ for $v \in [0, v_0]$, then $J'(v) < 0$ on $v \in [0, v_0]$, and there is $v^* \in (0, v_0)$ such as $J(v^*) = 0$.

(A6) The function $G_+(v) \equiv g(h_+(v), v)$ satisfies $G_+(v) > 0$ and $G'_+(v) < 0$ on $[0, v_0]$, while the function $G_-(v) \equiv g(h_-(v), v)$ satisfies either $G'_-(v) < 0$ or $G_-(v) \equiv 0$ on $[0, v_0]$.

Under these assumptions we can define traveling wave solutions of

$$\text{(2.1)} \qquad \begin{cases} \tau c_1(v)\phi_1' = \phi_1'' - \lambda_1 \phi_1' + f(\phi_1, v), & \xi \in (-\infty, \infty), \\ \phi_1(-\infty) = h_-(v), \quad \phi_1(+\infty) = h_+(v), \end{cases}$$

and

$$\text{(2.2)} \qquad \begin{cases} \tau c_2(v)\phi_2' = \phi_2'' - \lambda_1 \phi_2' + f(\phi_2, v), & \xi \in (-\infty, \infty), \\ \phi_2(-\infty) = h_+(v), \quad \phi_2(+\infty) = h_-(v) \end{cases}$$

for any $v \in [0, v_0]$, respectively ([7]). As stated in [7], ϕ_1, ϕ_2 are monotonic functions. Here the wave speeds are denoted by $c_i = c_i(v)$ $(i = 1, 2)$. The condition (A5) implies that $\tau c_1(v^*) + \lambda_1 = \tau c_2(v^*) + \lambda_1 = 0$, and $\tau c_1(v) + \lambda_1 > 0$ and $\tau c_2(v) + \lambda_1 < 0$ on $[0, v^*]$, while $\tau c_1(v) + \lambda_1 < 0$ and $\tau c_2(v) + \lambda_1 > 0$ on $[v^*, v_0]$. A traveling pulse is supposed to have two transition layers with a positive speed and be characterized by $c_1(v), c_2(v)$, and a function v. In this paper we call the transition layer ϕ_1 (ϕ_2) *a forward front* (*a backward front*). Since the pulse maintains its shape, these transition layers have to move with the same velocity, from which we need the following condition;

(A7) The wave speed $c_1(0)$ is positive. In addition, there is $v_{\max} \in (v_*, v_0]$ such that $c_1(0) = c_2(v_{\max})$.

(A8) For any $v \in [0, v_{\max}]$, ϕ_1, ϕ_2 satisfy

$$\int_{-\infty}^{\infty} f_v(\phi_1, v)\phi_1' e^{-(\tau c_1(v) + \lambda_1)\xi} d\xi < 0, \quad \int_{-\infty}^{\infty} f_v(\phi_2, v)\phi_2' e^{-(\tau c_2(v) + \lambda_1)\xi} d\xi > 0.$$

From (A7) and (A8), we know that for any $v_1 \in [0, v^*]$, there exists $v_2 = v_2(v_1) \in [v^*, v_0]$ such as $c_1(v_1) = c_2(v_2(v_1))$. The assumption (A8) implies the monotonicity of $c_1(v)$ and $c_2(v)$ with respect to v so that $v_2(v_1)$ is well-defined. This notation will be used in Section 3. In addition, we can obtain $c_1'(0) < 0$ due to (A8), which is a key fact to determine the stability of a planer traveling pulse. In the assumption (A7) we implicitly suppose that λ_1 must be small to some extent.

Since it is much complicated to directly treat (1.2) and prove the stability of a traveling pulse, we would like to simplify (1.2). Then we formally reduce the reaction-diffusion system (1.2) to a free boundary problem given in the following. Here we explain some notation included in the problem. Since we would like to study the stability of a traveling pulse, we only consider solutions which consist of two interfaces $\Gamma_1(t), \Gamma_2(t)$ given by

$$\Gamma_i(t) = \{(x, y) \in (-\infty, \infty) \times D \mid x = \gamma_i(y, t)\} \quad (i = 1, 2).$$

Here the functions $\gamma_1 = \gamma_1(y, t)$, $\gamma_2 = \gamma_2(y, t)$ are defined on $y \in D, t > 0$. Since we only consider the stability of a traveling pulse with two transition layers in this paper, we naturally assume that γ_1, γ_2 are almost constant at $t = 0$ and $\gamma_1(y, t) < \gamma_2(y, t)$ for any $y \in D$ and $t > 0$ if exist. If γ_i is sufficiently smooth, the mean curvature κ_i can be defined. Similarly, we are able to define the normal velocities of $\Gamma_i(t)$, denoted by $N_i(x, y, t)$, where the normal vector on $\Gamma_1(t)$ ($\Gamma_2(t)$) points from $\Omega_+(t)$ ($\Omega_-(t)$) to $\Omega_-(t)$ ($\Omega_+(t)$). By these notations, $G(v)$ is defined by

$$G(v) = \begin{cases} G_-(v), & (x, y) \in \Omega_-(t), \\ G_+(v), & (x, y) \in \Omega_+(t) \end{cases}$$

for a function $v = v(x, y, t)$ defined on $(-\infty, \infty) \times D$ and $t > 0$, where $\Omega_+(t)$ is a domain enclosed by $\Gamma_1(t)$, $\Gamma_2(t)$, and $(-\infty, \infty) \times \partial D$, and $\Omega_-(t) = ((-\infty, \infty) \times D) \setminus \overline{\Omega_+(t)}$. Set $D(t) = \Omega_+(t) \cup \Omega_-(t)$. Note that $D(t)$ is divided into three simply connected domains which are separated by the two interfaces $\Gamma_i(t)$ $(i = 1, 2)$.

Using the notation above, we formally introduce a free boundary problem given by (2.3)

$$\begin{cases} N_i = c_i(v) - \varepsilon\kappa_i, & (x, y) \in \Gamma_i(t), \quad t > 0, \\ \dfrac{\partial v}{\partial t} = d\Delta v - \lambda_2 \dfrac{\partial v}{\partial x} + G(v), & (x, y) \in (-\infty, \infty) \times D(t), \quad t > 0, \\ \dfrac{\partial v}{\partial n} = 0, & (x, y) \in (-\infty, \infty) \times \partial D, \quad t > 0, \\ \lim\limits_{x \to -\infty} v(x, y, t) = 0, \quad \lim\limits_{x \to \infty} |v(x, y, t)| < \infty, & y \in D, \quad t > 0 \end{cases}$$

(see [6]). In this paper we only consider this free boundary problem instead of (1.2) and prove the existence and stability of a traveling pulse although we do not derive (2.3) from (1.2) rigorously. Since a traveling pulse is a planar pattern, we neglect the variable

y in (2.3) to study the existence of a traveling pulse. For example, we let the mean curvature κ_i be 0 in this case.

Next we consider the existence of a traveling pulse (γ_1, γ_2, v). We say that $(\gamma_1, \gamma_2, v) = (\gamma_1(t), \gamma_2(t), v(x,t))$ is a traveling pulse in (2.3) if there are a wave speed c, a width of the pulse ρ, and a bounded function $V = V(z) \in C^1(-\infty,\infty)$ such that $(\gamma_1(t), \gamma_2(t), v(x,t)) = (-ct, -ct+\rho, V(x+ct))$ satisfies (2.3). The variable $z = x+ct$ is called a moving coordinate. From this definition, the function V must be a solution of

$$cV' = dV'' - \lambda_2 V' + G(V), \quad z \in (-\infty,\infty). \tag{2.4}$$

We impose boundary conditions on V such as

$$V(-\infty) = 0, \quad \lim_{z\to\infty} |V(z)| < \infty. \tag{2.5}$$

In (2.4), the nonlinear term $G(V)$ is more simply given by

$$G(V) = \begin{cases} G_-(V), & z \in (-\infty,0) \cup (\rho,\infty), \\ G_+(V), & z \in (0,\rho). \end{cases}$$

In the equation (2.4), we have to find a wave speed c, a width of the pulse ρ, and a function V. To begin with, we state the existence of a traveling pulse and determine the behavior of V as $d \to 0$.

Theorem 2.1. *Under Assumptions (A1)–(A8), there is a traveling pulse $(-ct, -ct+\rho, V)$ in (2.3) for small $d > 0$. In addition, the function V converges as $d \to 0$ in C^1-sense.*

In this theorem, we assume the smallness of d. In this paper we do not pay attention to the situation that the diffusive coefficient d is large because the traveling pulse may be unstable even in one dimension or never exists. According to the result in [9], there is $\tau_0 > 0$ such that a traveling pulse bifurcates from a standing pulse at $\tau = \tau_0$ in (1.2) with $\lambda_1 = \lambda_2 = 0$ and a nonlinearity of the FitzHugh-Nagumo type, and no traveling pulse exists for $\tau > \tau_0$ around the bifurcation point. Roughly speaking, τ plays the same role as $\sqrt{d}$, and then the traveling pulse may not exist for large d. This is why we do not treat the case that d is large.

The traveling pulse given by Theorem 2.1 can be a solution in the cylindrical domain $(-\infty,\infty) \times D$. We derive an eigenvalue problem with respect to the planar traveling pulse and study the distribution of eigenvalues. We substitute

$$\begin{aligned} \gamma_1(y,t) &= -ct + r_1\varphi_k(y)e^{\mu t}, \\ \gamma_2(y,t) &= -ct + \rho + r_2\varphi_k(y)e^{\mu t}, \\ v(x,y,t) &= V(x+ct) + \psi(x+ct)\varphi_k(y)e^{\mu t} \end{aligned}$$

into (2.3) and linearize it, where $(-ct, -ct+\rho, V)$ is the traveling pulse given in Theorem 2.1 and $\varphi_k = \varphi_k(y)$ is the k-th eigenfunction of $-\Delta_y = -\sum_{i=1}^N \partial^2/\partial^2 y_i$ in D with the Neumann boundary condition. More precisely, there is an eigenvalue ω_k $(k = 0, 1, \dots)$ such that (ω_k, φ_k) is a solution of

$$\begin{cases} -\Delta_y \varphi_k = \omega_k \varphi_k, & y \in D, \\ \dfrac{\partial \varphi_k}{\partial n} = 0, & y \in \partial D. \end{cases}$$

Without loss of generality, we assume that $0 = \omega_0 < \omega_1 \leq \omega_2 \leq \cdots$, and $\omega_k \to \infty$ as $k \to \infty$. Then the eigenvalue problem is given by

$$\text{(2.6)} \quad \begin{cases} \mu r_1 = -c_1'(v_1)(V'(0)r_1 + \psi(0)) - \varepsilon\omega_k r_1, \\ \mu r_2 = -c_2'(v_2)(V'(\rho)r_2 + \psi(\rho)) - \varepsilon\omega_k r_2, \\ \mu\psi = d\psi'' - d\omega_k\psi - (c + \lambda_2)\psi' + G'(V)\psi, \quad z \in (-\infty, 0) \cup (0, \rho) \cup (\rho, \infty), \\ d(\psi'(+0) - \psi'(-0)) = (G_+(v_1) - G_-(v_1))r_1, \\ d(\psi'(\rho + 0) - \psi'(\rho - 0)) = -(G_+(v_2) - G_-(v_2))r_2, \end{cases}$$

where $v_1 = V(0)$, $v_2 = V(\rho)$, and the variable $z = x + ct$ is the moving coordinate for the wave speed c given in Theorem 2.1. We show that the planar traveling pulse is unstable for small $d > 0$.

Theorem 2.2. *Fix $k \geq 1$ arbitrarily, and let $d > 0$ be small dependently on k. Then, under Assumptions (A1)–(A8), there is a positive eigenvalue in (2.6) for sufficiently small $\varepsilon > 0$, that is, the traveling pulse given in Theorem 2.1 is unstable.*

Here we point out that Theorem 2.2 is similar to previous results associated with the instability of interfaces in reaction-diffusion systems, which are composed of two components with positive diffusive coefficients. In [17], the stability analysis was succeeded for stationary solutions with pulse shape in higher dimension, where a free boundary problem was induced from the FitzHugh-Nagumo system with a piecewise linear term. The authors in [21] generalized a nonlinearity in [17] to a wider class, and characterized an eigenvalue with the largest positive real part and the corresponding wave number. Since this wave number becomes dominant for deformations of a destabilized planar interface, it is called the fastest growth wavelength. These works treat only stationary interfaces. Actually, it was shown in [20] that a traveling wave solution is unstable in higher dimension in a reaction-diffusion system with a bistable nonlinearity. These works strongly suggest that the planar interfaces be destabilized by the diffusive effect.

Theorem 2.2 shows the instability of the traveling pulse in higher dimension for small $d > 0$. This is similar to the results in [17], [21] and [20]. On the other hand, the traveling pulse is stable for $d = 0$ as shown in [2]. Hence the traveling pulse loses its stability immediately no matter how weakly the diffusive effect influences. The smallness of the diffusive coefficient d implies that the two interfaces in the pulse interact each other weakly. Then the instability of the traveling pulse in higher dimension results from that of the forward front. Indeed, the instability of the forward front will determine the existence of a positive eigenvalue in (2.6) (see Section 4).

§ 3. Existence of a traveling wave pulse

In this section we prove Theorem 2.1, which gives us a one-dimensional traveling pulse of (2.3). In the following we divide $(-\infty, \infty)$ into three parts, $(-\infty, 0)$, $(0, \rho)$, and (ρ, ∞), and construct a solution of (2.4) in each interval by applying a perturbation method. Since d is close to 0, it seems that a singular perturbation theory should be applied. In fact, it is unnecessary because the wave speed c is strictly positive due to (A7) and does not tend to 0 as $d \to 0$. Eventually we will show that $V(z)$ converges piecewisely in each interval in C^0-space and $V'(z)$ is uniformly bounded in d. In this section, C represents a general constant independent of d.

First we would like to find a bounded solution of

$$\begin{cases} dV'' - (c+\lambda_2)V' + G_-(V) = 0, \quad z \in (-\infty, 0), \\ V(-\infty) = 0, \quad V(0) = v_1 \end{cases} \tag{3.1}$$

for any given v_1 in C^1-space as $d \to 0$, denoted by $V_1 = V_1(z)$. Note that the wave speed c is determined by v_1, that is, $c = c_1(v_1)$. However it is hopeless to obtain a uniformly bounded solution in (3.1) in $d > 0$ in C^1-sense and we see that $V_1'(0)$ tends to ∞ as $d \to 0$ if the solution exists for $v_1 > 0$ independent of d. Then we replace v_1 in (3.1) into dv_1 so that V_1 and V_1' are uniformly bounded in $d > 0$.

In the case of $G_-(v) \equiv 0$ on $[0, v_0]$, $V_1 = dv_1 e^{(c+\lambda_2)z/d}$. Next we assume $G_-'(v) < 0$ on $[0, v_0]$. We define a function ψ by $\psi = dv_1 e^{\kappa z}$ with an exponent $\kappa = ((c+\lambda_2) + \sqrt{(c+\lambda_2)^2 - 4dG_-'(0)})/2d$. Then we set $V = \psi + R$, and find a solution R in

$$\begin{cases} LR \equiv dR'' - (c+\lambda_2)R' + G_-'(0)R = F(R, z) \quad z \in (-\infty, 0), \\ R(-\infty) = R(0) = 0, \end{cases} \tag{3.2}$$

where $F(R, z) = -(G_-(\psi + R) - G_-'(0)(\psi + R))$. If d is small, the function F satisfies

$$\|F(R_0, z)\|_{C^0_\kappa} \le Cd, \quad \|F(R_1, z) - F(R_2, z)\|_{C^0_\kappa} \le \frac{1}{2}\|R_1 - R_2\|_{C^0_\kappa} \tag{3.3}$$

for $\|R_i\|_{C^0_\kappa} \le \sqrt{d}$ $(i = 0, 1, 2)$, where $\|\cdot\|_{C^0_\kappa}$ is a norm defined by $\|\varphi\|_{C^0_\kappa} = \sup_{z \in (-\infty, 0)} |\varphi(z)e^{-\kappa z}|$ for a function φ. Also, we set $\|\varphi\|_{C^1_\kappa} = \|\varphi\|_{C^0_\kappa} + \|\varphi'\|_{C^0_\kappa}$.

Lemma 3.1. *Fix $v_1 > 0$ arbitrarily. If $d > 0$ is small and under Assumptions (A1)–(A3), (A6), and (A7), there is a solution R of (3.2) such that $\|R\|_{C^1_\kappa} \le Cd^2$ as $d \to 0$ for a constant $C > 0$ independent of $d > 0$.*

Proof. At first we prove the invertibility of L and $\|L^{-1}f\|_{C^0_\kappa} \le C\|f\|_{C^0_\kappa}$ for any continuous function f with $\|f\|_{C^0_\kappa} < \infty$. The negativity of G'_- enables us to apply the Lax-Milgram theorem to a differential equation

$$\begin{cases} L\varphi = f, & z \in (-\infty, 0), \\ \varphi(-\infty) = \varphi(0) = 0 \end{cases}$$

for any function f with $\|f\|_{C^0_\kappa} < \infty$. From the usual regularity argument, the solution φ belongs to C^2 space. Here we see $\|\varphi\|_{C^0_\kappa} \le C\|f\|_{C^0_\kappa}$. From the boundary conditions, the maximum of φ must be nonnegative. Suppose that there is a point $z_0 \in (-\infty, 0)$ such that φ attains the maximum at $z = z_0$. Thanks to $\varphi' = 0$ and $\varphi'' \le 0$ at $z = z_0$, we have $G'_-(0)\varphi \ge f$. Since $G'_-(0) < 0$ and $\varphi > 0$, f must be negative at $z = z_0$. Hence

$$0 < \varphi(z_0)e^{-\kappa z_0} \le C|f(z_0)|e^{-\kappa z_0} \le C\|f\|_{C^0_\kappa}$$

Similarly, we easily see $\varphi(z)e^{-\kappa z} \ge -C\|f\|_{C^0_\kappa}$ for any z. So $\|\varphi\|_{C^0_\kappa} \le C\|f\|_{C^0_\kappa}$.

By (3.3), the estimate of L^{-1} obtained above, and the contraction mapping principle, we easily find a fixed point $R = L^{-1}F(R, z)$. Then we see $\|R\|_{C^0_\kappa} \le Cd$ because of the estimate of F and L^{-1}. In addition, from this inequality and the fact of $|F(R, z)| \le C(|R|^2 + |\psi|^2)$ for any function R with $\|R\|_{C^0_\kappa} \le Cd$, the solution R satisfies $\|R\|_{C^0_\kappa} \le Cd^2$. Hence it suffices to estimate $\|R'\|_{C^0_\kappa}$ in order to conclude Lemma 3.1. We rewrite (3.2) into an integral form

$$\begin{aligned} R(z) = \alpha_+\varphi_+ &- \frac{d}{\sqrt{(c+\lambda_2)^2 - 4dG'_-(0)}}\varphi_- \int_{-\infty}^{z} e^{-\frac{c+\lambda_2}{d}s}\varphi_+ F(R, s)ds \\ &+ \frac{d}{\sqrt{(c+\lambda_2)^2 - 4dG'_-(0)}}\varphi_+ \int_0^z e^{-\frac{c+\lambda_2}{d}s}\varphi_- F(R, s)ds, \end{aligned}$$

where $\varphi_\pm(z) = e^{\beta_\pm z}$ for $\beta_\pm = (c + \lambda_2 \pm \sqrt{(c+\lambda_2)^2 - 4dG'_-(0)})/2d$, and the constant α_+ is given by

$$\alpha_+ = \frac{d}{\sqrt{(c+\lambda_2)^2 - 4dG'_-(0)}} \int_{-\infty}^{0} e^{-\frac{c+\lambda_2}{d}s}\varphi_+ F(R, s)ds$$

because of the boundary conditions in (3.2). Then we see

$$\begin{aligned} R'(z) = \alpha_+\varphi'_+ &- \frac{d}{\sqrt{(c+\lambda_2)^2 - 4dG'_-(0)}}\varphi'_- \int_{-\infty}^{z} e^{-\frac{c+\lambda_2}{d}s}\varphi_+ F(R, s)ds \\ &+ \frac{d}{\sqrt{(c+\lambda_2)^2 - 4dG'_-(0)}}\varphi'_+ \int_0^z e^{-\frac{c+\lambda_2}{d}s}\varphi_- F(R, s)ds. \end{aligned}$$

Estimating the three terms in the right-hand side of this integral form one by one, we show $|\alpha_+|\|\varphi'_+\|_{C^0_\kappa} \le Cd^2$, and

$$\left\|\varphi'_- \int_{-\infty}^{z} e^{-\frac{c+\lambda_2}{d}s}\varphi_+ F(R,s)ds\right\|_{C^0_\kappa} \le Cd^2, \quad \left\|\varphi'_+ \int_{0}^{z} e^{-\frac{c+\lambda_2}{d}s}\varphi_- F(R,s)ds\right\|_{C^0_\kappa} \le Cd^2,$$

where we used the fact that $\|\psi\|_{C^0_\kappa} \le Cd$, and $\|R\|_{C^0_\kappa} \le Cd^2$. These inequalities imply $\|R'\|_{C^0_\kappa} \le Cd^2$, which concludes Lemma 3.1. □

Next we construct a solution $V_3 = V_3(y)$ of

$$\begin{cases} dV'' - (c+\lambda_2)V' + G_-(V) = 0, & y \in (0,\infty), \\ V(0) = v_2(dv_1), \quad \lim_{z\to\infty} |V(z)| < \infty. \end{cases} \tag{3.4}$$

By using V_3, the solution in the third interval (ρ,∞) can be represented as $V_3(z-\rho)$. Here $v_2(v_1)$ has been defined in Section 2. In the following lemma we prove the existence of V_3 such that V_3 is uniformly bounded in $d > 0$ in C^1-sense.

As described in the proof of Lemma 3.1, we explicitly have the solution V_3 in the case of $G_-(v) \equiv 0$. So we only consider the case that $G'_-(v)$ is strictly negative. Let V_0 be a solution of

$$\begin{cases} -(c+\lambda_2)V' + G_-(V) = 0, \quad y > 0, \\ V(0) = v_2(dv_1). \end{cases} \tag{3.5}$$

By substituting $V = V_0 + R$ into (3.4), R must be a solution of

$$\begin{cases} LR \equiv dR'' - (c+\lambda_2)R' + G'_-(V_0)R = F(R,z), \quad y > 0, \\ R(0) = R(\infty) = 0, \end{cases} \tag{3.6}$$

where $F(R,z) \equiv -dV_0'' - (G_-(V_0+R) - G_-(V_0) - G'_-(V_0)R)$. The straightforward calculation gives us

$$\|F(R_0,z)\|_{C^0_\kappa} \le Cd, \quad \|F(R_1,z) - F(R_2,z)\|_{C^0_\kappa} \le \frac{1}{2}\|R_1 - R_2\|_{C^0_\kappa} \tag{3.7}$$

for $\|R_i\|_{C^0_\kappa} \le \sqrt{d}$ $(i = 0,1,2)$ if d is small, where $\|\cdot\|_{C^0_\kappa}$ is a norm defined by $\|\varphi\|_{C^0_\kappa} = \sup_{y\in(0,\infty)} |\varphi(y)e^{\kappa y}|$ for a small exponent $\kappa > 0$ fixed independently of d. Also we put $\|\varphi\|_{C^1_\kappa} = \|\varphi\|_{C^0_\kappa} + \|\varphi'\|_{C^0_\kappa}$. By the same argument as in the proof of Lemma 3.1, we have the following lemma.

Lemma 3.2. *Fix v_1 arbitrarily. Under Assumptions (A1)–(A3), (A6) and (A7), there is a solution R of (3.6) such that $\|R\|_{C^1_\kappa} \to 0$ as $d \to 0$.*

Finally we construct a solution in the second interval, denoted by V_2. Now we do formal calculations in

$$\begin{cases} dV'' - (c+\lambda_2)V' + G_+(V) = 0, & z \in (0,\rho), \\ V(0) = dv_1, \quad V(\rho) = v_2(dv_1), \quad V'(\rho) = V_3'(\rho), \end{cases} \tag{3.8}$$

and obtain an approximated solution in this equation. In (3.8), c and $v_2(dv_1)$ have been determined by v_1. The width of the pulse ρ is defined as follows. It is easy to see that the equation

$$\begin{cases} -(c+\lambda_2)V' + G_+(V) = 0, \quad z > 0, \\ V(0) = dv_1 \end{cases} \tag{3.9}$$

has a monotonically increasing solution $V_0 = V_0(z)$ due to (A6). Then V_0 achieves $v_2(dv_1)$ at some $z > 0$, denoted by ρ. Eventually, the free parameter in (3.8) is only v_1. Note that usual elliptic equations are supposed to have two boundary conditions in one-dimension while (3.8) has three boundary conditions.

By using the lowest order term V_0, the solution V is thought to be expressed as $V = V_0 + R$, where R is thought as a higher order term. However it cannot be because the function V_0 satisfies only the first and second boundary conditions in (3.8), and may not do the third one. Then we introduce, what we call, an inner solution at $z = \rho$. We define $\psi(z) = d(V_3'(\rho) - V_0'(\rho) - R'(\rho))\ (e^{(c+\lambda_2)(z-\rho)/d} - 1)/(c+\lambda_2)$ for a function $R \in C^1[0,\rho]$, which will be taken in Lemma 3.3. Note that ψ satisfies $\psi(\rho) = 0$, and can be estimated such as $\|\psi\|_{C^0} \le Cd(1 + \|R\|_{C^1})$ and $\|\psi'\|_{C^0} \le C(1 + \|R\|_{C^1})$, where $\|\cdot\|_{C^0}$ and $\|\cdot\|_{C^1}$ are usual norms in C^0 and C^1-spaces. Instead of $V_0 + R$, we set $V(z) = V_0(z) + \chi(z)\psi((z-\rho)/d) + R(z)$, and see that V satisfies all the boundary conditions of (3.8), where χ is a cut-off function defined by

$$\chi(z) = \begin{cases} 1, & z \in (\frac{3\rho}{4}, \rho), \\ 0, & z \in (0, \frac{\rho}{2}). \end{cases}$$

Without loss of generality, χ is supposed to be smooth.

Finally it is sufficient to find a solution R of

$$\begin{cases} LR \equiv dR'' - (c+\lambda_2)R' + G_+'(V_0)R = F(R,z), \quad z \in (0,\rho), \\ R(0) = R(\rho) = 0 \end{cases} \tag{3.10}$$

which satisfies $\|R\|_{C^1} \to 0$ as $d \to 0$. $F(R,z)$ is defined by

$$F(R,z) = -dV_0'' - d\psi'\chi' - d\psi\chi'' - (c+\lambda_2)\psi\chi' - (G_+(V_0+\chi\psi+R) - G_+(V_0) - G_+'(V_0)R).$$

Note that

$$\|F(R_0,z)\|_{C^0} \le Cd, \quad \|F(R_1,z)-F(R_2,z)\|_{C^0} \le \frac{1}{2}\|R_1-R_2\|_{C^1} \tag{3.11}$$

for $\|R_i\|_{C^1} \le \sqrt{d}$ $(i=0,1,2)$. By the similar argument to the proofs of the previous two lemmas and Assumption (A6), we easily find a solution of (3.10), and have the following lemma. So we omit the details of the proof.

Lemma 3.3. *Fix $v_1 \ge 0$ arbitrarily. If $d > 0$ is sufficiently small, then there exists a solution R in (3.10) such that $\|R\|_{C^1} \to 0$ as $d \to 0$.*

In order to obtain a smooth solution of (2.4) in $(-\infty,\infty)$, the functions V_1 and V_2 need to match smoothly at $z=0$. Owing to $V_1(0)=V_2(0)=dv_1$, we only have to check the condition $V_1'(0)=V_2'(0)$ by picking up v_1 suitably. Here we study the behavior of these terms as $d \to 0$. Owing to Lemma 3.1, we see that $\lim_{d\to 0} V_1'(0) = (c_1(0)+\lambda_2)v_1$. On the other hand, $V_2'(0)$ is close to $V_0'(0)$, where V_0 is given as a solution of (3.9). Because of $\lim_{d\to 0} V_2'(0) = G_+(0)/(c_1(0)+\lambda_2)$ and by the implicit function theorem, there exists a continuous function $v_1 = v_1(d)$ such as $v_1(0) = G_+(0)/(c_1(0)+\lambda_2)^2$, and $V_1'(0)=V_2'(0)$ for small $d>0$. Therefore a traveling pulse has been constructed.

§ 4. Stability of a planar traveling pulse

The stability of the traveling pulse given in Theorem 2.2 can be governed by the distribution of eigenvalues in the eigenvalue problem (2.6). To prove Theorem 2.2, we only have to find a positive eigenvalue in (2.6).

The eigenvalue problem includes the k-th eigenvalue of Laplacian $-\Delta_y$ with the Neumann boundary condition. As stated in Section 2, ω_k goes to infinity as $k\to\infty$. Then we let $X = d\omega_k$ and think of X as a small positive parameter independent of d. Actually, we can construct an eigenvalue $\mu=\mu(X)$ for small $X>0$ such that $\mu(0)=0$ and $\mu(X)$ is smooth in $X>0$. This fact can be rigorously proved by a similar technique in [11] and the implicit function theorem. So we omit the details of the proof. Note that (2.6) has a simple zero eigenvalue for $\omega_0 = 0$ because $(r_1, r_2, \psi) = (-1,-1,V')$ is a unique eigenfunction up to multiplications by constants.

Now we calculate $\mu'(0)$ and see that $\mu(X)$ is positive for small $X>0$. As seen in the statement of Theorem 2.2, ε is supposed to be sufficiently small so that we simply put $\varepsilon = 0$ in (2.6). Differentiating the both sides of (2.6) with respect to X, and setting

$X = 0$, we have

$$(4.1)\quad \begin{cases} -\mu' = -c_1'(dv_1)(V'(0)R_1 + \Psi(0)), \\ -\mu' = -c_2'(v_2(dv_1))(V'(\rho)R_2 + \Psi(\rho)), \\ (\mu'+1)V' = d\Psi'' - (c+\lambda_2)\Psi' + G'(V)\Psi, \quad z \in (-\infty,0)\cup(0,\rho)\cup(\rho,\infty), \\ d(\Psi'(+0) - \Psi(-0)) = (G_+(dv_1) - G_-(dv_1))R_1, \\ d(\Psi'(\rho+0) - \Psi(\rho-0)) = -(G_+(v_2(dv_1)) - G_-(v_2(dv_1)))R_2. \end{cases}$$

Here we set $R_1 = dr_1/dX|_{X=0}, R_2 = dr_2/dX|_{X=0}$, and $\Psi = \partial\psi/\partial X|_{X=0}$. In addition, Multiply $V'e^{-\frac{c+\lambda_2}{d}z}$ to the third equation and integrate it over $(-\infty,\infty)$ by parts. Using the first and second equation above, we have

$$\begin{aligned}&(\mu'+1) < V', V'e^{-\frac{c+\lambda_2}{d}z} > \\ &= -(G_+(dv_1) - G_-(dv_1))\frac{\mu'}{c_1'(dv_1)} + e^{-\frac{c+\lambda_2}{d}\rho}(G_+(v_2(dv_1)) - G_-(v_2(dv_1)))\frac{\mu'}{c_2'(v_2(dv_1))},\end{aligned}$$

where $< \cdot, \cdot >$ is the usual inner product in $L^2(-\infty,\infty)$. Then μ' is explicitly expressed by

$$(4.2)\quad \mu' = -\frac{< V', V'e^{-\frac{c+\lambda_2}{d}z} >}{< V', V'e^{-\frac{c+\lambda_2}{d}z} > + \frac{(G_+(dv_1) - G_-(dv_1))}{c_1'(dv_1)} - e^{-\frac{c+\lambda_2}{d}\rho}\frac{G_+(v_2(dv_1)) - G_-(v_2(dv_1))}{c_2'(v_2(dv_1))}}.$$

Since $< V', V'e^{-\frac{c+\lambda_2}{d}z} >$ goes to 0 as $d \to 0$ due to Lemmas 3.1, 3.2 and 3.3, we have

$$\lim_{d\to 0}\frac{\mu'}{< V', V'e^{-\frac{c+\lambda_2}{d}z} >} = -\frac{c_1'(0)}{(G_+(0) - G_-(0))} > 0.$$

Note that v_1 tends to 0 as $d \to 0$. Thanks to $\mu'(0) > 0$, we see that $\mu(X)$ is positive. In the calculations above, we assume $\varepsilon = 0$.

Actually, we also have the same result in $\varepsilon > 0$ by a simple argument. So we conclude Theorem 2.2.

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