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Nonplanar traveling waves of a bistable reaction-diffusion equation in the multi-dimensional space

By

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Abstract

We survey the existence result for nonplanar traveling waves of a bistable reaction-diffusion equation in the space $\mathbb{R}^{n+1}$, which gives a heteroclinic connection of the trivial solution and the unstable standing wave in $\mathbb{R}^n$. Since the problem can be formulated in a monostable case, the equation allows a continuous family of the traveling waves with the speed in a semi-infinite interval.

§1. Introduction

In this article we consider the following scalar reaction-diffusion equation in the multi-dimensional space $\mathbb{R}^{n+1}$:

\begin{equation}
(1.1) \quad u_t = \Delta u + u_{yy} + f(u),
\end{equation}

where

\[
x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \quad y \in \mathbb{R}, \quad u_t = \frac{\partial u}{\partial t}, \quad \Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}.
\]

We assume that $f(u)$ is $C^2$ on an open interval containing $[0,1]$ and that $f(u)$ satisfies

\begin{align}
(1.2) \quad & f(0) = f(a) = f(1) = 0, \quad f'(0) < 0, \quad f'(a) > 0, \quad f'(1) < 0, \\
(1.3) \quad & f(u) \neq 0 \text{ for } u \in (0, a) \cup (a, 1),
\end{align}

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where $0 < a < 1$, and

\begin{equation}
  f''(u) \geq 0 \quad (0 \leq u \leq \beta), \quad f''(u) \leq 0 \quad (\beta \leq u \leq 1)
\end{equation}

for a number $\beta \in (0, 1)$. The condition (1.2) implies that constant solutions $u = 0, 1$ are asymptotically stable equilibria of the diffusion free equation of (1.1) while $u = a$ is an unstable one. The equation (1.1) satisfying (1.2) and (1.3) is called a bistable reaction-diffusion equation. It is known that (1.1) with (1.2)-(1.3) allows a planar traveling wave, that is, there is a solution $u = \Phi(y - ct)$ satisfying

\begin{equation}
  \left\{ \begin{array}{l}
  \Phi_{zz} + c\Phi_z + f(\Phi) = 0, \quad \Phi(z) > 0 \quad (z \in \mathbb{R}), \\
  \lim_{z \to -\infty} \Phi(z) = 1, \quad \lim_{z \to \infty} \Phi(z) = 0.
  \end{array} \right.
\end{equation}

where $z = y - ct$ (for instance, see [7]). Since $\phi'(z) < 0$ holds for the solution $\phi(z)$, it is called a monotone planar traveling wave (solution). Henceforth we assume

\begin{equation}
  \int_0^1 f(u) du > 0,
\end{equation}

which implies that the speed $c$ is positive.

We note that the stationary equation

\begin{equation}
  \Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \quad v(x) > 0, \quad \lim_{|x| \to \infty} v(x) = 0
\end{equation}

allows a one hump solution $v(x)$ for $n = 1$ or for $n \geq 2$ a radially symmetric solution $v(x) = \Phi(r) \ (r = |x|)$

\begin{equation}
  \left\{ \begin{array}{l}
  \Phi_{rr} + \frac{n-1}{r} \Phi_r + f(\Phi) = 0, \quad \Phi(r) > 0 \quad (0 < r < \infty), \\
  \Phi_r(0) = 0, \quad \lim_{r \to \infty} \Phi(r) = 0
  \end{array} \right.
\end{equation}

with the monotone profile $\Phi_r(r) < 0 \ (0 < r < \infty)$. These solutions $v(x)$ are called standing wave solutions and they are unique up to translation. Moreover, those are unstable as equilibrium solutions to

\begin{equation}
  v_t = \Delta v + f(v) \quad (x \in \mathbb{R}^n)
\end{equation}

for $n = 1$ and $n \geq 2$ respectively.

We survey the results for the traveling wave connecting $u = 1$ (at $y = -\infty$) with $u = v(x)$ (at $y = \infty$). We look for the solution with the form $u = U(x, y - ct)$, namely, we consider the equation

\begin{equation}
  \left\{ \begin{array}{l}
  \Delta U + U_{zz} + cU_z + f(U) = 0 \quad ((x, z) \in \mathbb{R}^{n+1}), \\
  \lim_{z \to -\infty} U(x, z) = 1, \quad \lim_{z \to \infty} U(x, z) = v(x).
  \end{array} \right.
\end{equation}
Similarly, we can also consider the solution of
\[
\begin{aligned}
\Delta U + U_{zz} + cU_z + f(U) &= 0 \quad ((x, z) \in \mathbb{R}^{n+1}), \\
\lim_{z \to -\infty} U(x, z) &= 0, \quad \lim_{z \to \infty} U(x, z) = v(x).
\end{aligned}
\]

The same argument can apply to the latter case, so we omit the statement for this case, though the profile of $U(x, z)$ turns to be monotone increasing in $z$ while decreasing in (1.6).

The problem of (1.6) reminds us of the monostable case as in [15] and [1, 2]. Therefore, we can expect that there is a family of traveling waves with the speed $c$ in a semi-infinite interval. We show that this is certainly true in §3.

§ 2. Stationary problem in $\mathbb{R}^n$

Assume (1.2) and (1.3) and consider
\[
(2.1) \quad \Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \quad v(x) > 0, \quad \lim_{|x| \to \infty} v(x) = 0.
\]

For $n = 1$, the equation
\[
(2.2) \quad v_{xx} + f(v) = 0
\]
is converted to a Hamilton system and we can easily obtain a unique solution of (2.1) up to translation by a homoclinic orbit of the Hamiltonian system. Moreover, the linearized eigenvalue problem for such a solution $v(x)$
\[
\varphi_{xx} + f'(v(x)) \varphi = \mu \varphi, \quad \lim_{|x| \to \infty} \varphi(x) = 0
\]
tells that the first eigenvalue is simple and positive while the second one is zero with the corresponding eigenfunction $\varphi = v_x(x)$.

For $n \geq 2$ we further assume (1.4). Given a positive $R$, we first consider the following problem in the ball with the radius $R$:
\[
(2.3) \quad \Delta v + f(v) = 0 \quad (|x| < R), \quad v = 0 \quad (|x| = R)
\]
By the famous result [8] all positive solutions to (2.3) is radially symmetric, so put $v = v(r), r = |x|$, and they satisfy
\[
(2.4) \quad v'' + \frac{n-1}{r} v' + f(v) = 0 \quad (0 < r < R), \quad v'(0) = v(R) = 0.
\]
Then the solution satisfies $v'(r) < 0 \quad (0 < r < R)$. In fact, using the shooting method, we can classify the positive solutions to (2.4). In addition, the local bifurcation theorem applied to this case tells the following result:
Proposition 2.1. ([3], [19]) There is a number $R^* > 0$ such that (2.4) with (1.2)~(1.4) has no positive solution for $R < R^*$, exactly one positive solution for $R = R^*$ and for $R > R^*$ exactly two positive solutions, denoted by $v = v^+_R, v^-_R$, satisfying $0 < v^-_R(r) < v^+_R(r) < 1$.

We note that for the linearized eigenvalue problem of a positive solution $v_R$ to (2.4),

\begin{align*}
\mathcal{L}_R[w] := \Delta w + f'(v_R)w = \mu w \quad (|x| < R), \quad w = 0 \quad (|x| = R),
\end{align*}

the eigenfunction $w$ corresponding to a positive eigenvalue $\mu$ ($> 0$) is radially symmetric. This implies that the Morse index for the solution $v_R$ in (2.3) coincides with that in (2.4). Consequently, we have the next result.

Proposition 2.2. ([3], [19]) Assume $R > R^*$. Then the spectrum of the linearized operator $\mathcal{L}_R$ for $v = v^+_R$ consists of negative eigenvalues while for $v = v^-_R$ it consists of one positive eigenvalue and negative eigenvalues.

Now we consider the problem in the whole space.

\begin{align*}
\Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \quad v(x) > 0, \quad \lim_{|x| \to \infty} v(x) = 0.
\end{align*}

By the argument in [9] we see that a positive solution to (2.6) must be radially symmetric. In fact, we have more properties for the solution to (2.6).

Theorem 2.3. ([3], [19]) Assume (1.2)~(1.4). Then there is a positive solution to (2.6) which is radially symmetric and unique up to translation. For the solution denoted by $\Phi(|x|)$, consider the spectrum $\sigma(\mathcal{L})$ of

\begin{align*}
\mathcal{L} := \Delta + f'(\Phi) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).
\end{align*}

Then the following properties hold:

(i) $\sigma(\mathcal{L}) = \sigma_p(\mathcal{L}) \cup \sigma_e(\mathcal{L})$, where $\sigma_p(\mathcal{L})$ is the point spectrum and $\sigma_e(\mathcal{L})$ is the essential spectrum.

(ii) $\sigma_e(\mathcal{L}) = (-\infty, f'(0)], \quad \sigma_p(\mathcal{L}) \subset (f'(0), \infty)$.

(iii) For $\mu \in \sigma_p(\mathcal{L})$ the corresponding eigenfunction $\psi$ satisfies

\begin{align*}
|\psi(x)| \leq C_\varepsilon e^{-\sqrt{(-f'(0)+\mu+\varepsilon)/2}|x|}
\end{align*}

for any $\varepsilon > 0$ and a constant $C_\varepsilon > 0$ depending on $\varepsilon$. 
(iv) For $\mu \in (0, \infty) \cap \sigma_p(\mathcal{L})$ the corresponding eigenfunction is radially symmetric.

(v) The principal eigenvalue $\lambda_1 > 0$ is simple and the corresponding eigenfunction can be chosen positive.

(vi) The second eigenvalue is $\lambda_2 = 0$ and the eigenspace for $\lambda_2 = 0$ is spanned by

$$\frac{\partial \Phi}{\partial x_i} \quad (i = 1, 2, \cdots, n).$$

We remark that as $R \to \infty$, the solution $v_R^{+}$ to (2.4) converges to $v = 1$ uniformly in any compact set while $v_R^{-}$ converges to the solution $\Phi$ of (1.5).

§ 3. Nonplanar traveling waves

By [6], [10], [11, 12, 13] and [17, 18] the existence and the stability of nonplanar traveling waves, called $V$-shaped waves and conical waves, have been extensively studied. Moreover, the existence of pyramidal shaped fronts is also given in [20, 21]. Here we introduce another type of traveling waves studied in [14] and [16].

For $n = 1$, as mentioned in the previous section, the stationary problem (2.2) is converted to a Hamiltonian system, thus we see that it has a family of periodic solutions $\{v_p(x; T)\}_{T<\infty} (\omega := 2\pi \sqrt{f'(a)})$, where $T$ is the period of the solution, and a homoclinic solution $v_h(x)$. Then $v_p(\cdot; T) \to a$ as $T \to \omega + 0$, while as $T \to \infty$, $v_p(\cdot; T) \to v_h(\cdot)$ in any compact set.

The authors of [14] prove that (1.1) with (1.2) and (1.3) allows a heteroclinic connection between $u = 1$ ($y \to -\infty$) and $u = v_p(\cdot; T)$ ($y \to \infty$). More precisely, for each $T \in (\omega, \infty)$, there is a positive number $c_{min}(T)$ such that the equation allows a traveling wave solution given by solving (1.6) with $v = v_p(\cdot; T)$ if and only if $c \in [c_{min}(T), \infty)$. Moreover, the solution is unique up to translation.

They also prove that $c_{min}(T) \to c^* (T \to \omega + 0)$ for a number $c_{min} > 0$ and that for each $c > c^*$, the solution denoted by $U(x, z; c, T)$, converges to the planer traveling wave of (1.1), say $U^{1a}(z; c)$, connecting $u = 1$ ($z \to -\infty$) with $u = a$ ($z \to \infty$).

On the other hand, there is $c_{min} > 0$ such that $c_{min}(T) \to c_{min} (T \to \infty)$ and for each $c > c_{min}$ the solution $U(x, z; c, T)$ converges to $U(\cdot, \cdot; c)$ of (1.6) with $v = v_h$ as $T \to \infty$ with an appropriate shift.

In sequel they succeed to prove the existence of the traveling wave together with the characterization of the minimum speed for the solution to (1.6).

**Theorem 3.1.** ([14]) Assume (1.2) and (1.3). For $n = 1$ there is a solution $U = U(x, z; c)$ to (1.6) satisfying $U(-x, z; c) = U(x, z; c), U_z(\cdot, z; c) < 0$ if and only if $c \in [c_{min}, \infty)$. This solution is unique up to $z$ translation.
As for the higher dimensional case we have the next result.

**Theorem 3.2.** ([16]) Let $\Phi$ be the positive solution to (1.5) which is unique. Then there is a solution to (1.6) satisfying $U_z(x, z) \leq 0$ if $c \geq 2\sqrt{\kappa}$, where

$$\kappa := \max \left\{ \max_{0 \leq u \leq 1} |f'(u)|, \lambda_1 \right\}$$

($\lambda_1$ is the first eigenvalue of $\mathcal{L}$ defined in (2.7)).

The proof is done by the comparison method, that is, constructing appropriate subsolution and supersolution\(^1\). Unfortunately, this result is not so strong as Theorem 3.1 since the existence of the minimum speed is not discussed. Namely, $c \geq 2\sqrt{\kappa}$ is a technical condition to ensure the existence of the traveling wave. The uniqueness of the solution to (1.6) up to translation is neither discussed.

To apply the similar argument in [14] for $n \geq 2$, it is natural to consider the equation in the cylindrical domain given by

$$\left\{ \begin{array}{l} 
\Delta W + W_{zz} + c W_z + f(w) = 0 \quad ((x, z) \in \{|x| < R\} \times \mathbb{R}), \\
\lim_{z \to -\infty} W(x, z) = v_R^+(x), \quad \lim_{z \to \infty} W(x, z) = v_R^-(x), 
\end{array} \right.$$  

(3.1)

where $v_R^+(r)$ and $v_R^-(r)$ are the stable solution and the unstable solution of (2.4) with (1.2)~(1.4) respectively. As mentioned in the previous section, those solutions are radially symmetric and ordered as $0 < v_R^-(r) < v_R^+(r)$. Moreover,

$$\lim_{R \to \infty} v_R^-(r) = \Phi(r), \quad \lim_{R \to \infty} v_R^+(r) = 1 \quad \text{(in any bounded interval } [0, K])$$

hold. By the results of the previous section and applying the results found in [4] or [22, 23], we obtain the next result.

**Lemma 3.3.** Assume (1.2)~(1.4). Then there is a number $c_m^R > 0$ such that for each $c \in [c_m^R, \infty)$ the equation (3.1) allows a unique solution $W(r, z)$ ($r = |x|$) satisfying $W_z < 0$ up to translation in $z$.

Thus, the solution $u_R(x, y, t) := W(|x|, y - ct)$ gives the traveling wave connecting $u_R = v_R^+(y = -\infty)$ and $u_R = v_R^-(y \to \infty)$ in the cylindrical domain.

In conclusion we need a further discussion to prove the convergence of the minimum speed $c_m^R$ as $R \to \infty$, which is a future work.

\(^1\)The definition of $\kappa$ in [16] should be corrected as in the above theorem.
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