Quantum alpha-determinants and $q$-deformations of hypergeometric polynomials (New developments in group representation theory and non-commutative harmonic analysis)

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数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2012), B36: 97-111

URL: http://hdl.handle.net/2433/198108
Quantum $\alpha$-determinants and $q$-deformations of hypergeometric polynomials

By

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Abstract

The quantum $\alpha$-determinant is a parametric deformation of the ordinary quantum determinant. We study the cyclic $\mathcal{U}_q(sl_2)$-submodules of the quantum matrix algebra $\mathcal{A}_q(Mat_2)$ generated by the powers of the quantum $\alpha$-determinant. The irreducible decomposition of this cyclic module is explicitly described in terms of certain polynomials in the parameter $\alpha$, which is a $q$-deformation of the Gaussian hypergeometric polynomials.


§1. Background

As a parametric interpolation of the determinant and permanent, we define the $\alpha$-determinant of a matrix $X = (x_{ij})_{1 \leq i,j \leq n}$ by

$$\det^{(\alpha)}(X) := \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\nu(\sigma)} x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n},$$

where $\alpha$ is a complex parameter and $\nu(\sigma) = n - (m_1 + m_2 + \cdots + m_n)$ if the cycle type of a permutation $\sigma$ in the symmetric group $\mathfrak{S}_n$ of degree $n$ is $1^{m_1} 2^{m_2} \cdots n^{m_n}$ [13]. In fact, we have $\det(X) = \det^{(-1)}(X)$ and $\text{per}(X) = \det^{(1)}(X)$ by definition.

We are interested in the representation-theoretical properties of the $\alpha$-determinant. Let us set the stage to formulate the problem. Denote by $\mathcal{A}(\text{Mat}_n)$ the $\mathbb{C}$-algebra of polynomials in the $n^2$ commuting variables $\{x_{ij}\}_{1 \leq i,j \leq n}$, and $\mathcal{U}(\mathfrak{g}_l)$ the universal
enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$. We can introduce a $\mathcal{U}(\mathfrak{gl}_n)$-module structure on $\mathcal{A}(\text{Mat}_n)$ by

$$E_{ij} \cdot f := \sum_{r=1}^{n} x_{ir} \frac{\partial f}{\partial x_{jr}} \quad (f \in \mathcal{A}(\text{Mat}_n)),$$

where $\{E_{ij}\}_{1 \leq i,j \leq n}$ is the standard basis of $\mathfrak{gl}_n$. Recall that the cyclic submodules $\mathcal{U}(\mathfrak{gl}_n) \cdot \det(X)$ and $\mathcal{U}(\mathfrak{gl}_n) \cdot \text{per}(X)$ are equivalent to the skew-symmetric tensor product $\wedge^n(\mathbb{C}^n)$ and symmetric tensor product $\text{Sym}^n(\mathbb{C}^n)$ of the natural representation $\mathbb{C}^n$ respectively, which are both irreducible. Thus the cyclic module $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$ can be regarded as an ‘interpolating’ family of $\mathcal{U}(\mathfrak{gl}_n)$-submodules of the two irreducible submodules above, and it is natural and interesting to study the irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$. This is the starting point of our study.

**History of the representation-theoretical studies of $\alpha$-determinants**

Here we briefly explain the history of the studies of $\alpha$-determinants from the viewpoint of representation theory to clarify the position of the matter which we deal with in this note (see also Figure 1 for a graphical summary).
The problem to determine the irreducible decomposition of $\mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X)$ was raised and settled by Matsumoto and Wakayama [5]. They described the irreducible decomposition of $\mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X)$ explicitly as follows:

$$\mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X) \cong \bigoplus_{\lambda \vdash n} (\mathcal{M}_n^\lambda)^{\oplus f^\lambda_{\lambda(\alpha) \neq 0}}.$$  

Here we identify the dominant integral weights and partitions, and denote by $\mathcal{M}_n^\lambda$ the irreducible highest weight $\mathcal{U}(\mathfrak{g}l_n)$-module with highest weight $\lambda$. We also denote by $f_{\lambda}(x)$ the (modified) content polynomial of $\lambda$

$$f_{\lambda}(x) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i}(1+(j-i)x).$$  

At this point, there are at least three direction to proceed with the studies of $\alpha$-determinants.

(A) The result (1.1) (together with (1.2)) by Matsumoto and Wakayama implies that the structure of the cyclic module $\mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X)$ changes drastically when $\alpha = \pm 1/k$ for $k = 1, 2, \ldots, n - 1$. Study the $\alpha$-determinant for such special values.

(B) Study a $q$-analog of the problem above, that is, define a quantum version of the $\alpha$-determinant in the quantum matrix algebra $\mathcal{A}_q(\text{Mat}_n)$ suitably, and consider the cyclic $\mathcal{U}_q(\mathfrak{g}l_n)$-module generated by it.

(C) Study the cyclic $\mathcal{U}(\mathfrak{g}l_n)$-module generated by the powers $\det^{(\alpha)}(X)^m$ of the $\alpha$-determinant in general.

(A) Wreath determinant

When $\alpha = -1/k$ for $k = 1, 2, \ldots, n - 1$, the $\alpha$-determinant has a ‘$-1/k$-analog’ of the alternating property. Precisely, we have

$$\sum_{\sigma \in \mathfrak{S}_n(I)} \det^{(-1/k)}(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) = 0$$

for any $n$ by $n$ matrix $(a_1, a_2, \ldots, a_n)$ and any subset $I \subset \{1, 2, \ldots, n\}$ such that $\#I > k$. Here we put

$$\mathfrak{S}_n(I) = \{\sigma \in \mathfrak{S}_n; \sigma(x) = x, \forall x \in \{1, 2, \ldots, n\} \setminus I\}.$$  

This fact suggests that $\det^{(-1/k)}$ may have ‘determinant-like’ properties. Actually, if we define the $k$-wreath determinant $\text{wrdet}_k(a_1, \ldots, a_n)$ of a $kn$ by $n$ matrix $(a_1, \ldots, a_n)$
by
\[
\text{wrdet}_k(a_1, \ldots, a_n) := \det\left( (-1/k)^k \langle a_1, \ldots, a_1, \ldots, a_1, a_n, \ldots, a_n \rangle \right),
\]
then it satisfies a relative $GL_n$-invariance
\[
\text{wrdet}_k(AP) = \text{wrdet}_k(A) \det(P)^k, \quad P \in GL_n(\mathbb{C}).
\]
We also have wreath-determinant analog for Vandermonde and Cauchy determinants. It would be interesting to seek various special wreath determinant formulas like these. See [3] for more details.

(B) Quantum $\alpha$-determinant

Define the quantum $\alpha$-determinant as an element in the quantum matrix algebra $\mathcal{A}_q(\text{Mat}_n)$ by
\[
\det_q^{(\alpha)} := \sum_{\sigma \in \mathfrak{S}_n} q^{\ell(\sigma)} \alpha^{
u(\sigma)} x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)},
\]
where $\ell(\sigma)$ denotes the inversion number of a permutation $\sigma$. This is nothing but the ordinary quantum determinant $\det_q$ when $\alpha = -1$. Since the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ acts on $\mathcal{A}_q(\text{Mat}_n)$, we can consider the cyclic $\mathcal{U}_q(\mathfrak{gl}_n)$-submodule $\mathcal{U}_q(\mathfrak{gl}_n) \cdot \det_q^{(\alpha)}$. Thus the irreducible decomposition of this cyclic submodule is regarded as a $q$-analog of the first problem studied by Matsumoto and Wakayama. The structure of $\mathcal{U}_q(\mathfrak{gl}_n) \cdot \det_q^{(\alpha)}$ is, however, quite complicated, so that we have only several less explicit results at present. See [4] for more details.

(C) Cyclic modules generated by powers of the $\alpha$-determinant

Recently, Matsumoto, Wakayama and the author investigated the generalized case $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^m$ and proved that
\[
\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^m \cong \bigoplus_{\lambda \vdash mn, \ell(\lambda) \leq n} \mathcal{M}_n^{\lambda} \otimes \mathcal{F}_{n,m}^{\lambda}(\alpha)
\]
holds for certain square matrices $F_{n,m}^{\lambda}(\alpha)$ whose entries are polynomials in $\alpha$. In this direct sum, $\lambda$ runs over the partitions of $mn$ whose length are at most $n$. Remark that the matrices $F_{n,m}^{\lambda}(\alpha)$ are determined up to conjugacy and non-zero scalar factor. In the particular case where $m = 1$, we explicitly have $F_{n,1}^{\lambda}(\alpha) = f_\lambda(\alpha)I$, where $I$ is the identity matrix of size $f_\lambda$ and $f_\lambda(\alpha)$ is given by (1.2). It seems quite difficult to describe $F_{n,m}^{\lambda}(\alpha)$ in an explicit manner in general. However, when $n = 2$, all the matrices $F_{2,m}^{\lambda}(\alpha)$ are one by one, and they are explicitly given by
\[
F_{2,m}^{(m+s,m-s)}(\alpha) = (1 + \alpha)^s \mathcal{F}_1 \left( \begin{array}{c} s - m, s + 1 \\ -m \end{array} ; -\alpha \right) \quad (s = 0, 1, \ldots, m),
\]
where \(2F_1(a, b; c; x)\) is the Gaussian hypergeometric function [2].

**Goal of this note**

The problem we study here is a \(q\)-analog of the study of \(\mathcal{U}(\mathfrak{gl}_2)\cdot \det^{(\alpha)}(X)^m\). Namely, we investigate the cyclic \(\mathcal{U}_q(\mathfrak{sl}_2)\)-submodule (instead of \(\mathcal{U}_q(\mathfrak{gl}_2)\)-submodule just for simplicity of the description) of \(\mathcal{A}_q(\text{Mat}_2)\) defined by

\[V_q^m(\alpha) = \mathcal{U}_q(\mathfrak{sl}_2) \cdot (\det_q^{(\alpha)})^m.\]

We prove that there exists a collection of polynomials \(F_{m,j}(\alpha)\) \((j = 0, 1, \ldots, m)\) such that

\[V_q^m(\alpha) \cong \bigoplus_{0 \leq j \leq m} \mathcal{M}_q(2j + 1),\]

where \(\mathcal{M}_q(d)\) is the \(d\)-dimensional irreducible representation of \(\mathcal{U}_q(\mathfrak{sl}_2)\) given in the next section, and show that the polynomials \(F_{m,j}(\alpha)\) are written in terms of a certain \(q\)-deformation of the hypergeometric polynomials (Theorem 3.3). Taking a limit \(q \to 1\), we also obtain the formula (1.3) again (Corollary 3.6).

**§ 2. Preliminaries**

We first fix the convention on quantum groups (we basically follow to [7] and [11]). Assume that \(q \in \mathbb{C}^\times\) is not a root of unity. The quantum enveloping algebra \(\mathcal{U}_q(\mathfrak{sl}_2)\) is an associative algebra generated by \(k, k^{-1}, e, f\) with the fundamental relations

\[kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2e, \quad kfk^{-1} = q^{-2}f, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.\]

\(\mathcal{U}_q(\mathfrak{sl}_2)\) has a (coassociative) coproduct

\[\Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1},\]

\[\Delta(e) = e \otimes 1 + k \otimes e,\]

\[\Delta(f) = f \otimes k^{-1} + 1 \otimes f,\]

which enables us to define tensor products of \(\mathcal{U}_q(\mathfrak{sl}_2)\)-modules.

The quantum matrix algebra \(\mathcal{A}_q(\text{Mat}_2)\) is an associative algebra generated by \(x_{11}, x_{12}, x_{21}, x_{22}\) with the fundamental relations

\[
x_{11}x_{12} = qx_{12}x_{11}, \quad x_{21}x_{22} = qx_{22}x_{21},
\]

\[
x_{11}x_{21} = qx_{21}x_{11}, \quad x_{12}x_{22} = qx_{22}x_{12},
\]

\[
x_{12}x_{21} = x_{21}x_{12}, \quad x_{11}x_{22} - x_{22}x_{11} = (q - q^{-1})x_{12}x_{21}.
\]
For convenience, we put
\[ z_1 := x_{11} x_{22}, \quad z_2 := x_{12} x_{21}. \]
Notice that they \textit{commute}:
\[ z_1 z_2 = z_2 z_1. \]
The quantum $\alpha$-determinant of size two is then a linear polynomial
\[ \det_q^{(\alpha)} = x_{11} x_{22} + \alpha q x_{12} x_{21} = z_1 + \alpha q z_2 \]
in commuting variables $z_1, z_2$.

\textit{Remark.} The quantum $\alpha$-determinant of size two interpolates the quantum counterparts of the determinant and permanent. In fact, we have
\[ \det_q = x_{11} x_{22} - q x_{12} x_{21} = \det_q^{(-1)}, \quad \mathrm{per}_q = x_{11} x_{22} + q^{-1} x_{12} x_{21} = \det_q^{(q^{-2})}. \]
Here, in general, we define the quantum permanent of size $n$ by
\[ \mathrm{per}_q := \sum_{\sigma \in \mathfrak{S}_n} q^{-\ell(\sigma)} x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n} \in \mathcal{A}_q(\mathrm{Mat}_n), \]
which can be regarded as a $q$-analog of the usual permanent in the sense that the cyclic module $\mathcal{U}_q(\mathfrak{gl}_n) \cdot \mathrm{per}_q$ is equivalent to the $q$-analog of $n$-th symmetric tensor product of the natural representation $\mathbb{C}^n$ of $\mathcal{U}_q(\mathfrak{gl}_n)$. However, the quantum $\alpha$-determinant of size $n$ does not coincide with the quantum permanent for any $\alpha$ if $n \geq 3$. This is because $\nu(\cdot)$ is a class function on $\mathfrak{S}_n$ in general, whereas the inversion number $\ell(\cdot)$ is \textit{not} if $n \geq 3$.

We briefly recall necessary basic facts on representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$. Let \{\(e_1, e_2\)\} be the standard basis of the vector space $\mathbb{C}^2$. By defining
\[ k^{\pm 1} \cdot e_1 := q^{\pm 1} e_1, \quad e \cdot e_1 := 0, \quad f \cdot e_1 := e_2, \]
\[ k^{\pm 1} \cdot e_2 := q^{\mp 1} e_2, \quad e \cdot e_2 := e_1, \quad f \cdot e_2 := 0, \]
$\mathbb{C}^2$ becomes a $\mathcal{U}_q(\mathfrak{sl}_2)$-module. Put
\[ \mathcal{M}_q(l+1) := \mathcal{U}_q(\mathfrak{sl}_2) \cdot \mathbf{v}_0^{(l)} \subset (\mathbb{C}^2)^{\otimes l}, \quad \mathbf{v}_0^{(l)} := e_1 \otimes e_1 \otimes \cdots \otimes e_1 \]
for $l = 1, 2, \ldots$. We also put $\mathcal{M}_q(1) := \mathbb{C} \cdot \mathbf{v}_0^{(0)}$ (with $\mathbf{v}_0^{(0)} := 1 \in \mathbb{C}$) and define $e \cdot \mathbf{v}_0^{(0)} = f \cdot \mathbf{v}_0^{(0)} = 0$ and $k^{\pm 1} \cdot \mathbf{v}_0^{(0)} := \mathbf{v}_0^{(0)}$. If we set $\mathbf{v}_r^{(l)} := f^r \cdot \mathbf{v}_0^{(l)}$ ($r = 0, 1, \ldots, l$) and $\mathbf{v}_{-1}^{(l)} = \mathbf{v}_{l+1}^{(l)} = 0$, then it follows that $\mathcal{M}_q(l+1) = \bigoplus_{r=0}^{l} \mathbb{C} \cdot \mathbf{v}_r^{(l)}$ and
\[ k^{\pm 1} \cdot \mathbf{v}_r^{(l)} = q^{\pm(l-2r)} \mathbf{v}_r^{(l)}, \]
\[ e \cdot \mathbf{v}_r^{(l)} = [l-r+1]_q \mathbf{v}_{r-1}^{(l)}, \]
\[ f \cdot \mathbf{v}_r^{(l)} = [r+1]_q \mathbf{v}_{r+1}^{(l)} \]
for $r = 0, 1, \ldots, l$. This is an irreducible $(l+1)$-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module.
Remark. There exists a convolution \( \theta \) of \( \mathcal{U}_q(\mathfrak{s}l_2) \) such that
\[
\theta(e) = -e, \quad \theta(f) = f, \quad \theta(k^{\pm 1}) = k^{\mp 1}.
\]
Using this, we can introduce inequivalent \( \mathcal{U}_q(\mathfrak{s}l_2) \)-module structure on the vector space \( \mathcal{M}_q(l + 1) \), say \( \mathcal{M}_q(l + 1)^{\theta} \). However, such modules do not appear in the following discussion. Note that any finite dimensional irreducible \( \mathcal{U}_q(\mathfrak{s}l_2) \)-module is isomorphic to either \( \mathcal{M}_q(l + 1) \) or \( \mathcal{M}_q(l + 1)^{\theta} \) for some \( l = 0, 1, 2, \ldots \) (see, e.g. [7, 9]).

The algebra \( \mathcal{A}_q(\text{Mat}_2) \) becomes a \( \mathcal{U}_q(\mathfrak{s}l_2) \)-module by
\[
k^{\pm 1} \cdot x_{i1} := q^{\pm 1} x_{i1}, \quad e \cdot x_{i1} := 0, \quad f \cdot x_{i1} := x_{i2},
\]
\[
k^{\pm 1} \cdot x_{i2} := q^{\mp 1} x_{i2}, \quad e \cdot x_{i2} := x_{i1}, \quad f \cdot x_{i2} := 0 \quad (i = 1, 2).
\]
These are compatible with the fundamental relations (2.1) above. Notice that \( e \cdot \det_q^{r} = f \cdot \det_q^{r} = 0 \) and \( k^{\pm 1} \cdot \det_q^{r} = \det_q^{r} \), (i.e. \( \mathcal{U}_q(\mathfrak{s}l_2) \cdot \det_q^{r} \cong \mathcal{M}_q(1) \)). It then follows that \( X \cdot (v \det_q^{r}) = (X \cdot v) \det_q^{r} \) for \( X \in \mathcal{U}_q(\mathfrak{s}l_2) \) and \( v \in \mathcal{A}_q(\text{Mat}_2) \). We have
\[
(2.2) \quad \mathcal{U}_q(\mathfrak{s}l_2) \cdot (x_{11}x_{21})^s \det_q^{m-s} \cong \mathcal{U}_q(\mathfrak{s}l_2) \cdot (x_{11}x_{21})^s \cong \mathcal{M}_q(2s + 1) \quad (s = 0, 1, 2, \ldots).
\]
Actually, the linear map defined by \( v^{(l)} \mapsto f^r \cdot (x_{11}x_{21})^s \) gives a bijective intertwiner between \( \mathcal{M}_q(2s + 1) \) and \( \mathcal{U}_q(\mathfrak{s}l_2) \cdot (x_{11}x_{21})^s \).

Define \( q \)-analogs of numbers, factorials and binomial coefficients by
\[
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := \prod_{i=1}^{n} [i]_q, \quad \left[ \begin{array}{l} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.
\]

§ 3. Cyclic modules generated by the quantum alpha-determinant

We fix a positive integer \( m \) and discuss the irreducible decomposition of the cyclic module \( V_q^m(\alpha) := \mathcal{U}_q(\mathfrak{s}l_2) \cdot (\det_q^{(\alpha)})^m \). We refer to [1] for detailed discussion.

Put
\[
v_{m,j} := (f^j \cdot (x_{11}x_{21})^j) \det_q^{m-j} \quad (j = 0, 1, \ldots, m).
\]
It is easy to see that the cyclic module \( \mathcal{U}_q(\mathfrak{s}l_2) \cdot v_{m,j} \) is equivalent to \( \mathcal{M}_q(2j + 1) \). We show that \( v_{m,j} \) is a homogeneous polynomial in \( z_1 \) and \( z_2 \) of degree \( m \) and give an explicit expression of it. For this purpose, we need the following two lemmas.

Lemma 3.1. For each positive integer \( j \), it follows that
\[
f^j \cdot (x_{11}x_{21})^j = q^{-j(j-1)/2} [j]_q! \sum_{r=0}^{j} q^{-r^2} \left[ \begin{array}{l} j \\ r \end{array} \right]_q \sum_{r=0}^{j} q^{-r^2} \left[ \begin{array}{l} j \\ r \end{array} \right]_q x_{11}^{j-r} x_{22}^{j-r} (x_{12}x_{21})^r.
\]
Sketch of proof. For $1 \leq i \leq 2j$, put

$$f_j(i) := \underbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes f \otimes \underbrace{k^{-1} \otimes \cdots \otimes k^{-1}}^{2j-i} \in \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2j}.$$ 

Then it follows that

$$\Delta^{2j-1}(f)^j = \sum_{1 \leq n_1, \ldots, n_j \leq 2j} f_j(n_1) \cdots f_j(n_j).$$

Since $f_j(m)f_j(n) = q^{-2}f_j(n)f_j(m)$ if $m > n$ and $f^2 \cdot x_{11} = f^2 \cdot x_{21} = 0$, we have

$$\Delta^{2j-1}(f)^j = q^{-j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_j \leq 2j} f_j(n_1) \cdots f_j(n_j) + R,$n

where $R$ is a certain element in $\mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2j}$ such that $R \cdot (x_{11}^j x_{21}^j) = 0$. Here we also use the well-known identity

$$\sum_{\sigma \in \mathfrak{S}_j} x^{\ell(\sigma)} = (1+x)(1+x+x^2) \cdots (1+x+\cdots+x^{j-1})$$

with $x = q^{-2}$. For given $n_1, \ldots, n_j$ ($1 \leq n_1 < \cdots < n_j \leq 2j$), we get

$$f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{21}^j) = q^{-r^2+j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_j \leq 2j} f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{21}^j)$$

by a careful calculation (see [1] for detail). Using this, we have

$$f^j \cdot (x_{11}^j x_{22}^j) = q^{-j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_j \leq 2j} f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{21}^j)$$

$$= [j]_q! \sum_{r=0}^{j} q^{-r^2} \sum_{1 \leq n_1 < \cdots < n_r \leq j} \sum_{1 \leq m_1 < \cdots < m_r \leq j} q^{-2(n_1 + \cdots + n_r)} q^{-2(m_1 + \cdots + m_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r$$

$$= [j]_q! \sum_{r=0}^{j} q^{-r^2} e_r(1, q^2, \ldots, q^{2j-2}) e_r(1, q^{-2}, \ldots, q^{-2j+2}) x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r,$n

where $e_r(x_1, x_2, \ldots, x_j)$ is the rth elementary symmetric polynomial in $x_1, x_2, \ldots, x_j$. By the identity (see, e.g. [10])

$$e_r(1, q^2, \ldots, q^{2j-2}) = q^{r(j-1)/2} [j]_q$$

together with the symmetry $[j]_q = [j]_q^{-1}$, we obtain

$$f^j \cdot (x_{11}^j x_{21}^j) = [j]_q! \sum_{r=0}^{j} q^{-r^2} [j]_q^2 \sum_{r=0}^{j} q^{j-r} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r.$$

Since $(x_{11} x_{21})^j = q^{-j(j-1)/2} x_{11}^j x_{21}^j$, we have the desired conclusion.
Lemma 3.2. For each positive integer \( l \), it follows that
\[
x_{11}^l x_{22}^l = \prod_{s=1}^{l} (z_1 + (q^{2s-1} - q)z_2) = \sum_{j=0}^{l} q^{l(l-j)} \left[ \begin{array}{c} l \\ j \end{array} \right]_q (z_1 - qz_2)^j z_2^{l-j}.
\]

Proof. The first equality is proved by induction on \( l \) by using the relation
\[
(z_1 + (q^{2r-1} - q)z_2)x_{22} = x_{22}(z_1 + (q^{2r+1} - q)z_2).
\]
The second equality is a specialization of the \( q \)-binomial theorem
\[
\prod_{i=1}^{n} (x + q^{2i-n-1}y) = \sum_{r=0}^{n} \left[ \begin{array}{c} n \\ r \end{array} \right]_q x^r y^{n-r},
\]
which is indeed applicable since \( z_1 \) and \( z_2 \) commute.
\[\square\]

As a result, we get the explicit expression
\[
(3.1) \quad v_{m,j} = \left\{ q^{-j(j-1)/2} [j]_q! \sum_{r=0}^{j} q^{-r^2} \left[ \begin{array}{c} j \\ r \end{array} \right]_q z_2^r \prod_{s=1}^{j-r} (z_1 + (q^{2s-1} - q)z_2) \right\} (z_1 - qz_2)^{m-j},
\]
which is indeed a homogeneous polynomial in \( z_1 \) and \( z_2 \) of degree \( m \).

The vectors \( \{v_{m,j}\}_{j=0}^{m} \) are linearly independent since they belong to inequivalent representations, and hence form a basis of the space consisting of homogeneous polynomials in \( z_1 \) and \( z_2 \) of degree \( m \), whose dimension is \( m + 1 \). Thus we conclude that
\[
(\det_q^{(\alpha)})^m (\in \mathcal{A}_q(\text{Mat}_2) \otimes_{\mathbb{C}} \mathbb{C}[\alpha])) \text{ can be expressed as a linear combination}
\]
\[
(3.2) \quad (\det_q^{(\alpha)})^m = \sum_{j=0}^{m} F_{m,j}(\alpha) v_{m,j}
\]
of the vectors \( v_{m,j} \), where \( F_{m,j}(\alpha) \) are certain polynomial functions in \( \alpha \). This implies that the cyclic module \( V_q^m(\alpha) \) contains an irreducible submodule equivalent to \( \mathcal{M}_q(2j+1) \) (with multiplicity one) if \( F_{m,j}(\alpha) \neq 0 \). Consequently, it follows that
\[
V_q^m(\alpha) \cong \bigoplus_{0 \leq j \leq m} \mathcal{M}_q(2j+1).
\]

Let us determine the functions \( F_{m,j}(\alpha) \) explicitly. The conditions (3.2) for the functions \( F_{m,j}(\alpha) \) are given in terms of polynomials in commuting variables \( z_1, z_2 \), so that it is meaningful to consider the specialization \( z_1 = z, z_2 = 1 \) in (3.2), where \( z \) is a
new variable. Put

$$g_j(z) := \prod_{i=1}^{j} (z + q^{2i-1} - q) = \sum_{i=0}^{j} q^{i(j-i)} \left[ \begin{array}{c} j \\ i \end{array} \right]_q (z - q)^i,$$

$$v_j(z) := q^{-\frac{j(j-1)}{2}} [j]_q! \sum_{r=0}^{j} q^{-r^2} \left[ \begin{array}{c} j \\ r \end{array} \right]_q^2 g_{j-r}(z).$$

Then (3.2) together with Lemmas 3.1 and 3.2 yields

$$(z + qa)^m = \sum_{j=0}^{m} F_{m,j}(\alpha) v_j(z) (z - q)^{m-j}. $$

If we take the $l$th derivative of this formula with respect to $z$ ($l = 0, 1, \ldots, m$) and substitute $z = q$, then we get the relation

$$(3.4) \quad \binom{m}{l} q^{m-l}(1 + \alpha)^{m-l} = \sum_{j=m-l}^{m} F_{m,j}(\alpha) \frac{v_j^{(l-m+j)}(q)}{(l-m+j)!} = \sum_{s=0}^{l} F_{m,m-s}(\alpha) \frac{v_{m-s}^{(l-s)}(q)}{(l-s)!}. $$

Since

$$v_j(z) = q^{i(j+1)/2} [j]_q! \sum_{i=0}^{j} q^{-i} \left[ \begin{array}{c} j \\ i \end{array} \right]_q \sum_{r=0}^{j} q^{r(i-2j)} \left[ \begin{array}{c} j \\ r \end{array} \right]_q \left[ \begin{array}{c} j - i \\ r \end{array} \right]_q (z - q)^i,$$

we have

$$v_j(z) = q^{-\frac{j(j-1)}{2}} [j]_q! \sum_{i=0}^{j} \frac{[2j-i]_q!}{[j-i]_q!^{2}} (z - q)^i,$$

or

$$\frac{v_j^{(i)}(q)}{i!} = q^{-\frac{i(i+1)}{2}} \frac{[2j-i]_q!}{[j-i]_q!^{2}}.$$

Here we use the $q$-Chu-Vandermonde formula

$$\sum_{r=0}^{j-i} q^{r(i-2j)} \left[ \begin{array}{c} j \\ j - r \end{array} \right]_q \left[ \begin{array}{c} j - i \\ r \end{array} \right]_q = q^{j(i-j)} \left[ \begin{array}{c} 2j - i \\ j \end{array} \right]_q.$$

Thus the formula (3.4) is rewritten more explicitly as

$$(3.5) \quad [m-l]_q!^2 \binom{m}{l} q^{m-l}(1 + \alpha)^{m-l} = \sum_{s=0}^{l} q^{-\frac{m-s}{2}} \frac{[m-s]_q!}{[l-s]_q!} [2m-l-s]_q! F_{m,m-s}(\alpha).$$
This also implies that the polynomial $F_{m,j}(\alpha)$ is divisible by $(1+\alpha)^j$, that is

\begin{equation}
F_{m,j}(\alpha) = (1+\alpha)^j Q_{m,j}(\alpha) \tag{3.6}
\end{equation}

for some $Q_{m,j}(\alpha) \in \mathbb{C}[\alpha]$. By (3.6) and (3.5), we have

\begin{equation}
\begin{align*}
\binom{2m-2i}{m-i}^{-1} q^{m-i} & \\
\sum_{j=0}^{i} & \left[\begin{array}{ll}
2i-2m-1 \\
\quad i-j
\end{array}\right]_{q} (-1)^{i-j} (1+\alpha)^{i-j} \cdot q^{-\left[\begin{array}{l}
\quad m-j
\end{array}\right]_q} [m-j]_q! Q_{m,m-j}(\alpha).
\end{align*}
\tag{3.7}
\end{equation}

Now we define the mixed hypergeometric series by

\begin{equation}
\Phi\left(^{a_1, \ldots, a_k}_{b_1, \ldots, b_l}^{c_1, \ldots, c_m} ; d_1, \ldots, d_m ; q ; x\right) = \sum_{i=0}^{\infty} \frac{(a_1; i) \cdots (a_k; i) (c_1; i) \cdots (c_m; i) x^i}{(b_1; i) \cdots (b_l; i) (d_1; i) \cdots (d_m; i) [i]_q!},
\end{equation}

where $(a; i) = a(a+1) \cdots (a+i-1)$ and $(a; i)_q = [a]_q [a+1]_q \cdots [a+i-1]_q$ (cf. [8]).

**Theorem 3.3.** For $s = 0, 1, \ldots, m$,

\begin{equation}
F_{m,s}(\alpha) = q^{\left(\begin{array}{l}
s+1 \\
\quad 2
\end{array}\right)} m! [2m-2i+1]_q
\end{equation}

holds.

**Sketch of proof.** We can prove the identity

\begin{equation}
\begin{align*}
\left(\begin{array}{l}
2i-2m-1 \\
\quad i-j
\end{array}\right)_{q}^{-1} & \\
\sum_{j=0}^{i} & \left(\begin{array}{l}
2m-2i+1 \\
\quad 2m-2j+1
\end{array}\right)_{q}^{-1} \left(\begin{array}{l}
m-j \\
\quad i-j
\end{array}\right)_{q}^{-1} (1+\alpha)^{i-j}.
\end{align*}
\end{equation}

Using this, we solve (3.7) and find that

\begin{equation}
Q_{m,m-i}(\alpha) = \frac{q\left[\begin{array}{l}
m-i \\
\quad m-i
\end{array}\right]_q
\end{equation}

\begin{equation}
\times \sum_{j=0}^{i} \left(\begin{array}{l}
2m-2i+1 \\
\quad m-j
\end{array}\right)_{q}^{-1} \left(\begin{array}{l}
m \\
\quad j
\end{array}\right) (1+\alpha)^{i-j}
\end{equation}

\begin{equation}
= q^{\left(\begin{array}{l}
m-i+1 \\
\quad m-i
\end{array}\right)} m! [2m-2i+1]_q
\end{equation}

\begin{equation}
\times \sum_{r=0}^{i} \frac{(-q)^r [m-i+r]_q!^2}{(m-i+r)! (i-r)! [2m-2i+r+1]_q! [r]_q!} (1+\alpha)^r.
\end{equation}
where we set $r = i - j$. Since

$$(i - r)! = (-1)^r \frac{i!}{(-i; r)}, \quad (n + r)! = n! (n + 1; r), \quad [n + r]_q! = [n]_q! (n + 1; r)_q,$$

we have

$$Q_{m,m-i}(\alpha) = \frac{q \binom{n-i+1}{2} m! [m-i]_q!}{i! (m-i)! [2m-2i]_q!} \sum_{r=0}^{i} \frac{(-i;r)(m-i+1;r)_q^2}{(m-i+1;r)(2m-2i+2;r)_q} \frac{(q(1+\alpha))^r}{[r]_q!} \Phi(i;m-i+1,m-i+1;2m-2i+2;q;q(1+\alpha)).$$

If we substitute this into (3.6) and replace $m - i$ by $s$, then we have the conclusion. \(\square\)

**Example 3.4** \((m = 1)\). We have

$$F_{1,0}(\alpha) = \Phi\left(\begin{array}{l}
-1 \\
1 \\
2
\end{array} ; q; q(1+\alpha)\right) = \frac{1 - \alpha q^2}{1 + q^2},$$

$$F_{1,1}(\alpha) = q \Phi\left(\begin{array}{l}
1 \\
2
\end{array} ; q; q(1+\alpha)\right) = \frac{q}{[2]_q} (1 + \alpha).$$

Thus it follows that

$$\mathcal{U}_q(\mathfrak{sl}_2) \cdot \det_q(\alpha) \cong \begin{cases} 
\mathcal{M}_q(3) & \alpha = q^{-2}, \\
\mathcal{M}_q(1) & \alpha = -1, \\
\mathcal{M}_q(1) \oplus \mathcal{M}_q(3) & \text{otherwise.}
\end{cases}$$

Notice that $q^{-2} \neq -1$ since we assume that $q$ is not a root of unity.

**Example 3.5** \((m = 2)\). We have

$$F_{2,0}(\alpha) = \Phi\left(\begin{array}{l}
-2 \\
1 \\
2
\end{array} ; q; q(1+\alpha)\right) = C_0(q) \left((q^6 + q^4)\alpha^2 - 2q^2\alpha + q^4 + 1\right),$$

$$F_{2,1}(\alpha) = 2q^2 \frac{[1]_q!}{[2]_q!} (1 + \alpha) \Phi\left(\begin{array}{l}
-1 \\
2 \\
4
\end{array} ; q; q(1+\alpha)\right) = C_1(q)(1 + \alpha) \left((q^4 + q^2)\alpha - q^4 + q^2 - 2\right),$$

$$F_{2,2}(\alpha) = q^3 \frac{[2]_q!}{[4]_q!} (1 + \alpha)^2 \Phi\left(\begin{array}{l}
0 \\
3 \\
6
\end{array} ; q; q(1+\alpha)\right) = C_2(q)(1 + \alpha)^2,$$
where $C_0(q), C_1(q), C_2(q)$ are certain rational functions in $q$. Hence, if we assume that $q$ is transcendental (we have only to assume that $(q^4 - q^2 + 1)^2 + q^2(q^4 + q^2 + 1) \neq 0$ practically), then we see that

$$
\mathcal{U}_q(\mathfrak{sl}_2) \cdot \left( \det_{q}^{(\alpha)} \right)^2 \cong \begin{cases} 
\mathcal{M}_q(1) & \alpha = -1, \\
\mathcal{M}_q(1) \oplus \mathcal{M}_q(5) & \alpha = \frac{q^4 - q^2 + 2}{q^4 + q^2}, \\
\mathcal{M}_q(3) \oplus \mathcal{M}_q(5) & \alpha = \frac{1 \pm \sqrt{-1}}{q^4 + q^2}, \\
\mathcal{M}_q(1) \oplus \mathcal{M}_q(3) \oplus \mathcal{M}_q(5) & \text{otherwise}.
\end{cases}
$$

When $(q^4 - q^2 + 1)^2 + q^2(q^4 + q^2 + 1) = 0$ (this does not imply that $q$ is a root of unity), $\frac{q^4 - q^2 + 2}{q^4 + q^2}$ becomes a common root of $F_{2,0}(\alpha)$ and $F_{2,1}(\alpha)$, so that we have

$$
\mathcal{U}_q(\mathfrak{sl}_2) \cdot \left( \det_{q}^{(\alpha)} \right)^2 \cong \mathcal{M}_q(5), \quad \alpha = \frac{q^4 - q^2 + 2}{q^4 + q^2}.
$$

**Remark.** The mixed hypergeometric series (3.8) can be regarded as a common generalization of the generalized hypergeometric series and basic hypergeometric series as we see below:

$$_k\Phi_1 \left( \begin{array}{c}
a_1, \ldots, a_k \\
b_1, \ldots, b_l, 1
\end{array} ; q, x \right) = {}_kF_1 \left( \begin{array}{c}
a_1, \ldots, a_k \\
b_1, \ldots, b_l
\end{array} ; q, x \right),
$$

$$_m\phi_n \left( \begin{array}{c}
c_1, \ldots, c_m \\
d_1, \ldots, d_n
\end{array} ; q, x \right) = {}_m\phi_n \left( \begin{array}{c}
c_1, \ldots, c_m \\
d_1, \ldots, d_n
\end{array} ; q^2, (-1)^{1+n-m}q^{1+d-c}x \right),
$$

where $c = c_1 + \ldots + c_m$ and $d = d_1 + \ldots + d_n$.

**Remark.** The function $\Phi$ given by (3.8) satisfies the difference-differential equation

$$
\left\{ -(E + a_1) \cdots (E + a_k) [E + c_1]_q \cdots [E + c_m]_q \\
+ \partial_q(E + b_1 - 1) \cdots (E + b_l - 1) [E + d_1 - 1]_q \cdots [E + d_n - 1]_q \right\} \Phi = 0,
$$

where we put

$$
E = x \frac{d}{dx}, \quad [E + a]_q = \frac{q^{E+a} - q^{-E-a}}{q - q^{-1}}, \quad \partial_q f(x) = \frac{f(qx) - f(q^{-1}x)}{qx - q^{-1}x}.
$$

If we take a limit $q \to 1$, then the equation above becomes a hypergeometric differential equation for $F_{l+n}(a_1, \ldots, a_k, c_1, \ldots, c_m; b_1, \ldots, b_l, d_1, \ldots, d_n; x)$.

All the discussion above also work in the classical case (i.e. the case where $q = 1$), thus, by taking a limit $q \to 1$ in Theorem 3.3, we will obtain Theorem 4.1 in [2] (or
(1.3) up to constant) again. We abuse the same notations used in the discussion of quantum case above to indicate the classical counterparts. From (3.9), we have

\[ Q_{m,s}(\alpha) = \frac{m!}{(m-s)!(2s)!} {}_{3}F_{2}\left(\begin{array}{c}s-m, s+1, s+1 \\ s+1, 2s+2\end{array}; 1+\alpha\right) \]
\[ = \frac{m!}{(m-s)!(2s)!} {}_{2}F_{1}\left(\begin{array}{c}s-m, s+1 \\ 2s+2\end{array}; 1+\alpha\right). \]

Notice that

\[ {}_{2}F_{1}\left(\begin{array}{c}s-m, s+1 \\ 2s+2\end{array}; 1-x\right) = \frac{m!(2s+1)!}{s!(m+s+1)!} {}_{2}F_{1}\left(\begin{array}{c}s-m, s+1 \\ -m\end{array}; x\right). \]

Thus we also get

\[ Q_{m,s}(\alpha) = \frac{m!^{2}(2s+1)}{(m-s)!s!(m+s+1)!} {}_{2}F_{1}\left(\begin{array}{c}s-m, s+1 \\ -m\end{array}; -\alpha\right) \quad (s=0,1,\ldots,m). \]

Summarizing these, we have the

**Corollary 3.6** (Classical case). It follows that

\[ F_{m,s}(\alpha) = \frac{m!}{(m-s)!(2s)!} (1+\alpha)^{s} {}_{2}F_{1}\left(\begin{array}{c}s-m, s+1 \\ 2s+2\end{array}; 1+\alpha\right) \]
\[ = \frac{(2m)!}{m!} \frac{2^{m}}{s!} (1+\alpha)^{s} {}_{2}F_{1}\left(\begin{array}{c}s-m, s+1 \\ -m\end{array}; -\alpha\right) \]

for \( s = 0, 1, \ldots, m \).

**References**


