Quantum alpha-determinants and $q$-deformations of hypergeometric polynomials (New developments in group representation theory and non-commutative harmonic analysis)

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Quantum $\alpha$-determinants and $q$-deformations of hypergeometric polynomials

By

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Abstract

The quantum $\alpha$-determinant is a parametric deformation of the ordinary quantum determinant. We study the cyclic $\mathcal{U}_q(\mathfrak{sl}_2)$-submodules of the quantum matrix algebra $\mathcal{A}_q(\text{Mat}_2)$ generated by the powers of the quantum $\alpha$-determinant. The irreducible decomposition of this cyclic module is explicitly described in terms of certain polynomials in the parameter $\alpha$, which is a $q$-deformation of the Gaussian hypergeometric polynomials.

This note is a summary of the author’s recent paper [Kimoto, K., “Quantum alpha-determinants and $q$-deformed hypergeometric polynomials,” Int. Math. Res. Not.].

§ 1. Background

As a parametric interpolation of the determinant and permanent, we define the $\alpha$-determinant of a matrix $X = (x_{ij})_{1 \leq i, j \leq n}$ by

$$\det^{(\alpha)}(X) := \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\nu(\sigma)} x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n},$$

where $\alpha$ is a complex parameter and $\nu(\sigma) = n - (m_1 + m_2 + \cdots + m_n)$ if the cycle type of a permutation $\sigma$ in the symmetric group $\mathfrak{S}_n$ of degree $n$ is $1^{m_1}2^{m_2} \cdots n^{m_n}$ [13]. In fact, we have $\det(X) = \det^{(-1)}(X)$ and $\text{per}(X) = \det^{(1)}(X)$ by definition.

We are interested in the representation-theoretical properties of the $\alpha$-determinant. Let us set the stage to formulate the problem. Denote by $\mathcal{A}(\text{Mat}_n)$ the $\mathbb{C}$-algebra of polynomials in the $n^2$ commuting variables $\{x_{ij}\}_{1 \leq i,j \leq n}$, and $\mathcal{U}(\mathfrak{gl}_n)$ the universal
enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$. We can introduce a $\mathcal{U}(\mathfrak{gl}_n)$-module structure on $\mathcal{A}(\text{Mat}_n)$ by

$$E_{ij} \cdot f := \sum_{r=1}^{n} x_{ir} \frac{\partial f}{\partial x_{jr}} \quad (f \in \mathcal{A}(\text{Mat}_n)),$$

where $\{E_{ij}\}_{1 \leq i, j \leq n}$ is the standard basis of $\mathfrak{gl}_n$. Recall that the cyclic submodules $\mathcal{U}(\mathfrak{gl}_n) \cdot \det(X)$ and $\mathcal{U}(\mathfrak{gl}_n) \cdot \text{per}(X)$ are equivalent to the skew-symmetric tensor product $\wedge^n(\mathbb{C}^n)$ and symmetric tensor product $\text{Sym}^n(\mathbb{C}^n)$ of the natural representation $\mathbb{C}^n$ respectively, which are both irreducible. Thus the cyclic module $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$ can be regarded as an ‘interpolating’ family of $\mathcal{U}(\mathfrak{gl}_n)$-submodules of the two irreducible submodules above, and it is natural and interesting to study the irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$. This is the starting point of our study.

**History of the representation-theoretical studies of $\alpha$-determinants**

Here we briefly explain the history of the studies of $\alpha$-determinants from the viewpoint of representation theory to clarify the position of the matter which we deal with in this note (see also Figure 1 for a graphical summary).
The problem to determine the irreducible decomposition of $\mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X)$ was raised and settled by Matsumoto and Wakayama [5]. They described the irreducible decomposition of $\mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X)$ explicitly as follows:

\begin{equation}
\mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X) \cong \bigoplus_{f_\lambda(\alpha) \neq 0} (\mathcal{M}_n^\lambda)^{\oplus f_\lambda(\alpha)}.
\end{equation}

Here we identify the dominant integral weights and partitions, and denote by $\mathcal{M}_n^\lambda$ the irreducible highest weight $\mathcal{U}(\mathfrak{g}l_n)$-module with highest weight $\lambda$. We also denote by $f_\lambda(x)$ the (modified) content polynomial of $\lambda$

\begin{equation}
f_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (1 + (j - i)x).
\end{equation}

At this point, there are at least three direction to proceed with the studies of $\alpha$-determinants.

(A) The result (1.1) (together with (1.2)) by Matsumoto and Wakayama implies that the structure of the cyclic module $\mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X)$ changes drastically when $\alpha = \pm 1/k$ for $k = 1, 2, \ldots, n - 1$. Study the $\alpha$-determinant for such special values.

(B) Study a $q$-analog of the problem above, that is, define a quantum version of the $\alpha$-determinant in the quantum matrix algebra $\mathcal{A}_q(\text{Mat}_n)$ suitably, and consider the cyclic $\mathcal{U}_q(\mathfrak{g}l_n)$-module generated by it.

(C) Study the cyclic $\mathcal{U}(\mathfrak{g}l_n)$-module generated by the powers $\det^{(\alpha)}(X)^m$ of the $\alpha$-determinant in general.

(A) Wreath determinant

When $\alpha = -1/k$ for $k = 1, 2, \ldots, n - 1$, the $\alpha$-determinant has a ‘$-1/k$-analog’ of the alternating property. Precisely, we have

$$
\sum_{\sigma \in \mathfrak{S}_n(I)} \det^{(-1/k)}(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) = 0
$$

for any $n$ by $n$ matrix $(a_1, a_2, \ldots, a_n)$ and any subset $I \subset \{1, 2, \ldots, n\}$ such that $\#I > k$. Here we put

$$
\mathfrak{S}_n(I) = \{ \sigma \in \mathfrak{S}_n; \sigma(x) = x, \forall x \in \{1, 2, \ldots, n\} \setminus I \}.
$$

This fact suggests that $\det^{(-1/k)}$ may have ‘determinant-like’ properties. Actually, if we define the $k$-wreath determinant $\text{wrdet}_k(a_1, \ldots, a_n)$ of a $kn$ by $n$ matrix $(a_1, \ldots, a_n)$
by

\[ \text{wrdet}_k(a_1, \ldots, a_n) := \det(-1/k)(a_1, \ldots, a_1, \ldots, a_n, \ldots, a_n), \]

then it satisfies a relative \( GL_n \)-invariance

\[ \text{wrdet}_k(AP) = \text{wrdet}_k(A) \det(P)^k, \quad P \in GL_n(\mathbb{C}). \]

We also have wreath-determinant analog for Vandermonde and Cauchy determinants. It would be interesting to seek various special wreath determinant formulas like these. See [3] for more details.

(B) Quantum \( \alpha \)-determinant

Define the quantum \( \alpha \)-determinant as an element in the quantum matrix algebra \( A_q(\text{Mat}_n) \) by

\[ \det_q^{(\alpha)} := \sum_{\sigma \in S_n} q^{\ell(\sigma)} \alpha^{\nu(\sigma)} x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n}, \]

where \( \ell(\sigma) \) denotes the inversion number of a permutation \( \sigma \). This is nothing but the ordinary quantum determinant \( \det_q \) when \( \alpha = -1 \). Since the quantum enveloping algebra \( \mathcal{U}_q(\mathfrak{g}l_n) \) acts on \( A_q(\text{Mat}_n) \), we can consider the cyclic \( \mathcal{U}_q(\mathfrak{g}l_n) \)-submodule \( \mathcal{U}_q(\mathfrak{g}l_n) \cdot \det_q^{(\alpha)} \). Thus the irreducible decomposition of this cyclic submodule is regarded as a \( q \)-analog of the first problem studied by Matsumoto and Wakayama. The structure of \( \mathcal{U}_q(\mathfrak{g}l_n) \cdot \det_q^{(\alpha)} \) is, however, quite complicated, so that we have only several less explicit results at present. See [4] for more details.

(C) Cyclic modules generated by powers of the \( \alpha \)-determinant

Recently, Matsumoto, Wakayama and the author investigated the generalized case \( \mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X)^m \) and proved that

\[ \mathcal{U}(\mathfrak{g}l_n) \cdot \det^{(\alpha)}(X)^m \simeq \bigoplus_{\lambda \vdash mn, \ell(\lambda) \leq n} (\mathcal{M}_n^\lambda) \oplus \text{rk} F_{n,m}^\lambda(\alpha) \]

holds for certain square matrices \( F_{n,m}^\lambda(\alpha) \) whose entries are polynomials in \( \alpha \). In this direct sum, \( \lambda \) runs over the partitions of \( mn \) whose length are at most \( n \). Remark that the matrices \( F_{n,m}^\lambda(\alpha) \) are determined up to conjugacy and non-zero scalar factor. In the particular case where \( m = 1 \), we explicitly have \( F_{n,1}^\lambda(\alpha) = f_\lambda(\alpha)I \), where \( I \) is the identity matrix of size \( f_\lambda \) and \( f_\lambda(\alpha) \) is given by (1.2). It seems quite difficult to describe \( F_{n,m}^\lambda(\alpha) \) in an explicit manner in general. However, when \( n = 2 \), all the matrices \( F_{2,m}^\lambda(\alpha) \) are one by one, and they are explicitly given by

\[ F_{2,m}^{(m+s,m-s)}(\alpha) = (1 + \alpha)^s \binom{s-m, s+1}{-m; -\alpha} \quad (s = 0, 1, \ldots, m), \]
where \( {}_2F_1(a, b; c; x) \) is the Gaussian hypergeometric function \([2]\).

**Goal of this note**

The problem we study here is a \( q \)-analog of the study of \( \mathcal{U}(\mathfrak{gl}_2) \cdot \det(\alpha) (X)^m \). Namely, we investigate the cyclic \( \mathcal{U}_q(\mathfrak{sl}_2) \)-submodule (instead of \( \mathcal{U}_q(\mathfrak{gl}_2) \)-submodule just for simplicity of the description) of \( \mathcal{A}_q(\text{Mat}_2) \) defined by

\[
V_q^{m}(\alpha) = \mathcal{U}_q(\mathfrak{sl}_2) \cdot (\det_q^{(\alpha)})^m.
\]

We prove that there exists a collection of polynomials \( F_{m,j}(\alpha) \) \((j=0,1,\ldots,m)\) such that

\[
V_q^{m}(\alpha) \cong \bigoplus_{0 \leq j \leq m} \mathcal{M}_q(2j+1),
\]

where \( \mathcal{M}_q(d) \) is the \( d \)-dimensional irreducible representation of \( \mathcal{U}_q(\mathfrak{sl}_2) \) given in the next section, and show that the polynomials \( F_{m,j}(\alpha) \) are written in terms of a certain \( q \)-deformation of the hypergeometric polynomials (Theorem 3.3). Taking a limit \( q \to 1 \), we also obtain the formula (1.3) again (Corollary 3.6).

**§ 2. Preliminaries**

We first fix the convention on quantum groups (we basically follow to \([7]\) and \([11]\)). Assume that \( q \in \mathbb{C}^\times \) is not a root of unity. The quantum enveloping algebra \( \mathcal{U}_q(\mathfrak{sl}_2) \) is an associative algebra generated by \( k, k^{-1}, e, f \) with the fundamental relations

\[
kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2e, \quad kfk^{-1} = q^{-2}f, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.
\]

\( \mathcal{U}_q(\mathfrak{sl}_2) \) has a (coassociative) coproduct

\[
\Delta(k^\pm 1) = k^\pm 1 \otimes k^\pm 1,
\]

\[
\Delta(e) = e \otimes 1 + k \otimes e,
\]

\[
\Delta(f) = f \otimes k^{-1} + 1 \otimes f,
\]

which enables us to define tensor products of \( \mathcal{U}_q(\mathfrak{sl}_2) \)-modules.

The quantum matrix algebra \( \mathcal{A}_q(\text{Mat}_2) \) is an associative algebra generated by \( x_{11}, x_{12}, x_{21}, x_{22} \) with the fundamental relations

\[
\begin{align*}
x_{11}x_{12} &= qx_{12}x_{11}, & x_{21}x_{22} &= qx_{22}x_{21}, \\
x_{11}x_{21} &= qx_{21}x_{11}, & x_{12}x_{22} &= qx_{22}x_{12}, \\
x_{12}x_{21} &= x_{21}x_{12}, & x_{11}x_{22} - x_{22}x_{11} &= (q - q^{-1})x_{12}x_{21}.
\end{align*}
\]
For convenience, we put

\[ z_1 := x_{11}x_{22}, \quad z_2 := x_{12}x_{21}. \]

Notice that they commute:

\[ z_1z_2 = z_2z_1. \]

The quantum \(\alpha\)-determinant of size two is then a linear polynomial

\[ \det_q^{(\alpha)} = x_{11}x_{22} + \alpha qx_{12}x_{21} = z_1 + \alpha qz_2 \]

in commuting variables \(z_1, z_2\).

Remark. The quantum \(\alpha\)-determinant of size two interpolates the quantum counterparts of the determinant and permanent. In fact, we have

\[ \det_q = x_{11}x_{22} - qx_{12}x_{21} = \det_q^{(-1)}, \quad \text{per}_q = x_{11}x_{22} + q^{-1}x_{12}x_{21} = \det_q^{(q^{-2})}. \]

Here, in general, we define the quantum permanent of size \(n\) by

\[ \text{per}_q := \sum_{\sigma \in \mathfrak{S}_n} q^{-\ell(\sigma)} x_{\sigma(1)1}x_{\sigma(2)2} \cdots x_{\sigma(n)n} \in \mathcal{A}_q(\text{Mat}_n), \]

which can be regarded as a \(q\)-analog of the usual permanent in the sense that the cyclic module \(\mathcal{U}_q(\mathfrak{gl}_n) \cdot \text{per}_q\) is equivalent to the \(q\)-analog of \(n\)-th symmetric tensor product of the natural representation \(\mathbb{C}^n\) of \(\mathcal{U}_q(\mathfrak{gl}_n)\). However, the quantum \(\alpha\)-determinant of size \(n\) does not coincide with the quantum permanent for any \(\alpha\) if \(n \geq 3\). This is because \(\nu(\cdot)\) is a class function on \(\mathfrak{S}_n\) in general, whereas the inversion number \(\ell(\cdot)\) is not if \(n \geq 3\).

We briefly recall necessary basic facts on representation theory of \(\mathcal{U}_q(\mathfrak{sl}_2)\). Let \(\{e_1, e_2\}\) be the standard basis of the vector space \(\mathbb{C}^2\). By defining

\[ k^\pm e_1 := q^{\pm 1}e_1, \quad e \cdot e_1 := 0, \quad f \cdot e_1 := e_2, \]
\[ k^\pm e_2 := q^{\mp 1}e_2, \quad e \cdot e_2 := e_1, \quad f \cdot e_2 := 0, \]

\(\mathbb{C}^2\) becomes a \(\mathcal{U}_q(\mathfrak{sl}_2)\)-module. Put

\[ \mathcal{M}_q(l+1) := \mathcal{U}_q(\mathfrak{sl}_2) \cdot \nu_0^{(l)} \subset (\mathbb{C}^2)^{\otimes l}, \quad \nu_0^{(l)} := e_1 \otimes e_1 \otimes \cdots \otimes e_1 \]

for \(l = 1, 2, \ldots\). We also put \(\mathcal{M}_q(1) := \mathbb{C} \cdot \nu_0^{(0)}\) (with \(\nu_0^{(0)} := 1 \in \mathbb{C}\)) and define \(e \cdot \nu_0^{(0)} = f \cdot \nu_0^{(0)} = 0\) and \(k^\pm \nu_0^{(0)} := \nu_0^{(0)}\). If we set \(\nu_r^{(l)} := f^r \cdot \nu_0^{(l)}\) (for \(r = 0, 1, \ldots, l\)) and \(\nu_l^{(l)} = \nu_{l+1}^{(l)} = 0\), then it follows that \(\mathcal{M}_q(l+1) = \bigoplus_{r=0}^l \mathbb{C} \cdot \nu_r^{(l)}\) and

\[ k^\pm \nu_r^{(l)} = q^{\pm (l-2r)} \nu_r^{(l)}, \]

\[ e \cdot \nu_r^{(l)} = [l-r+1]_q \nu_{r-1}^{(l)}, \]

\[ f \cdot \nu_r^{(l)} = [r+1]_q \nu_{r+1}^{(l)} \]

for \(r = 0, 1, \ldots, l\). This is an irreducible \((l+1)\)-dimensional \(\mathcal{U}_q(\mathfrak{sl}_2)\)-module.
Remark. There exists a convolution $\theta$ of $\mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2)$ such that
$$\theta(e) = -e, \quad \theta(f) = f, \quad \theta(k^{\pm 1}) = k^{\mp 1}. $$

Using this, we can introduce inequivalent $\mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2)$-module structure on the vector space $\mathcal{M}_q(l + 1)$, say $\mathcal{M}_q(l + 1)^{\theta}$. However, such modules do not appear in the following discussion. Note that any finite dimensional irreducible $\mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2)$-module is isomorphic to either $\mathcal{M}_q(l + 1)$ or $\mathcal{M}_q(l + 1)^{\theta}$ for some $l = 0, 1, 2, \ldots$ (see, e.g. [7, 9]).

The algebra $\mathcal{A}_q(\text{Mat}_2)$ becomes a $\mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2)$-module by
$$k^{\pm 1} \cdot x_{i1} := q^{\pm 1}x_{i1}, \quad e \cdot x_{i1} := 0, \quad f \cdot x_{i1} := x_{i2},$$
$$k^{\pm 1} \cdot x_{i2} := q^{\mp 1}x_{i2}, \quad e \cdot x_{i2} := x_{i1}, \quad f \cdot x_{i2} := 0 \quad (i = 1, 2).$$

These are compatible with the fundamental relations (2.1) above. Notice that $e \cdot \det_q^r = f \cdot \det_q^r = 0$ and $k^{\pm 1} \cdot \det_q^r = \det_q^r$, (i.e. $\mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2) \cdot \det_q^r \cong \mathcal{M}_q(1)$). It then follows that $X \cdot (v \det_q^r) = (X \cdot v) \det_q^r$ for $X \in \mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2)$ and $v \in \mathcal{A}_q(\text{Mat}_2)$. We have

(2.2) $\mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2) \cdot (x_{11}x_{21})^s \det_q^{m-s} \cong \mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2) \cdot (x_{11}x_{21})^s \cong \mathcal{M}_q(2s + 1) \quad (s = 0, 1, 2, \ldots).$

Actually, the linear map defined by $v_{l} \mapsto f^r \cdot (x_{11}x_{21})^s$ gives a bijective intertwiner between $\mathcal{M}_q(2s + 1)$ and $\mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2) \cdot (x_{11}x_{21})^s$.

Define $q$-analogs of numbers, factorials and binomial coefficients by
$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := \prod_{i=1}^{n} [i]_q, \quad \left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

§ 3. Cyclic modules generated by the quantum alpha-determinant

We fix a positive integer $m$ and discuss the irreducible decomposition of the cyclic module $V_{q}^{m}((\alpha)) := \mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2) \cdot (\det_q^{(\alpha)})^m$. We refer to [1] for detailed discussion.

Put
$$v_{m,j} := (f^j \cdot (x_{11}x_{21})^j) \det_q^{m-j} \quad (j = 0, 1, \ldots, m).$$

It is easy to see that the cyclic module $\mathcal{U}_q(\mathfrak{s} \mathfrak{l}_2) \cdot v_{m,j}$ is equivalent to $\mathcal{M}_q(2j + 1)$. We show that $v_{m,j}$ is a homogeneous polynomial in $z_1$ and $z_2$ of degree $m$ and give an explicit expression of it. For this purpose, we need the following two lemmas.

Lemma 3.1. For each positive integer $j$, it follows that
$$f^j \cdot (x_{11}x_{21})^j = q^{-j(j-1)/2} [j]_q! \sum_{r=0}^{j} q^{-r^2} \left[ \begin{array}{c} j \\ r \end{array} \right]_q^2 x_{11}^{j-r} x_{22}^{j-r} (x_{12}x_{21})^r.$$
Sketch of proof. For $1 \leq i \leq 2j$, put

$$f_j(i) := \underbrace{1 \otimes \cdots \otimes 1}_i \otimes f \otimes \underbrace{k^{-1} \otimes \cdots \otimes k^{-1}}_{2j-i} \in U_q(sl_2)^{\otimes 2j}.$$

Then it follows that

$$\Delta^{2j-1}(f)^j = \sum_{1 \leq n_1, \ldots, n_j \leq 2j} f_j(n_1) \cdots f_j(n_j).$$

Since $f_j(m)f_j(n) = q^{-2}f_j(n)f_j(m)$ if $m > n$ and $f^2 \cdot x_{11} = f^2 \cdot x_{21} = 0$, we have

$$\Delta^{2j-1}(f)^j = q^{-j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_j \leq 2j} f_j(n_1) \cdots f_j(n_j) + R,$$

where $R$ is a certain element in $U_q(sl_2)^{\otimes 2j}$ such that $R \cdot (x_{11}^j x_{21}^j) = 0$. Here we also use the well-known identity

$$\sum_{\sigma \in S_j} x^{\ell(\sigma)} = (1+x)(1+x+x^2) \cdots (1+x+\cdots+x^{j-1})$$

with $x = q^{-2}$. For given $n_1, \ldots, n_j$ ($1 \leq n_1 < \cdots < n_j \leq 2j$), we get

$$f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{21}^j) = q^{-r^2+j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_j \leq 2j} q^{2(n_1+\cdots+n_r) - 2(m_1+\cdots+m_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r$$

by a careful calculation (see [1] for detail). Using this, we have

$$f^j \cdot (x_{11}^j x_{22}^j) = q^{-r^2+j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_j \leq 2j} q^{2(n_1+\cdots+n_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r$$

where $e_r(x_1, x_2, \ldots, x_j)$ is the $r$th elementary symmetric polynomial in $x_1, x_2, \ldots, x_j$. By the identity (see, e.g. [10])

$$e_r(1, q^2, \ldots, q^{2j-2}) = q^{r(j-1)} \left[j \atop r \right]_q$$

together with the symmetry $[j]_q = [j]_q^{-1}$, we obtain

$$f^j \cdot (x_{11}^j x_{21}^j) = [j]_q! \sum_{r=0}^j q^{-r^2} \left[j \atop r \right]_q^2 x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r.$$

Since $(x_{11} x_{21})^j = q^{-j(j-1)/2} x_{11}^j x_{21}^j$, we have the desired conclusion. \qed
Lemma 3.2. For each positive integer \( l \), it follows that

\[
x_{11}^{l}x_{22}^{l} = \prod_{s=1}^{l} (z_{1} + (q^{2s-1} - q)z_{2}) = \sum_{j=0}^{l} q^{l(l-j)} \left\{ \begin{array}{l} l \\ j \end{array} \right\} q (z_{1} - qz_{2})^{j} z_{2}^{l-j}.
\]

Proof. The first equality is proved by induction on \( l \) by using the relation

\[
(z_{1} + (q^{2r-1} - q)z_{2})x_{22} = x_{22}(z_{1} + (q^{2r+1} - q)z_{2}).
\]

The second equality is a specialization of the \( q \)-binomial theorem

\[
\prod_{i=1}^{n}(x + q^{2i-n-1}y) = \sum_{r=0}^{n} \left\{ \begin{array}{l} n \\ r \end{array} \right\} x^{r}y^{n-r},
\]

which is indeed applicable since \( z_{1} \) and \( z_{2} \) commute. \( \square \)

As a result, we get the explicit expression

\[
v_{m,j} = \left\{ q^{-j(j-1)/2} [j]_{q}! \sum_{r=0}^{j} q^{-r^{2}} \left\{ \begin{array}{l} j \\ r \end{array} \right\} z_{2}^{r} \prod_{s=1}^{j-r} (z_{1} + (q^{2s-1} - q)z_{2}) \right\} (z_{1} - qz_{2})^{m-j},
\]

which is indeed a homogeneous polynomial in \( z_{1} \) and \( z_{2} \) of degree \( m \).

The vectors \( \{v_{m,j}\}_{j=0}^{m} \) are linearly independent since they belong to inequivalent representations, and hence form a basis of the space consisting of homogeneous polynomials in \( z_{1} \) and \( z_{2} \) of degree \( m \), whose dimension is \( m + 1 \). Thus we conclude that \((\det_{q}^{(\alpha)})^{m}(\in \mathcal{A}_{q}(\text{Mat}_{2}) \otimes_{\mathbb{C}} \mathbb{C}[\alpha])\) can be expressed as a linear combination

\[
(\det_{q}^{(\alpha)})^{m} = \sum_{j=0}^{m} F_{m,j}(\alpha)v_{m,j}
\]

of the vectors \( v_{m,j} \), where \( F_{m,j}(\alpha) \) are certain polynomial functions in \( \alpha \). This implies that the cyclic module \( V_{q}^{m}(\alpha) \) contains an irreducible submodule equivalent to \( \mathcal{M}_{q}(2j+1) \) (with multiplicity one) if \( F_{m,j}(\alpha) \neq 0 \). Consequently, it follows that

\[
V_{q}^{m}(\alpha) \cong \bigoplus_{0 \leq j \leq m, F_{m,j}(\alpha) \neq 0} \mathcal{M}_{q}(2j+1).
\]

Let us determine the functions \( F_{m,j}(\alpha) \) explicitly. The conditions (3.2) for the functions \( F_{m,j}(\alpha) \) are given in terms of polynomials in commuting variables \( z_{1}, z_{2} \), so that it is meaningful to consider the specialization \( z_{1} = z, z_{2} = 1 \) in (3.2), where \( z \) is a
new variable. Put

$$g_j(z) := \prod_{i=1}^{j} (z + q^{2i-1} - q) = \sum_{i=0}^{j} q^{j(i-1)} \left[ \begin{array}{l} j \\ i \end{array} \right]_q (z - q)^i,$$

$$v_j(z) := q^{-j(j-1)/2} [j]_q! \sum_{r=0}^{j} q^{-r^2} \left[ \begin{array}{l} j \\ r \end{array} \right]_q g_{j-r}(z).$$

Then (3.2) together with Lemmas 3.1 and 3.2 yields

$$(z + q\alpha)^m = \sum_{j=0}^{m} F_{m,j}(\alpha) v_j(z)(z - q)^{m-j}.$$

If we take the lth derivative of this formula with respect to z (l = 0, 1, ..., m) and substitute z = q, then we get the relation

$$(3.4) \binom{m}{l} q^{m-l} (1 + \alpha)^{m-l} = \sum_{j=m-l}^{m} F_{m,j}(\alpha) v_j^{(l-m+j)}(q) = \sum_{s=0}^{l} F_{m,m-s}(\alpha) \frac{v_{m-s}^{(l-s)}(q)}{(l-s)!}.$$

Since

$$v_j(z) = q^{j(j+1)/2} [j]_q! \sum_{i=0}^{j} q^{-ij} \left[ \begin{array}{l} j \\ i \end{array} \right]_q \sum_{r=0}^{j-i} q^{r(i-2j)} \left[ \begin{array}{l} j \\ j-r \end{array} \right]_q \left[ \begin{array}{l} 2j-i \\ j \end{array} \right]_q (z - q)^i,$$

we have

$$v_j(z) = q^{-j(j-1)/2} [j]_q! \sum_{i=0}^{j} \frac{[2j-i]_q!}{[i]_q! [j-i]_q!^2} (z - q)^i,$$

or

$$\frac{v_j^{(i)}(q)}{i!} = q^{-\binom{j}{2}} \frac{[j]_q! [2j-i]_q!}{[i]_q! [j-i]_q!^2}.$$

Here we use the q-Chu-Vandermonde formula

$$\sum_{r=0}^{j-i} q^{r(i-2j)} \left[ \begin{array}{l} j \\ j-r \end{array} \right]_q \left[ \begin{array}{l} j-i \\ r \end{array} \right]_q = q^{j(j-1)/2} \left[ \begin{array}{l} 2j-i \\ j \end{array} \right]_q.$$

Thus the formula (3.4) is rewritten more explicitly as

$$(3.5) [m-l]_q!^2 \binom{m}{l} q^{m-l} (1 + \alpha)^{m-l} = \sum_{s=0}^{l} q^{-\binom{m-s}{2}} \frac{[m-s]_q! [2m-l-s]_q!}{[l-s]_q!} F_{m,m-s}(\alpha).$$
This also implies that the polynomial \( F_{m,j}(\alpha) \) is divisible by \((1 + \alpha)^j\), that is

\[
F_{m,j}(\alpha) = (1 + \alpha)^j Q_{m,j}(\alpha)
\]

for some \( Q_{m,j}(\alpha) \in \mathbb{C}[\alpha] \). By (3.6) and (3.5), we have

\[
\sum_{j=0}^{i} \begin{bmatrix} 2m - 2i - 1 \\ m - i \end{bmatrix}_{q}^{-1} \begin{bmatrix} m \\ i \end{bmatrix}_{q} q^{m-i} (1 + \alpha)^{i-j} \cdot q^{-\binom{m-j}{2}} [m-j]_q! Q_{m,m-j}(\alpha).
\]

Now we define the mixed hypergeometric series by

\[
\Phi\left(\begin{array}{l}
{a_1, \ldots, a_k} \\
b_1, \ldots, b_l
\end{array} ; c_1, \ldots, c_m ; \frac{x}{q} \right) = \sum_{i=0}^{[s+1]/2} \frac{(a_1;i) \cdots (a_k;i) (c_1;i) \cdots (c_m;i) x^i}{(b_1;i) \cdots (b_l;i) (d_1;i) \cdots (d_m;i) [i]_q!},
\]

where \((a;i) = a(a+1) \cdots (a+i-1)\) and \((a;i)_q = [a]_q [a+1]_q \cdots [a+i-1]_q\) (cf. [8]).

**Theorem 3.3.** For \( s = 0, 1, \ldots, m \),

\[
F_{m,s}(\alpha) = q^{\binom{s+1}{2}} \left( \begin{array}{l}
m \\
2
\end{array} \right) \left( \begin{array}{l}
s+1 \\
2
\end{array} \right) \frac{[s]_q!}{[2s]_q!} (1 + \alpha)^s \Phi\left(\begin{array}{l}
{s+1} \\
2s+2
\end{array} ; \frac{s+1, s+1}{q} ; q(1 + \alpha) \right)
\]

holds.

**Sketch of proof.** We can prove the identity

\[
\left( \begin{bmatrix} 2i - 2m - 1 \\ i - j \end{bmatrix}_{q} \right)_{0 \leq i, j \leq m}^{-1} = \left( \begin{bmatrix} 2m - 2i + 1 \\ 2m - 2j + 1 \end{bmatrix}_{q} \right)_{0 \leq i, j \leq m}^{-1}.
\]

Using this, we solve (3.7) and find that

\[
Q_{m,m-i}(\alpha) = \frac{q^{\binom{m-i}{2}} [2m-2i+1]_q}{[m-i]_q!}
\]

\[
\times \sum_{j=0}^{i} \frac{(-1)^{i-j} q^{m-j} [2m-2j+1]_q}{[2m-2j+1]_q} \left( \begin{bmatrix} 2m-2j+1 \\ m-j \end{bmatrix}_{q} \right)^{-1} (1 + \alpha)^{i-j}
\]

\[
= \frac{q^{\binom{m-i+1}{2}} m! [2m-2i+1]_q}{[m-i]_q!}
\]

\[
\times \sum_{r=0}^{i} \frac{(-q)^r [m-i+r]_q!^2}{(m-i+r)!(i-r)! [2m-2i+r+1]_q!} (1 + \alpha)^r.
\]
where we set $r = i - j$. Since

$$(i - r)! = (-1)^r \frac{i!}{(-i; r)}, \quad (n + r)! = n!(n + 1; r), \quad [n + r]_q! = [n]_q!(n + 1; r)_q,$$

we have

$$Q_{m, m-i}(\alpha) = \frac{q \left(\begin{array}{l}
7n-i+1\
2
\end{array}\right)m![m-i]_q!}{i!(m-i)! [2m-2i]_q!} \sum_{r=0}^{i} \frac{(-i;r)(m-i+1;r)_q^2}{(m-i+1;r)(2m-2i+2;r)_q} \frac{(q(1+\alpha))^r}{[r]_q!}$$

$$= q^{\frac{m-i+1}{2}} \left(\begin{array}{l}
m\
i
\end{array}\right) \frac{[m-i]_q!}{[2m-2i]_q!} \Phi(-i; m-i+1, m-i+1; 2m-2i+2; q; q(1+\alpha)).$$

If we substitute this into (3.6) and replace $m-i$ by $s$, then we have the conclusion. \qed

**Example 3.4**  $(m = 1)$. We have

$$F_{1,0}(\alpha) = \Phi(-1; 1, 2; q; q(1+\alpha)) = \frac{1 - \alpha q^2}{1 + q^2},$$

$$F_{1,1}(\alpha) = q \frac{[1]_q!}{[2]_q!} (1+\alpha) \Phi(-1; 2, 4; q; q(1+\alpha)) = \frac{q}{[2]_q} (1+\alpha).$$

Thus it follows that

$$\mathcal{U}_q(\mathfrak{sl}_2) \cdot \det_q^{(\alpha)} \cong \begin{cases}
\mathcal{M}_q(3) & \alpha = q^{-2}, \\
\mathcal{M}_q(1) & \alpha = -1, \\
\mathcal{M}_q(1) \oplus \mathcal{M}_q(3) & \text{otherwise}.
\end{cases}$$

Notice that $q^{-2} \neq -1$ since we assume that $q$ is not a root of unity.

**Example 3.5**  $(m = 2)$. We have

$$F_{2,0}(\alpha) = \Phi(-2; 1, 2; q; q(1+\alpha)) = C_0(q) \left((q^6 + q^4)\alpha^2 - 2q^2\alpha + q^4 + 1\right),$$

$$F_{2,1}(\alpha) = 2q^2 \frac{[1]_q!}{[2]_q!} (1+\alpha) \Phi(-1; 2, 4; q; q(1+\alpha)) = C_1(q)(1 + \alpha) \left((q^4 + q^2)\alpha - q^4 + q^2 - 2\right),$$

$$F_{2,2}(\alpha) = q^3 \frac{[2]_q!}{[4]_q!} (1+\alpha)^2 \Phi(0; 3, 6; q; q(1+\alpha)) = C_2(q)(1 + \alpha)^2,$$
where $C_0(q), C_1(q), C_2(q)$ are certain rational functions in $q$. Hence, if we assume that $q$ is transcendental (we have only to assume that $(q^4 - q^2 + 1)^2 + q^2(q^4 + q^2 + 1) \neq 0$ practically), then we see that

$$
\mathcal{U}_q(\mathfrak{sl}_2) \cdot \left( \det_q^{(\alpha)} \right)^2 \cong \begin{cases} 
\mathcal{M}_q(1) & \alpha = -1, \\
\mathcal{M}_q(1) \oplus \mathcal{M}_q(5) & \alpha = \frac{q^4 - q^2 + 2}{q^4 + q^2}, \\
\mathcal{M}_q(3) \oplus \mathcal{M}_q(5) & \alpha = \frac{1 + q\sqrt{-q^4 - q^2 - 1}}{q^4 + q^2}, \\
\mathcal{M}_q(1) \oplus \mathcal{M}_q(3) \oplus \mathcal{M}_q(5) & \text{otherwise.}
\end{cases}
$$

When $(q^4 - q^2 + 1)^2 + q^2(q^4 + q^2 + 1) = 0$ (this does not imply that $q$ is a root of unity), $\frac{q^4 - q^2 + 2}{q^4 + q^2}$ becomes a common root of $F_{2,0}(\alpha)$ and $F_{2,1}(\alpha)$, so that we have

$$
\mathcal{U}_q(\mathfrak{sl}_2) \cdot \left( \det_q^{(\alpha)} \right)^2 \cong \mathcal{M}_q(5), \quad \alpha = \frac{q^4 - q^2 + 2}{q^4 + q^2}.
$$

Remark. The mixed hypergeometric series (3.8) can be regarded as a common generalization of the generalized hypergeometric series and basic hypergeometric series as we see below:

$$
\Phi \left( \begin{array}{llllllllll}
\begin{array}{llllllllll}
\alpha_1 & \cdots & \alpha_k & 1 \\
\beta_1 & \cdots & \beta_l, 1 & \end{array} & q ; x \\
\begin{array}{llllllllll}
\delta_1 & \cdots & \delta_n & x
\end{array}
\end{array} \right) = k \Phi \left( \begin{array}{llllllllll}
\begin{array}{llllllllll}
\alpha_1 & \cdots & \alpha_k & 1 \\
\beta_1 & \cdots & \beta_l, 1 & \end{array} & q ; x \\
\begin{array}{llllllllll}
\delta_1 & \cdots & \delta_n & x
\end{array}
\end{array} \right) = m \Phi \left( \begin{array}{llllllllll}
\begin{array}{llllllllll}
\alpha_2c_1 & \cdots & \alpha_2c_m & q^2; x \\
\beta_1 & \cdots & \beta_l & q^2d_1, \ldots, q^2d_n; q^2,(-1)^{1+n-m}q^{1+d-c}x
\end{array} \right),
$$

where $c = c_1 + \cdots + c_m$ and $d = d_1 + \cdots + d_n$.

Remark. The function $\Phi$ given by (3.8) satisfies the difference-differential equation

$$
\left\{ -(E + a_1) \cdots (E + a_k) [E + c_1]_q \cdots [E + c_m]_q \\
+ \partial_q(E + b_1 - 1) \cdots (E + b_l - 1) [E + d_1 - 1]_q \cdots [E + d_n - 1]_q \right\} \Phi = 0,
$$

where we put

$$
E = x \frac{d}{dx}, \quad [E + a]_q = \frac{q^{E+a} - q^{-E-a}}{q - q^{-1}}, \quad \partial_q f(x) = \frac{f(qx) - f(q^{-1}x)}{qx - q^{-1}x}.
$$

If we take a limit $q \to 1$, then the equation above becomes a hypergeometric differential equation for $k+mF_{l+n}(a_1, \ldots, a_k, c_1, \ldots, c_m; b_1, \ldots, b_l, d_1, \ldots, d_n; x)$.

All the discussion above also work in the classical case (i.e. the case where $q = 1$). Thus, by taking a limit $q \to 1$ in Theorem 3.3, we will obtain Theorem 4.1 in [2] (or...
(1.3) up to constant) again. We abuse the same notations used in the discussion of quantum case above to indicate the classical counterparts. From (3.9), we have
\[
Q_{m,s}(\alpha) = \frac{m!}{(m-s)!(2s)!} {}_3F_2\left( \begin{array}{c} \alpha, \alpha, \alpha \\ s-m, s+1, s+1 \end{array} ; 1+\alpha \right) \\
= \frac{m!}{(m-s)!(2s)!} {}_2F_1\left( \begin{array}{c} \alpha, \alpha \\ s-m, 2s+2 \end{array} ; 1+\alpha \right).
\]

Notice that
\[
{}_2F_1\left( \begin{array}{c} s-m, s+1 \\ 2s+2 \end{array} ; 1-x \right) = \frac{m!(2s+1)!}{s!(m+s+1)!} {}_2F_1\left( \begin{array}{c} s-m, 2s+2 \\ s-m \end{array} ; x \right).
\]

Thus we also get
\[
Q_{m,s}(\alpha) = \frac{m!^2(2s+1)}{(m-s)!s!(m+s+1)!} {}_2F_1\left( \begin{array}{c} s-m, s-m \\ -m \end{array} ; -\alpha \right) \quad (s=0,1,\ldots,m).
\]

Summarizing these, we have the

**Corollary 3.6 (Classical case).** It follows that
\[
F_{m,s}(\alpha) = \frac{m!}{(m-s)!(2s)!} (1+\alpha)^s {}_2F_1\left( \begin{array}{c} s-m, s+1 \\ 2s+2 \end{array} ; 1+\alpha \right) \\
= \frac{m!}{(m-s)!(2s)!} (1+\alpha)^s \left( \begin{array}{c} 2m \\ m-s \end{array} \right)^2 {}_2F_1\left( \begin{array}{c} s-m, s+1 \\ -m \end{array} ; -\alpha \right)
\]
for \(s=0,1,\ldots,m\).

**References**


