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Quantum α-determinants and q-deformations of hypergeometric polynomials

By

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Abstract

The quantum α-determinant is a parametric deformation of the ordinary quantum determinant. We study the cyclic $\mathcal{U}_q(\mathfrak{sl}_2)$-submodules of the quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_2)$ generated by the powers of the quantum α-determinant. The irreducible decomposition of this cyclic module is explicitly described in terms of certain polynomials in the parameter α, which is a q-deformation of the Gaussian hypergeometric polynomials.

This note is a summary of the author's recent paper [Kimoto, K., "Quantum alpha-determinants and q-deformed hypergeometric polynomials," *Int. Math. Res. Not.*].

§ 1. Background

As a parametric interpolation of the determinant and permanent, we define the *α-determinant* of a matrix $X = (x_{ij})_{1\le i,j\le n}$ by

$$\det{}^{(\alpha)}(X) := \sum_{\sigma\in\mathfrak{S}_n} \alpha^{\nu(\sigma)} x_{\sigma(1)1}x_{\sigma(2)2}\cdots x_{\sigma(n)n},$$

where α is a complex parameter and $\nu(\sigma) = n-(m_1+m_2+\cdots+m_n)$ if the cycle type of a permutation σ in the symmetric group $\mathfrak{S}_n$ of degree n is $1^{m_1}2^{m_2}\cdots n^{m_n}$ [13]. In fact, we have $\det(X) = \det^{(-1)}(X)$ and $\mathrm{per}(X) = \det^{(1)}(X)$ by definition.

We are interested in the representation-theoretical properties of the α-determinant. Let us set the stage to formulate the problem. Denote by $\mathcal{A}(\mathrm{Mat}_n)$ the $\mathbb{C}$-algebra of polynomials in the n^2 commuting variables $\{x_{ij}\}_{1\le i,j\le n}$, and $\mathcal{U}(\mathfrak{gl}_n)$ the universal

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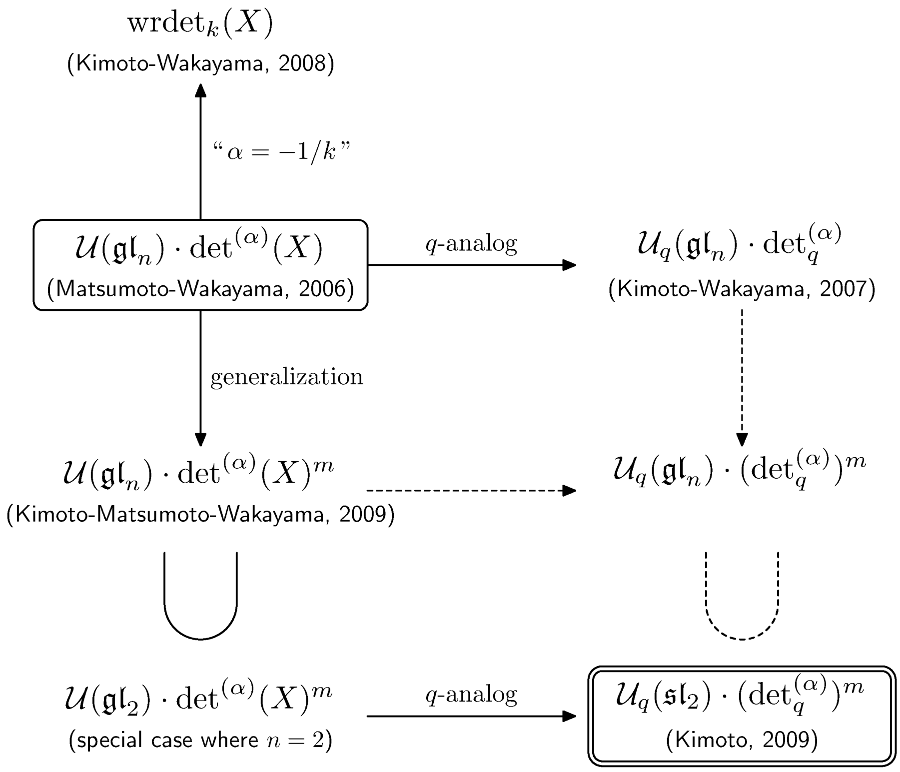

Figure 1. Representation-theoretical studies on α-determinants

enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$. We can introduce a $\mathcal{U}(\mathfrak{gl}_n)$-module structure on $\mathcal{A}(\mathrm{Mat}_n)$ by

$$E_{ij} \cdot f := \sum_{r=1}^{n} x_{ir} \frac{\partial f}{\partial x_{jr}} \quad (f \in \mathcal{A}(\mathrm{Mat}_n)),$$

where $\{E_{ij}\}_{1 \le i,j \le n}$ is the standard basis of $\mathfrak{gl}_n$. Recall that the cyclic submodules $\mathcal{U}(\mathfrak{gl}_n) \cdot \det(X)$ and $\mathcal{U}(\mathfrak{gl}_n) \cdot \mathrm{per}(X)$ are equivalent to the skew-symmetric tensor product $\bigwedge^n(\mathbb{C}^n)$ and symmetric tensor product $\mathrm{Sym}^n(\mathbb{C}^n)$ of the natural representation $\mathbb{C}^n$ respectively, which are both irreducible. Thus the cyclic module $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$ can be regarded as an 'interpolating' family of $\mathcal{U}(\mathfrak{gl}_n)$-submodules of the two irreducible submodules above, and it is natural and interesting to study the irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$. This is the starting point of our study.

History of the representation-theoretical studies of α-determinants

Here we briefly explain the history of the studies of α-determinants from the viewpoint of representation theory to clarify the position of the matter which we deal with in this note (see also Figure 1 for a graphical summary).

The problem to determine the irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(\alpha)}(X)$ was raised and settled by Matsumoto and Wakayama [5]. They described the irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(\alpha)}(X)$ explicitly as follows:

$$\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(\alpha)}(X)\cong\bigoplus_{\substack{\lambda\vdash n\\ f_\lambda(\alpha)\neq 0}}\left(\mathcal{M}_n^\lambda\right)^{\oplus f^\lambda}. \tag{1.1}$$

Here we identify the dominant integral weights and partitions, and denote by $\mathcal{M}_n^\lambda$ the irreducible highest weight $\mathcal{U}(\mathfrak{gl}_n)$-module with highest weight λ. We also denote by $f_\lambda(x)$ the (modified) content polynomial of λ

$$f_\lambda(x)=\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}(1+(j-i)x). \tag{1.2}$$

At this point, there are at least three direction to proceed with the studies of α-determinants.

(A) The result (1.1) (together with (1.2)) by Matsumoto and Wakayama implies that the structure of the cyclic module $\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(\alpha)}(X)$ changes drastically when $\alpha=\pm 1/k$ for $k=1,2,\dots,n-1$. Study the α-determinant for such special values.

(B) Study a q-analog of the problem above, that is, define a quantum version of the α-determinant in the quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_n)$ suitably, and consider the cyclic $\mathcal{U}_q(\mathfrak{gl}_n)$-module generated by it.

(C) Study the cyclic $\mathcal{U}(\mathfrak{gl}_n)$-module generated by the powers $\det^{(\alpha)}(X)^m$ of the α-determinant in general.

(A) Wreath determinant

When $\alpha=-1/k$ for $k=1,2,\dots,n-1$, the α-determinant has a '$-1/k$-analog' of the alternating property. Precisely, we have

$$\sum_{\sigma\in\mathfrak{S}_n(I)}\det^{(-1/k)}(\boldsymbol{a}_{\sigma(1)},\boldsymbol{a}_{\sigma(2)},\dots,\boldsymbol{a}_{\sigma(n)})=0$$

for any n by n matrix $(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)$ and any subset $I\subset\{1,2,\dots,n\}$ such that $\#I>k$. Here we put

$$\mathfrak{S}_n(I)=\{\sigma\in\mathfrak{S}_n\,;\,\sigma(x)=x,\ \forall x\in\{1,2,\dots,n\}\setminus I\}.$$

This fact suggests that $\det^{(-1/k)}$ may have 'determinant-like' properties. Actually, if we define the *k-wreath determinant* $\mathrm{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)$ of a kn by n matrix $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)$

by

$$\mathrm{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n) := \det{}^{(-1/k)}(\overbrace{\boldsymbol{a}_1,\dots,\boldsymbol{a}_1}^{k},\dots,\overbrace{\boldsymbol{a}_n,\dots,\boldsymbol{a}_n}^{k}),$$

then it satisfies a relative GL_n-invariance

$$\mathrm{wrdet}_k(AP) = \mathrm{wrdet}_k(A)\det(P)^k, \quad P \in GL_n(\mathbb{C}).$$

We also have wreath-determinant analog for Vandermonde and Cauchy determinants. It would be interesting to seek various special wreath determinant formulas like these. See [3] for more details.

(B) Quantum α-determinant

Define the *quantum α-determinant* as an element in the quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_n)$ by

$$\det{}_q^{(\alpha)} := \sum_{\sigma\in\mathfrak{S}_n} q^{\ell(\sigma)}\alpha^{\nu(\sigma)} x_{\sigma(1)1}x_{\sigma(2)2}\cdots x_{\sigma(n)n},$$

where $\ell(\sigma)$ denotes the inversion number of a permutation σ. This is nothing but the ordinary quantum determinant $\det_q$ when $\alpha = -1$. Since the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ acts on $\mathcal{A}_q(\mathrm{Mat}_n)$, we can consider the cyclic $\mathcal{U}_q(\mathfrak{gl}_n)$-submodule $\mathcal{U}_q(\mathfrak{gl}_n)\cdot \det_q^{(\alpha)}$. Thus the irreducible decomposition of this cyclic submodule is regarded as a q-analog of the first problem studied by Matsumoto and Wakayama. The structure of $\mathcal{U}_q(\mathfrak{gl}_n)\cdot\det_q^{(\alpha)}$ is, however, quite complicated, so that we have only several less explicit results at present. See [4] for more details.

(C) Cyclic modules generated by powers of the α-determinant

Recently, Matsumoto, Wakayama and the author investigated the generalized case $\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(\alpha)}(X)^m$ and proved that

$$\mathcal{U}(\mathfrak{gl}_n)\cdot\det{}^{(\alpha)}(X)^m \cong \bigoplus_{\substack{\lambda\vdash mn\\ \ell(\lambda)\le n}} \left(\mathcal{M}_n^\lambda\right)^{\oplus \operatorname{rk} F_{n,m}^\lambda(\alpha)}$$

holds for certain square matrices $F_{n,m}^\lambda(\alpha)$ whose entries are polynomials in α. In this direct sum, λ runs over the partitions of mn whose length are at most n. Remark that the matrices $F_{n,m}^\lambda(\alpha)$ are determined up to conjugacy and non-zero scalar factor. In the particular case where $m = 1$, we explicitly have $F_{n,1}^\lambda(\alpha) = f_\lambda(\alpha)I$, where I is the identity matrix of size f^λ and $f_\lambda(\alpha)$ is given by (1.2). It seems quite difficult to describe $F_{n,m}^\lambda(\alpha)$ in an explicit manner in general. However, when $n = 2$, all the matrices $F_{2,m}^\lambda(\alpha)$ are *one by one*, and they are explicitly given by

$$F_{2,m}^{(m+s,m-s)}(\alpha) = (1+\alpha)^s {}_2F_1\left(\begin{matrix} s-m, s+1 \\ -m \end{matrix}; -\alpha\right) \qquad (s=0,1,\dots,m), \tag{1.3}$$

where ${}_2F_1(a,b;c;x)$ is the *Gaussian hypergeometric function* [2].

Goal of this note

The problem we study here is a q-analog of the study of $\mathcal{U}(\mathfrak{gl}_2)\cdot\det^{(\alpha)}(X)^m$. Namely, we investigate the cyclic $\mathcal{U}_q(\mathfrak{sl}_2)$-submodule (instead of $\mathcal{U}_q(\mathfrak{gl}_2)$-submodule just for simplicity of the description) of $\mathcal{A}_q(\mathrm{Mat}_2)$ defined by

$$V_q^m(\alpha) = \mathcal{U}_q(\mathfrak{sl}_2)\cdot(\det_q^{(\alpha)})^m.$$

We prove that there exists a collection of polynomials $F_{m,j}(\alpha)$ $(j=0,1,\dots,m)$ such that

$$V_q^m(\alpha) \cong \bigoplus_{\substack{0\le j\le m \\ F_{m,j}(\alpha)\neq 0}} \mathcal{M}_q(2j+1),$$

where $\mathcal{M}_q(d)$ is the d-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ given in the next section, and show that the polynomials $F_{m,j}(\alpha)$ are written in terms of a certain q-deformation of the hypergeometric polynomials (Theorem 3.3). Taking a limit $q\to 1$, we also obtain the formula (1.3) again (Corollary 3.6).

§ 2. Preliminaries

We first fix the convention on quantum groups (we basically follow to [7] and [11]). Assume that $q\in\mathbb{C}^\times$ is not a root of unity. The quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is an associative algebra generated by k, k^{-1}, e, f with the fundamental relations

$$kk^{-1}=k^{-1}k=1,\qquad kek^{-1}=q^2e,\qquad kfk^{-1}=q^{-2}f,\qquad ef-fe=\frac{k-k^{-1}}{q-q^{-1}}.$$

$\mathcal{U}_q(\mathfrak{sl}_2)$ has a (coassociative) coproduct

$$\begin{aligned}\Delta(k^{\pm1}) &= k^{\pm1}\otimes k^{\pm1},\\ \Delta(e) &= e\otimes 1 + k\otimes e,\\ \Delta(f) &= f\otimes k^{-1} + 1\otimes f,\end{aligned}$$

which enables us to define tensor products of $\mathcal{U}_q(\mathfrak{sl}_2)$-modules.

The quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_2)$ is an associative algebra generated by $x_{11},x_{12},x_{21},x_{22}$ with the fundamental relations

$$\begin{aligned}&x_{11}x_{12}=qx_{12}x_{11}, && x_{21}x_{22}=qx_{22}x_{21},\\ &x_{11}x_{21}=qx_{21}x_{11}, && x_{12}x_{22}=qx_{22}x_{12},\\ &x_{12}x_{21}=x_{21}x_{12}, && x_{11}x_{22}-x_{22}x_{11}=(q-q^{-1})x_{12}x_{21}.\end{aligned}\tag{2.1}$$

For convenience, we put

$$z_1 := x_{11}x_{22}, \quad z_2 := x_{12}x_{21}.$$

Notice that they *commute*:

$$z_1z_2 = z_2z_1.$$

The quantum α-determinant of size two is then a linear polynomial

$$\det_q^{(\alpha)} = x_{11}x_{22} + \alpha q x_{12}x_{21} = z_1 + \alpha q z_2$$

in commuting variables z_1, z_2.

Remark. The quantum α-determinant of size two interpolates the quantum counterparts of the determinant and permanent. In fact, we have

$$\det_q = x_{11}x_{22} - q x_{12}x_{21} = \det_q^{(-1)}, \quad \operatorname{per}_q = x_{11}x_{22} + q^{-1}x_{12}x_{21} = \det_q^{(q^{-2})}.$$

Here, in general, we define the *quantum permanent* of size n by

$$\operatorname{per}_q := \sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)} x_{\sigma(1)1}x_{\sigma(2)2}\cdots x_{\sigma(n)n} \in \mathcal{A}_q(\mathrm{Mat}_n),$$

which can be regarded as a q-analog of the usual permanent in the sense that the cyclic module $\mathcal{U}_q(\mathfrak{gl}_n)\cdot\operatorname{per}_q$ is equivalent to the q-analog of n-th symmetric tensor product of the natural representation $\mathbb{C}^n$ of $\mathcal{U}_q(\mathfrak{gl}_n)$. However, the quantum α-determinant of size n does not coincide with the quantum permanent for any α if $n \geq 3$. This is because $\nu(\cdot)$ is a class function on $\mathfrak{S}_n$ in general, whereas the inversion number $\ell(\cdot)$ is *not* if $n \geq 3$.

We briefly recall necessary basic facts on representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the standard basis of the vector space $\mathbb{C}^2$. By defining

$$\begin{aligned} k^{\pm1}\cdot\boldsymbol{e}_1 &:= q^{\pm1}\boldsymbol{e}_1, & e\cdot\boldsymbol{e}_1 &:= \boldsymbol{0}, & f\cdot\boldsymbol{e}_1 &:= \boldsymbol{e}_2, \\ k^{\pm1}\cdot\boldsymbol{e}_2 &:= q^{\mp1}\boldsymbol{e}_2, & e\cdot\boldsymbol{e}_2 &:= \boldsymbol{e}_1, & f\cdot\boldsymbol{e}_2 &:= \boldsymbol{0}, \end{aligned}$$

$\mathbb{C}^2$ becomes a $\mathcal{U}_q(\mathfrak{sl}_2)$-module. Put

$$\mathcal{M}_q(l+1) := \mathcal{U}_q(\mathfrak{sl}_2)\cdot\boldsymbol{v}_0^{(l)} \subset (\mathbb{C}^2)^{\otimes l}, \quad \boldsymbol{v}_0^{(l)} := \underbrace{\boldsymbol{e}_1\otimes\boldsymbol{e}_1\otimes\cdots\otimes\boldsymbol{e}_1}_{l}$$

for $l = 1, 2, \dots$. We also put $\mathcal{M}_q(1) := \mathbb{C}\cdot\boldsymbol{v}_0^{(0)}$ (with $\boldsymbol{v}_0^{(0)} := 1 \in \mathbb{C}$) and define $e\cdot\boldsymbol{v}_0^{(0)} = f\cdot\boldsymbol{v}_0^{(0)} := 0$ and $k^{\pm1}\cdot\boldsymbol{v}_0^{(0)} := \boldsymbol{v}_0^{(0)}$. If we set $\boldsymbol{v}_r^{(l)} := f^r\cdot\boldsymbol{v}_0^{(l)}$ $(r = 0, 1, \dots, l)$ and $\boldsymbol{v}_{-1}^{(l)} = \boldsymbol{v}_{l+1}^{(l)} = \boldsymbol{0}$, then it follows that $\mathcal{M}_q(l+1) = \bigoplus_{r=0}^{l}\mathbb{C}\cdot\boldsymbol{v}_r^{(l)}$ and

$$\begin{aligned} k^{\pm1}\cdot\boldsymbol{v}_r^{(l)} &= q^{\pm(l-2r)}\boldsymbol{v}_r^{(l)}, \\ e\cdot\boldsymbol{v}_r^{(l)} &= [l-r+1]_q\,\boldsymbol{v}_{r-1}^{(l)}, \\ f\cdot\boldsymbol{v}_r^{(l)} &= [r+1]_q\,\boldsymbol{v}_{r+1}^{(l)} \end{aligned}$$

for $r = 0, 1, \dots, l$. This is an irreducible $(l+1)$-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module.

Remark. There exists a convolution θ of $\mathcal{U}_q(\mathfrak{sl}_2)$ such that

$$\theta(e) = -e, \quad \theta(f) = f, \quad \theta(k^{\pm 1}) = k^{\mp 1}.$$

Using this, we can introduce inequivalent $\mathcal{U}_q(\mathfrak{sl}_2)$-module structure on the vector space $\mathcal{M}_q(l+1)$, say $\mathcal{M}_q(l+1)^\theta$. However, such modules do not appear in the following discussion. Note that any finite dimensional irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module is isomorphic to either $\mathcal{M}_q(l+1)$ or $\mathcal{M}_q(l+1)^\theta$ for some $l = 0, 1, 2, \ldots$ (see, e.g. [7, 9]).

The algebra $\mathcal{A}_q(\mathrm{Mat}_2)$ becomes a $\mathcal{U}_q(\mathfrak{sl}_2)$-module by

$$\begin{matrix} k^{\pm 1} \cdot x_{i1} := q^{\pm 1} x_{i1}, & e \cdot x_{i1} := 0, & f \cdot x_{i1} := x_{i2}, \\ k^{\pm 1} \cdot x_{i2} := q^{\mp 1} x_{i2}, & e \cdot x_{i2} := x_{i1}, & f \cdot x_{i2} := 0 \end{matrix} \qquad (i = 1, 2).$$

These are compatible with the fundamental relations (2.1) above. Notice that $e \cdot \det_q^r = f \cdot \det_q^r = 0$ and $k^{\pm 1} \cdot \det_q^r = \det_q^r$, (i.e. $\mathcal{U}_q(\mathfrak{sl}_2) \cdot \det_q^r \cong \mathcal{M}_q(1)$). It then follows that $X \cdot (v \det_q^r) = (X \cdot v) \det_q^r$ for $X \in \mathcal{U}_q(\mathfrak{sl}_2)$ and $v \in \mathcal{A}_q(\mathrm{Mat}_2)$. We have

$$\text{(2.2)} \quad \mathcal{U}_q(\mathfrak{sl}_2) \cdot (x_{11}x_{21})^s \det_q^{m-s} \cong \mathcal{U}_q(\mathfrak{sl}_2) \cdot (x_{11}x_{21})^s \cong \mathcal{M}_q(2s+1) \quad (s = 0, 1, 2, \ldots).$$

Actually, the linear map defined by $\boldsymbol{v}_r^{(l)} \mapsto f^r \cdot (x_{11}x_{21})^s$ gives a bijective intertwiner between $\mathcal{M}_q(2s+1)$ and $\mathcal{U}_q(\mathfrak{sl}_2) \cdot (x_{11}x_{21})^s$.

Define q-analogs of numbers, factorials and binomial coefficients by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]_q! := \prod_{i=1}^{n} [i]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! \, [n-k]_q!}.$$

§ 3. Cyclic modules generated by the quantum alpha-determinant

We fix a positive integer m and discuss the irreducible decomposition of the cyclic module $V_q^m(\alpha) := \mathcal{U}_q(\mathfrak{sl}_2) \cdot (\det_q^{(\alpha)})^m$. We refer to [1] for detailed discussion.

Put

$$v_{m,j} := \left(f^j \cdot (x_{11}x_{21})^j\right) \det_q^{m-j} \qquad (j = 0, 1, \ldots, m).$$

It is easy to see that the cyclic module $\mathcal{U}_q(\mathfrak{sl}_2) \cdot v_{m,j}$ is equivalent to $\mathcal{M}_q(2j+1)$. We show that $v_{m,j}$ is a homogeneous polynomial in z_1 and z_2 of degree m and give an explicit expression of it. For this purpose, we need the following two lemmas.

Lemma 3.1. *For each positive integer j, it follows that*

$$f^j \cdot (x_{11}x_{21})^j = q^{-j(j-1)/2} \, [j]_q! \sum_{r=0}^{j} q^{-r^2} \begin{bmatrix} j \\ r \end{bmatrix}_q^2 x_{11}^{j-r} x_{22}^{j-r} (x_{12}x_{21})^r.$$

Sketch of proof. For $1 \le i \le 2j$, put

$$f_j(i) := \overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes f \otimes \overbrace{k^{-1} \otimes \cdots \otimes k^{-1}}^{2j-i} \in \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2j}.$$

Then it follows that

$$\Delta^{2j-1}(f)^j = \sum_{1 \le n_1, \dots, n_j \le 2j} f_j(n_1) \cdots f_j(n_j).$$

Since $f_j(m)f_j(n) = q^{-2} f_j(n) f_j(m)$ if $m > n$ and $f^2 \cdot x_{11} = f^2 \cdot x_{21} = 0$, we have

$$\Delta^{2j-1}(f)^j = q^{-j(j-1)/2} \, [j]_q! \sum_{1 \le n_1 < \cdots < n_j \le 2j} f_j(n_1) \cdots f_j(n_j) + R,$$

where R is a certain element in $\mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2j}$ such that $R \cdot (x_{11}^j x_{21}^j) = 0$. Here we also use the well-known identity

$$\sum_{\sigma \in \mathfrak{S}_j} x^{\ell(\sigma)} = (1+x)(1+x+x^2) \cdots (1+x+\cdots+x^{j-1})$$

with $x = q^{-2}$. For given $n_1, \dots, n_j$ $(1 \le n_1 < \cdots < n_j \le 2j)$, we get

$$f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{21}^j) = q^{-r^2 + j(j-1)/2 + 2(n_1 + \cdots + n_r) - 2(m_1 + \cdots + m_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r$$

by a careful calculation (see [1] for detail). Using this, we have

$$\begin{aligned}
&f^j \cdot (x_{11}^j x_{22}^j) \\
&= q^{-j(j-1)/2} \, [j]_q! \sum_{1 \le n_1 < \cdots < n_j \le 2j} f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{21}^j) \\
&= [j]_q! \sum_{r=0}^{j} q^{-r^2} \sum_{\substack{1 \le n_1 < \cdots < n_r \le j \\ 1 \le m_1 < \cdots < m_r \le j}} q^{2(n_1 + \cdots + n_r)} q^{-2(m_1 + \cdots + m_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r \\
&= [j]_q! \sum_{r=0}^{j} q^{-r^2} e_r(1, q^2, \dots, q^{2(j-1)}) e_r(1, q^{-2}, \dots, q^{-2(j-1)}) x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r,
\end{aligned}$$

where $e_r(x_1, x_2, \dots, x_j)$ is the rth elementary symmetric polynomial in $x_1, x_2, \dots, x_j$. By the identity (see, e.g. [10])

$$e_r(1, q^2, \dots, q^{2j-2}) = q^{r(j-1)} \begin{bmatrix} j \\ r \end{bmatrix}_q$$

together with the symmetry $\begin{bmatrix} j \\ r \end{bmatrix}_q = \begin{bmatrix} j \\ r \end{bmatrix}_{q^{-1}}$, we obtain

$$f^j \cdot (x_{11}^j x_{21}^j) = [j]_q! \sum_{r=0}^{j} q^{-r^2} \begin{bmatrix} j \\ r \end{bmatrix}_q^2 x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r.$$

Since $(x_{11} x_{21})^j = q^{-j(j-1)/2} x_{11}^j x_{21}^j$, we have the desired conclusion. □

Lemma 3.2. *For each positive integer l, it follows that*

$$x_{11}^l x_{22}^l = \prod_{s=1}^{l} \left(z_1 + (q^{2s-1} - q)z_2\right) = \sum_{j=0}^{l} q^{l(l-j)} \begin{bmatrix} l \\ j \end{bmatrix}_q (z_1 - qz_2)^j z_2^{l-j}.$$

Proof. The first equality is proved by induction on l by using the relation

$$(z_1 + (q^{2r-1} - q)z_2)x_{22} = x_{22}(z_1 + (q^{2r+1} - q)z_2).$$

The second equality is a specialization of the q-binomial theorem

$$\prod_{i=1}^{n}(x + q^{2i-n-1}y) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q x^r y^{n-r},$$

which is indeed applicable since z_1 and z_2 commute. □

As a result, we get the explicit expression

(3.1)
$$v_{m,j} = \left\{ q^{-j(j-1)/2} [j]_q! \sum_{r=0}^{j} q^{-r^2} \begin{bmatrix} j \\ r \end{bmatrix}_q^2 z_2^r \prod_{s=1}^{j-r} \left(z_1 + (q^{2s-1} - q)z_2\right) \right\} (z_1 - qz_2)^{m-j},$$

which is indeed a homogeneous polynomial in z_1 and z_2 of degree m.

The vectors $\{v_{m,j}\}_{j=0}^m$ are linearly independent since they belong to inequivalent representations, and hence form a basis of the space consisting of homogeneous polynomials in z_1 and z_2 of degree m, whose dimension is $m+1$. Thus we conclude that $(\det_q^{(\alpha)})^m (\in \mathcal{A}_q(\mathrm{Mat}_2) \otimes_{\mathbb{C}} \mathbb{C}[\alpha]))$ can be expressed as a linear combination

$$\left(\det_q^{(\alpha)}\right)^m = \sum_{j=0}^{m} F_{m,j}(\alpha) v_{m,j} \tag{3.2}$$

of the vectors $v_{m,j}$, where $F_{m,j}(\alpha)$ are certain polynomial functions in α. This implies that the cyclic module $V_q^m(\alpha)$ contains an irreducible submodule equivalent to $\mathcal{M}_q(2j+1)$ (with multiplicity one) if $F_{m,j}(\alpha) \neq 0$. Consequently, it follows that

$$V_q^m(\alpha) \cong \bigoplus_{\substack{0 \le j \le m \\ F_{m,j}(\alpha) \neq 0}} \mathcal{M}_q(2j+1). \tag{3.3}$$

Let us determine the functions $F_{m,j}(\alpha)$ explicitly. The conditions (3.2) for the functions $F_{m,j}(\alpha)$ are given in terms of polynomials in commuting variables z_1, z_2, so that it is meaningful to consider the specialization $z_1 = z$, $z_2 = 1$ in (3.2), where z is a new variable. Put

$$g_j(z) := \prod_{i=1}^{j} (z + q^{2i-1} - q) = \sum_{i=0}^{j} q^{j(j-i)} \begin{bmatrix} j \\ i \end{bmatrix}_q (z-q)^i,$$
$$v_j(z) := q^{-j(j-1)/2} [j]_q! \sum_{r=0}^{j} q^{-r^2} \begin{bmatrix} j \\ r \end{bmatrix}_q^2 g_{j-r}(z).$$

Then (3.2) together with Lemmas 3.1 and 3.2 yields

$$(z+q\alpha)^m = \sum_{j=0}^{m} F_{m,j}(\alpha) v_j(z)(z-q)^{m-j}.$$

If we take the lth derivative of this formula with respect to z ($l = 0, 1, \ldots, m$) and substitute $z = q$, then we get the relation

$$\binom{m}{l} q^{m-l}(1+\alpha)^{m-l} = \sum_{j=m-l}^{m} F_{m,j}(\alpha) \frac{v_j^{(l-m+j)}(q)}{(l-m+j)!} = \sum_{s=0}^{l} F_{m,m-s}(\alpha) \frac{v_{m-s}^{(l-s)}(q)}{(l-s)!}. \tag{3.4}$$

Since

$$v_j(z) = q^{j(j+1)/2} [j]_q! \sum_{i=0}^{j} q^{-ij} \left\{ \begin{bmatrix} j \\ i \end{bmatrix}_q \sum_{r=0}^{j-i} q^{r(i-2j)} \begin{bmatrix} j \\ j-r \end{bmatrix}_q \begin{bmatrix} j-i \\ r \end{bmatrix}_q \right\} (z-q)^i$$
$$= q^{-j(j-1)/2} [j]_q! \sum_{i=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} 2j-i \\ j \end{bmatrix}_q (z-q)^i,$$

we have

$$v_j(z) = q^{-j(j-1)/2} \sum_{i=0}^{j} \frac{[j]_q! \, [2j-i]_q!}{[i]_q! \, [j-i]_q!^2} (z-q)^i,$$

or

$$\frac{v_j^{(i)}(q)}{i!} = q^{-\binom{j}{2}} \frac{[j]_q! \, [2j-i]_q!}{[i]_q! \, [j-i]_q!^2}.$$

Here we use the q-Chu-Vandermonde formula

$$\sum_{r=0}^{j-i} q^{r(i-2j)} \begin{bmatrix} j \\ j-r \end{bmatrix}_q \begin{bmatrix} j-i \\ r \end{bmatrix}_q = q^{j(i-j)} \begin{bmatrix} 2j-i \\ j \end{bmatrix}_q.$$

Thus the formula (3.4) is rewritten more explicitly as

$$[m-l]_q!^2 \binom{m}{l} q^{m-l}(1+\alpha)^{m-l} = \sum_{s=0}^{l} q^{-\binom{m-s}{2}} \frac{[m-s]_q! \, [2m-l-s]_q!}{[l-s]_q!} F_{m,m-s}(\alpha). \tag{3.5}$$

This also implies that the polynomial $F_{m,j}(\alpha)$ is divisible by $(1+\alpha)^j$, that is

$$F_{m,j}(\alpha) = (1+\alpha)^j Q_{m,j}(\alpha) \tag{3.6}$$

for some $Q_{m,j}(\alpha) \in \mathbb{C}[\alpha]$. By (3.6) and (3.5), we have

$$\begin{aligned} &\begin{bmatrix} 2m-2i \\ m-i \end{bmatrix}_q^{-1} \binom{m}{i} q^{m-i} \\ &= \sum_{j=0}^{i} \begin{bmatrix} 2i-2m-1 \\ i-j \end{bmatrix}_q (-1)^{i-j}(1+\alpha)^{i-j} \cdot q^{-\binom{m-j}{2}} [m-j]_q! Q_{m,m-j}(\alpha). \end{aligned} \tag{3.7}$$

Now we define the *mixed hypergeometric series* by

$$\Phi\left(\begin{matrix} a_1, \dots, a_k \\ b_1, \dots, b_l \end{matrix} ; \begin{matrix} c_1, \dots, c_m \\ d_1, \dots, d_n \end{matrix} ; q ; x \right) = \sum_{i=0}^{\infty} \frac{(a_1; i) \cdots (a_k; i)}{(b_1; i) \dots (b_l; i)} \frac{(c_1; i)_q \cdots (c_m; i)_q}{(d_1; i)_q \cdots (d_m; i)_q} \frac{x^i}{[i]_q!}, \tag{3.8}$$

where $(a; i) = a(a+1) \cdots (a+i-1)$ and $(a; i)_q = [a]_q [a+1]_q \cdots [a+i-1]_q$ (cf. [8]).

Theorem 3.3. *For $s = 0, 1, \dots, m$,*

$$F_{m,s}(\alpha) = q^{\binom{s+1}{2}} \binom{m}{s} \frac{[s]_q!}{[2s]_q!} (1+\alpha)^s \, \Phi\left(\begin{matrix} s-m \\ s+1 \end{matrix} ; \begin{matrix} s+1, s+1 \\ 2s+2 \end{matrix} ; q ; q(1+\alpha) \right)$$

holds.

Sketch of proof. We can prove the identity

$$\left(\begin{bmatrix} 2i-2m-1 \\ i-j \end{bmatrix}_q \right)_{0 \le i,j \le m}^{-1} = \left(\frac{[2m-2i+1]_q}{[2m-2j+1]_q} \begin{bmatrix} 2m-2j+1 \\ i-j \end{bmatrix}_q \right)_{0 \le i,j \le m}.$$

Using this, we solve (3.7) and find that

$$\begin{aligned} Q_{m,m-i}(\alpha) &= \frac{q^{\binom{m-i}{2}} [2m-2i+1]_q}{[m-i]_q!} \\ &\quad \times \sum_{j=0}^{i} \frac{(-1)^{i-j} q^{m-j}}{[2m-2j+1]_q} \begin{bmatrix} 2m-2j+1 \\ i-j \end{bmatrix}_q \begin{bmatrix} 2m-2j \\ m-j \end{bmatrix}_q^{-1} \binom{m}{j} (1+\alpha)^{i-j} \\ &= \frac{q^{\binom{m-i+1}{2}} m! \, [2m-2i+1]_q}{[m-i]_q!} \\ &\quad \times \sum_{r=0}^{i} \frac{(-q)^r [m-i+r]_q!^2}{(m-i+r)!(i-r)! \, [2m-2i+r+1]_q!} \frac{(1+\alpha)^r}{[r]_q!}, \end{aligned}$$

where we set $r = i - j$. Since

$$(i-r)! = (-1)^r \frac{i!}{(-i;r)}, \quad (n+r)! = n!\,(n+1;r)\,, \quad [n+r]_q! = [n]_q!\,(n+1;r)_q\,,$$

we have

$$\begin{aligned} &Q_{m,m-i}(\alpha) \\ &= \frac{q^{\binom{m-i+1}{2}} m!\,[m-i]_q!}{i!(m-i)!\,[2m-2i]_q!} \sum_{r=0}^{i} \frac{(-i;r)\,(m-i+1;r)_q^2}{(m-i+1;r)\,(2m-2i+2;r)_q} \frac{(q(1+\alpha))^r}{[r]_q!} \qquad (3.9)\\ &= q^{\binom{m-i+1}{2}} \binom{m}{i} \frac{[m-i]_q!}{[2m-2i]_q!} \Phi\left(\begin{matrix} -i \\ m-i+1 \end{matrix} ; \begin{matrix} m-i+1, m-i+1 \\ 2m-2i+2 \end{matrix} ; q\,;\, q(1+\alpha) \right). \end{aligned}$$

If we substitute this into (3.6) and replace $m - i$ by s, then we have the conclusion. □

Example 3.4 ($m = 1$). We have

$$\begin{aligned} F_{1,0}(\alpha) &= \Phi\left(\begin{matrix} -1 \\ 1 \end{matrix} ; \begin{matrix} 1,1 \\ 2 \end{matrix} ; q\,;\, q(1+\alpha) \right) = \frac{1-\alpha q^2}{1+q^2}, \\ F_{1,1}(\alpha) &= q\frac{[1]_q!}{[2]_q!}(1+\alpha)\Phi\left(\begin{matrix} 0 \\ 2 \end{matrix} ; \begin{matrix} 2,2 \\ 4 \end{matrix} ; q\,;\, q(1+\alpha) \right) = \frac{q}{[2]_q}(1+\alpha). \end{aligned}$$

Thus it follows that

$$\mathcal{U}_q(\mathfrak{sl}_2) \cdot \det_q^{(\alpha)} \cong \begin{cases} \mathcal{M}_q(3) & \alpha = q^{-2}, \\ \mathcal{M}_q(1) & \alpha = -1, \\ \mathcal{M}_q(1) \oplus \mathcal{M}_q(3) & \text{otherwise}. \end{cases}$$

Notice that $q^{-2} \neq -1$ since we assume that q is not a root of unity.

Example 3.5 ($m = 2$). We have

$$\begin{aligned} F_{2,0}(\alpha) &= \Phi\left(\begin{matrix} -2 \\ 1 \end{matrix} ; \begin{matrix} 1,1 \\ 2 \end{matrix} ; q\,;\, q(1+\alpha) \right) \\ &= C_0(q)\left((q^6+q^4)\alpha^2 - 2q^2\alpha + q^4 + 1\right), \\ F_{2,1}(\alpha) &= 2q^2\frac{[1]_q!}{[2]_q!}(1+\alpha)\Phi\left(\begin{matrix} -1 \\ 2 \end{matrix} ; \begin{matrix} 2,2 \\ 4 \end{matrix} ; q\,;\, q(1+\alpha) \right) \\ &= C_1(q)(1+\alpha)\left((q^4+q^2)\alpha - q^4 + q^2 - 2\right), \\ F_{2,2}(\alpha) &= q^3\frac{[2]_q!}{[4]_q!}(1+\alpha)^2\Phi\left(\begin{matrix} 0 \\ 3 \end{matrix} ; \begin{matrix} 3,3 \\ 6 \end{matrix} ; q\,;\, q(1+\alpha) \right) \\ &= C_2(q)(1+\alpha)^2, \end{aligned}$$

where $C_0(q), C_1(q), C_2(q)$ are certain rational functions in q. Hence, if we assume that q is *transcendental* (we have only to assume that $(q^4-q^2+1)^2+q^2(q^4+q^2+1)\neq 0$ practically), then we see that

$$\mathcal{U}_q(\mathfrak{sl}_2)\cdot\left(\det_q^{(\alpha)}\right)^2 \cong \begin{cases} \mathcal{M}_q(1) & \alpha=-1,\\ \mathcal{M}_q(1)\oplus\mathcal{M}_q(5) & \alpha=\frac{q^4-q^2+2}{q^4+q^2},\\ \mathcal{M}_q(3)\oplus\mathcal{M}_q(5) & \alpha=\frac{1\pm q\sqrt{-q^4-q^2-1}}{q^4+q^2},\\ \mathcal{M}_q(1)\oplus\mathcal{M}_q(3)\oplus\mathcal{M}_q(5) & \text{otherwise.}\end{cases}$$

When $(q^4-q^2+1)^2+q^2(q^4+q^2+1)=0$ (this does not implies that q is a root of unity), $\frac{q^4-q^2+2}{q^4+q^2}$ becomes a common root of $F_{2,0}(\alpha)$ and $F_{2,1}(\alpha)$, so that we have

$$\mathcal{U}_q(\mathfrak{sl}_2)\cdot\left(\det_q^{(\alpha)}\right)^2 \cong \mathcal{M}_q(5), \qquad \alpha=\frac{q^4-q^2+2}{q^4+q^2}.$$

Remark. The mixed hypergeometric series (3.8) can be regarded as a common generalization of the generalized hypergeometric series and basic hypergeometric series as we see below:

$$\Phi\left(\begin{matrix}a_1,\dots,a_k\\ b_1,\dots,b_l,1\end{matrix};\begin{matrix}1\\ \end{matrix};q;x\right) = {}_kF_l\left(\begin{matrix}a_1,\dots,a_k\\ b_1,\dots,b_l\end{matrix};x\right),$$

$$\Phi\left(;\begin{matrix}c_1,\dots,c_m\\ d_1,\dots,d_n\end{matrix};q;x\right) = {}_m\phi_n\left(\begin{matrix}q^{2c_1},\dots,q^{2c_m}\\ q^{2d_1},\dots,q^{2d_n}\end{matrix};q^2,(-1)^{1+n-m}q^{1+d-c}x\right),$$

where $c=c_1+\cdots+c_m$ and $d=d_1+\cdots+d_n$.

Remark. The function Φ given by (3.8) satisfies the difference-differential equation

$$\Big\{-(E+a_1)\cdots(E+a_k)\,[E+c_1]_q\cdots[E+c_m]_q \\ +\partial_q(E+b_1-1)\cdots(E+b_l-1)\,[E+d_1-1]_q\cdots[E+d_n-1]_q\Big\}\Phi=0,$$

where we put

$$E=x\frac{d}{dx},\quad [E+a]_q=\frac{q^{E+a}-q^{-E-a}}{q-q^{-1}},\quad \partial_q f(x)=\frac{f(qx)-f(q^{-1}x)}{qx-q^{-1}x}.$$

If we take a limit $q\to 1$, then the equation above becomes a hypergeometric differential equation for ${}_{k+m}F_{l+n}(a_1,\dots,a_k,c_1,\dots,c_m;b_1,\dots,b_l,d_1,\dots,d_n;x)$.

All the discussion above also work in the classical case (i.e. the case where $q=1$). Thus, by taking a limit $q\to 1$ in Theorem 3.3, we will obtain Theorem 4.1 in [2] (or

(1.3) up to constant) again. We abuse the same notations used in the discussion of quantum case above to indicate the classical counterparts. From (3.9), we have

$$\begin{aligned} Q_{m,s}(\alpha) &= \frac{m!}{(m-s)!(2s)!}\, {}_3F_2\left(\begin{matrix} s-m, s+1, s+1 \\ s+1, 2s+2 \end{matrix}; 1+\alpha\right) \\ &= \frac{m!}{(m-s)!(2s)!}\, {}_2F_1\left(\begin{matrix} s-m, s+1 \\ 2s+2 \end{matrix}; 1+\alpha\right). \end{aligned}$$

Notice that

$${}_2F_1\left(\begin{matrix} s-m, s+1 \\ 2s+2 \end{matrix}; 1-x\right) = \frac{m!(2s+1)!}{s!(m+s+1)!}\, {}_2F_1\left(\begin{matrix} s-m, s+1 \\ -m \end{matrix}; x\right).$$

Thus we also get

$$Q_{m,s}(\alpha) = \frac{m!^2(2s+1)}{(m-s)!s!(m+s+1)!}\, {}_2F_1\left(\begin{matrix} s-m, s+1 \\ -m \end{matrix}; -\alpha\right) \qquad (s=0,1,\dots,m).$$

Summarizing these, we have the

Corollary 3.6 (Classical case)**.** *It follows that*

$$\begin{aligned} F_{m,s}(\alpha) &= \frac{m!}{(m-s)!(2s)!}(1+\alpha)^s\, {}_2F_1\left(\begin{matrix} s-m, s+1 \\ 2s+2 \end{matrix}; 1+\alpha\right) \\ &= \frac{\binom{2m}{m-s} - \binom{2m}{m-s-1}}{\binom{2m}{m} s!}(1+\alpha)^s\, {}_2F_1\left(\begin{matrix} s-m, s+1 \\ -m \end{matrix}; -\alpha\right) \end{aligned}$$

for $s = 0, 1, \dots, m$.

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