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MONOTONICITY IN STEEPEST ASCENT ALGORITHMS FOR POLYHEDRAL L-CONCAVE FUNCTIONS

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Abstract For the minimum cost flow problem, Hassin (1983) proposed a dual algorithm, which iteratively updates dual variables in a steepest ascent manner. This algorithm is generalized to the minimum cost submodular flow problem by Chung and Tcha (1991). In discrete convex analysis, the dual of the minimum cost flow problem is known to be formulated as maximization of a polyhedral L-concave function. It is recently pointed out by Murota and Shioura (2014) that Hassin's algorithm can be recognized as a steepest ascent algorithm for polyhedral L-concave functions. The objective of this paper is to show some monotonicity properties of the steepest ascent algorithm for polyhedral L-concave functions. We show that the algorithm shares a monotonicity property of Hassin's algorithm. Moreover, the algorithm finds the "nearest" optimal solution to a given initial solution, and the trajectory of the solutions generated by the algorithm is a "shortest" path from the initial solution to the "nearest" optimal solution. The algorithm and its properties can be extended for polyhedral L$^\natural$-concave functions.

Keywords: Combinatorial optimization, discrete concave function, steepest ascent algorithm, minimum cost flow, discrete optimization

1. Introduction

Among many algorithms for the minimum cost flow problem (see, e.g., [1, 17]), Hassin's dual algorithm [6] is unique in that it maintains only dual variables, while most of the other algorithms use primal (i.e., flow) variables. Hassin's algorithm iteratively chooses a subset of dual variables that corresponds to a graph cut and increments them so that the dual objective function increases strictly. It is shown in [6] that the sequence of solutions generated by the algorithm has a certain monotonicity property, from which the finite termination of the algorithm follows. Hassin's algorithm is later generalized to the minimum cost submodular flow problem by Chung and Tcha [2].

In discrete convex analysis [9, 10], the dual of the minimum cost (submodular) flow problem is known to be formulated as the maximization of a polyhedral L-concave function. The concept of polyhedral L-concave functions in real variables was introduced by Murota and Shioura [12] as a variant of L-concave functions originally defined for functions on integer lattice points. It is pointed out in [13] that Hassin's algorithm as well as Chung and Tcha's algorithm can be recognized as a steepest ascent algorithm for polyhedral L-concave functions, where the steepest ascent direction is chosen from a finite set of 0-1 vectors. This observation indicates that the steepest ascent algorithm for polyhedral L-concave functions is fundamental in combinatorial optimization.

In this paper, we investigate the behavior of the steepest ascent algorithm for polyhedral L-concave function maximization and show its monotonicity properties. First, it is endowed with the same monotonicity property as that of Hassin's algorithm, which guarantees its finite termination. Second, for any initial solution, the algorithm finds the smallest optimal solution that is not smaller than the initial solution. Third, the trajectory of the solutions generated by the algorithm is a "shortest" path from the initial solution to the smallest optimal solution in the sense that the total sum of the step lengths is equal to the ℓ_∞-distance from the initial solution to the smallest optimal solution. Fourth, the function g restricted on the trajectory of the solutions generated by the algorithm is a concave function. Our second and third results imply, in particular, that Hassin's and Chung and Tcha's algorithms are efficient in terms of the number of iterations. The steepest ascent algorithm for polyhedral L-concave functions can naturally be adapted to polyhedral L$^\natural$-concave functions. The algorithm outputs the optimal solution that is "nearest" with respect to a variant of the ℓ_∞-distance.

In each iteration of the steepest ascent algorithm discussed in this paper, there may be several choices of steepest ascent directions, and from among them, a steepest ascent direction satisfying a certain "minimality" condition is chosen. We consider a variant of the steepest ascent algorithm which chooses an arbitrary steepest ascent direction in each iteration, and show that the modified algorithm still satisfies some of the monotonicity properties. We also prove that the modified algorithm terminates if the input function satisfies a certain "rationality" condition. On the other hand, we show by giving a bad instance for the modified algorithm that the "minimality" condition of a steepest ascent direction is essential for the finite termination of the algorithm if a polyhedral L-concave function is not "rational."

The organization of this paper is as follows. We review the concept of polyhedral L-concave function in Section 2, and Hassin's and Chung and Tcha's algorithms for the dual of the minimum cost (submodular) flow problems in Section 3. Monotonicity properties of the steepest ascent algorithm for polyhedral L-concave functions are shown in Section 4. The algorithm and its properties can be extended for polyhedral L$^\natural$-concave functions in Section 5. Finally, a variant of the steepest ascent algorithm is discussed in Section 6.

2. Preliminaries on L-concave Functions

We review the concept of polyhedral L-concave functions. Throughout this paper, let V be a finite set. For a function $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$, its *effective domain* is defined as

$$\mathrm{dom}\, g = \{p \in \mathbb{R}^V \mid g(p) > -\infty\}.$$

A function $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ is said to be a *polyhedral concave function* if the set

$$\{(p, \alpha) \in \mathbb{R}^V \times \mathbb{R} \mid p \in \mathrm{dom}\, g,\ \alpha \leq g(p)\}$$

is a (nonempty) polyhedron. Equivalently, $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ is a polyhedral concave function if there exist a nonempty polyhedron $S \subseteq \mathbb{R}^V$, a finite number of vectors $a_1, \ldots, a_t \in \mathbb{R}^V$, and scalars $b_1, \ldots, b_t \in \mathbb{R}$ such that

$$\mathrm{dom}\, g = S, \qquad g(p) = \min_{1 \leq i \leq t}\{a_i^{\mathrm{T}} p + b_i\} \qquad (p \in S). \tag{2.1}$$

For $p \in \mathrm{dom}\, g$ and $d \in \mathbb{R}^V$, the *directional derivative* of g at p in direction d is defined as the limit

$$g'(p; d) = \lim_{\lambda \downarrow 0} \frac{g(p + \lambda d) - g(p)}{\lambda},$$

if it exists. We also define $g'(p; d) = -\infty$ if $p + \lambda d \notin \mathrm{dom}\, g$ for all $\lambda > 0$. Moreover, when $g'(p; d) > -\infty$, define the value $\bar{c}(p; d) \in \mathbb{R} \cup \{+\infty\}$ by

$$\bar{c}(p; d) = \sup\{\lambda \in \mathbb{R}_+ \mid g(p + \lambda d) - g(p) = \lambda\, g'(p; d)\}. \tag{2.2}$$

If function g is given in the form (2.1), then we have

$$g'(p;d) = a_i^{\mathrm{T}} d, \qquad \bar{c}(p;d) = \sup\{\lambda \in \mathbb{R}_+ \mid g(p+\lambda d) = a_i^{\mathrm{T}}(p+\lambda d) + b_i\}$$

for some i with $1 \le i \le t$. Note that $\bar{c}(p;d) > 0$ and $g(p+\lambda d) - g(p) = \lambda\, g'(p;d)$ holds for every λ with $0 \le \lambda \le \bar{c}(p;d)$. Hence, if $g'(p;d) > 0$ and g is bounded from above, then $\bar{c}(p;d) < +\infty$.

A polyhedral concave function $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ is said to be *polyhedral L-concave* [12] if it satisfies the following conditions:

(LF1) $g(p) + g(q) \le g(p \wedge q) + g(p \vee q) \quad (\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbb{R}$ s.t. $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \ (\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbb{R})$,

where $p \wedge q, p \vee q\ (\in \mathbb{R}^V)$ denote the vectors with

$$(p \wedge q)(v) = \min\{p(v), q(v)\}, \quad (p \vee q)(v) = \max\{p(v), q(v)\} \quad (v \in V),$$

and $\mathbf{1}$ $(\in \mathbb{R}^V)$ is the vector with each component being equal to one. Note that $r = 0$ is assumed in (LF2) whenever we consider maximization of a polyhedral L-concave function since otherwise there exists no maximizer.

A typical example of a polyhedral L-concave function arises from the maximum weight tension problem. For a directed graph $G = (V, E)$, let $\varphi_{uv} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be an edge weight function for $(u, v) \in E$. We assume that functions φ_{uv} $((u, v) \in E)$ are polyhedral (or piecewise-linear) concave functions. The maximum weight tension problem is formulated as follows:

$$\textbf{(MWT)} \quad \left|\ \begin{array}{ll} \text{Maximize} & \displaystyle\sum_{(u,v)\in E} \varphi_{uv}(p(u) - p(v)) \\ \text{subject to} & p \in \mathbb{R}^V. \end{array}\right.$$

We denote by $g_{\mathrm{T}} : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ the objective function of the problem (MWT). Note that $\mathrm{dom}\, g_{\mathrm{T}} \ne \emptyset$ if and only if (MWT) has a feasible solution, i.e., there exists some $p \in \mathbb{R}^V$ such that $\varphi_{uv}(p(u) - p(v)) > -\infty$ for all $(u, v) \in E$.

Proposition 2.1 ([12, Example 2.4])**.** *Suppose that* $\mathrm{dom}\, g_{\mathrm{T}} \ne \emptyset$*. Then, the function* g_{T} *is polyhedral L-concave with* $r = 0$ *in* (LF2).

Another example of a polyhedral L-concave function comes from the so-called Lovász extension of a submodular function. A set function $\rho : 2^V \to \mathbb{R}$ is said to be *submodular* if it satisfies

$$\rho(X) + \rho(Y) \ge \rho(X \cap Y) + \rho(X \cup Y) \qquad (\forall X, Y \subseteq V).$$

Given a set function $\rho : 2^V \to \mathbb{R}$ with $\rho(\emptyset) = 0$, define a function $\hat{\rho} : \mathbb{R}^V \to \mathbb{R}$ by

$$\hat{\rho}(p) = \sum_{i=1}^{h-1} (\tilde{p}_i - \tilde{p}_{i+1}) \rho(L_i) + \tilde{p}_h \rho(L_h) \qquad (p \in \mathbb{R}^V), \tag{2.3}$$

where $\tilde{p}_1 > \tilde{p}_2 > \cdots > \tilde{p}_h$ are the distinct values of components of p and

$$L_i \ = \ \{v \in V \mid p(v) \ge \tilde{p}_i\} \qquad (i = 1, 2, \ldots, h).$$

The function $\hat{\rho}$ is called the *Lovász extension* of ρ.

Proposition 2.2 ([12, Theorem 4.36])**.** *For a submodular set function* $\rho : 2^V \to \mathbb{R}$ *with* $\rho(\emptyset) = 0$*, the function* $-\hat{\rho}$ *is a polyhedral L-concave function. If* $\rho(V) = 0$*, then* $-\hat{\rho}$ *satisfies property* (LF2) *with* $r = 0$.

3. Hassin's and Chung and Tcha's Algorithms

As the motivation of the present paper, we review the dual algorithm for the minimum cost flow problem by Hassin [6] and that for the minimum cost submodular flow problem by Chung and Tcha [2]. We also observe the polyhedral L-concavity of the dual objective functions of the problems in [6] and in [2]. In the following, we denote by $\chi_X \in \{0,1\}^V$ the characteristic vector of $X \subseteq V$, i.e., $\chi_X(v) = 1$ if $v \in X$ and $\chi_X(v) = 0$ if $v \in V \setminus X$.

3.1. Hassin's algorithm

For a directed graph $G = (V, E)$ with nonnegative edge capacity $c(e)$ and edge cost $\gamma(e)$ for $e \in E$, the minimum cost flow problem treated in [6] is formulated as follows:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{(u,v)\in E} \gamma(u,v)x(u,v) \\ \text{subject to} & \partial x(u) = 0 \quad (u \in V), \\ & 0 \le x(u,v) \le c(u,v) \quad ((u,v) \in E), \end{array}$$

where

$$\partial x(u) = \sum_{v:(u,v)\in E} x(u,v) - \sum_{v:(v,u)\in E} x(v,u) \qquad (u \in V).$$

The dual problem is given as

$$\begin{array}{ll} \text{Maximize} & g_{\mathrm{H}}(p) \equiv \displaystyle\sum_{(u,v)\in E} c(u,v)\min\{0, p(u) - p(v) + \gamma(u,v)\} \\ \text{subject to} & p \in \mathbb{R}^V. \end{array} \tag{3.1}$$

The function g_{H} is polyhedral L-concave since it is a special case of the function g_{T} in Proposition 2.1.

Hassin's algorithm is described as follows. For $p \in \mathbb{R}^V$ and $X \subseteq V$, we define

$$I(p, X) = \sum_{(u,v)\in E^<_{\mathrm{out}}(p,X)} c(u,v) - \sum_{(u,v)\in E^\le_{\mathrm{in}}(p,X)} c(u,v), \tag{3.2}$$

where

$$\begin{array}{rcl} E^<_{\mathrm{out}}(p,X) & = & \{(u,v) \in E \mid p(u) - p(v) + \gamma(u,v) < 0,\ u \in X,\ v \in V \setminus X\}, \\ E^\le_{\mathrm{in}}(p,X) & = & \{(u,v) \in E \mid p(u) - p(v) + \gamma(u,v) \le 0,\ u \in V \setminus X,\ v \in X\}. \end{array}$$

We also define $\lambda(p, X)$ by

$$\lambda(p,X) = \min\left\{|p(u) - p(v) + \gamma(u,v)| \;\middle|\; (u,v) \in E^<_{\mathrm{out}}(p,X) \cup E^>_{\mathrm{in}}(p,X)\right\}, \tag{3.3}$$

where

$$E^>_{\mathrm{in}}(p,X) = \{(u,v) \in E \mid p(u) - p(v) + \gamma(u,v) > 0,\ u \in V \setminus X, v \in X\}.$$

Then, it holds that

$$g_{\mathrm{H}}(p + \alpha\chi_X) - g_{\mathrm{H}}(p) = \alpha I(p,X) \qquad (0 \le \forall\alpha \le \lambda(p,X)).$$

Hassin's Algorithm

Step 0: Set $p := p^\circ$, where p° is an initial vector chosen from $\mathbb{R}^V$.

Step 1: Find $X \subseteq V$ that maximizes $I(p, X)$; if there exists more than one such X, then take a (unique) minimal one.
Step 2: If $I(p, X) \leq 0$, then stop; p is a maximizer of g_{H}.
Step 3: Set $p := p + \lambda(p, X)\,\chi_X$. Go to Step 1. □

We note that the set X in Step 1 can be obtained by solving a minimum cut problem.

For each positive integer k, we denote by X_k and p_k, respectively, the set X and the vector p just after Step 1 in the k-th iteration. The next property shows that the value $I(p_k, X_k)$ is monotone nonincreasing.

Proposition 3.1 ([6])**.** *For $k = 1, 2, \ldots$, $I(p_k, X_k) \geq I(p_{k+1}, X_{k+1})$ holds. Moreover, if $I(p_k, X_k) = I(p_{k+1}, X_{k+1})$, then we have $X_k \subsetneq X_{k+1}$.*

It is observed in [6] that the set of possible values of $I(p, X)$ is finite, and hence the algorithm terminates in a finite number of iterations by Proposition 3.1 (see [6] for details).

3.2. Chung and Tcha's algorithm

Suppose now that a submodular function $\rho : 2^V \to \mathbb{R}$ with $\rho(\emptyset) = \rho(V) = 0$ is given, in addition to a directed graph $G = (V, E)$ with nonnegative edge capacity $c(e)$ and edge cost $\gamma(e)$ for $e \in E$. Then, the minimum cost submodular flow problem is formulated as follows:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{(u,v)\in E} \gamma(u,v)x(u,v) \\ \text{subject to} & \displaystyle\sum_{u\in Y} \partial x(u) \leq \rho(Y) \quad (Y \subsetneq V), \qquad \sum_{u\in V} \partial x(u) = \rho(V), \\ & 0 \leq x(u,v) \leq c(u,v) \quad ((u,v) \in E). \end{array}$$

The linear programming dual is given as

$$\begin{array}{ll} \text{Maximize} & \displaystyle -\sum_{(u,v)\in E} c(u,v)s(u,v) - \sum_{Y\subseteq V} \rho(Y)t(Y) \\ \text{subject to} & \displaystyle -s(u,v) - \sum_{Y:u\in Y} t(Y) + \sum_{Y:v\in Y} t(Y) \leq \gamma(u,v) \quad ((u,v) \in E), \\ & s(u,v) \geq 0 \qquad ((u,v) \in E), \\ & t(Y) \geq 0 \quad (Y \subsetneq V), \qquad t(V) \in \mathbb{R}. \end{array}$$

It is known that for every vector $p \in \mathbb{R}^V$, the real numbers $s_p(u,v)$ $((u,v) \in E)$ and $t_p(Y)$ $(Y \subseteq V)$ defined by

$$\begin{array}{rcl} s_p(u,v) & = & -\min\{0, p(u) - p(v) + \gamma(u,v)\} \qquad ((u,v) \in E), \\ t_p(Y) & = & \begin{cases} \tilde{p}_i - \tilde{p}_{i+1} & (\text{if } Y = L_i,\ 1 \leq i \leq h-1), \\ \tilde{p}_h & (\text{if } Y = L_h), \\ 0 & (\text{otherwise}) \end{cases} \end{array} \tag{3.4}$$

provide a feasible solution of the dual problem, where $\tilde{p}_1 > \tilde{p}_2 > \cdots > \tilde{p}_h$ are the distinct values of components of p and $L_i = \{v \in V \mid p(v) \geq \tilde{p}_i\}$ $(i = 1, 2, \ldots, h)$. Moreover, some optimal solution of the dual problem can be represented in the form of (3.4) for some p (see [2, 3]; see also Theorem 5.6 and its proof in [4]). Hence, the dual problem is rewritten as follows:

$$\begin{array}{ll} \text{Maximize} & \displaystyle g_{\mathrm{CT}}(p) \equiv \sum_{(u,v)\in E} c(u,v)\min\{0, p(u) - p(v) + \gamma(u,v)\} - \hat{\rho}(p) \\ \text{subject to} & p \in \mathbb{R}^V, \end{array} \tag{3.5}$$

where $\hat{\rho} : \mathbb{R}^V \to \mathbb{R}$ is the Lovász extension of ρ given by (2.3). It is observed that the objective function g_{CT} is expressed as $g_{\mathrm{CT}} = g_{\mathrm{H}} - \hat{\rho}$, which implies that g_{CT} is polyhedral L-concave since both of g_{H} and $-\hat{\rho}$ are polyhedral L-concave functions and polyhedral L-concavity is closed under addition.

Chung and Tcha's algorithm is described as follows. Recall the definitions of $I(p, X)$ and $\lambda(p, X)$ in (3.2) and in (3.3), respectively. We also define

$$\mu(p, X) = \min\{\tilde{p}_i - \tilde{p}_{i+1} \mid 1 \le i \le h-1,\ (L_{i+1} \setminus L_i) \cap X \neq \emptyset,\ (L_i \setminus L_{i-1}) \setminus X \neq \emptyset\}, \quad (3.6)$$

where L_0 is defined to be the empty set. Then, it holds that

$$g_{\mathrm{CT}}(p + \alpha\chi_X) - g_{\mathrm{CT}}(p) = \alpha(I(p, X) - \hat{\rho}'(p; \chi_X))$$

for every $\alpha \in \mathbb{R}$ with $0 \le \alpha \le \min\{\lambda(p, X), \mu(p, X)\}$, where $\hat{\rho}'(p; \chi_X)$ is the directional derivative* of $\hat{\rho}$ at p in direction χ_X.

Chung and Tcha's Algorithm

Step 0: Set $p := p^\circ$, where p° is an initial vector chosen from $\mathbb{R}^V$.

Step 1: Find $X \subseteq V$ that maximizes $I(p, X) - \hat{\rho}'(p; \chi_X)$.

Step 2: If $I(p, X) \le \hat{\rho}'(p; \chi_X)$, then stop; p is a maximizer of g_{CT}.

Step 3: Set $p := p + \min\{\lambda(p, X), \mu(p, X)\}\, \chi_X$. Go to Step 1. □

Chung and Tcha derive a pseudo-polynomial bound on the number of iterations of the algorithm by assuming that the edge costs $\gamma(e)$ are all integer-valued [2]. We note that the set X in Step 1 can be obtained by solving a maximum submodular flow problem; see, e.g., [4, Section 5.5] and [5] for algorithms of the maximum submodular flow problem.

4. Steepest Ascent Algorithm for Polyhedral L-concave Functions

4.1. Algorithm

We consider the following steepest ascent algorithm for the maximization of a polyhedral L-concave function $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$, where it is assumed that $\arg\max g$ is nonempty. Since g is a polyhedral concave function, the nonemptiness of $\arg\max g$ is equivalent to the condition that g is bounded from above; in particular, g satisfies property (LF2) with $r = 0$.

Whereas a standard steepest ascent algorithm iteratively updates a current solution p by using a direction $d \in \mathbb{R}^V$ which maximizes the value of directional derivative $g'(p; d)$, our algorithm uses a restricted class of directions given by 0-1 vectors. Recall the definition of $\bar{c}(p; d)$ in (2.2). It is assumed that we have an oracle for computing the value $\bar{c}(p; d)$ exactly; see Section 4.4 for more details on such an oracle.

Steepest Ascent Algorithm for Polyhedral L-concave Functions

Step 0: Set $k := 1$ and $p_1 := p^\circ$, where p° is an initial vector chosen from $\mathrm{dom}\, g$.

Step 1: Let $X_k \subseteq V$ be a set maximizing the value $g'(p_k; \chi_{X_k})$; if there exists more than one such X_k, then take a (unique) minimal one.

Step 2: If $g'(p_k; \chi_{X_k}) \le 0$, then output the current vector p_k and stop† (p_k is a maximizer of g).

Step 3: Set $\lambda_k := \bar{c}(p_k; \chi_{X_k})$, $p_{k+1} := p_k + \lambda_k \chi_{X_k}$, and $k := k + 1$. Go to Step 1. □

We note that the minimal X_k that maximizes $g'(p_k; \chi_{X_k})$ in Step 1 is uniquely determined by the following property:

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*$\hat{\rho}'(p; \chi_X)$ admits an explicit formula [2], which is omitted here.

†If the algorithm stops, we have $X_k = \emptyset$ since it is the smallest set with $g'(p_k; \chi_{X_k}) \le 0$.

Proposition 4.1. *Let $p \in \mathrm{dom}\, g$. If $X, Z \in \arg\max\{g'(p; \chi_Y) \mid Y \subseteq V\}$, then it holds that $X \cap Z, X \cup Z \in \arg\max\{g'(p; \chi_Y) \mid Y \subseteq V\}$.*

Proof. By (LF1) for g, we have $g'(p; \chi_X) + g'(p; \chi_Z) \leq g'(p; \chi_{X\cap Z}) + g'(p; \chi_{X\cup Z})$. Hence, $X \cap Z, X \cup Z \in \arg\max\{g'(p; \chi_Y) \mid Y \subseteq V\}$ holds if $X, Z \in \arg\max\{g'(p; \chi_Y) \mid Y \subseteq V\}$. □

The validity of the steepest ascent algorithm follows immediately from the following proposition, stating that maximizers of a polyhedral L-concave function are characterized by a local property.

Proposition 4.2 ([12, Theorem 4.29])**.** *Let $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ be a polyhedral L-concave function. Then, $p \in \mathrm{dom}\, g$ is a maximizer of g if and only if $g'(p; \chi_X) \leq 0$ for every $X \subseteq V$.*

Hence, the output of the algorithm is a maximizer of function g.

Remark 4.3. It is easy to see that the steepest ascent algorithm described above coincides with Hassin's and Chung and Tcha's algorithms when applied to polyhedral L-concave functions g_{H} in (3.1) and g_{CT} in (3.5), respectively. Our algorithm is different from Chung and Tcha's algorithm in the choice of X in Step 1. The unique minimal maximizer X of $g'(p; \chi_X)$ is chosen in our algorithm to guarantee the finite termination (see Theorem 4.7), whereas Chung and Tcha's algorithm takes an arbitrary maximizer and imposes an integrality assumption on the input to show the finite termination. □

Remark 4.4. The steepest ascent algorithm presented in this section can also be seen as a natural generalization of the steepest ascent algorithm for L-concave functions defined on integer lattice points (see [10, 11]), for which an analysis of the number of iterations is given in [14]. □

4.2. Monotonicity properties

In the analysis of the steepest ascent algorithm, the smallest maximizer that is not smaller than the initial vector p° plays an important role. Note that a maximizer p satisfying $p \geq p^\circ$ always exists by property (LF2) with $r = 0$. Moreover, the existence of the unique minimal maximizer p satisfying $p \geq p^\circ$ follows from the closedness of $\mathrm{dom}\, g$ and the following property.

Proposition 4.5. *Let $p, q \in \mathrm{dom}\, g$ be maximizers of g satisfying $p \geq p^\circ$ and $q \geq p^\circ$. Then, $p \wedge q$ is also a maximizer of g and satisfies $p \wedge q \geq p^\circ$.*

Proof. It follows from $p \geq p^\circ$ and $q \geq p^\circ$ that $p \wedge q \geq p^\circ$. By property (LF1), we have

$$g(p \wedge q) + g(p \vee q) \geq g(p) + g(q) = 2 \max\{g(p') \mid p' \in \mathbb{R}^V\},$$

from which it follows that both of $p \wedge q$ and $p \vee q$ are maximizers of g. □

Denote by $\hat{p}$ the unique minimal maximizer of g such that $\hat{p} \geq p^\circ$. Note that the maximizer $\hat{p}$ is "nearest" to p° in the sense that

$$\|\hat{p} - p^\circ\|_\infty = \min\{\|p - p^\circ\|_\infty \mid p \in \arg\max g,\ p \geq p^\circ\}. \tag{4.1}$$

We now present the main theorem of this paper. For $k = 1, 2, \ldots$, we define real numbers

α_k, β_k and sets $A_k, B_k \subseteq V$ by[‡]

$$\begin{aligned} \alpha_k &= \min\{\hat{p}(i) - p_k(i) \mid i \in X_k\}, \\ \beta_k &= \max\{\hat{p}(i) - p_k(i) \mid i \in V\} - \max\{\hat{p}(i) - p_k(i) \mid i \in V \setminus X_k\}, \\ A_k &= \arg\max\{\hat{p}(i) - p_k(i) \mid i \in V\}, \\ B_k &= \{i \in V \mid \hat{p}(i) = p_k(i)\}, \end{aligned}$$

where p_k and X_k are the variables in the k-th iteration of the steepest ascent algorithm. Note that set B_1 is nonempty due to the choice of $\hat{p}$ and property (LF2) of the function g.

Theorem 4.6. *In the steepest ascent algorithm for a polyhedral L-concave function g, the following hold for $k = 1, 2, \ldots$:*

$$p_k \leq \hat{p}, \tag{4.2}$$

$$\lambda_k \leq \min\{\alpha_k, \beta_k\}, \tag{4.3}$$

$$A_k \subseteq X_k \subseteq V \setminus B_k, \tag{4.4}$$

$$A_k \subseteq A_{k+1};\ \textit{moreover, } A_k \subsetneq A_{k+1} \textit{ if } \lambda_k = \beta_k \textit{ and } A_k = A_{k+1} \textit{ if } \lambda_k < \beta_k, \tag{4.5}$$

$$B_k \subseteq B_{k+1};\ \textit{moreover, } B_k \subsetneq B_{k+1} \textit{ if } \lambda_k = \alpha_k \textit{ and } B_k = B_{k+1} \textit{ if } \lambda_k < \alpha_k, \tag{4.6}$$

$$g'(p_k; \chi_{X_k}) \geq g'(p_{k+1}; \chi_{X_{k+1}});\ \textit{moreover, } X_k \subsetneq X_{k+1} \textit{ if } g'(p_k; \chi_{X_k}) = g'(p_{k+1}; \chi_{X_{k+1}}). \tag{4.7}$$

The proof of Theorem 4.6 is given in Section 4.3.

From this technical theorem we obtain various nice properties which are peculiar to the steepest ascent algorithm described above for polyhedral L-concave functions and are not shared by the ordinary steepest ascent algorithm for general concave functions.

The first property is the finite termination of the algorithm.

Theorem 4.7. *The steepest ascent algorithm for polyhedral L-concave functions terminates in a finite number of iterations.*

Proof. By (4.7) in Theorem 4.6, it suffices to show that $g'(p_k; \chi_{X_k})$ takes a value in a finite set of real numbers. Let

$$D = \{g'(p; \chi_X) \mid p \in \operatorname{dom} g,\ X \subseteq V,\ g'(p; \chi_X) > -\infty\}.$$

Since g is a polyhedral concave function, it can be represented as $g(p) = \min_{1 \leq i \leq t}\{a_i^{\mathrm{T}} p + b_i\}$ ($p \in \operatorname{dom} g$) for some $a_i \in \mathbb{R}^V$ and $b_i \in \mathbb{R}$ ($i = 1, 2, \ldots, t$). Hence, if $g'(p; \chi_X) > -\infty$, then we have $g'(p; \chi_X) = a_i^{\mathrm{T}} \chi_X$ for some i. This implies that D is a finite set. □

In the following, we denote by m the total number of iterations executed in the algorithm. The next property is that the algorithm outputs the maximizer $\hat{p}$ that is smallest with $\hat{p} \geq p^\circ$.

Theorem 4.8. *The steepest ascent algorithm outputs the maximizer $\hat{p}$.*

Proof. By (4.2) in Theorem 4.6, we have $p_k \leq \hat{p}$ for all k. Since $p_k \geq p^\circ$, if $p_k \neq \hat{p}$, then p_k is not a maximizer of g, and the algorithm continues to the next iteration. Hence, the algorithm outputs the vector $\hat{p}$ when it terminates. □

The third property is that the trajectory of the solutions generated by the algorithm is a "shortest" path from the initial solution p° to the "nearest" maximizer $\hat{p}$ in the sense that the total sum of the step lengths is equal to the ℓ_∞-distance $\|\hat{p} - p^\circ\|_\infty$ from the initial solution p° to the nearest optimal solution $\hat{p}$.

[‡]It should be understood that α_k, β_k, A_k, and B_k are defined for each k which is less than the total number of the iterations executed in the algorithm.

Theorem 4.9. *The total sum* $\sum_{k=1}^{m-1} \lambda_k$ *of the step lengths is equal to* $\|\hat{p} - p^\circ\|_\infty$.

Proof. By (4.4) in Theorem 4.6, we have $\|\hat{p}-p_{k+1}\|_\infty = \|\hat{p}-p_k\|_\infty - \lambda_k$ for $k = 1, 2, \ldots, m-1$. This implies $\sum_{k=1}^{m-1} \lambda_k = \|\hat{p} - p^\circ\|_\infty$. □

The fourth property is the concavity of the function g on the trajectory of the solutions generated by the algorithm.

Theorem 4.10. *Let* $\psi : [0, \Lambda_{m-1}] \to \mathbb{R}$ *be a function defined by*

$$\psi(\lambda) = g(p_k + (\lambda - \Lambda_{k-1})\chi_{X_k}) \qquad (k = 1, 2, \ldots, m-1,\ \Lambda_{k-1} \le \lambda \le \Lambda_k),$$

where $\Lambda_k = \sum_{j=1}^{k} \lambda_j$ $(k = 0, 1, \ldots, m-1)$*. Then,* ψ *is a piecewise-linear increasing concave function.*

Proof. For each $k = 1, 2, \ldots, m-1$, the function ψ is linear in each subinterval and its slope is given by $g'(p_k; \chi_{X_k})$, which is a positive number due to the choice of X_k. Hence, the claim follows from (4.7) in Theorem 4.6. □

Remark 4.11. We see from Theorem 4.6 that the monotonicity property of Hassin's algorithm extends to the steepest ascent algorithm for L-concave functions. Indeed, Proposition 3.1 for Hassin's algorithm can be obtained as a special case of Theorem 4.6 applied to the polyhedral L-concave functions g_{H} given by (3.1), where $g'(p_k; \chi_{X_k}) = I(p_k, X_k)$ for each k. □

Remark 4.12. The results in this section can be naturally extended to locally polyhedral concave functions with L-concavity (i.e., (LF1) and (LF2)); a function $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ is said to be a *locally polyhedral concave function* if for every bounded interval $[a, b]$ with $\mathrm{dom}\, g \cap [a, b] \neq \emptyset$, the restriction of the effective domain of g to $\mathrm{dom}\, g \cap [a, b]$ is a polyhedral concave function. We note that the concave closure of an L-concave function defined on integer lattice points (see [10, 11]) is a locally polyhedral concave function with L-concavity. □

4.3. Proof

In this section we give a proof of Theorem 4.6. We assume that $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ is a polyhedral L-concave function that has a maximizer.

Lemma 4.13 ([12, Lemma 4.28])**.** *It holds that*

$$g(p) + g(q) \le g(p + \lambda\chi_X) + g(q - \lambda\chi_X)$$

for every $p, q \in \mathrm{dom}\, g$ *and* $\lambda \in \mathbb{R}$ *with* $0 \le \lambda \le \lambda' - \lambda''$*, where*

$$\lambda' = \max_{i \in V}\{q(i) - p(i)\}, \quad X = \arg\max\{q(i) - p(i) \mid i \in V\},$$
$$\lambda'' = \max_{i \in V \setminus X}\{q(i) - p(i)\} \quad (\lambda'' = -\infty \text{ if } V \setminus X = \emptyset).$$

The following two lemmas show two properties for the proof of Theorem 4.6. We say that $X \subseteq V$ is a *steepest ascent direction* of function g at $p \in \mathrm{dom}\, g$ if

$$g'(p; \chi_X) = \max\{g'(p; \chi_Y) \mid Y \subseteq V\}.$$

By Proposition 4.2 and property (LF2) with $r = 0$, every steepest ascent direction X is a nonempty proper subset of V (i.e., $\emptyset \subsetneq X \subsetneq V$) if p is not a maximizer of g. Note that for $p \in \mathrm{dom}\, g$ with $p \notin \arg\max g$ and every steepest ascent direction X at p, we have $\bar{c}(p; \chi_X) < +\infty$ since $g'(p; \chi_X) > 0$ holds by Proposition 4.2 and there exists a maximizer of g by our assumption.

Lemma 4.14. *Let $p \in \mathrm{dom}\, g$ be a vector with $p^\circ \leq p \leq \hat{p}$ and $p \neq \hat{p}$, and X be a steepest ascent direction of g at p. Put*

$$A = \arg\max_{i \in V}\{\hat{p}(i) - p(i)\}, \qquad B = \{i \in V \mid \hat{p}(i) = p(i)\}.$$

(i) *$A \subseteq X$ holds.*
(ii) *$X \setminus B$ is also a steepest ascent direction at p.*
(iii) *$X \cap B = \emptyset$ holds if X is the unique minimal steepest ascent direction at p.*

Proof. We first note that p is not a maximizer of g. Let ε be a sufficiently small positive real number such that

$$g(p + \varepsilon\chi_X) - g(p) = \varepsilon g'(p; \chi_X). \tag{4.8}$$

To prove (i), assume, to the contrary, that $A \setminus X \neq \emptyset$ holds. Then, we have

$$\arg\max_{i \in V}\{\hat{p}(i) - (p + \varepsilon\chi_X)(i)\} = A \setminus X.$$

Hence, Lemma 4.13 implies that

$$\begin{aligned} g(\hat{p}) + g(p + \varepsilon\chi_X) &\leq g(\hat{p} - \varepsilon\chi_{A\setminus X}) + g(p + \varepsilon\chi_X + \varepsilon\chi_{A\setminus X}) \\ &= g(\hat{p} - \varepsilon\chi_{A\setminus X}) + g(p + \varepsilon\chi_{X\cup A}). \end{aligned} \tag{4.9}$$

Since $\hat{p} \geq p$ and $\hat{p} \neq p$, we have $A \setminus X \subseteq \{i \in V \mid \hat{p}(i) > p(i)\}$. Therefore, we may assume that ε is chosen so that

$$\hat{p} \geq \hat{p} - \varepsilon\chi_{A\setminus X} \geq p \tag{4.10}$$

holds. It follows from (4.10) and the choice of $\hat{p}$ that $g(\hat{p}) > g(\hat{p} - \varepsilon\chi_{A\setminus X})$, which, together with (4.9), implies $g(p + \varepsilon\chi_{X\cup A}) > g(p + \varepsilon\chi_X)$. From this inequality and (4.8) it follows that

$$\varepsilon g'(p; \chi_{X\cup A}) \geq g(p + \varepsilon\chi_{X\cup A}) - g(p) > g(p + \varepsilon\chi_X) - g(p) = \varepsilon g'(p; \chi_X),$$

where the first inequality is by the concavity of g. This, however, is a contradiction to the choice of X. Hence, we have $A \setminus X = \emptyset$, i.e., $A \subseteq X$.

We then show that $X \setminus B$ is also a steepest ascent direction at p. We may assume that $X \cap B \neq \emptyset$ since otherwise the claim holds immediately. Then, we have

$$\arg\max_{i \in V}\{(p + \varepsilon\chi_X)(i) - \hat{p}(i)\} = X \cap B.$$

It follows from Lemma 4.13 that

$$\begin{aligned} g(p + \varepsilon\chi_X) + g(\hat{p}) &\leq g(p + \varepsilon\chi_X - \varepsilon\chi_{X\cap B}) + g(\hat{p} + \varepsilon\chi_{X\cap B}) \\ &= g(p + \varepsilon\chi_{X\setminus B}) + g(\hat{p} + \varepsilon\chi_{X\cap B}). \end{aligned} \tag{4.11}$$

Since $\hat{p}$ is a maximizer of g, we have $g(\hat{p}) \geq g(\hat{p} + \varepsilon\chi_{X\cap B})$, which, together with (4.11), implies $g(p + \varepsilon\chi_X) \leq g(p + \varepsilon\chi_{X\setminus B})$. Hence, it follows that

$$\varepsilon g'(p, \chi_X) = g(p + \varepsilon\chi_X) - g(p) \leq g(p + \varepsilon\chi_{X\setminus B}) - g(p) \leq \varepsilon g'(p, \chi_{X\setminus B}),$$

i.e., $X \setminus B$ is also a steepest ascent direction at p. Hence Claim (ii) holds.

Finally, (iii) follows easily from (ii) since (ii) implies $X \setminus B \supseteq X$ if X is the unique minimal steepest ascent direction at p. □

Lemma 4.15. *Let $p \in \operatorname{dom} g$ be a vector with $p \notin \arg\max g$, $X \subseteq V$ be a steepest ascent direction of g at p, and $\lambda \in \mathbb{R}$ be a real number with $0 < \lambda \leq \bar{c}(p; \chi_X)$. Put $q = p + \lambda\chi_X$, and let $Y \subseteq V$ be a steepest ascent direction of g at q. Then, the following three properties hold:*
(i) $g'(p; \chi_X) \geq g'(q; \chi_Y)$.
(ii) *If $g'(p; \chi_X) = g'(q; \chi_Y)$, then $X \cap Y$ is also a steepest ascent direction at p.*
(iii) *If $\lambda < \bar{c}(p; \chi_X)$, then X is a steepest ascent direction at q.*
Moreover, under the assumption that X (resp., Y) is the unique minimal steepest ascent direction at p (resp., at q), the following two properties also hold:
(iv) *If $g'(p; \chi_X) = g'(q; \chi_Y)$, then $X \subseteq Y$ holds.*
(v) *If $\lambda < \bar{c}(p; \chi_X)$, then $X = Y$ holds.*

Proof. By the choice of λ and concavity of g, we have

$$g(p + \lambda\chi_X) - g(p) = \lambda\, g'(p; \chi_X). \tag{4.12}$$

Let $\varepsilon \in \mathbb{R}$ be a positive real number with $\varepsilon < \lambda$ such that

$$g(q + \varepsilon\chi_Y) - g(q) = \varepsilon g'(q; \chi_Y). \tag{4.13}$$

Put $\hat{q} = q + \varepsilon\chi_Y$. By (4.12) and (4.13), we have

$$g(\hat{q}) - g(p) = \lambda g'(p; \chi_X) + \varepsilon g'(q; \chi_Y). \tag{4.14}$$

Note that $\hat{q}$ can be represented as

$$\hat{q} = p + \varepsilon\chi_{X\cup Y} + (\lambda - \varepsilon)\chi_X + \varepsilon\chi_{X\cap Y}.$$

Claim: The following inequalities hold:

$$g(\hat{q}) - g(\hat{q} - \varepsilon\chi_{X\cap Y}) \leq \varepsilon g'(p; \chi_X), \tag{4.15}$$
$$g(p + \varepsilon\chi_{X\cup Y} + (\lambda - \varepsilon)\chi_X) - g(p + \varepsilon\chi_{X\cup Y}) \leq (\lambda - \varepsilon) g'(p; \chi_X), \tag{4.16}$$
$$g(p + \varepsilon\chi_{X\cup Y}) - g(p) \leq \varepsilon g'(p; \chi_X). \tag{4.17}$$

[Proof of Claim] Inequality (4.17) can be shown as follows:

$$g(p + \varepsilon\chi_{X\cup Y}) - g(p) \leq \varepsilon g'(p; \chi_{X\cup Y}) \leq \varepsilon g'(p; \chi_X),$$

where the first inequality is by the concavity of g and the second inequality follows from the fact that X is a steepest ascent direction of g at p.

We then prove (4.15). It may be assumed that $X \cap Y \neq \emptyset$ since otherwise

$$g(\hat{q}) - g(\hat{q} - \varepsilon\chi_{X\cap Y}) = g(\hat{q}) - g(\hat{q}) = 0 < \varepsilon g'(p; \chi_X)$$

holds, where the inequality follows from $p \notin \arg\max g$ and Proposition 4.2. Since $\arg\max\{\hat{q}(i) - p(i) \mid i \in V\} = X \cap Y$, Lemma 4.13 implies that

$$g(p) + g(\hat{q}) \leq g(p + \varepsilon\chi_{X\cap Y}) + g(\hat{q} - \varepsilon\chi_{X\cap Y}),$$

from which it follows that

$$g(\hat{q}) - g(\hat{q} - \varepsilon\chi_{X\cap Y}) \leq g(p + \varepsilon\chi_{X\cap Y}) - g(p) \leq \varepsilon g'(p; \chi_{X\cap Y}) \leq \varepsilon g'(p; \chi_X). \tag{4.18}$$

Inequality (4.16) can be shown similarly to (4.15) as follows. Lemma 4.13 implies that

$$\begin{aligned}
& g(p) + g(p + \varepsilon\chi_{X\cup Y} + (\lambda-\varepsilon)\chi_X) \\
& \quad \le \; g(p + (\lambda-\varepsilon)\chi_X) + g((p + \varepsilon\chi_{X\cup Y} + (\lambda-\varepsilon)\chi_X) - (\lambda-\varepsilon)\chi_X) \\
& \quad = \; g(p + (\lambda-\varepsilon)\chi_X) + g(p + \varepsilon\chi_{X\cup Y}),
\end{aligned}$$

from which it follows that

$$g(p + \varepsilon\chi_{X\cup Y} + (\lambda-\varepsilon)\chi_X) - g(p + \varepsilon\chi_{X\cup Y}) \le g(p + (\lambda-\varepsilon)\chi_X) - g(p) \le (\lambda-\varepsilon)g'(p;\chi_X).$$

[End of Claim]

Inequalities (4.15), (4.16), and (4.17) imply

$$g(\hat{q}) - g(p) \le (\lambda+\varepsilon)g'(p;\chi_X). \tag{4.19}$$

Then, Claim (i) follows from (4.14) and (4.19).

To prove Claims (ii) and (iv), assume that $g'(p;\chi_X) = g'(q;\chi_Y)$. It follows from (4.14) that $g(\hat{q}) - g(p) = (\lambda+\varepsilon)g'(p;\chi_X)$, which, together with the inequalities (4.15), (4.16), and (4.17), implies that all the inequalities (4.15), (4.16), and (4.17) hold with equality. In particular, we have

$$g(\hat{q}) - g(\hat{q} - \varepsilon\chi_{X\cap Y}) = \varepsilon g'(p;\chi_X), \tag{4.20}$$

from which it follows that $X \cap Y \neq \emptyset$ since $g'(p;\chi_X) > 0$ by Proposition 4.2. By (4.18) and (4.20), we have

$$\varepsilon g'(p;\chi_X) = g(\hat{q}) - g(\hat{q} - \varepsilon\chi_{X\cap Y}) \le g(p + \varepsilon\chi_{X\cap Y}) - g(p) \le \varepsilon g'(p;\chi_{X\cap Y}).$$

This shows that $X \cap Y$ is also a steepest ascent direction of g at p, i.e., (ii) holds. If X is the unique minimal steepest ascent direction at p, then we have $X \subseteq X \cap Y$, i.e., $X \subseteq Y$. Thus, (iv) holds.

We finally prove Claims (iii) and (v). For $\lambda < \bar{c}(p;\chi_X)$, we have $g'(q;\chi_X) = g'(p;\chi_X)$, which, together with (i), implies

$$g'(q;\chi_X) = g'(p;\chi_X) \ge g'(q;\chi_Y).$$

This shows that X is also a steepest ascent direction at q and that $g'(p;\chi_X) = g'(q;\chi_Y)$. If Y is the unique minimal steepest ascent direction at q, we have $X \supseteq Y$, which, combined with Claim (iv), implies $X = Y$. □

We now give a proof of Theorem 4.6.

[Proofs of (4.2) and (4.3)] It suffices to show that for $k = 1, 2, \ldots, m-1$, if $p_k \le \hat{p}$ then $\lambda_k \le \min\{\alpha_k, \beta_k\}$ and $p_{k+1} \le \hat{p}$ hold.

Assume that $p_k \le \hat{p}$ holds. First suppose, to the contrary, that $\alpha_k < \lambda_k$. Let $p' = p_k + \alpha_k\chi_{X_k}$. Since $p_k \le \hat{p}$, the definition of p' implies $p' \le \hat{p}$. Note that p' is not a maximizer of g since

$$g(p_{k+1}) - g(p') = (\lambda_k - \alpha_k)g'(p_k;\chi_{X_k}) > 0.$$

Hence, $p' \neq \hat{p}$ holds. Since p' is given as $p' = p_k + \alpha_k\chi_{X_k}$ with $\alpha_k < \lambda_k = \bar{c}(p_k;\chi_{X_k})$, the set X_k is also the unique minimal steepest ascent direction at p' by Lemma 4.15 (v). By Lemma 4.14, we have $X_k \cap B' = \emptyset$, where $B' = \{i \in V \mid \hat{p}(i) = p'(i)\}$. On the other hand, the definition of p' implies that B' contains some element in X_k, a contradiction. Hence, we have $\alpha_k \geq \lambda_k$.

Next suppose that $\beta_k < \lambda_k$. Let $p'' = p_k + \beta_k \chi_{X_k}$. In a similar way as in the previous case, we can show that $p'' \leq \hat{p}$, $p'' \neq \hat{p}$, and that the set X_k is also the unique minimal steepest ascent direction at p''. By Lemma 4.14, we have $A'' \subseteq X_k$, where $A'' = \arg\max_{i \in V}\{\hat{p}(i) - p''(i)\}$. On the other hand, the definition of p'' implies that A'' contains some element in $V \setminus X_k$, a contradiction. Hence, we have $\beta_k \geq \lambda_k$.

Finally, the inequality $p_{k+1} \leq \hat{p}$ follows immediately from $p_k \leq \hat{p}$ and $\lambda_k \leq \alpha_k$.

[Proof of (4.4)] By Lemma 4.14 applied to $p = p_k$ and $X = X_k$, we obtain $A_k \subseteq X_k \subseteq V \setminus B_k$, i.e., (4.4) holds.

[Proofs of (4.5) and (4.6)] Claims (4.5) and (4.6) follow from (4.3), (4.4), and definitions of α_k, β_k, and p_{k+1}.

[Proof of (4.7)] The inequality $g'(p_k; \chi_{X_k}) \geq g'(p_{k+1}; \chi_{X_{k+1}})$ follows immediately from Lemma 4.15 (i). Suppose that $g'(p_k; \chi_{X_k}) = g'(p_{k+1}; \chi_{X_{k+1}})$ holds. By the definition of p_{k+1} (and $\bar{c}(p_k; \chi_{X_k})$), we have $g'(p_{k+1}; \chi_{X_k}) < g'(p_k; \chi_{X_k})$, and therefore $X_k \neq X_{k+1}$ holds. This, together with Lemma 4.15 (iv), implies $X_k \subsetneq X_{k+1}$.

This completes the proof of Theorem 4.6.

4.4. Computation of step size

So far we have assumed that the value $\bar{c}(p; \chi_X)$ is given by some "oracle," i.e., that the exact value of $\bar{c}(p; \chi_X)$ can be computed efficiently for every $p \in \mathrm{dom}\, g$ and $X \subseteq V$. Here we discuss the computation of the value $\bar{c}(p; \chi_X)$ for a polyhedral L-concave function g.

We can indeed compute the value $\bar{c}(p; \chi_X)$ efficiently for the special case of polyhedral L-concave functions arising from the minimum cost (submodular) flow problems discussed in Section 3. For the minimum cost flow problem considered in Section 3.1, the value $\bar{c}(p; \chi_X)$ is equal to $\lambda(p, X)$ given by (3.3), which can be computed in linear time in the size of the edge set E. In the minimum cost submodular flow problem considered in Section 3.2, the value $\bar{c}(p; \chi_X)$ is equal to $\min\{\lambda(p, X), \mu(p, X)\}$, where $\mu(p, X)$ given by (3.6) can also be computed in polynomial time in the size of the set V.

For a general polyhedral L-concave function g, we can still compute the value $\bar{c}(p; \chi_X)$ efficiently if the function g has a certain "integrality" property. For a polyhedral concave function $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ and $x \in \mathbb{R}^V$, we denote

$$\arg\max g[-x] = \arg\max\{g(p) - p^{\mathrm{T}} x \mid p \in \mathrm{dom}\, g\}. \tag{4.21}$$

A polyhedral L-concave function $g : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ is said to be *integral* if for every $x \in \mathbb{R}^V$ with $\arg\max g[-x] \neq \emptyset$, the polyhedron $\arg\max g[-x]$ is integral. Note that the polyhedral L-concave function g_{H} in (3.1) is integral if $\gamma(e) \in \mathbb{Z} \cup \{+\infty\}$ for all $e \in E$, and g_{CT} in (3.5) is integral if $\gamma(e) \in \mathbb{Z} \cup \{+\infty\}$ for all $e \in E$ and ρ is an integer-valued function [10].

It is known (see, e.g., [10, Chapter 7]) that for every integral polyhedral L-concave function g and every $x \in \mathbb{R}^V$ with $\arg\max g[-x] \neq \emptyset$, there exists an integer-valued function $\mu : V \times V \to \mathbb{Z} \cup \{+\infty\}$ such that

$$\arg\max g[-x] = \{p \in \mathbb{R}^V \mid p(v) - p(u) \leq \mu(u, v)\ (u, v \in V)\}.$$

It follows from this fact that the value $\bar{c}(p; \chi_X)$ is an integer for every $p \in \mathrm{dom}\, g \cap \mathbb{Z}^V$ and $X \subseteq V$. Hence, $\bar{c}(p; \chi_X)$ can be computed exactly by binary search, and its running time is bounded by $\log \Phi$, where

$$\Phi = \max_{i,j \in V} \max\{|(p(i) - p(j)) - (q(i) - q(j))| \mid p, q \in \mathrm{dom}\, g\}.$$

Even for a non-integral polyhedral L-concave function g, we can compute an approximate value of $\bar{c}(p;\chi_X)$ with additive error at most ε in time polynomial in $\log\Phi$ and $\log(1/\varepsilon)$. Note that if an approximate value of $\bar{c}(p;\chi_X)$ with additive error at most ε is used in the steepest ascent algorithm, we can compute a vector $p \in \mathrm{dom}\, g$ satisfying

$$\inf\{\|p-p^*\|_\infty \mid p^* \in \arg\max g\} \leq |V|\cdot\varepsilon$$

(i.e., p is in the neighbor of an optimal solution) by the proximity result shown in [8].

5. Algorithm for Polyhedral L$^\natural$-concave Functions

The steepest ascent algorithm for maximization of polyhedral L-concave functions is naturally adapted to polyhedral L$^\natural$-concave functions. A polyhedral concave function $g : \mathbb{R}^V \to \mathbb{R}\cup\{-\infty\}$ is said to be *polyhedral L$^\natural$-concave* if the function $\tilde{g} : \mathbb{R}^{\tilde{V}} \to \mathbb{R}\cup\{-\infty\}$ defined by

$$\tilde{g}(\eta, p) = g(p - \eta\mathbf{1}) \quad ((\eta, p) \in \mathbb{R}\times\mathbb{R}^V = \mathbb{R}^{\tilde{V}}) \tag{5.1}$$

is polyhedral L-concave, where $\tilde{V} = \{v_0\}\cup V$. Polyhedral L$^\natural$-concavity of g is characterized by the following "translation-supermodularity" [12, Theorem 4.39]:

$$g(p) + g(q) \leq g(p \vee (q - \lambda\mathbf{1})) + g((p+\lambda\mathbf{1})\wedge q) \quad (\forall p, q \in \mathrm{dom}\, g,\ \forall\lambda \geq 0). \tag{5.2}$$

We now consider the maximization of a polyhedral L$^\natural$-concave function $g : \mathbb{R}^V \to \mathbb{R}\cup\{-\infty\}$, where it is assumed that $\arg\max g$ is nonempty. The steepest ascent algorithm in Section 4.1 applied to the polyhedral L-concave function $\tilde{g}$ given by (5.1) with an initial vector $(0, p^\circ) \in \mathbb{R}^{\tilde{V}}$ yields the following algorithm for the polyhedral L$^\natural$-concave function g and the initial vector p° through the following correspondence (see also [10, Section 10.3.1]):

$$\begin{array}{ccc} \tilde{g} & & g \\ \hline (\eta,p) & \Longleftrightarrow & q = p - \eta\mathbf{1} \\ (\eta,p)+\lambda(0,\chi_X) & \Longleftrightarrow & q + \lambda\chi_X \\ (\eta,p)+\lambda(1,\chi_X) & \Longleftrightarrow & q - \lambda\chi_{V\setminus X} \\ \hline \end{array} \tag{5.3}$$

Steepest Ascent Algorithm for Polyhedral L$^\natural$-concave Functions

Step 0: Set $k := 1$ and $p_1 := p^\circ$, where p° is an initial vector chosen from $\mathrm{dom}\, g$.

Step 1: Let $\sigma_k \in \{+1, -1\}$ and $X_k \subseteq V$ be a pair of a sign and a set maximizing the value $g'(p_k; \sigma_k\,\chi_{X_k})$; if there exists more than one such pair, then choose σ_k and X_k according to the following rule:
(i) if there exists such (σ_k, X_k) with $\sigma_k = +1$, then set $\sigma_k = +1$ and take a (unique) minimal X_k.
(ii) otherwise, set $\sigma_k = -1$ and take a (unique) maximal X_k.

Step 2: If $g'(p_k; \sigma_k\,\chi_{X_k}) \leq 0$, then output the current vector p_k and stop (p_k is a maximizer of g).

Step 3: Set $\lambda_k := \bar{c}(p_k; \sigma_k\,\chi_{X_k})$, $p_{k+1} := p_k + \lambda_k\,\sigma_k\,\chi_{X_k}$, and $k := k+1$. Go to Step 1. □

The finite termination of the algorithm follows immediately from Theorem 4.7.

Corollary 5.1. *The steepest ascent algorithm for polyhedral L$^\natural$-concave functions terminates in a finite number of iterations.*

In the following, we denote by m the total number of iterations executed in the algorithm.

The properties of the steepest ascent algorithm for polyhedral L$^\natural$-concave functions in this section can be derived from the corresponding results for polyhedral L-concave functions in Section 4.1 through rather mechanical translations based on the correspondence (5.3). Although the concept of polyhedral L$^\natural$-concave function is equivalent to that of polyhedral L-concave function by definition, it is emphasized that the class of polyhedral L$^\natural$-concave functions contains that of polyhedral L-concave functions as a special case, and accordingly, the results to be established in this section have wider applicability than those in Section 4.

Remark 5.2. If the steepest ascent algorithm in this section is applied to a polyhedral L-concave function g, then its behavior coincides with that of the steepest ascent algorithm for polyhedral L-concave functions in Section 4.1. $\square$

For a vector $q \in \mathbb{R}^V$, define

$$\|q\|_\infty^+ = \max\left[0, \max_{i\in V} q(i)\right], \qquad \|q\|_\infty^- = \max\left[0, \max_{i\in V}\{-q(i)\}\right].$$

Note that $\|q\|_\infty^+ + \|q\|_\infty^-$ serves as a norm of q (satisfying the axioms of norms), and accordingly, the value $\|p-q\|_\infty^+ + \|p-q\|_\infty^-$ represents a “distance” between two vectors p and q.

Let

$$\hat{\eta} = \min\{\eta \mid \eta \in \mathbb{R}_+,\ \exists p \in \arg\max g \text{ s.t. } p \geq p^\circ - \eta \mathbf{1}\}, \tag{5.4}$$

and denote by $\hat{p}$ the unique minimal maximizer of g under the condition $\hat{p} \geq p^\circ - \hat{\eta}\mathbf{1}$. Note that the vector $(\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1})$ is the unique minimal maximizer of the associated L-concave function $\tilde{g}$ under the condition $(\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1}) \geq (0, p^\circ)$, and satisfies

$$\|(\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1}) - (0, p^\circ)\|_\infty = \|\hat{p} - p^\circ\|_\infty^+ + \|\hat{p} - p^\circ\|_\infty^-. \tag{5.5}$$

The maximizer $\hat{p}$ is “nearest” to the initial vector p° in the following sense.

Proposition 5.3. *The maximizer $\hat{p}$ satisfies*

$$\|\hat{p} - p^\circ\|_\infty^+ + \|\hat{p} - p^\circ\|_\infty^- = \min\{\|p - p^\circ\|_\infty^+ + \|p - p^\circ\|_\infty^- \mid p \in \arg\max g\}. \tag{5.6}$$

Proof. Let $\hat{\eta}$ be the real number defined by (5.4). Since the vector $(\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1})$ is the unique minimal maximizer of $\tilde{g}$ under the condition $(\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1}) \geq (0, p^\circ)$, we have

$$\begin{aligned}&\|(\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1}) - (0, p^\circ)\|_\infty \\ &\ = \min\{\|(\zeta, p + \zeta\mathbf{1}) - (0, p^\circ)\|_\infty \mid (\zeta, p + \zeta\mathbf{1}) \in \arg\max \tilde{g},\ (\zeta, p + \zeta\mathbf{1}) \geq (0, p^\circ)\}.\end{aligned} \tag{5.7}$$

The left-hand side of (5.7) can be rewritten by (5.5). On the right-hand side of (5.7), we have

$$\begin{aligned}(\zeta, p + \zeta\mathbf{1}) \in \arg\max \tilde{g} &\iff p \in \arg\max g, \\ (\zeta, p + \zeta\mathbf{1}) \geq (0, p^\circ) &\iff \zeta \geq \|p - p^\circ\|_\infty^-,\end{aligned}$$

and the latter implies

$$\|(\zeta, p + \zeta\mathbf{1}) - (0, p^\circ)\|_\infty = \|p - p^\circ\|_\infty^+ + \zeta.$$

Therefore, the right-hand side of (5.7) is equal to

$$\min\{\|p - p^\circ\|_\infty^+ + \|p - p^\circ\|_\infty^- \mid p \in \arg\max g\}.$$

Hence, the equation (5.6) follows from (5.7). $\square$

The next property, which follows from Theorem 4.8, states that the algorithm outputs the "nearest" maximizer $\hat{p}$.

Corollary 5.4. *The steepest ascent algorithm for polyhedral* $L^\natural$*-concave functions outputs the maximizer* $\hat{p}$.

As in Theorem 4.9, the trajectory of the solutions generated by the algorithm is a "shortest" path from the initial solution p° to the "nearest" maximizer $\hat{p}$.

Corollary 5.5. *The total sum* $\sum_{k=1}^{m-1} \lambda_k$ *of the step lengths is equal to* $\|\hat{p}-p^\circ\|_\infty^+ + \|\hat{p}-p^\circ\|_\infty^-$.

Proof. By Theorem 4.9 applied to the L-concave function $\tilde{g}$ and the initial vector $(0, p^\circ)$, it holds that

$$\sum_{k=1}^{m-1} \lambda_k = \|(\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1}) - (0, p^\circ)\|_\infty = \|\hat{p} - p^\circ\|_\infty^+ + \|\hat{p} - p^\circ\|_\infty^-,$$

where the second equality is due to (5.5). □

The next property is the concavity of the function g on the trajectory of the solutions generated by the algorithm.

Corollary 5.6. *Let* $\psi : [0, \Lambda_{m-1}] \to \mathbb{R}$ *be a function defined by*

$$\psi(\lambda) = g(p_k + (\lambda - \Lambda_{k-1})\sigma_k \chi_{X_k}) \qquad (k = 1, 2, \ldots, m-1,\ \Lambda_{k-1} \leq \lambda \leq \Lambda_k),$$

where $\Lambda_k = \sum_{j=1}^{k} \lambda_j$ $(k = 0, 1, \ldots, m-1)$*. Then,* ψ *is a piecewise-linear increasing concave function.*

Proof. The claim follows from Theorem 4.10 and the correspondence (5.3) between the two algorithms. □

We finally show some monotonicity properties of the algorithm by using Theorem 4.6. For $q \in \mathbb{R}^V$, we denote

$$\mathrm{supp}^+(q) = \{i \in V \mid q(i) > 0\}, \qquad \mathrm{supp}^-(q) = \{i \in V \mid q(i) < 0\}.$$

For $k = 1, 2, \ldots, m$, we define sets $A_k, B_k \subseteq V$ by

$$\begin{aligned}
A_k &= \arg\max\{\hat{p}(i) - p_k(i) \mid i \in V,\ \hat{p}(i) \geq p_k(i)\} \\
&= \begin{cases} \arg\max\{\hat{p}(i) - p_k(i) \mid i \in V\} & (\text{if } \mathrm{supp}^+(\hat{p} - p_k) \neq \emptyset), \\ \{i \in V \mid \hat{p}(i) = p_k(i)\} & (\text{if } \mathrm{supp}^+(\hat{p} - p_k) = \emptyset), \end{cases} \\
B_k &= \arg\min\{\hat{p}(i) - p_k(i) \mid i \in V,\ \hat{p}(i) \leq p_k(i)\} \\
&= \begin{cases} \arg\min\{\hat{p}(i) - p_k(i) \mid i \in V\} & (\text{if } \mathrm{supp}^-(\hat{p} - p_k) \neq \emptyset), \\ \{i \in V \mid \hat{p}(i) = p_k(i)\} & (\text{if } \mathrm{supp}^-(\hat{p} - p_k) = \emptyset), \end{cases} \\
\eta_k &= \sum\{\lambda_j \mid 1 \leq j \leq k-1,\ \sigma_j = -1\},
\end{aligned}$$

where p_k is the vector in the k-th iteration of the steepest ascent algorithm.

Corollary 5.7. *In the steepest ascent algorithm for a polyhedral* $L^\natural$*-concave function* g*, the following hold for* $k = 1, 2, \ldots, m-1$:

(i) $(\eta_k, p_k + \eta_k \mathbf{1}) \leq (\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1})$.

(ii) $A_k \subseteq X_k \subseteq V \setminus B_k$ *if* $\sigma_k = +1$; $B_k \subseteq X_k \subseteq V \setminus A_k$ *if* $\sigma_k = -1$.

(iii) $A_k \subseteq A_{k+1}$ *and* $B_k \subseteq B_{k+1}$.

(iv) $g'(p_k; \sigma_k \chi_{X_k}) \geq g'(p_{k+1}; \sigma_{k+1} \chi_{X_{k+1}})$; *moreover, if* $g'(p_k; \sigma_k \chi_{X_k}) = g'(p_{k+1}; \sigma_{k+1} \chi_{X_{k+1}})$, *then we have the following:*

(iv-a) *If* $\sigma_k = -1$*, then* $\sigma_{k+1} = -1$*.*

(iv-b) *If* $\sigma_k = \sigma_{k+1} = +1$*, then* $X_k \subsetneq X_{k+1}$*.*

(iv-c) *If* $\sigma_k = \sigma_{k+1} = -1$*, then* $X_k \supsetneq X_{k+1}$*.*

(iv-d) *If* $\sigma_k = +1$ *and* $\sigma_{k+1} = -1$*, then* $X_k \subseteq V \setminus X_{k+1}$*.*

Proof. For $k = 1, 2, \ldots, m$, let $\tilde{p}_k \in \mathbb{R}^{\tilde{V}}$ and $\tilde{X}_k \subseteq \tilde{V}$ be the variables obtained in the k-th iteration of the steepest ascent algorithm in Section 4.1 applied to the L-concave function $\tilde{g}$ and the initial vector $\tilde{p}^\circ = (0, p^\circ)$. Note that

$$\tilde{p}_1 = \tilde{p}^\circ, \qquad \tilde{p}_m = (\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1}).$$

We denote $\tilde{p}^* = (\hat{\eta}, \hat{p} + \hat{\eta}\mathbf{1})$, and define

$$\begin{aligned} \tilde{A}_k &= \arg\max\{\tilde{p}^*(i) - \tilde{p}_k(i) \mid i \in \tilde{V}\}, \\ \tilde{B}_k &= \{i \in \tilde{V} \mid \tilde{p}^*(i) - \tilde{p}_k(i) = 0\}. \end{aligned}$$

Then, Theorem 4.6 implies the following properties:

(i$'$) $\tilde{p}_k \leq \tilde{p}^*$.

(ii$'$) $\tilde{A}_k \subseteq \tilde{X}_k \subseteq \tilde{V} \setminus \tilde{B}_k$.

(iii$'$) $\tilde{A}_k \subseteq \tilde{A}_{k+1}$ and $\tilde{B}_k \subseteq \tilde{B}_{k+1}$.

(iv$'$) $\tilde{g}'(\tilde{p}_k; \chi_{\tilde{X}_k}) \geq \tilde{g}'(\tilde{p}_{k+1}; \chi_{\tilde{X}_{k+1}})$; moreover, $\tilde{X}_k \subsetneq \tilde{X}_{k+1}$ if $\tilde{g}'(\tilde{p}_k; \chi_{\tilde{X}_k}) = \tilde{g}'(\tilde{p}_{k+1}; \chi_{\tilde{X}_{k+1}})$.

We show below that Claims (i), (ii), (iii), and (iv) follow from (i$'$), (ii$'$), (iii$'$), and (iv$'$), respectively.

By the correspondence (5.3) between the two algorithms, the following property holds:

$$\left.\begin{array}{l} \text{if } \sigma_k = +1, \text{ then } v_0 \notin \tilde{X}_k \text{ and } X_k = \tilde{X}_k; \\ \text{if } \sigma_k = -1, \text{ then } v_0 \in \tilde{X}_k \text{ and } X_k = V \setminus \tilde{X}_k. \end{array}\right\} \tag{5.8}$$

Hence, we have $\eta_k = \tilde{p}_k(v_0)$ for all k, and therefore $\tilde{p}_k = (\eta_k, p_k + \eta_k\mathbf{1})$ holds. From this and (i$'$), Claim (i) follows.

We then prove (ii) and (iii). By (ii$'$), (iii$'$), and (5.8), it suffices to show that $\tilde{A}_k \cap V = A_k$ and $\tilde{B}_k \cap V = B_k$. Since

$$\begin{aligned} \max\{\tilde{p}^*(i) - \tilde{p}_k(i) \mid i \in \tilde{V}\} &= \max\left[\hat{\eta} - \eta_k,\ \max\{(\hat{p}(i) + \hat{\eta}) - (p_k(i) + \eta_k) \mid i \in V\}\right] \\ &= \hat{\eta} - \eta_k + \max\left[0, \max\{\hat{p}(i) - p_k(i) \mid i \in V\}\right], \end{aligned}$$

it holds that

$$\tilde{A}_k = \left\{\begin{array}{ll} \arg\max\{\hat{p}(i) - p_k(i) \mid i \in V\} & (\text{if } \mathrm{supp}^+(\hat{p} - p_k) \neq \emptyset), \\ \{i \in V \mid \hat{p}(i) = p_k(i)\} \cup \{v_0\} & (\text{if } \mathrm{supp}^+(\hat{p} - p_k) = \emptyset). \end{array}\right. \tag{5.9}$$

Hence, we have $\tilde{A}_k \cap V = A_k$.

Note that $\tilde{B}_k$ is rewritten as

$$\tilde{B}_k = \arg\min\{\tilde{p}^*(i) - \tilde{p}_k(i) \mid i \in \tilde{V}\}$$

since $\tilde{B}_k \neq \emptyset$ and $\tilde{p}^* \geq \tilde{p}_k$ for each k. Hence, we can show the following equation in the same way as (5.9):

$$\tilde{B}_k = \left\{\begin{array}{ll} \arg\min\{\hat{p}(i) - p_k(i) \mid i \in V\} & (\text{if } \mathrm{supp}^-(\hat{p} - p_k) \neq \emptyset), \\ \{i \in V \mid \hat{p}(i) = p_k(i)\} \cup \{v_0\} & (\text{if } \mathrm{supp}^-(\hat{p} - p_k) = \emptyset). \end{array}\right. \tag{5.10}$$

This implies $\tilde{B}_k \cap V = B_k$, and therefore Claims (ii) and (iii) hold.

Finally, Claim (iv) follows from (iv$'$) and (5.8). □

6. Variant of Steepest Ascent Algorithm

We discuss a variant of the steepest ascent algorithm for polyhedral L-concave functions, where the set X_k in Step 1 is chosen arbitrarily from among the sets maximizing the value $g'(p_k; \chi_{X_k})$ (i.e., not necessarily minimal); in this section we call this variant the *modified (steepest ascent) algorithm.*

We see from Proposition 4.2 that the modified algorithm still outputs a maximizer of a polyhedral L-concave function g if it terminates. While the output of the algorithm is not necessarily the unique minimal maximizer $\hat{p}$ of g under the condition $\hat{p} \geq p^\circ$, the output inherits a nice property of $\hat{p}$ as shown in Section 6.1 (see Theorems 6.1 and 6.2). In addition, it is shown that the modified algorithm terminates if the function g has a certain "rationality" property (see Theorem 6.3).

On the other hand, we demonstrate in Section 6.2 that the modified algorithm may not terminate in any finite number of iterations if the function g is not "rational." This fact shows that the choice of the unique minimal steepest ascent direction is essential for the finite termination of the steepest ascent algorithm for polyhedral L-concave functions.

6.1. Monotonicity properties of modified algorithm

We show three properties of the modified algorithm. The first property is that the output is the "nearest" maximizer of g from the initial vector p°. This property is a generalization of Theorem 4.8 (see (4.1)).

Theorem 6.1. *Suppose that the modified steepest ascent algorithm for a polyhedral L-concave function g with an initial solution $p^\circ \in \operatorname{dom} g$ terminates. Then, the output p^* of the algorithm is a maximizer of g satisfying*

$$\|p^* - p^\circ\|_\infty = \min\{\|p - p^\circ\|_\infty \mid p \in \arg\max g,\ p \geq p^\circ\}. \tag{6.1}$$

Note that (6.1) with (4.1) shows $\|p^* - p^\circ\|_\infty = \|\hat{p} - p^\circ\|_\infty$, which, however, does not imply $p^* = \hat{p}$.

The second property, which is a generalization of Theorem 4.9, is that the trajectory of the solutions generated by the modified algorithm is a "shortest" path from the initial solution p° to the "nearest" maximizer p^*.

Theorem 6.2. *Suppose that the modified steepest ascent algorithm for a polyhedral L-concave function g with an initial solution $p^\circ \in \operatorname{dom} g$ terminates after m iterations. Then, the total sum $\sum_{k=1}^{m-1} \lambda_k$ of the step lengths is equal to $\|p^* - p^\circ\|_\infty$.*

The third property is that the modified algorithm terminates if the function g has a certain "rationality" property. Recall the definition of $\arg\max g[-x]$ for $x \in \mathbb{R}^V$ in (4.21).

Theorem 6.3. *The modified steepest ascent algorithm for a polyhedral L-concave function g with an initial solution $p^\circ \in \operatorname{dom} g$ terminates in a finite number of iterations if the following conditions hold:*

- *for every $x \in \mathbb{R}^V$ with $\arg\max g[-x] \neq \emptyset$, the polyhedron $\arg\max g[-x]$ is rational,*
- *the initial solution p° is a rational vector.*

Proofs of Theorems 6.1, 6.2, and 6.3 are given below, where the following lemma is crucial.

Lemma 6.4. *Let $p \in \operatorname{dom} g$ be a vector that is not a maximizer of g, and $X \subseteq V$ be a steepest ascent direction at p. Also, let $\bar{p}$ be the unique minimal maximizer of g under the condition $\bar{p} \geq p$. Then, the following properties hold with $\bar{\lambda} = \bar{c}(p; \chi_X)$:*

(i) *For every $\lambda \in [0, \bar{\lambda}]$, the vector $\bar{p}_\lambda = \bar{p} \vee (p + \lambda\chi_X)$ is the unique minimal maximizer of g under the condition $\bar{p}_\lambda \geq p + \lambda\chi_X$.*

(ii) *For every $\lambda \in [0, \bar{\lambda}]$, it holds that $\|\bar{p}_\lambda - (p + \lambda\chi_X)\|_\infty = \|\bar{p} - p\|_\infty - \lambda$.*

Proof. [Proof of (i)] Let $\lambda^* \in [0, \bar{\lambda}]$ be the maximum real number such that $\bar{p}_\lambda \in \arg\max g$ for every $\lambda \in [0, \lambda^*]$. In the following, we assume, to the contrary, that $\lambda^* < \bar{\lambda}$ and derive a contradiction.

Put $q = p + \lambda^* \chi_X$ and $q_\varepsilon = p + (\lambda^* + \varepsilon)\chi_X$ with a sufficiently small positive real number ε. Then, for $Y = \{i \in X \mid \bar{p}(i) \leq p(i) + \lambda^*\}$, we have

$$q_\varepsilon \wedge \bar{p}_{\lambda^*} = q + \varepsilon \chi_{X \setminus Y}, \qquad q_\varepsilon \vee \bar{p}_{\lambda^*} = \bar{p}_{\lambda^* + \varepsilon}.$$

By (LF1) for g, it holds that

$$g(q_\varepsilon) + g(\bar{p}_{\lambda^*}) \leq g(q_\varepsilon \wedge \bar{p}_{\lambda^*}) + g(q_\varepsilon \vee \bar{p}_{\lambda^*}) = g(q + \varepsilon \chi_{X \setminus Y}) + g(\bar{p}_{\lambda^* + \varepsilon}). \tag{6.2}$$

By Lemma 4.15 (iii), X is a steepest ascent direction at q since $\lambda^* < \bar{\lambda}$. Hence, we have

$$g'(q; \chi_X) \geq g'(q; \chi_{X \setminus Y}),$$

from which it follows that

$$g(q_\varepsilon) - g(q) = \varepsilon g'(q; \chi_X) \geq \varepsilon g'(q; \chi_{X \setminus Y}) \geq g(q + \varepsilon \chi_{X \setminus Y}) - g(q), \tag{6.3}$$

where the second inequality is by the concavity of g. By (6.2) and (6.3), it holds that $g(\bar{p}_{\lambda^*}) \leq g(\bar{p}_{\lambda^* + \varepsilon})$, i.e., $\bar{p}_{\lambda^* + \varepsilon} \in \arg\max g$.

We can choose ε sufficiently small so that

$$\bar{p}_\lambda = \bar{p}_{\lambda^*} + (\lambda - \lambda^*)\chi_Y \qquad (\lambda^* \leq \forall \lambda \leq \lambda^* + \varepsilon)$$

holds. This, together with the concavity of g, implies that $\bar{p}_\lambda \in \arg\max g$ for every $\lambda \in [\lambda^*, \lambda^* + \varepsilon]$ since $\bar{p}_{\lambda^*}, \bar{p}_{\lambda^* + \varepsilon} \in \arg\max g$. Hence, we have $\bar{p}_\lambda \in \arg\max g$ for every $\lambda \in [0, \lambda^* + \varepsilon]$, a contradiction to the definition of λ^*. Hence, we have $\lambda^* = \bar{\lambda}$ and $\bar{p}_\lambda \in \arg\max g$ for every $\lambda \in [0, \bar{\lambda}]$.

It remains to show the minimality of $\bar{p}_\lambda$, i.e., that for every $\lambda \in [0, \bar{\lambda}]$, if q is a maximizer of g with $q \geq p + \lambda \chi_X$, then it satisfies $q \geq \bar{p}_\lambda$. Since q is a maximizer of g with $q \geq p + \lambda \chi_X \geq p$, we have $q \geq \bar{p}$ by the definition of $\bar{p}$. This inequality and $q \geq p + \lambda \chi_X$ imply that $q \geq \bar{p} \vee (p + \lambda \chi_X) = \bar{p}_\lambda$.

[Proof of (ii)] Let λ be a real number with $0 \leq \lambda < \bar{\lambda}$. To prove

$$\|\bar{p}_\lambda - (p + \lambda \chi_X)\|_\infty = \|\bar{p} - p\|_\infty - \lambda, \tag{6.4}$$

it suffices to show the following:

$$\|\bar{p}_\lambda - (p + \lambda \chi_X)\|_\infty = \max\{\bar{p}(i) - p(i) \mid i \in X\} - \lambda, \tag{6.5}$$

$$\|\bar{p}_\lambda - (p + \lambda \chi_X)\|_\infty \geq \max\{\bar{p}(i) - p(i) \mid i \in V \setminus X\} - \lambda, \tag{6.6}$$

since

$$\|\bar{p} - p\|_\infty = \max\left[\max\{\bar{p}(i) - p(i) \mid i \in X\},\ \max\{\bar{p}(i) - p(i) \mid i \in V \setminus X\}\right].$$

We first prove (6.5). By Lemma 4.15 (iii) and $\lambda < \bar{\lambda}$, the set X is a steepest ascent direction at $p + \lambda \chi_X$. Hence, Lemma 4.14 (i) implies that

$$\arg\max\{\bar{p}_\lambda(i) - (p + \lambda \chi_X)(i) \mid i \in V\} \subseteq X, \tag{6.7}$$

since $\bar{p}_\lambda$ is the unique minimal maximizer of g under the condition $\bar{p}_\lambda \geq p + \lambda\chi_X$ by Claim (i) of this lemma. From (6.7) it follows that

$$\begin{aligned}\|\bar{p}_\lambda - (p + \lambda\chi_X)\|_\infty &= \max\{\bar{p}_\lambda(i) - (p + \lambda\chi_X)(i) \mid i \in X\} \\ &= \max\left\{ \max(\bar{p}(i) - p(i) - \lambda, 0) \mid i \in X\right\} \\ &= \max\left[\max\{\bar{p}(i) - p(i) \mid i \in X\} - \lambda, 0\right] \\ &= \max\left[\text{RHS of (6.5)},\ 0\right],\end{aligned}$$

where $\bar{p}_\lambda = \bar{p} \vee (p + \lambda\chi_X)$ is used. In addition, we have $\|\bar{p}_\lambda - (p + \lambda\chi_X)\|_\infty \neq 0$ since $p + \lambda\chi_X$ is not a maximizer of g by:

$$g(p + \lambda\chi_X) = g(p + \bar{\lambda}\chi_X) - (\bar{\lambda} - \lambda)g'(p; \chi_X) < g(p + \bar{\lambda}\chi_X).$$

Therefore, we have (6.5). The inequality (6.6) can be shown easily as follows:

$$\begin{aligned}\|\bar{p}_\lambda - (p + \lambda\chi_X)\|_\infty &\geq \max\{\bar{p}_\lambda(i) - (p + \lambda\chi_X)(i) \mid i \in V \setminus X\} \\ &= \max\{\bar{p}(i) - p(i) \mid i \in V \setminus X\} \\ &\geq \max\{\bar{p}(i) - p(i) \mid i \in V \setminus X\} - \lambda.\end{aligned}$$

From (6.5) and (6.6) it follows (6.4) for every λ with $0 \leq \lambda < \bar{\lambda}$. By the continuity of the norm, the equation (6.4) also holds with $\lambda = \bar{\lambda}$. This concludes the proof of (ii). $\square$

We now prove Theorems 6.1 and 6.2. Suppose that the algorithm terminates in m iterations, and let p^* be the output of the algorithm, i.e., $p^* = p_m$. Since $p^* \geq p^\circ$, we have

$$\|p^* - p^\circ\|_\infty \geq \min\{\|p - p^\circ\|_\infty \mid p \in \arg\max g,\ p \geq p^\circ\} = \|\hat{p} - p^\circ\|_\infty,$$

where the equality is by (4.1). We also have

$$\|p^* - p^\circ\|_\infty = \|(p^\circ + \sum_{k=1}^{m-1} \lambda_k \chi_{X_k}) - p^\circ\|_\infty \leq \sum_{k=1}^{m-1} \lambda_k.$$

Hence, it suffices to show that

$$\|\hat{p} - p^\circ\|_\infty = \sum_{k=1}^{m-1} \lambda_k. \tag{6.8}$$

We prove (6.8). Repeated application of Lemma 6.4 implies that for $k = 1, 2, \ldots, m$, the vector $\hat{p} \vee p_k$ is the unique minimal maximizer of g that is lower-bounded by p_k, and satisfies

$$\|(\hat{p} \vee p_k) - p_k\|_\infty = \|(\hat{p} \vee p_{k-1}) - p_{k-1}\|_\infty - \lambda_{k-1} = \|\hat{p} - p^\circ\|_\infty - \sum_{j=1}^{k-1} \lambda_j.$$

We have $\hat{p} \vee p_m = p_m$ since $p_m \in \arg\max g$. Therefore, (6.8) holds. This concludes the proofs of Theorems 6.1 and 6.2.

We next prove Theorem 6.3. We first give a proof for the special case where

- g is an integral polyhedral L-concave function, i.e., for every $x \in \mathbb{R}^V$ with $\arg\max g[-x] \neq \emptyset$, the polyhedron $\arg\max g[-x]$ is integral (see Section 4.4 for the definition of integral polyhedral L-concave function),
- the initial solution p° is an integral vector.

Then, the value $\bar{c}(p;\chi_X)$ is an integer for every $p \in \mathrm{dom}\, g \cap \mathbb{Z}^V$ and $X \subseteq V$, as shown in Section 4.4. Hence, we can inductively show that the vector p_k as well as the step size λ_k are integral for each k. This fact and Lemma 6.4 (ii) imply that

$$\|\hat{p}_{k+1} - p_{k+1}\|_\infty = \|\hat{p}_k - p_k\|_\infty - \lambda_k \le \|\hat{p}_k - p_k\|_\infty - 1 \qquad (k = 1, 2, \ldots),$$

where $\hat{p}_k$ denotes the unique minimal maximizer of g under the condition that $\hat{p}_k \ge p_k$. Hence, the number of iterations is at most $\|\hat{p}_1 - p_1\|_\infty = \|\hat{p} - p^\circ\|_\infty$.

We then consider the general case. Since g is a polyhedral concave functions, there exist a finite number of distinct sets of the form $\arg\max g[-x]$. Hence, there exists a positive integer τ such that

- the function $g_\tau : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ given by $g_\tau(p) = g(\tau p)$ $(p \in \mathbb{R}^V)$ is an integral polyhedral L-concave function,
- τp° is an integral vector with $\tau p^\circ \in \mathrm{dom}\, g_\tau$.

We see that the behavior of the modified steepest ascent algorithm applied to g with the initial vector p° is essentially the same as that of the modified steepest ascent algorithm applied to g_τ with the initial vector τp°, which terminates in a finite number of iterations by the discussion above. This concludes the proof of Theorem 6.3.

6.2. A bad instance for the modified algorithm

We show that the modified algorithm may not terminate in any finite number of iterations by using the polyhedral L-concave function g_{T} arising from the maximum weight tension problem (MWT) described in Section 2. In particular, we consider a special case of the problem (MWT), which we call the *maximum linear-weight tension problem*, where the weight function φ_{uv} for $(u, v) \in E$ is of the form

$$\varphi_{uv}(\alpha) = \begin{cases} \kappa(u,v)\,\alpha & (\text{if } \underline{\mu}(u,v) \le \alpha \le \overline{\mu}(u,v)), \\ -\infty & (\text{otherwise}) \end{cases}$$

with $\kappa(u, v) \in \mathbb{R}$ and $\overline{\mu}(u, v), \underline{\mu}(u, v) \in \mathbb{R}$. The function $g_{\mathrm{T}} : \mathbb{R}^V \to \mathbb{R} \cup \{-\infty\}$ associated with the maximum linear-weight tension problem is represented as

$$g_{\mathrm{T}}(p) = \begin{cases} \displaystyle\sum_{(u,v)\in E} \kappa(u,v)(p(u) - p(v)) & (\text{if } p \in P), \\ -\infty & (\text{otherwise}), \end{cases} \tag{6.9}$$

where

$$P = \{p \in \mathbb{R}^V \mid \underline{\mu}(u,v) \le p(u) - p(v) \le \overline{\mu}(u,v)\ (\forall (u,v) \in E)\}. \tag{6.10}$$

Note that g_{T} is a linear function on $\mathrm{dom}\, g_{\mathrm{T}} = P$.

We consider the function g_{T} in (6.9) associated with an instance of the maximum linear-weight tension problem given in Figure 1, which is a modification of the one in McCormick and Shioura [7] (see also Queyranne [15]). Only the five broken edges have nonzero value of $\kappa(u, v)$; $\kappa(s, t) = +1$, and each of four arcs $(u, v) = (s, 7), (7, 3), (8, 4)$, and $(4, t)$ has $\kappa(u, v) = K$ with a sufficiently large positive number K. We set $\underline{\mu}(u, v) = 0$ for all $(u, v) \in E$. The value $\overline{\mu}(u, v)$ of each edge (u, v) is indicated in the figure, where

$$r = (\sqrt{5} - 1)/2, \quad S_1 = (1 + r)/2, \quad S_2 = 1/2, \quad S_3 = r/2,$$

and M is a sufficiently large positive number. These numbers satisfy the identities

$$\begin{aligned} &1 = r + r^2, \\ &S_1 - r = r^3 S_1, \quad S_2 - r^2 = r^3 S_2, \quad S_3 - r^3 = r^3 S_3, \end{aligned}$$

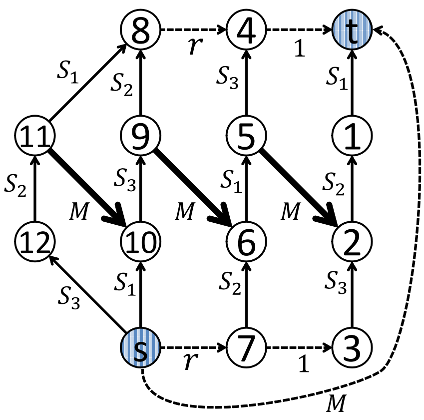

Figure 1: A bad instance for the modified algorithm. The number associated with an edge (u,v) means the upper capacity $\overline{\mu}(u,v)$.

and the inequalities $1 > S_1 > r > S_2 > r^2 > S_3 > r^3 > S_1 - r = r^3 S_1$. The three thick edges $(11,10)$, $(9,6)$, and $(5,2)$ are *special edges*, which control the behavior of the algorithm.

Suppose that the modified steepest ascent algorithm is applied to this function with the initial vector $p_1 = \mathbf{0}$. Set $X_1 = \{s, 8, 9, 10, 11, 12\}$ is a steepest ascent direction at p_1, where $(g_{\rm T})'(p_1; \chi_{X_1}) = 2K + 1$. We assume that this steepest ascent direction is selected in the first iteration. Then, each component $p_1(i)$ for $i \in X_1$ is incremented by $\bar{c}(p_1; \chi_{X_1}) = r$, i.e.,

$$p_2(i) = \begin{cases} r & (\text{if } i \in X_1), \\ 0 & (\text{otherwise}). \end{cases}$$

In the second iteration, the set $X_2 = X_1 \cup \{4, 5, 6, 7\}$ is a steepest ascent direction, where $(g_{\rm T})'(p_2; \chi_{X_2}) = 2K + 1$. We assume that this steepest ascent direction is selected in the second iteration. Then, each component $p_2(i)$ for $i \in X_2$ is incremented by $\bar{c}(p_2; \chi_{X_2}) = 1$, i.e.,

$$p_3(i) = \begin{cases} 1 + r & (\text{if } i \in X_1), \\ 1 & (\text{if } i \in X_2 \setminus X_1), \\ 0 & (\text{otherwise}). \end{cases}$$

The values of $p_k(u) - p_k(v)$ at the beginning of the k-th iteration with $k = 1, 2, 3$ are shown in Table 1. In the analysis below, we mainly consider changes of values $p_k(u) - p_k(v)$ for edges (u,v) rather than the components $p_k(u)$ for $u \in V$ of the vector p_k.

Note that in the following iterations, every steepest ascent direction X must satisfy the following conditions:

$$s \in X,\ t \in V \setminus X, \quad \{s,7\} \cup \{7,3\} \subseteq X, \quad \{8,4\} \cup \{4,t\} \subseteq V \setminus X.$$

Hence, every steepest ascent direction X has the value of the directional derivative equal to 1 in the following iterations.

Repeated choice of the same steepest ascent directions starts from the third iteration. In the 3ℓ-th iteration with $\ell \geq 1$, we can select a steepest ascent direction

$$X_{3\ell} = \{s, 1, 2, 3, 6, 7, 11, 12\}, \text{ with } (g_{\rm T})'(p_{3\ell}; \chi_{X_{3\ell}}) = 1,\ \bar{c}(p_{3\ell}; \chi_{X_{3\ell}}) = r^{3\ell-2}.$$

Note that the set $X_{3\ell}$ cuts all edges in G with capacity S_1, i.e., $u \in X_{3\ell}$ and $v \notin X_{3\ell}$ for each edge (u,v) with capacity S_1. Note also that $X_{3\ell}$ is not a minimal steepest ascent direction at $p_{3\ell}$; the minimal steepest ascent direction is given by $\{s, 3, 7\}$.

Table 1: The values $p_k(u) - p_k(v)$ for $(u, v) \in E$ at the beginning of the k-th iteration.

k	$(4,t)$ $(7,3)$	$(8,4)$ $(s,7)$	special edges		
			$(11,10)$	$(9,6)$	$(5,2)$
1	0	0	0	0	0
2	0	r	0	r	0
3ℓ	1	r	0	$r^{3\ell-2}$	$r^{3\ell-3}$
$3\ell+1$	1	r	$r^{3\ell-2}$	0	$r^{3\ell-1}$
$3\ell+2$	1	r	$r^{3\ell}$	$r^{3\ell-1}$	0

k	edges with		
	$\overline{\mu}(u,v) = S_1$	$\overline{\mu}(u,v) = S_2$	$\overline{\mu}(u,v) = S_3$
1	0	0	0
2	0	0	0
3ℓ	$S_1(1 - r^{3\ell-3})$	$S_2(1 - r^{3\ell-3})$	$S_3(1 - r^{3\ell-3})$
$3\ell+1$	$S_1(1 - r^{3\ell})$	$S_2(1 - r^{3\ell-3})$	$S_3(1 - r^{3\ell-3})$
$3\ell+2$	$S_1(1 - r^{3\ell})$	$S_2(1 - r^{3\ell})$	$S_3(1 - r^{3\ell-3})$

Similarly, in the $(3\ell + 1)$-st and $(3\ell + 2)$-nd iterations, we can select steepest ascent directions $X_{3\ell+1} = \{s, 2, 3, 7, 9, 10, 12\}$ and $X_{3\ell+2} = \{s, 3, 5, 6, 7, 10\}$, respectively, as

$$(g_{\mathrm{T}})'(p_{3\ell+1}; \chi_{X_{3\ell+1}}) = 1, \qquad \bar{c}(p_{3\ell+1}; \chi_{X_{3\ell+1}}) = r^{3\ell-1},$$
$$(g_{\mathrm{T}})'(p_{3\ell+2}; \chi_{X_{3\ell+2}}) = 1, \qquad \bar{c}(p_{3\ell+2}; \chi_{X_{3\ell+2}}) = r^{3\ell}.$$

This shows that in the following iterations, there always exists a steepest ascent direction with a positive value of the directional derivative. Hence, the modified steepest ascent algorithm applied to the function g_{T} given above does not terminate in any finite number of iterations. It should be noted that this bad instance also shows that even if a given function is linear on its effective domain, the modified steepest ascent algorithm may not terminate in any finite number of iterations.

Remark 6.5. We see that the sequence of vector p_k generated by the modified steepest ascent algorithm applied to the bad instance converges to an optimal solution of the maximum linear-weight tension problem. This instance can be modified so that p_k does not converge to any optimal solution (and the algorithm does not terminate in any finite number of iterations).

Let us consider a directed graph obtained from the one in Figure 1 by adding a new vertex t' and a new edge (t, t'), where $\overline{\mu}(t, t') = 1$, $\underline{\mu}(t, t') = 0$, and $\kappa(t, t')$ is a real number with $0 < \kappa(t, t') < 1$. If we apply the modified steepest ascent algorithm to the new instance, then the same sequence of steepest ascent directions can be selected since $\kappa(t, t') < 1$. Therefore, we have $p_k(t) = p_k(t') = 0$ for all k, while every optimal solution p^* of this instance satisfies $p^*(t) - p^*(t') = \overline{\mu}(t, t') > 0$. Hence, p_k does not converge to any optimal solution. □

6.3. A bad instance for the modified Hassin's algorithm

In each iteration of Hassin's algorithm in Section 3.1, the unique minimal maximizer $X \subseteq V$ of the value $I(p, X)$ is chosen and used to update the vector p. We point out that the choice of the unique minimal set is essential for the finite termination of Hassin's algorithm. We show that a modified version of Hassin's algorithm, where X is not necessarily the minimal maximizer of the value $I(p, X)$, does not terminate in any finite number of iterations for some instance of the dual of the minimum cost flow problem. This modified version of Hassin's algorithm coincides with a special case of the modified steepest ascent algorithm considered in this section applied to the polyhedral L-concave function in (3.1).

In the following, we show that every instance of the maximum linear-weight tension problem can be transformed to an instance of the dual of a minimum cost flow problem. According to this transformation, the bad instance in Section 6.2 yields a bad instance for the modified version of Hassin's algorithm.

Consider an instance of the maximum linear-weight tension problem given by a directed graph $G = (V, E)$ and values $\kappa(u, v), \overline{\mu}(u, v), \underline{\mu}(u, v) \in \mathbb{R}$ for $(u, v) \in E$. Recall that the set P of feasible vectors is given by (6.10). We assume that $P \neq \emptyset$. We define an instance of the dual of a minimum cost flow problem on a directed graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with $\tilde{V} = V$ and

$$\tilde{E} = \{e', e'' \mid e', e'' \text{ are copies of } e \in E\} \cup \{\bar{e} \mid \bar{e} \text{ is the reverse edge of } e \in E\}.$$

With sufficiently large positive numbers Γ and C, we set

$$\begin{aligned} \gamma(e') &= -\Gamma, & c(e') &= \kappa(e), \\ \gamma(e'') &= -\underline{\mu}(e), & c(e'') &= C, \\ \gamma(\bar{e}) &= \overline{\mu}(e), & c(\bar{e}) &= C. \end{aligned}$$

Then, the objective function of the dual minimum cost flow problem in (3.1) is given as follows:

$$\begin{aligned} g_{\mathrm{H}}(p) = &\sum_{(u,v)\in E} \kappa(u, v) \min\{0, p(u) - p(v) - \Gamma\} \\ &+ \sum_{(u,v)\in E} C\bigl[\min\{0, p(u) - p(v) - \underline{\mu}(u, v)\} + \min\{0, p(v) - p(u) + \overline{\mu}(u, v)\}\bigr]. \end{aligned}$$

We show that the resulting instance of the dual minimum cost flow problem is equivalent to the given instance of the maximum linear-weight tension problem. Since C is sufficiently large, every maximizer p of the function g_{H} satisfies

$$p(u) - p(v) - \underline{\mu}(u, v) \geq 0, \quad p(v) - p(u) + \overline{\mu}(u, v) \geq 0 \qquad ((u, v) \in E),$$

i.e., $\arg\max g_{\mathrm{H}} \subseteq P$. Since Γ is also sufficiently large, we have

$$\min\{0, p(u) - p(v) - \Gamma\} = p(u) - p(v) - \Gamma$$

for $p \in P$, which implies that

$$\begin{aligned} g_{\mathrm{H}}(p) &= \sum_{(u,v)\in E} \kappa(u, v)(p(u) - p(v) - \Gamma) \\ &= \sum_{(u,v)\in E} \kappa(u, v)(p(u) - p(v)) - \Gamma \sum_{(u,v)\in E} \kappa(u, v) \qquad (p \in P), \end{aligned}$$

where $\sum_{(u,v)\in E} \kappa(u, v)(p(u) - p(v))$ is equal to the objective function value of the maximum linear-weight tension problem, and the term $\Gamma \sum_{(u,v)\in E} \kappa(u, v)$ is a constant. Hence, the set of optimal solutions of the dual minimum cost flow problem coincides with that of the maximum linear-weight tension problem. Since C is sufficiently large, every steepest ascent direction X of g_{H} at $p \in P$ satisfies the condition that $p + \varepsilon' \chi_X \in P$ for $\varepsilon' \in [0, \varepsilon]$ with a sufficiently small $\varepsilon > 0$. This fact implies that the behavior of the modified Hassin's algorithm for the instance of the maximum linear-weight tension problem coincides with that of the modified steepest ascent algorithm for the given instance of the dual minimum cost flow problem.

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