THE $E_6$ STATE SUM INVARIANT OF LENS SPACES

KENTA OKAZAKI

Abstract. In this paper, we calculate the values of the $E_6$ state sum invariant for the lens spaces $L(p, q)$. In particular, we show that the values of the invariant are determined by $p \mod 12$ and $q \mod (p, 12)$. As a corollary, we show that the $E_6$ state sum is a homotopy invariant for the oriented lens spaces.

1. Introduction

In [5], Turaev and Viro constructed a state sum invariant of 3-manifolds based on their triangulations, by using the $6j$-symbols of representations of the quantum group $U_q(sl_2)$. Further, Ocneanu [2] generalized the construction to the case of other types of $6j$-symbols, say, the $6j$-symbols of subfactors. The $E_6$ state sum invariant is the state sum invariant constructed from the $6j$-symbols of the $E_6$ subfactor, which we denote by $Z$. Suzuki and Wakui [4] calculated the $E_6$ state sum invariant for some of the lens spaces, where they used the representation of the mapping class group of a torus $SL(2, \mathbb{Z})$.

In this paper, we calculate the $E_6$ state sum invariant for all of the lens spaces, as follows. For integers $m, n$, we denote by $(m, n)$ the greatest common divisor of $m$ and $n$. We put $\zeta = \exp(\pi \sqrt{-1}/12)$ and $[n] = (\zeta^n - \zeta^{-n})/(\zeta - \zeta^{-1})$ for an integer $n$, noting that

- $[12 - n] = [n], \quad [n + 12] = -[n],$

1
Theorem 1.1. For coprime integers $p$ and $q$, the $E_6$ state sum invariant of the lens space $L(p,q)$ is given as

$$Z(L(p,q)) = \begin{cases} 
[p] & \text{if } (p,12) = 1, \\
[4][3]/[2] & \text{if } (p,12) = 2, 6, \\
\zeta^{\pm 3}[4] & \text{if } (p,12) = 3 \text{ and } q \equiv \pm 1 \mod 3, \\
2\zeta^{\pm 2}[3] & \text{if } (p,12) = 4 \text{ and } q \equiv \pm 1 \mod 4, \\
2[4][3]/[2] & \text{if } 12|p \text{ and } q \equiv \pm 1 \mod 12, \\
0 & \text{if } 12|p \text{ and } q \equiv \pm 5 \mod 12. 
\end{cases}$$

(1.1)

In particular, the value of $Z(L(p,q))$ is determined by $p \mod 12$ and $q \mod (p,12)$.

We note that we normalize the invariant so that $Z(S^3) = 1$. Thus, our $Z$ is equal to $wZ$ in [4], where we put $w = 2 + [3] = 6 + 2\sqrt{3}$.

Corollary 1.2. If there exists an orientation-preserving homotopy equivalence between the two lens spaces $L(p,q)$ and $L(p',q')$, then $Z(L(p,q)) = Z(L(p',q'))$.

We note that from Theorem 1.1 the $E_6$ state sum invariant distinguishes $L(p,q)$ from $L(p,-q)$ if and only if $(p,12) = 3$ or 4. This is a generalization of [4, Corollary 4.3].

The author would like to thank Tomotada Ohtsuki for his encouragement and comments. The author is also grateful to Michihisa Wakui for his helpful comments.

2. The Calculation of the $E_6$ State Sum Invariant

In this section, we briefly review the calculation of the $E_6$ state sum invariant for the lens spaces. Suzuki and Wakui [4] defined the representation $\rho : SL(2,\mathbb{Z}) \rightarrow GL_{10}(\mathbb{C})$ by

$\rho(S)$
THE $E_6$ STATE SUM INVARIANT OF LENS SPACES

\[
\begin{pmatrix}
0 & 0 & -[3] & -[3] & 0 & 0 & 0 & 0 & 0 & 1  \\
0 & -[4][3] & [4][3] & 0 & 0 & 0 & 0 & 0 & 0 & 0  \\
[3][3] & [-3] & [-3] & 0 & 0 & 0 & 0 & 0 & 0 & 0  \\
\end{pmatrix}
\]

\[
\frac{1}{w} \rho(T) = \frac{1}{w} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \\
0 & -\zeta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \\
0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0  \\
0 & 0 & 0 & 0 & 0 & -\zeta^2 & 0 & 0 & 0 & 0  \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0  \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^8 & 0 & 0  \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^{-4} & 0  \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1  \\
\end{pmatrix},
\]

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are generators of $SL(2, \mathbb{Z})$.

Let $p, q$ be coprime integers. We choose a continued fraction expansion of $p/q$:

\[
p = c_1 - \frac{1}{c_2 - \frac{1}{c_3} - \frac{1}{\cdots - \frac{1}{c_n}}}
\]

From this, we have that

\[
\begin{pmatrix} a & p \\ b & q \end{pmatrix} = T^{c_1} S T^{c_2} \cdots S T^{c_{n-1}} S T^{c_n} \in SL(2, \mathbb{Z})
\]

for some integers $a, b$. From [4, Lemma 4.2], one can derive that the $E_6$ state sum invariant of lens spaces are given as

\[
Z(L(p, q)) = w^t \epsilon \rho(ST^{c_1} ST^{c_2} \cdots ST^{c_{n-1}} ST^{c_n} S) \epsilon
\]

\[
= w^t \epsilon \rho\left(\begin{pmatrix} -q & b \\ p & -a \end{pmatrix}\right) \epsilon,
\]
where we put \( e = \mathcal{L}(1, 0, 0, 0, 0, 0, 0, 0, 0) \). We note that this value does not depend on the choice of continued fraction expansions of \( p/q \) (and that of \( a \) and \( b \)).

3. Proof of the theorem

In this section, we prove Theorem 1.1 and Corollary 1.2. In order to show Theorem 1.1, we show Proposition 3.4, which says that the values of \( Z(L(p, q)) \) have period 12 for \( p \) and \( q \). In order to show Proposition 3.4, we show Lemmas 3.1 and 3.3, as follows.

**Lemma 3.1.** Let \( p, q, p', q' \) be integers satisfying \( (p, q) = 1 \), \( (p', q') = 1 \) and \( p \equiv p' \mod 12 \), \( q \equiv q' \mod 12 \). Then, there exist integers \( a, b, a', b' \) such that

\[
aq - bp = 1, \quad a'q' - b'p' = 1 \quad \text{and} \quad a \equiv a', \; b \equiv b' \mod 12.
\]

**Proof.** We put integers \( a \) and \( b \) satisfying \( aq - bp = 1 \). Further, we put

\[
\begin{pmatrix}
a' & p' \\
b' & q'
\end{pmatrix} = \begin{pmatrix} a & p \\ b & q \end{pmatrix} + 12 \begin{pmatrix} x & z \\ y & w \end{pmatrix},
\]

noting that, by assumption, \( z \) and \( w \) are determined uniquely. It is sufficient to show that there exist integers \( x \) and \( y \) satisfying \( a'q' - b'p' = 1 \). The determinant of the right-hand side of the above formula is equal to

\[
(a + 12x)(q + 12w) - (p + 12z)(b + 12y) = a(q + 12w) + 12xq' - (p + 12z)b - 12p'y = 1 + 12(axq' - zb - p'y).
\]

Since \( (p', q') = 1 \), there exists integers \( x \) and \( y \) satisfying

\[
p'y - q'x = aw - zb.
\]

Then, the last term of (3.1) is equal to 1. Thus, we have \( a'q' - b'p' = 1 \), as required. \( \square \)

We denote by \( I_n \) the \( n \)-by-\( n \) identity matrix. We put

\[
\Gamma = \{ P \in SL(2, \mathbb{Z}) \mid P \equiv I_2 \mod 12 \}.\]
Lemma 3.2. $\Gamma$ is a normal closure of the set of the following 19 matrices.

\[
\begin{align*}
P_0 &= \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix}, & P_1 &= \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}, & P_2 &= \begin{pmatrix} -143 & 12 \\ -12 & 1 \end{pmatrix}, \\
P_3 &= \begin{pmatrix} -155 & 84 \\ -24 & 13 \end{pmatrix}, & P_4 &= \begin{pmatrix} -191 & 156 \\ -60 & 49 \end{pmatrix}, & P_5 &= \begin{pmatrix} -443 & 120 \\ -48 & 13 \end{pmatrix}, \\
P_6 &= \begin{pmatrix} -467 & 360 \\ -48 & 37 \end{pmatrix}, & P_7 &= \begin{pmatrix} -299 & 108 \\ -36 & 13 \end{pmatrix}, & P_8 &= \begin{pmatrix} -311 & 216 \\ -36 & 25 \end{pmatrix}, \\
P_9 &= \begin{pmatrix} 937 & -396 \\ 168 & -71 \end{pmatrix}, & P_{10} &= \begin{pmatrix} 157 & -36 \\ 48 & -11 \end{pmatrix}, & P_{11} &= \begin{pmatrix} 157 & -48 \\ 36 & -11 \end{pmatrix}, \\
P_{12} &= \begin{pmatrix} 205 & -84 \\ 144 & -59 \end{pmatrix}, & P_{13} &= \begin{pmatrix} 157 & -72 \\ 24 & -11 \end{pmatrix}, & P_{14} &= \begin{pmatrix} 229 & -132 \\ 144 & -83 \end{pmatrix}, \\
P_{15} &= \begin{pmatrix} 169 & -108 \\ 36 & -23 \end{pmatrix}, & P_{16} &= \begin{pmatrix} 181 & -132 \\ 48 & -35 \end{pmatrix}, & P_{17} &= \begin{pmatrix} 589 & -108 \\ 60 & -11 \end{pmatrix}, \\
P_{18} &= \begin{pmatrix} 649 & -384 \\ 120 & -71 \end{pmatrix}.
\end{align*}
\]

Proof. The GAP package Congruence [1] shows that $\Gamma$ is generated by 97 matrices, as follows.

```gap
gap> LoadPackage("congruence");
true
gap> G:=PrincipalCongruenceSubgroup(12);
GeneratorsOfGroup(G); <principal congruence subgroup of level 12 in SL_2(Z)>
gap> GeneratorsOfGroup(G);
#I Using the Congruence package for GeneratorsOfGroup ... 
[ [ [ [ 1, 12 ] ], [ 0, 1 ] ], [ [ -143, 12 ] ], [ -12, 1 ] ],
 [ [ 589, -108 ], [ 60, -11 ] ], [ [ 157, -36 ], [ 48, -11 ] ],
 [ [ -299, 108 ], [ -36, 13 ] ], [ [ 205, -84 ], [ 144, -59 ] ],
 [ [ 937, -396 ], [ 168, -71 ] ], [ [ 157, -72 ], [ 24, -11 ] ],
 [ [ -155, 84 ], [ -24, 13 ] ], [ [ 229, -132 ], [ 144, -83 ] ],
 [ [ 649, -384 ], [ 120, -71 ] ], [ [ 169, -108 ], [ 36, -23 ] ],
 [ [ -311, 216 ], [ -36, 25 ] ], [ [ 181, -132 ], [ 48, -35 ] ],
 [ [ -467, 360 ], [ -48, 37 ] ], [ [ -191, 156 ], [ -60, 49 ] ],
 [ [ 13, -12 ], [ 12, -11 ] ], [ [ 649, -768 ], [ 60, -71 ] ],
 [ [ 205, -252 ], [ 48, -59 ] ], [ [ -491, 624 ], [ -48, 61 ] ],
```
We can verify that the set of the above 97 matrices is equal to
\[
\{ P_0 \} \cup \{ T^j P T^{-j} | j = 1, 2, \ldots, 11 \} \cup \bigcup_{n=2}^{18} \{ T^j P_n T^{-j} | j = 0, 1, \ldots, m_n \},
\]
where we put
\[m_2 = 0, \ m_3 = 5, \ m_4 = 8, \ m_5 = 2, \ m_6 = 2, \ m_7 = 3, \]
\[m_8 = 3, \ m_9 = 0, \ m_{10} = 8, \ m_{11} = 7, \ m_{12} = 4, \ m_{13} = 5, \]
\[m_{14} = 4, \ m_{15} = 7, \ m_{16} = 8, \ m_{17} = 2, \ m_{18} = 0, \]
completing the proof.

\[\square\]

**Lemma 3.3.** \(\Gamma \subset \ker \rho\), that is, \(\rho(P) = I_{10}\) for any \(P \in \Gamma\).

**Proof.** From Lemma 3.2, it is sufficient to show that \(\rho(P_n) = I_{10}\) for \(n = 0, 1, \ldots, 18\). We can verify that these matrices is presented as the products of \(S\) and \(T\), as follows.

\[
P_0 = T^{12}, \quad P_1 = S^3T^{-12}S,
\]
\[
P_2 = S^2T^{12}ST^{12}S, \quad P_3 = S^2T^7ST^7ST^7ST^2S,
\]
\[
P_4 = S^2T^3ST^{-5}ST^3ST^{-4}ST^S, \quad P_5 = T^9ST^{-4}ST^3ST^4S,
\]
\[
P_6 = T^{10}ST^4ST^3ST^{-3}STS, \quad P_7 = T^8ST^{-3}ST^4ST^3S,
\]
\[
P_8 = T^9ST^3ST^4ST^{-2}STS, \quad P_9 = T^5ST^{-2}ST^{-4}ST^{-4}ST^{-3}ST^2S,
\]
\[
P_{10} = T^3ST^{-4}ST^{-3}ST^4S, \quad P_{11} = T^4ST^{-3}ST^{-4}ST^3S,
\]
\[
P_{12} = TST^{-2}ST^3ST^4ST^{-2}ST^2S, \quad P_{13} = T^6ST^{-2}ST^{-6}ST^2S,
\]
\[
P_{14} = TST^{-2}ST^{-3}ST^4ST^2S, \quad P_{15} = S^2T^5ST^3ST^{-3}ST^2ST^2S,
\]
\[
P_{16} = T^4ST^3ST^{-3}ST^3STS, \quad P_{17} = S^2T^{10}ST^5ST^{-2}ST^5S,
\]
\[
P_{18} = T^5ST^{-2}ST^3ST^{-4}ST^3ST^2S.
\]

By using these formulae, we can verify that \(\rho\) takes to each of the matrices to \(I_{10}\), completing the proof of the lemma.

\[\square\]

**Proposition 3.4.** Let \(p, q, p', q'\) be integers satisfying \((p, q) = 1, (p', q') = 1\) and \(p \equiv p', q \equiv q' \mod 12\). Then, \(Z(L(p, q)) = Z(L(p', q'))\).

**Proof.** From Lemma 3.1, there exist matrices
\[
A = \begin{pmatrix} -q & b \\ p & -a \end{pmatrix}, \quad A' = \begin{pmatrix} -q' & b' \\ p' & -a' \end{pmatrix} \in SL(2, \mathbb{Z})
\]
such that \( A \equiv A' \mod 12 \). We put \( P = I_2 + A^{-1}(A' - A) \). By definition, \( P \in \Gamma \) and \( A' = AP \). Thus, by (2.1) and Lemma 3.3, we have
\[
Z(L(p', q')) = w^t \rho(A') e = w^t \rho(A) \rho(P) e = w^t \rho(A) e = Z(L(p, q)),
\]
completing the proof. \( \square \)

**Proof of Theorem 1.1.** By Proposition 3.4, the left-hand side of (1.1) has period 12 for \( p \) and \( q \). On the other hand, the right-hand side of (1.1) also has period 12 for \( p \) and \( q \). By [4, Appendix D], we can verify that (1.1) holds for coprime integers \( p \) and \( q \) with \( 1 \leq p \leq 12, q < p \). Therefore, (1.1) holds for any coprime integers \( p \) and \( q \). \( \square \)

**Proof of Corollary 1.2.** It is known, see [3, Remark 3], that there exists an orientation-preserving homotopy equivalence between the two lens spaces \( L(p, q) \) and \( L(p', q') \) if and only if \( p = p' \) and \( q \equiv n^2 q' \mod p \) for some integer \( n \).

When \( (p, 12) \neq 3, 4, 12 \), by Theorem 1.1 the value \( Z(L(p, q)) \) does not depend on \( q \). Thus, \( Z(L(p, q)) = Z(L(p', q')) \).

When \( (p, 12) = k \) with \( k \in \{3, 4, 12\} \), we have \( q \equiv n^2 q' \mod k \) for some integer \( n \). Thus, it is enough to show that \( q \equiv q' \mod k \).

(a) When \( k \in \{3, 4\} \), we have \( \{n^2 \mid n \in \mathbb{Z} / k \mathbb{Z} \} = \{0, 1\} \). Since \( (p, q) = 1 \), it implies that \( q \equiv q' \mod k \).

(b) When \( k = 12 \), we have \( \{n^2 \mid n \in \mathbb{Z} / 12 \mathbb{Z} \} = \{0, 1, 4, 9\} \). Again, since \( (p, q) = 1 \), it implies that \( q \equiv q' \mod k \).

Thus, by Theorem 1.1, \( Z(L(p, q)) = Z(L(p', q')) \). \( \square \)

**References**


Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan

E-mail address: junes@kurims.kyoto-u.ac.jp