

Schnorr triviality and its equivalent notions

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August 21, 2013

Abstract

We give some characterizations of Schnorr triviality. In concrete terms, we introduce a reducibility related to decidable prefix-free machines and show the equivalence with Schnorr reducibility. We also give a uniform-Schnorr-randomness version of the equivalence of LR-reducibility and LK-reducibility. Finally we prove a base-type characterization of Schnorr triviality.

Keywords: algorithmic randomness, Schnorr randomness, Schnorr triviality, uniform relativization, lowness for uniform Schnorr randomness

1 Introduction

The theory of algorithmic randomness [23, 9] defines some randomness notions and studies the properties of them. At the same time, the theory also studies a set that is “far from random” or “randomness theoretically weak”.

Martin-Löf randomness [18] is the randomness notion that has been studied the most. One formulation of “far from ML-random” is K -triviality and this notion has many characterizations. For instance, the following equivalence for a set A [22, 12]:

- (i) A is K -trivial, that is, $K(A \upharpoonright n) \leq K(n) + O(1)$,
- (ii) A is low for Martin-Löf randomness, that is, each Martin-Löf random set is Martin-Löf random relative to A ,
- (iii) A is low for prefix-free Kolmogorov complexity, that is, $K(\sigma) \leq K^A(\sigma) + O(1)$,
- (iv) A is a base for Martin-Löf randomness, that is, A is Turing reducible to a set Martin-Löf random relative to A .

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Note that (i) means that A is far from random by the Levin-Schnorr theorem [9, Theorem 6.2.3]. The equivalence between (ii) and (iii) was strengthened to the equivalence between LR-reducibility and LK-reducibility [15].

Schnorr randomness [24] is another important randomness notion. Similarly, the lowness notions for Schnorr randomness have been studied [7, 8, 16, 6]. Unlike the case of Martin-Löf randomness, lowness for Schnorr randomness is not equivalent to Schnorr triviality, which is a Schnorr-randomness version of K -triviality [8].

Franklin and Stephan [10] and Miyabe [20] claimed that truth-table reducibility is more suitable than Turing reducibility when studying a notion of relativized Schnorr randomness and showed the equivalence of the following:

- (i) Schnorr triviality,
- (ii) computable tt-traceability,
- (iii) tt-lowness for Schnorr randomness,
- (iv) lowness for truth-table reducible measure machines.

Franklin and Stephan [10] also tried to give a base-type characterization of Schnorr triviality, and found that the straight-forward analogue does not work (see Section 6).

Miyabe [20] claimed that truth-table Schnorr randomness is the proper relativization of Schnorr randomness and that it is different from the usual relativization. Subsequently, Miyabe and Rute [21] introduced *uniform relativization*. See Section 6 in Miyabe and Rute [21] for the discussion of the change of the terminology from “tt” to “uniform”.

We also started unusual usage about a randomness notion. For instance, if we say “ML-randomness”, it means that a unrelativized randomness notion without an oracle. However, if we say that “van Lambalgen’s theorem holds for ML-randomness”, we automatically think about the relativized randomness notion. Similarly, we would like to say that “van Lambalgen’s theorem holds for uniform Schnorr randomness” and “van Lambalgen’s theorem does not hold for Schnorr randomness”. Hence, it is better to think that

- a randomness notion is defined including its relativization,
- the lowness of the randomness notion is defined automatically.

This is why we prefer “low for uniform Schnorr randomness” to “uniformly low for Schnorr randomness”. Then, *uniform Schnorr randomness* is equivalent to tt-Schnorr randomness, which means that, for each $A, B \in 2^\omega$, B is Schnorr random uniformly relative to A if and only if B is Schnorr random tt-relative to A . Here, “uniform” modifies “randomness” because uniform Schnorr randomness is the uniform version of the randomness notion that is called Schnorr randomness.

In this paper we give some new characterizations of Schnorr triviality. In Section 3 we give a characterization of Schnorr reducibility via decidable prefix-free machines. Schnorr reducibility is defined via computable measure machines, with which Schnorr randomness has a characterization. Since we have a characterization of Schnorr randomness via decidable prefix-free machines, such a characterization should be studied. In Section 5 we study a uniform-Schnorr-randomness version of the equivalence between LK-reducibility and LR-reducibility, using the techniques of open covers. The proof highly depends on Bienvenu and Miller [2]. In Section 6 we give a base-type characterization of Schnorr triviality. More precisely, we will show the equivalence among the following:

- (i) Schnorr triviality,
- (ii) being a base for uniform Schnorr tests,
- (iii) being a base for uniformly computable martingales.

The main tool is a uniformly-computable-martingale version of the Kucěra-Gács Theorem, the proof of which make use of the lemma in Merkle and Mihailović [19]. It should be noted that Franklin, Stephan and Yu [11] studied a base for Schnorr randomness, which is a notion different from the one considered in our study.

2 Preliminary

We refer to the books [23, 9] for the notions in algorithmic randomness. By $2^{<\omega}$, we denote the set of binary strings. Cantor space 2^ω is the set of infinite binary sequences equipped with the canonical topology. By \preceq , we denote the prefix relation. For $\sigma \in 2^{<\omega}$, we denote by $[\sigma]$ the set of all extensions of σ in 2^ω , that is, $[\sigma] = \{X \in 2^\omega : \sigma \prec X\}$. For a set $W \subseteq 2^{<\omega}$, $\llbracket W \rrbracket = \bigcup_{\sigma \in W} [\sigma]$. The uniform measure on 2^ω is denoted by μ . We identify a set $X \subseteq \mathbb{N}$ with the binary sequence X by $n \in X \iff X(n) = 1$ where $X(n)$ is the n -th bit of X .

2.1 Schnorr randomness

The following are basic definitions and results on Schnorr randomness.

A *Schnorr test* is a sequence $\{U_n\}$ of uniformly c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ for all n and the measure $\mu(U_n)$ is uniformly computable in n . A set $X \in 2^\omega$ *passes* a Schnorr test $\{U_n\}$ if $x \notin \bigcap_n U_n$. A set is called *Schnorr random* if it passes all Schnorr tests.

A *machine* is a partial computable function from $2^{<\omega}$ to $2^{<\omega}$. For a machine M , *Kolmogorov complexity* K of $\tau \in 2^{<\omega}$ with respect to M is defined by $K_M(\tau) = \min\{|\sigma| : M(\sigma) = \tau\}$. A set of strings is *prefix-free* if, for two disjoint strings in the set, one is not a prefix of the other. A machine is called *prefix-free* if its domain is prefix-free. A *computable measure machine* is a prefix-free machine M such that $\mu(\llbracket \text{dom}(M) \rrbracket) = \sum_{\sigma \in \text{dom}(M)} 2^{-|\sigma|}$ is computable. A

set $X \in 2^\omega$ is Schnorr random iff $K_M(X \upharpoonright n) \geq n - O(1)$ for all computable measure machines M [7].

A machine M is called *decidable* if $\text{dom}(M)$ is computable. An *order* is an unbounded nondecreasing function from \mathbb{N} to \mathbb{N} .

Theorem 2.1 (Bienvenu and Merkle [1]). *A set X is Schnorr random iff for all decidable prefix-free machines M and all computable orders g , we have*

$$K_M(X \upharpoonright n) \geq n - g(n) - O(1).$$

A *martingale* is a function $d : 2^{<\omega} \rightarrow \mathbb{R}^+$ such that $2d(\sigma) = d(\sigma 0) + d(\sigma 1)$ for all $\sigma \in 2^{<\omega}$. A set X is Schnorr random iff $d(X \upharpoonright n) \leq g(n)$ for all but finitely many n for each computable martingale d and each computable order g .

2.2 Uniform Schnorr and computable randomness

When relativizing a randomness notion, we need to be careful. One criterion of naturalness of a relativization is whether van Lambalgen's theorem holds or not [20, 5]. Miyabe [20] and Miyabe and Rute [21] showed that van Lambalgen theorem holds for uniform Schnorr randomness. The following definitions and results in this subsection are essentially from Miyabe [20] and Miyabe and Rute [21] although the formulations are slightly different.

Let \mathcal{O} be the set of open sets on 2^ω . We refer to [26, 3, 4, 27] for computability from 2^ω to \mathcal{O} , from 2^ω to \mathbb{R} and so on.

Definition 2.2. *A uniform Schnorr test is a computable function $f : 2^\omega \times \omega \rightarrow \mathcal{O}$ such that the function $\langle X, n \rangle \mapsto \mu(f(X, n))$ is computable and $\mu(f(X, n)) \leq 2^{-n}$ for all $X \in 2^\omega$ and $n \in \omega$. We say that a sequence $\{U_n^A\}$ is a Schnorr test uniformly relative to A if $U_n^A = f(A, n)$ for all n for some uniform Schnorr test f . A set B is called Schnorr random uniformly relative to A if $B \notin \bigcap_n U_n^A$ for each Schnorr test $\{U_n^A\}$ uniformly relative to A .*

As already mentioned in Section 1, Miyabe and Rute [21] showed that uniform Schnorr randomness is equivalent to tt-Schnorr randomness studied in [10, 20].

Uniform Schnorr randomness has characterizations via complexity and martingales.

Definition 2.3. *An oracle prefix-free machine M is a uniformly computable measure machine if the function $X \mapsto \mu(\llbracket \text{dom}(M^X) \rrbracket)$ is computable.*

A uniformly computable measure machine is called a *truth-table reducible measure machine* in [20].

Theorem 2.4 ([20]). *A set B is Schnorr random uniformly relative to A iff $K_{M^A}(B \upharpoonright n) > n - O(1)$ for each uniformly computable measure machine M .*

Definition 2.5. *A uniform martingale test is a computable map $d : 2^\omega \times 2^{<\omega} \rightarrow \mathbb{R}^+$ such that $d^Z = d(Z, \cdot)$ is a martingale for each $Z \in 2^\omega$. In this case, we say that such a martingale d^A is computable uniformly relative to A .*

See also the definition in Franklin and Stephan [10].

Theorem 2.6 ([10, 20]). *A set B is Schnorr random uniformly relative to A iff $d^A(B \upharpoonright n) \leq h(n)$ for almost all n for each martingale d^A computable uniformly relative to A and each computable order h .*

2.3 Lowness notions for uniform Schnorr randomness

Here we review lowness notions for uniform Schnorr randomness. Notice that we modify the terminology according to the change of the name of uniform Schnorr randomness.

Definition 2.7 (Downey et al. [7]). *A set A is Schnorr reducible to a set B (denoted by $A \leq_{\text{Sch}} B$) if for each computable measure machine M there is a computable measure machine N such that $K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1)$. A set A is Schnorr trivial if $A \leq_{\text{Sch}} \emptyset$.*

Definition 2.8. *A trace is a sequence $\{T_n\}$ of sets. A trace $\{T_n\}$ is a trace for a total function f if $f(n) \in T_n$ for all n . We say a trace $\{T_n\}$ traces a function f if $\{T_n\}$ is a trace for f . A set A is computably tt-traceable via a computable order h if all functions $f \leq_{\text{tt}} A$ are traced by an h -bounded computable trace $\{T_n\}$. We say that a set A is computably tt-traceable if it is computable tt-traceable for some computable order h .*

Note that a set is computably tt-traceable iff it is computably tt-traceable for all computable orders. For details, see [25], [23, Theorem 8.2.3] and [13].

Definition 2.9. *A set A is low for uniform Schnorr randomness if every Schnorr random set is Schnorr random uniformly relative to A . A set A is low for uniformly computable measure machines if for each uniformly computable measure machine M there is a computable measure machine N such that $K_N(\sigma) \leq K_{M^A}(\sigma) + O(1)$.*

The following is due to Franklin and Stephan [10, Theorem 3.2] and Miyabe [20, Theorem 7.3].

Theorem 2.10. *The following are equivalent for a set $A \in 2^\omega$:*

- (i) *A is Schnorr trivial,*
- (ii) *A is computably tt-traceable,*
- (iii) *A is low for uniform Schnorr randomness,*
- (iv) *A is low for uniformly computable measure machines.*

We also use the following result.

Definition 2.11 (Hölzl and Merkle [13]). *A set A is called totally infinitely often complex (or totally i.o. complex) if there is a computable function h such that for all total machines M there are infinitely many n where $K_M(A \upharpoonright h(n)) \geq n$.*

Theorem 2.12 (Hölzl and Merkle [13]). *A set is not totally i.o. complex iff it is Schnorr trivial.*

3 Characterization of Schnorr triviality via decidable prefix-free machines

Schnorr randomness has a characterization via decidable prefix-free machines (Theorem 2.1). Then the following reducibility should have a strong relation to Schnorr randomness.

Definition 3.1. *A set A is called weakly decidable prefix-free machine reducible to a set B (denoted by $A \leq_{\text{wdm}} B$) if for each decidable prefix-free machine M and a computable order g there exists a decidable prefix-free machine N such that*

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + g(n) + O(1).$$

First we see basic properties.

Proposition 3.2. *The relation \leq_{wdm} is reflexive and transitive.*

Proof. The reflexivity is immediate.

Let $A \leq_{\text{wdm}} B$ and $B \leq_{\text{wdm}} C$. Let g be a computable order. For a decidable prefix-free machine M and the order $g/2$ there exists a decidable prefix-free machine N such that $K_N(B \upharpoonright n) \leq K_M(C \upharpoonright n) + g(n)/2 + O(1)$. For this N and the order $g/2$ there exists a decidable prefix-free machine L such that $K_L(A \upharpoonright n) \leq K_N(B \upharpoonright n) + g(n)/2 + O(1)$. Hence $K_L(A \upharpoonright n) \leq K_M(C \upharpoonright n) + g(n) + O(1)$. \square

The following is also immediate.

Theorem 3.3. *If a set A is Schnorr random and $A \leq_{\text{wdm}} B$, then B is Schnorr random.*

Proof. Let M be a decidable prefix-free machine and g be a computable order. Then there exists a decidable prefix-free machine N such that $K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + g(n)/2 + O(1)$. Since A is Schnorr random, $K_N(A \upharpoonright n) \geq n - g(n)/2 - O(1)$. Then it follows that $K_M(B \upharpoonright n) \geq n - g(n) - O(1)$. Since M and g are arbitrary, B is Schnorr random by Theorem 2.1. \square

We give a characterization of Schnorr triviality via the relation \leq_{wdm} .

Theorem 3.4. *A set A is Schnorr trivial iff $A \leq_{\text{wdm}} \emptyset$.*

This theorem follows from Theorem 3.5 but we give a direct short proof here.

Proof. (“if” direction) Suppose $A \leq_{\text{wdm}} \emptyset$. Let h be a computable order. Then there exists a computable order g such that $g \circ h(n) \leq n/2$. Note that $g \circ h$ is a computable order. Let M be a decidable prefix-free machine such that $K_M(h(n)) \leq 2 \log n + O(1)$ for almost all n . Since $A \leq_{\text{wdm}} \emptyset$, there exists a decidable prefix-free machine N such that

$$K_N(A \upharpoonright h(n)) \leq K_M(h(n)) + g(h(n)) + O(1) \leq 2 \log n + n/2 + O(1) < n$$

for almost all n . By Theorem 2.12, A is Schnorr trivial.

(“only if” direction) Suppose that A is Schnorr trivial. Then A is computably tt-traceable by Theorem 2.10.

We show $A \leq_{\text{wdm}} \emptyset$. Let M be a decidable prefix-free machine and g be a computable order. Let L be a decidable prefix-free machine such that $K_L(n) \leq 2 \log n$ for almost all n where we identify a natural number n with the string 0^n . Then

$$\max\{K_L(i) \mid 1 \leq i \leq n\} \leq 2 \log n$$

for almost all n . Let $f(n) = A \upharpoonright n$. Then $f \leq_{tt} A$. Hence there exists a computable trace $\{T_n\}$ such that $|T_n| \leq g(n)$ and $f(n) \in T_n$ for all n .

Let N be a decidable prefix-free machine such that $N(\sigma\tau)$ is the $L(\tau)$ -th element in $T_{M(\sigma)}$. Then

$$K_N(A \upharpoonright n) = K_N(f(n)) \leq K_M(n) + \max\{K_L(i) \mid 1 \leq i \leq g(n)\} \leq K_M(n) + g(n)$$

for almost all n . Then $A \leq_{\text{wdm}} \emptyset$. \square

Actually, wdm-reducibility is equivalent to Schnorr reducibility.

Theorem 3.5. *For all sets A and B ,*

$$A \leq_{\text{Sch}} B \iff A \leq_{\text{wdm}} B.$$

The theorem follows from the following four lemmas.

Let us recall the KC theorem [9, Theorem 3.6.1]. A KC set is a c.e. sequence of pairs $\{\langle d_i, \tau_i \rangle\}$ with $d_i \in \mathbb{N}$ and $\tau_i \in 2^{<\omega}$ such that $\sum_i 2^{-d_i} \leq 1$. Then there is a prefix-free machine M and strings σ_i of length d_i such that $M(\sigma_i) = \tau_i$ for all i and $\text{dom}(M) = \{\sigma_i : i \in \mathbb{N}\}$. we call a pair $\langle d, \tau \rangle$ a *request*. The *weight* of a request $\langle d, \tau \rangle$ is 2^{-d} . The *weight* of a computable sequence of requests $\{\langle d_i, \tau_i \rangle\}$ is the sum of the weights of the requests. Thus a KC set is a c.e. sequence of requests with the weight less than or equal to 1.

Lemma 3.6. *For every decidable prefix-free machine M and every computable order g , there exists a computable measure machine N such that*

$$K_N(\sigma) \leq K_M(\sigma) + g(|\sigma|) + O(1).$$

Proof. Let M be a decidable prefix-free machine and g be a computable order. We define a KC set S by

$$S = \{\langle |\tau| + g(|\sigma|) + 1, \sigma \rangle : M(\tau) = \sigma \text{ and } |\tau| \leq 2|\sigma| + 2\}.$$

Clearly S is a c.e. set. Furthermore

$$\sum_{\langle \rho, \sigma \rangle \in S} 2^{-|\rho|} \leq \sum_{M(\tau)=\sigma} 2^{-|\tau|-g(|\sigma|)-1} \leq \sum_{M(\tau)=\sigma} 2^{-|\tau|-1} \leq 2^{-1}.$$

We claim that the weight of the KC set is a computable real. For each $n \in \mathbb{N}$, define a uniformly c.e. set S_n by

$$S_n = \{ \langle |\tau| + g(|\sigma|) + 1, \sigma \rangle : M(\tau) = \sigma, |\tau| \leq 2|\sigma| + 2 \text{ and } |\tau| < 2n + 2 \}.$$

Then $S_n \uparrow S$. Since M is decidable and S_n is a finite set, the weight of S_n is computable for each n . If $\langle |\tau| + g(|\sigma|) + 1, \sigma \rangle \in S \setminus S_n$, then $|\sigma| \geq n$. Hence

$$\sum_{\langle \rho, \sigma \rangle \in S \setminus S_n} 2^{-|\rho|} \leq \sum_{M(\tau) = \sigma, |\sigma| \geq n} 2^{-|\tau| - g(|\sigma|) - 1} \leq 2^{-g(n) - 1}$$

where the last inequality holds by prefix-freeness of M . It follows that the weight of S is computable.

We define another KC set T by

$$T = \{ \langle 2|\sigma| + 2, \sigma \rangle : \sigma \in 2^* \}.$$

Then T is a c.e. set and the weight of T is

$$\sum_{\sigma \in 2^*} 2^{-2|\sigma| - 2} = \sum_{n=0}^{\infty} \sum_{\sigma \in 2^n} 2^{-2|\sigma| - 2} = \sum_{n=0}^{\infty} 2^n \cdot 2^{-2n - 2} = \sum_{n=0}^{\infty} 2^{-n - 2} = 2^{-1}.$$

Hence $U = S \cup T$ is a KC set with a computable weight.

Let N be a computable measure machine constructed from U by the KC theorem. For each σ , let τ_σ be a string such that

$$M(\tau_\sigma) = \sigma \text{ and } K_M(\sigma) = |\tau_\sigma|.$$

If $|\tau_\sigma| \leq 2|\sigma| + 2$, then

$$K_N(\sigma) \leq K_M(\sigma) + g(|\sigma|) + 1.$$

If $|\tau_\sigma| > 2|\sigma| + 2$, then

$$K_N(\sigma) \leq 2|\sigma| + 2 < K_M(\sigma) \leq K_M(\sigma) + g(|\sigma|) + 1.$$

Thus the lemma is proved. \square

Proposition 3.7. *Let A, B be sets. If $A \leq_{\text{Sch}} B$, then $A \leq_{\text{wdm}} B$.*

Proof. Suppose that $A \leq_{\text{Sch}} B$. Let M be a decidable prefix-free machine M and g be a computable order.

By Lemma 3.6, there exists a computable measure machine M' such that

$$K_{M'}(B \upharpoonright n) \leq K_M(B \upharpoonright n) + g(n) + O(1). \quad (1)$$

Since $A \leq_{\text{Sch}} B$, there exists a computable measure machine N such that

$$K_N(A \upharpoonright n) \leq K_{M'}(B \upharpoonright n) + O(1). \quad (2)$$

Then N is a decidable prefix-free machine and, by (1) and (2), we have

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + g(n) + O(1).$$

Since M and g are arbitrary, we have $A \leq_{\text{wdm}} B$. \square

Lemma 3.8. *For every computable measure machine M , there exists a decidable prefix-free machine N and a computable order g such that*

$$K_N(\sigma) \leq K_M(\sigma) - g(|\sigma|) + O(1).$$

Proof. We can assume that $\text{dom}(M)$ is not a finite set without loss of generality. We further assume that $|\sigma| \leq 2|\tau| + 2$ for each σ, τ such that $M(\tau) = \sigma$.

For each m , we define a set S_m by

$$S_m = \{\tau : M(\tau) \downarrow \text{ and } |\tau| < m\}.$$

Then $S_m \uparrow \text{dom}(M)$ as $m \rightarrow \infty$. It should be noted that $\mu(\llbracket \text{dom}(M) \setminus S_m \rrbracket) > 0$ for all m by the assumption. Then there exists a strictly increasing function $h : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$m > h(n) \implies \mu(\llbracket \text{dom}(M) \setminus S_m \rrbracket) < 2^{-n}.$$

Since M is a computable measure machine, h can be computable. We can further assume that $h(0) = 0$. Then there exists a computable order g such that

$$2^{g(2h(n+1)+1)} \leq n \text{ for all } n \text{ and } g(2k+2) \leq k/2 \text{ for all } k.$$

Let $C \in \omega$ be sufficiently large. We will be more precise later. We define a KC set W as follows: if $M(\tau) = \sigma$, then we enumerate the pair $\langle |\tau| - g(|\sigma|) + C, \sigma \rangle$ into W . Then the weight of W is

$$\begin{aligned} \sum_{M(\tau)=\sigma} 2^{-|\tau|+g(|\sigma|)-C} &= \sum_{m=0}^{\infty} \sum_{M(\tau)=\sigma, |\tau|=m} 2^{-|\tau|+g(|\sigma|)-C} \\ &= \sum_{n=0}^{\infty} \sum_{m=h(n)}^{h(n+1)-1} \sum_{M(\tau)=\sigma, |\tau|=m} 2^{-|\tau|+g(|\sigma|)-C} \\ &\leq \sum_{n=0}^{\infty} 2^{g(2h(n+1)+1)-C} \mu(\llbracket \text{dom}(M) \setminus S_m \rrbracket) \\ &< \sum_{n=0}^{\infty} n \cdot 2^{-n-C} \leq 1 \end{aligned}$$

for sufficiently large C .

Let N be the prefix-free machine constructed from W by the KC theorem. We claim that N is decidable. Let $m \in \mathbb{N}$. Suppose that $M(\tau) = \sigma$ and $0 \leq |\tau| - g(|\sigma|) + C < m$. It should be noted that

$$|\tau| - g(|\sigma|) \geq |\tau| - g(2|\tau| + 2) \geq |\tau| - |\tau|/2 = |\tau|/2.$$

Then $|\tau| < 2(m - C)$. Hence, by enumerating τ such that $|\tau| < 2(m - C)$, we can enumerate all ρ such that $\rho \in \text{dom}(N)$ and $|\rho| < m$.

For each $\sigma \in \text{rng}(M)$, let τ_σ be such that $M(\tau_\sigma) = \sigma$ and $K_M(\sigma) = |\tau_\sigma|$. Then

$$K_N(\sigma) \leq |\tau_\sigma| - g(|\sigma|) + C = K_M(\sigma) - g(|\sigma|) + C.$$

Thus the lemma is proved. \square

Proposition 3.9. *Let A, B be sets. If $A \leq_{\text{wdm}} B$, then $A \leq_{\text{Sch}} B$.*

Proof. Suppose that $A \leq_{\text{wdm}} B$. Let M be a computable measure machine. By Lemma 3.8, there exists a decidable prefix-free machine M' and a computable order g such that

$$K_{M'}(B \upharpoonright n) \leq K_M(B \upharpoonright n) - g(n) + O(1).$$

Since $A \leq_{\text{wdm}} B$, there exists a decidable prefix-free machine N' such that

$$K_{N'}(A \upharpoonright n) \leq K_{M'}(B \upharpoonright n) + g(n)/2 + O(1).$$

Again, by Lemma 3.6, there exists a computable measure machine N such that

$$K_N(A \upharpoonright n) \leq K_{N'}(A \upharpoonright n) + g(n)/2 + O(1).$$

By combining these, we have

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1).$$

Thus the proposition is proved. \square

4 Tails of random sets

Every tail of a ML-random set is ML-random, and every tail of a non-ML-random set is not ML-random. This property also holds for other appropriate randomness notions such as Schnorr randomness and Kurtz randomness. In this section, we will see some formalizations of this idea to use it in the later section.

4.1 Kučera's characterization and Merkle's criterion

For ML-randomness, the following two facts should be noted. An open set is called *bounded* if its measure is strictly less than 1.

Theorem 4.1 (Kučera [17]; see also Bienvenu and Miller [2, Theorem 8]). *The following are equivalent for a set $X \in 2^\omega$:*

- (i) X is not ML-random.
- (ii) There is a bounded c.e. open set U such that all tails of X belong to U .
- (iii) $X \in U^\omega$ for some bounded c.e. prefix-free subset U of $2^{<\omega}$.

Theorem 4.2 (Merkle's criterion of Martin-Löf randomness; [23, Proposition 3.2.17]). *The following are equivalent for a set $X \in 2^\omega$:*

- (i) X is not Martin-Löf random.
- (ii) $X = x_0x_1x_2\cdots$ for a sequence $\{x_i\}$ of strings such that $K(x_i) \leq |x_i| - 1$ for all i .

- (iii) *There is a prefix-free machine M such that $X = x_0x_1x_2 \cdots$ for a sequence $\{x_i\}$ of strings such that $K_M(x_i) \leq |x_i| - 1$ for all i .*

A Schnorr randomness version of Theorem 4.1 has been given in [2]. We say that an open set is a *Schnorr set* if it is c.e. and its measure is computable.

Theorem 4.3 (Bienvenu and Miller [2, Theorem 9]). *The following are equivalent for a set $X \in 2^\omega$:*

- (i) *X is not Schnorr random.*
- (ii) *There is a bounded Schnorr open set U such that all tails of X belong to U .*
- (iii) *$X \in S^\omega$ for some bounded Schnorr prefix-free subset S of $2^{<\omega}$.*

Here, we give a Schnorr randomness version of Merkle's criterion.

Theorem 4.4. *The following are equivalent for a set X .*

- (i) *X is not Schnorr random.*
- (ii) *There is a computable measure machine M such that $X = x_0x_1x_2 \cdots$, for a sequence of strings $\{x_i\}$ such that $(\forall i)K_M(x_i) \leq |x_i| - 1$.*

We will prove this later. A Kurtz randomness version of Merkle's criterion will be given in Kihara and Miyabe [14].

4.2 Uniform relativization

The (usual) relativization of Theorem 4.3 to A gives the equivalence among the following.

- (i) *X is not Schnorr random relative to A .*
- (ii) *There is a bounded A -c.e. open set U such that $\mu(U)$ is computable from A and all tails of X belong to U .*
- (iii) *$X \in U^\omega$ for some bounded A -c.e. prefix-free subset U of $2^{<\omega}$ such that $\mu(\llbracket U \rrbracket)$ is computable from A .*

In contrast, the uniform relativization of Theorem 4.3 to A gives the following. We say that a computable function $g : 2^\omega \rightarrow \mathcal{O}$ is a *uniformly Schnorr function* if the function $X \mapsto \mu(g(X))$ is computable. A computable function $g : 2^\omega \rightarrow \mathcal{O}$ is *strictly bounded* if $\sup_{X \in 2^\omega} \mu(g(X)) < 1$.

Proposition 4.5. *Let $A \in 2^\omega$. Then the following are equivalent for $X \in 2^\omega$:*

- (i) *X is not Schnorr random uniformly relative to A .*
- (ii) *There is a strictly bounded and uniformly Schnorr function g such that all tails of X belong to $g(A)$.*

- (iii) $X \in (G(A))^\omega$ for some c.e. function G from 2^ω to subsets of $2^{<\omega}$ such that $G(Z)$ is prefix-free for all Z and the function $Z \mapsto \llbracket G(Z) \rrbracket$ is a strictly bounded and uniformly Schnorr function.

Here a function G from 2^ω to some discrete space is c.e. if and only if there is a computable function h from $2^\omega \times \mathbb{N}$ such that $G(X) = \sigma \downarrow \iff h(X, s) = \sigma \downarrow$ for some s .

We introduce a notion that will be useful to understand the proof. A *test* is a non-increasing sequence $\{U_n\}$ of open sets such that $\bigcap_n U_n$ has measure 0.

Definition 4.6. Let S be the shift operator on 2^ω , that is $X = (X \upharpoonright 1)S(X)$ for each $X \in 2^\omega$. A subset $U \subseteq 2^\omega$ is closed under the shift operator if, for each $X \in 2^\omega$,

$$X \in U \Rightarrow S(X) \in U.$$

A test $\{U_n\}$ is closed under the shift operator if the induced null set $\bigcap_n U_n$ is closed under the shift operator.

Clearly, every universal ML-test $\{U_n\}$ is closed under the shift operator. Although there does not exist a universal Schnorr test, there exists a Schnorr test that is closed under the shift operator.

Lemma 4.7. Each Schnorr test is covered by a Schnorr test that is closed under the shift operator. Furthermore, for each uniform Schnorr test f , there is a uniform Schnorr test g such that $\bigcap_n f(Z, n) \subseteq \bigcap_n g(Z, n)$ and $\bigcap_n g(Z, n)$ is closed under the shift operator for each Z .

Proof. For a subset U of $2^{\geq k}$, let

$$S^k(U) = \{S^k(\sigma) : \sigma \in U\}.$$

Let f be a uniform Schnorr test. Let $F(Z, n)$ be a uniformly c.e. prefix-free subset of $2^{<\omega}$ such that $\llbracket F(Z, n) \rrbracket = f(Z, n)$. Since $\mu(f(Z, n)) \leq 2^{-n}$, we have $F(Z, n) \subseteq 2^{\geq n}$. Let

$$G(Z, n) = \bigcup_{k \geq 0} S^k(F(Z, n + 1 + 2k))$$

and $g(Z, n) = \llbracket G(Z, n) \rrbracket$. Then

$$\mu(g(Z, n)) \leq \sum_{k \geq 0} 2^k \cdot 2^{-n-1-2k} \leq 2^{-n}.$$

We show that $\mu(g(Z, n))$ is uniformly computable. Since $\mu(f(Z, n))$ is uniformly computable, the measure of $\llbracket \bigcup_{k=0}^N S^k(F(Z, n + 1 + 2k)) \rrbracket$ is computable uniformly in N . Furthermore,

$$\mu(\llbracket \bigcup_{k > N} S^k(F(Z, n + 1 + 2k)) \rrbracket) \leq \sum_{k > N} 2^{-n-1-k} = 2^{-n-1-N} < 2^{-N}.$$

Then, g is a uniform Schnorr test.

Since $F(Z, n+1) \subseteq G(Z, n)$, we have $f(Z, n+1) = \llbracket F(Z, n+1) \rrbracket \subseteq \llbracket G(Z, n) \rrbracket = g(Z, n)$, whence $\bigcap_n f(Z, n) \subseteq \bigcap_n g(Z, n)$.

Suppose that $X \in \bigcap_n g(Z, n)$. Then, for each $n \in \omega$, there are $k_n, l_n \in \omega$ such that $X \upharpoonright l_n = S^{k_n}(F(Z, (n+3) + 1 + 3k_n))$. Thus, $S(X) \upharpoonright (l_n - 1) = S^{k_n+1}(F(Z, n+1 + 3(k_n + 1))) \subseteq G(Z, n)$. Hence, $S(X) \in \bigcap_n g(Z, n)$. \square

Proof of Theorem 4.5. (i) \Rightarrow (ii). Suppose that $X \in \bigcap_n f(A, n)$ for a uniform Schnorr test f . Then, there is a uniform Schnorr test g such that $\bigcap_n f(A, n) \subseteq \bigcap_n g(A, n)$ and $\bigcap_n g(A, n)$ is closed under the shift operator. Consider the function $h : 2^\omega \rightarrow \mathcal{O}$ defined by $h(X) = g(X, 1)$. Then, h is a strictly bounded and uniformly Schnorr function. Since $\bigcap_n g(A, n)$ is closed under the shift operator and $X \in \bigcap_n f(A, n) \subseteq \bigcap_n g(A, n)$, we have $S^k(X) \in \bigcap_n g(A, n)$ for each $k \in \omega$, whence $S^k(X) \in h(A)$. In other words, all tails of X belong to $h(A)$.

(ii) \Rightarrow (iii). Let g be a strictly bounded and uniformly Schnorr function such that all tails of X belong to $g(A)$. Then, there exists a c.e. function h from 2^ω to subsets of $2^{<\omega}$ such that $h(Z)$ is prefix-free and $\llbracket h(Z) \rrbracket = g(Z)$ for each $Z \in 2^\omega$. We show by induction on n that $X \in \llbracket (h(A))^n \rrbracket$. The case of $n = 1$ holds because $X \in g(A) = \llbracket h(A) \rrbracket$. Assume that $X \in \llbracket (h(A))^n \rrbracket$ for some n . Then there exists $\sigma \in h(A)^n$ such that $X = \sigma X'$. Since X' is a tail of X , we have $X' \in g(A) = \llbracket h(A) \rrbracket$, which implies $X \in \llbracket (h(A))^{n+1} \rrbracket$. Therefore, $X \in (h(A))^\omega$.

(iii) \Rightarrow (i). By replacing $h(Z)$ with $(h(Z))^k$ for some $k \in \omega$, we can assume that $\mu(\llbracket h(Z) \rrbracket) \leq 1/2$ for each $Z \in 2^\omega$. Let $f(Z, n) = \llbracket (h(Z))^n \rrbracket$. Then, f is a uniform Schnorr test and $X \in \bigcap_n f(A, n) = \bigcap_n \llbracket (h(A))^n \rrbracket = (h(A))^\omega$. \square

Proof of Theorem 4.4. (i) \Rightarrow (ii). Suppose that $X \in \bigcap_n V_n$ for a Schnorr test $\{V_n\}$. By the proof of Proposition 4.5, there exists a prefix-free c.e. subset U of $2^{<\omega}$ such that $\mu(\llbracket U \rrbracket)$ is computable and $\leq 1/2$, and all tails of X belong to $\llbracket U \rrbracket$. Consider the KC set $\{|\sigma| - 1, \sigma\}_{\sigma \in U}$. Let M be a machine constructed by this KC set via the KC theorem. Since $\mu(\llbracket U \rrbracket)$ is computable, M is a computable measure machine. Furthermore, $K_M(\sigma) \leq |\sigma| - 1$ for each $\sigma \in U$. Since all tails of X belong to $\llbracket U \rrbracket$, there is a sequence $\{x_n\}$ of strings such that $X = x_0 x_1 x_2 \dots$.

(ii) \Rightarrow (i). We show that (ii) implies (iii) of Theorem 4.3. Suppose that there is a computable measure machine M such that $X = x_0 x_1 \dots$, for a sequence $\{x_n\}$ of strings such that $K_M(x_n) \leq |x_n| - 1$ for all n . Let $S = \{\sigma : K_M(\sigma) \leq |\sigma| - 1\}$. Then, S is a prefix-free c.e. set. Since M is prefix-free, we have $\mu(\llbracket S \rrbracket) < 1$. Since M is a computable measure machine, $\mu(\llbracket S \rrbracket)$ is computable. \square

5 Preorderings related to uniform Schnorr randomness

In this section, we study the analogue of the LR-reducibility for Schnorr randomness and show that this reducibility can similarly be stated either in an LR form or in an LK form.

First recall the following definitions and facts. The reducibility $A \leq_{LK} B$ is defined by

$$K^B(\sigma) \leq K^A(\sigma) + O(1).$$

The reducibility $A \leq_{LR} B$ is defined by that every Martin-Löf random set relative to B is Martin-Löf random relative to A . Recall that a set is called low for K if $A \leq_{LK} \emptyset$, and that a set is called low for Martin-Löf randomness if $A \leq_{LR} \emptyset$, and these notions are equivalent. Kjos-Hanssen et al. [15] strengthened this equivalence to the equivalence between two reducibilities of \leq_{LR} and \leq_{LK} .

By the results in Franklin and Stephan [10] and Miyabe [20], lowness for uniformly computable measure machines and lowness for uniform Schnorr randomness are equivalent. In the following we will strengthen this equivalence to the equivalence between reducibilities.

Theorem 5.1. *The following are equivalent for $A, B \in 2^\omega$.*

- (i) *Every set that is Schnorr random uniformly relative to B is Schnorr random uniformly relative to A .*
- (ii) *Every Schnorr test uniformly relative to A is covered by a Schnorr test uniformly relative to B .*
- (iii) *For each uniformly computable measure machine M , there exists a uniformly computable measure machine N such that*

$$K_{NB}(\sigma) \leq K_{MA}(\sigma) + O(1).$$

- (iv) *Each strictly bounded and uniformly Schnorr function $g : 2^\omega \rightarrow \mathcal{O}$, there is a strictly bounded and uniformly Schnorr function $h : 2^\omega \rightarrow \mathcal{O}$ such that $g(A) \subseteq h(B)$.*
- (v) *For every computable function $f : 2^\omega \times \mathbb{N} \rightarrow \mathbb{R}^+$ such that the function $X \mapsto \sum_{n=0}^{\infty} f(X, n)$ is computable, there exists a computable function $g : 2^\omega \times \mathbb{N} \rightarrow \mathbb{R}^+$ such that the function $X \mapsto \sum_{n=0}^{\infty} g(X, n)$ is computable and $f(A, n) \leq g(B, n)$ for all n .*

We use $A \leq_{LUS} B$ to mean that one of the above statements holds (and thus all statements above hold). Here, LUS means “low for uniform Schnorr randomness”. By (i), the relation \leq_{LUS} is reflexive and transitive.

We show the equivalence among these by showing

$$(iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (iii).$$

Proof of (iii)⇒(ii). Let f be a uniform Schnorr test. Then there exists a uniformly computable measure machine M such that

$$\bigcap_n f(A, n) \subseteq \{X : (\forall d)(\exists n)K_{M^A}(X \upharpoonright n) < n - d\}.$$

By the hypothesis of (iii), there exists a uniformly computable measure machine N such that

$$K_{N^B}(\sigma) \leq K_{M^A}(\sigma) + O(1).$$

Then there exists a uniform Schnorr test f' such that

$$\{X : (\forall d)(\exists n)K_{N^B}(X \upharpoonright n) < n - d\} \subseteq \bigcap_n f'(B, n).$$

Thus the Schnorr null set $\bigcap_n f(A, n)$ uniformly relative to A is covered by the Schnorr null set $\bigcap_n f'(B, n)$ uniformly relative to B . Since f is arbitrary, (iii) implies (ii). \square

Proof of (ii)⇒(i). Suppose that X is not Schnorr random uniformly relative to A . Then X is covered by a Schnorr test uniformly relative to A , which is covered by a Schnorr test uniformly relative to B by the hypothesis of (ii). Then X is not Schnorr random uniformly relative to B . \square

Our remaining proofs mostly follow the argument in [2]

5.1 Proof of (i)⇒(iv)

We use the following lemma to show the implication. Recall that a *test* is a non-increasing sequence $\{U_n\}$ of open sets such that $\bigcap_n U_n$ has measure 0. For $U \subseteq 2^\omega$ and $\sigma \in 2^{<\omega}$, we set

$$(U|\sigma) = \{X \in 2^\omega : \sigma X \in U\}.$$

Lemma 5.2 (Bienvenu and Miller [2]). *Let \mathcal{C} be a class of bounded open subsets of 2^ω . Let $\{T_n^e\}_{e,n \in \mathbb{N}}$ be a countable family of tests (i.e., for all e , $\{T_n^e\}_{n \in \mathbb{N}}$ is a test) such that $\{T_n^e\}$ belongs to \mathcal{C} for all e, n . Suppose we have the following closure properties.*

- (P1) *For all $U \in \mathcal{C}$ and $\sigma \in 2^{<\omega}$, if $\mu(U|\sigma) < 1$, then there exists a $V \in \mathcal{C}$ such that $(U|\sigma) \subseteq V$.*
- (P2) *For all $U \in \mathcal{C}$, there exists a $V \in \mathcal{C}$ such that $U \subseteq V$, and for all $\sigma \in 2^{<\omega}$, if $\mu(U|\sigma) = 1$, then $[\sigma] \subseteq V$.*
- (P3) *For all $U \in \mathcal{C}$, and $\sigma \in 2^{<\omega}$, if $\mu(U|\sigma) < 1$, then for all $e \in \mathbb{N}$, there exists $n_e \in \mathbb{N}$ and $V \in \mathcal{C}$ such that $(U \cup T_{n_e}^e) \subseteq V$ and $\mu(V|\sigma) < 1$.*

Finally, let W be a prefix-free subset of $2^{<\omega}$ such that $\llbracket W \rrbracket$ cannot be covered by any open set $U \in \mathcal{C}$. Then there exists $X \in W^\omega$ that passes all tests T^e .

Bienvendu and Miller [2] uses the following proposition to study lowness for Schnorr randomness.

Proposition 5.3 (Bienvendu and Miller [2]). *Let \mathcal{C} be the class of bounded Schnorr open sets and T^e be the family of Schnorr tests. Then the hypotheses (P1,P2,P3) of Lemma 5.2 are satisfied for these classes.*

The uniform relativization gives the following.

Proposition 5.4. *Let A be a set. Let \mathcal{C} be the class of open sets $g(A)$ such that $g : 2^\omega \rightarrow \mathcal{O}$ is a strictly bounded and uniformly Schnorr function. Let T^e be the family of Schnorr tests uniformly relative to A . Then the hypotheses (P1,P2,P3) of Lemma 5.2 are satisfied for these classes.*

The following proof is almost identical to the unrelativized one.

Proof. (P1) Let $U = g(A)$ for a strictly bounded and uniformly Schnorr function g and $\sigma \in 2^{<\omega}$ be a string such that $\mu(U|\sigma) < 1$. Pick up $q \in \mathbb{Q}$ such that $\mu(U|\sigma) < q < 1$. Let $h(X)$ be $(g(X)|\sigma)$ enumerated as long as its measure is less than or equal to q . Then h is a strictly bounded function. Since $X \mapsto \mu(g(X))$ is computable, so is $X \mapsto \mu(g(X)|\sigma)$, whence so is $X \mapsto \mu(h(X))$. Finally, since $\mu(U|\sigma) < q$, $h(A) = (g(A)|\sigma) = (U|\sigma)$.

(P3) Let $U = g(A)$ for a strictly bounded and uniformly Schnorr function g . For each e and n , there exists a strictly bounded and uniformly Schnorr function h_n^e such that $U \cup T_n^e = h_n^e(A)$. If $\mu(U|\sigma) < 1$, then $\mu(h_n^e(A)|\sigma) < 1$ for a sufficiently large n .

(P2) Let $U = g(A)$ for a strictly bounded and uniformly Schnorr function g and let k be large enough that $\mu(g(X)) < 1 - 2^{-k}$ for each $X \in 2^\omega$. We define a computable function h from 2^ω to a computable set of strings such that $\llbracket h(A) \rrbracket \supseteq g(A)$ and $X \mapsto \llbracket h(X) \rrbracket$ is a strictly bounded and uniformly Schnorr function.

The algorithm of h is as follows. For every $\sigma \in 2^{<\omega}$, look for a stage s such that $\mu(g(X) \setminus g_s(X)) < 2^{-2^{|\sigma|-k-1}}$. If $\mu(g_s(X)|\sigma) > 1 - 2^{-|\sigma|-k-1}$, then put σ into $h(X)$.

It is clear that $h(X)$ is computable. If $\mu(g(X)|\sigma) = 1$, then it must be the case that $\mu(g_s(X)|\sigma) > 1 - 2^{-|\sigma|-k-1}$; otherwise

$$\mu(g(X) \setminus g_s(X)) \geq \mu([\sigma] \setminus g_s(X)) \geq 2^{-|\sigma|} - 2^{-|\sigma|}(1 - 2^{-|\sigma|-k-1}) = 2^{-2^{|\sigma|-k-1}},$$

which contradicts the choice of s . Therefore, $\mu(U|\sigma) = 1$ implies that $\sigma \in h(A)$, so $[\sigma] \subseteq \llbracket h(A) \rrbracket$ as required. This also implies that $U \subseteq \llbracket h(A) \rrbracket$.

It remains to show that h is a strictly bounded and uniformly Schnorr function. By adding σ to $h(X)$, we are increasing the measure of $\llbracket h(X) \rrbracket \setminus g(X)$ by less than $2^{-|\sigma|}2^{-|\sigma|-k-1} = 2^{-2^{|\sigma|-k-1}}$. Therefore,

$$\mu(\llbracket h(X) \rrbracket \setminus g(X)) < \sum_{\sigma \in 2^*} 2^{-2^{|\sigma|-k-1}} = \sum_{n \in \mathbb{N}} 2^n 2^{-2n-k-1} = 2^{-k}.$$

This implies that $\mu(\llbracket h(X) \rrbracket) < \mu(g(X)) + 2^{-k} < 1$ for each X , so h is a strictly bounded function. Similarly, for every m ,

$$\begin{aligned} \mu(\llbracket h(X) \rrbracket \setminus (\llbracket h(X) \cap 2^{<m} \rrbracket \cup g(X))) &< \sum_{\sigma \in 2^{\geq m}} 2^{-2|\sigma|-k-1} \\ &= \sum_{n \geq m} 2^n 2^{-2n-k-1} = 2^{-m-k}. \end{aligned}$$

But $\mu(\llbracket h(X) \cap 2^{<m} \rrbracket \cup g(X))$ is computable uniformly in m , so $\mu(\llbracket h(X) \rrbracket)$ is also computable. \square

Proof of (i) \Rightarrow (iv) of Theorem 5.1. Suppose that there exists a strictly bounded and uniformly Schnorr function $g : 2^\omega \rightarrow \mathcal{O}$ such that $g(A) \not\subseteq h(B)$ for all strictly bounded and uniformly Schnorr function h . Let W be a prefix-free subset of $2^{<\omega}$ such that $[W] = g(A)$. Apply Lemma 5.2 for this W . Then there exists $X \in W^\omega$ that passes all Schnorr tests uniformly relative to B . Then X is Schnorr random uniformly relative to B and is not Schnorr random uniformly relative to A by the uniform relativization of Theorem 4.3. \square

5.2 Proof of (iv) \Rightarrow (v)

We use the same techniques as the proof of Proposition 5.1 in the revised version of [2] in arXiv.

Proof of (iv) \Rightarrow (v) of Theorem 5.1. By the usual identification of ω to $\omega \times \omega$ and of 2^ω to $[0, 1]^\omega$, we identify 2^ω with $[0, 1]^\omega$. For all $n \in \omega$ and $\alpha \in [0, 1]$, set

$$\mathcal{B}_{n,\alpha} = \{X \in [0, 1]^\omega : X_n \in [0, \alpha]\}.$$

Let $f : 2^\omega \times \omega \rightarrow \mathbb{R}^+$ be a computable function such that $X \mapsto \sum_{n=0}^\infty f(X, n)$ is computable. Since 2^ω is compact, $\sup_{X \in 2^\omega} \sum_{n=0}^\infty f(X, n)$ is finite. Without loss of generality, we can assume that $\sum_{n=0}^\infty f(X, n) \leq 1$. Consider the computable function $h : 2^\omega \rightarrow \mathcal{O}$ defined by

$$h(X) = \bigcup_n \mathcal{B}_{n, f(X, n)}.$$

Then

$$\mu(h(X)) = 1 - \prod_n (1 - \mu(\mathcal{B}_{n, f(X, n)})) = 1 - \prod_n (1 - f(X, n)).$$

This can be reformulated as

$$\log(1 - \mu(h(X))) = \sum_n \log(1 - f(X, n)).$$

Since $f(X, n)$ tends to 0, $X \mapsto \sum_n \log(1 - f(X, n))$ is computable. Thus, h is a strictly bounded and uniformly Schnorr function. By hypothesis (iv), there

is a strictly bounded and uniformly Schnorr function $k : 2^\omega \rightarrow \mathcal{O}$ such that $h(A) \subseteq k(B)$.

Let $\delta > 0$ be such that $\sup_{X \in 2^\omega} \mu(k(X)) < 1 - \delta$. For all $n \in \omega$, let $k(X, n)$ be the approximation of $k(X)$ with precision 2^{-n-c} where $c \in \omega$ to be specified shortly. That is, $k(X, n)$ is a clopen set for which an exact index can be uniformly computed from X and n , and $\mu(k(X) \setminus k(X, n)) < 2^{-n-c}$. Now define the function $g : 2^\omega \times \omega \rightarrow \mathbb{R}$ by

$$g(X, n) = \max\{\alpha \in [0, 1] : \mu(\mathcal{B}_{n,\alpha} \setminus k(X, n)) \leq 2^{-n-c}\}$$

Note that g is computable. Also note that $g(B, n) \geq f(A, n)$ because $\mathcal{B}_{n,f(A,n)} \subseteq h(A) \subseteq k(B)$ and

$$\mu(\mathcal{B}_{n,f(A,n)} \setminus k(B, n)) \leq \mu(k(B) \setminus k(B, n)) \leq 2^{-n-c}.$$

Now it suffices to show that the function $X \mapsto \sum_n g(X, n)$ is computable. Let m be a fixed integer. Since $k(X, m)$ is a clopen set, one can effectively find an integer $N = N(m)$ such that $k(X, m)$ is independent from $\{\mathcal{B}_{n,g(n)} : n \geq N\}$. By this independence, we have

$$\begin{aligned} \mu\left(\bigcup_{n>N} \mathcal{B}_{n,g(X,n)} \setminus k(X, m)\right) &= (1 - \mu(k(X, m)))\mu\left(\bigcup_{n>N} \mathcal{B}_{n,g(X,n)}\right) \\ &> \delta \cdot \mu\left(\bigcup_{n>N} \mathcal{B}_{n,g(X,n)}\right). \end{aligned} \quad (3)$$

On the other hand, we have

$$\begin{aligned} \mu\left(\bigcup_{n>N} \mathcal{B}_{n,g(X,n)} \setminus k(X)\right) &\leq \sum_{n>N} \mu(\mathcal{B}_{n,g(X,n)} \setminus k(X)) \\ &\leq \sum_{n>N} \mu(\mathcal{B}_{n,g(X,n)} \setminus k(X, n)) \\ &\leq \sum_{n>N} 2^{-n-c} = 2^{-N-c}. \end{aligned} \quad (4)$$

By combining (3) and (4) and the fact that $\mu(k(X) \setminus k(X, m)) < 2^{-m-c}$, we obtain

$$1 - \prod_{n>N} (1 - g(X, n)) = \mu\left(\bigcup_{n>N} \mathcal{B}_{n,g(X,n)}\right) \leq \frac{2^{-N-c} + 2^{-m-c}}{\delta}.$$

We assume that $2^{-c+1} < \delta$ and $N \geq m$. Then

$$1 - \prod_{n>N} (1 - g(X, n)) < 2^{-m}.$$

(By considering the case that $m = 0$ and $N = 0$, we have $1 - \prod_n (1 - g(X, n)) < 1$, which implies $\sum_n g(X, n)$ is finite.) Composing with $-\log$ on both sides, we obtain

$$\sum_{n>N} g(X, n) < -\log(1 - 2^{-m}) = 2^{-m+o(m)}.$$

Hence $\sum_n g(X, n)$ is computable from X . \square

5.3 Proof of (v) \Rightarrow (iii)

We follow the proof of Proposition 27 in [2].

Proof of (iv) \Rightarrow (iii) of Theorem 5.1. Let M be a uniformly computable measure machine. The goal is to show the existence of a uniformly computable measure machine N such that $K_{NB}(\sigma) \leq K_{MA}(\sigma) + O(1)$. Let $f : 2^\omega \times 2^{<\omega} \rightarrow \mathbb{R}^+$ be the function defined by

$$f(X, \sigma) = 2^{-K_{M^X}(\sigma)}.$$

Then f is lower semicomputable. Since the function $X \mapsto \sum_{\sigma \in \text{dom}(M^X)} 2^{-|\sigma|}$ is computable, the function $X \mapsto \sum_{\sigma \in 2^{<\omega}} f(X, \sigma)$ is computable. By the assumption of (iv), there is a computable function $g : 2^\omega \times 2^{<\omega} \rightarrow \mathbb{R}^+$ such that $X \mapsto \sum_{\sigma \in 2^{<\omega}} g(X, \sigma)$ is computable and $f(A, \sigma) \leq g(B, \sigma)$ for all n .

Since 2^ω is compact, $\sup_{X \in 2^\omega} \sum_{\sigma} g(X, \sigma)$ is finite. Let $c \in \omega$ be a constant such that $\sum_{\sigma} g(X, \sigma) \leq 2^c$ for all $X \in 2^\omega$. Let

$$L^X = \{\langle k, \sigma \rangle : g(X, \sigma) \geq 2^{-k+c+1}\}.$$

Then

$$\sum_{\langle k, \sigma \rangle \in L^X} 2^{-k} = \sum_{\sigma} 2^{-\lceil \log g(X, \sigma) - c - 1 \rceil + 1} \leq 2^{-c} \sum_{\sigma} g(X, \sigma) \leq 1.$$

Thus, L is a KC-set. Apply the KC theorem to construct a oracle prefix-free machine N whose domain is a prefix-free set $\{p_{k, \sigma}^X : \langle k, \sigma \rangle \in L^X\}$ with $|p_{k, \sigma}^X| = k$ and $N^X(p_{k, \sigma}^X) = \sigma$. Since the function $X \mapsto \sum_{\sigma} g(X, \sigma)$ is computable, the function $X \mapsto \sum_{\sigma} 2^{-\lceil \log g(X, \sigma) - c - 1 \rceil + 1}$ is computable. Finally notice that

$$K_{NB}(\sigma) \leq -\log g(B, \sigma) + c + 1 \leq -\log f(A, \sigma) + c + 1 \leq K_{MA}(\sigma) + c + 1.$$

□

6 Base for uniform Schnorr tests

We say that a set A is a base for ML-randomness if $A \leq_T X$ for some set X that is ML-random relative to A . It is known that being a base for ML-randomness is equivalent to K -triviality. Then we would like to have a base-type characterization of Schnorr triviality. One candidate is like this: a set A such that $A \leq_{\text{tt}} X$ for some set X that is uniform Schnorr random relative to A . However this notion is not equivalent to Schnorr triviality. Actually Franklin and Stephan [10] showed that there exists a Schnorr trivial set that is not truth-table reducible to any Schnorr random set.

It should be noted that the following notions are equivalent:

- (i) Schnorr triviality (Definition 2.7),
- (ii) computable tt-traceability (Definition 2.8),

(iii) non-totally i.o. complexity (Definition 2.11).

Notice that all of them have the following form: a set A is Schnorr trivial iff for any computable object, there exists another computable object such that A is in some object. Then we define a base for uniform Schnorr tests in a similar manner and show the equivalence to Schnorr triviality.

Notice that a set A is a *base for ML-randomness* if and only if $A \leq_T X$ for some set X such that $\sup_n d(X \upharpoonright n) < \infty$ for each A -c.e. martingale d if and only if, for each A -c.e. martingale d , there is a set X such that $A \leq_T X$ and $\sup_n d(X \upharpoonright n) < \infty$. Here, this equivalence follows from the universality. In the following we give an analogue of the latter statement.

Definition 6.1. *Let d be a computable martingale uniformly relative to A . A set X is computably random uniformly relative to A for d if $d(X \upharpoonright n) \leq O(1)$. A set X is Schnorr random uniformly relative to A for d if $d(X \upharpoonright n) \leq h(n) + O(1)$ for each computable order h .*

Note that a computable set can be Schnorr random for a computable martingale d . Schnorr randomness for d means that it looks random with respect to d .

Definition 6.2. *A set A is a base for uniformly computable martingales if, for each computable martingale d uniformly relative to A , there exists a set B such that $A \leq_{tt} B$ and B is computably random uniformly relative to A for d .*

A set A is a base for uniform Schnorr tests if, for each computable martingale d uniformly relative to A , there exists a set B such that $A \leq_{tt} B$ and B is Schnorr random uniformly relative to A for d .

Remark 6.3. *The definition of a base for uniform Schnorr tests uses a uniformly computable martingale rather than a uniform Schnorr test. The terminology makes sense because of the correspondence of tests and martingales.*

Then we show that being a base for uniform Schnorr tests is equivalent to Schnorr triviality. Further we show that being a base for uniformly computable martingales is also equivalent.

Theorem 6.4. *The following are equivalent for a set A :*

- (i) A is Schnorr trivial,
- (ii) A is a base for uniformly computable martingales,
- (iii) A is a base for uniform Schnorr tests.

First note that (ii) \Rightarrow (iii) is immediate. We prove the remaining implications by showing Lemma 6.5 and Lemma 6.6.

Lemma 6.5. *If a set is a base for uniform Schnorr tests, then it is Schnorr trivial.*

Franklin and Stephan [10] stated as a proposition that if $A \leq_{\text{tt}} B$ and B is Schnorr random uniformly relative to A , then A is Schnorr trivial. The proof actually showed something stronger: for any $A, B \in 2^\omega$ such that $A \leq_{\text{tt}} B$, at least one of the following holds:

- (i) There exists a computable martingale d uniformly relative to A such that B is not Schnorr random uniformly relative to A for d .
- (ii) A is Schnorr trivial.

Hence, if A is not Schnorr trivial, then A is not a base for uniform Schnorr tests.

Lemma 6.6. *If a set is Schnorr trivial, then it is a base for uniformly computable martingales.*

For the proof, we use the Space Lemma.

Lemma 6.7 (Space Lemma; see Merkle and Mihailović [19]). *Given a rational $\delta > 1$ and integer $k > 0$, we can compute a length $l(\delta, k)$ such that, for any martingale d and any σ ,*

$$|\{\tau \in 2^{l(\delta, k)} \mid d(\sigma\tau) \leq \delta d(\sigma)\}| \geq k.$$

Proof of Lemma 6.6. Let A be a Schnorr trivial set and d be a uniform martingale test. We assume that d is \mathbb{Q}_2 -valued without loss of generality. We will construct a set B such that $A \leq_{\text{tt}} B$ and B is computably random uniformly relative to A for d^A .

Let $r_1 > r_2 > \dots > 1$ be a computable sequence of rationals such that, letting $\beta_i = \prod_{j \leq i} r_j$, the sequence $\{\beta_i\}$ converges to some value β . Let $l_s = l(r_s, s)$ be as in the Space Lemma and $\hat{l}_s = \sum_{i=1}^s l_i$.

Let u be the use function of d^A . Then $n \mapsto \max\{u(\sigma) \mid \sigma \in 2^n\}$ is dominated by a computable function $r(n)$. We assume $u(n) \leq r(n)$ for all n . Since A is computably tt-traceable, there exists a trace $\{T_n\}$ such that $|T_n| \leq n$ and $A \upharpoonright r(\hat{l}_n) \in T_n$ for all n . We further assume that any two elements in T_n are distinct.

We construct $B = \lim_s \sigma_s$ as follows. Let σ_0 be the empty string. Then $d^A(\sigma_0) \leq 1$. Note that $\sigma_0 = B \upharpoonright 0$. For $s \geq 1$, we assume by induction that $d^A(\sigma_{s-1}) \leq \beta_{s-1}$ where $\sigma_{s-1} = B \upharpoonright \hat{l}_{s-1}$. Then there are at least s strings τ such that $d^A(\sigma_{s-1}\tau) \leq \beta_s$. Let τ_m be the lexicographically m -th least string among such strings. Let $\sigma_{s+1} = \sigma_{s-1}\tau_m$ when $A \upharpoonright r(\hat{l}_s)$ is the m -th element in T_s . Then $d^A(B \upharpoonright \hat{l}_s) < \beta$ for all s . It follows that B is computably random uniformly relative to A for d^A .

Next we construct a reduction Φ to X such that $A = \Phi^B$. We assume by induction that we have already known $\Phi^X \upharpoonright r(\hat{l}_s)$ for $s \geq 1$ where $\Phi^X \upharpoonright r(\hat{l}_1) = A \upharpoonright r(\hat{l}_1)$. Since r dominates the use function u of d , we have $u(\hat{l}_s) \leq r(\hat{l}_s)$. Hence we can compute $d^{\Phi^X}((X \upharpoonright \hat{l}_{s-1})\tau)$ for all τ such that $|\tau| = l_s$. Decide whether a string τ satisfy $|\tau| = l_s$ and $d^{\Phi^X}((X \upharpoonright \hat{l}_{s-1})\tau) \leq \beta_s$. Let τ_m be

the lexicographically m -th least string among such strings. Search m satisfying $X \upharpoonright \hat{l}_s = (X \upharpoonright \hat{l}_{s-1})\tau_m$. If such an m is found, then let $\Phi^X \upharpoonright r(\hat{l}_{s+1})$ be the m -th element in T_{s+1} . Otherwise let $\Phi^X = (\Phi^X \upharpoonright r(\hat{l}_s))0^\omega$. Then Φ is a tt-reduction.

Finally we show $A = \Phi^B$. We prove $\Phi^B \upharpoonright r(\hat{l}_{s+1}) = A \upharpoonright r(\hat{l}_{s+1})$ by induction on s . For $s = 0$, this is true. Suppose this is true for $s - 1$. Let τ_m be the lexicographically m -th least string τ such that $d^A(\sigma_{s-1}\tau) \leq \beta_s$. By the inductive hypothesis, $\Phi^B \upharpoonright r(\hat{l}_s) = A \upharpoonright r(\hat{l}_s)$. Then

$$d^{\Phi^B}((B \upharpoonright \hat{l}_{s-1})\tau) \leq \beta_s \iff d^A(\sigma_{s-1}\tau) \leq \beta_s.$$

Hence there exists only one m_0 such that $B \upharpoonright \hat{l}_s = (B \upharpoonright \hat{l}_{s-1})\tau_{m_0}$ and $A \upharpoonright r(\hat{l}_{s+1}) = \Phi^B \upharpoonright r(\hat{l}_{s+1})$ is the m_0 -th element in T_{s+1} . Then the equation is true for s . \square

Acknowledgement

The author thanks Takayuki Kihara triggering this study and Laurent Bienvenu for crucial discussion on the proof of (iv) \Rightarrow (v) of Theorem 5.1. This work was partly supported by GCOE, Kyoto University and JSPS KAKENHI 23740072.

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