The T-equivariant Integral Cohomology Ring of F_4/T

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Abstract

We determine the T-equivariant integral cohomology of F_4/T combinatorially by the GKM theory, where T is a maximal torus of the exceptional Lie group F_4 and acts on F_4/T by the left multiplication.

1 Introduction and statement of the result

Let G be a compact, connected Lie group and T its maximal torus. The homogeneous space G/T is a flag variety and it plays an important role in topology, algebraic geometry, representation theory, and combinatorics. In particular, the T-equivariant integral cohomology ring $H_T^*(G/T) = H^*(ET \times_T G/T)$ is especially important, where T acts on G/T by the left multiplication.

Goresky, Kottwitz, and MacPherson [GKM] gave a powerful method to determine the equivariant cohomology with $\mathbb{Q}$ coefficients of some good spaces. It is called the GKM theory. Let us explain how the GKM theory works in our situation. Since the fixed points set $(G/T)^T$ is identified with the Weyl group $W(G)$, the inclusion $i\colon (G/T)^T \to G/T$ induces the map

$$i^*\colon H_T^*(G/T) \to H_T^*((G/T)^T) = \prod_{W(G)} H^*(BT) = \mathrm{Map}(W(G), H^*(BT)).$$

Tensoring with $\mathbb{Q}$, i^* is injective by the localization theorem (cf. [H, Theorem (III.1)]). The GKM theory gives a way to describe the image of this map i^*, which is restated by Guillemin and Zara [GZ] as follows. The image of i^* is completely determined by a graph with additional data obtained from G. Precisely they defined the "cohomology" ring of the graph as a subring of $\mathrm{Map}(W(G), H^*(BT))$ and showed that it coincides with the image of i^*. This graph is called a GKM graph. Harada, Henriques, and Holm [HHH] showed that i^* is injective with integer coefficient when G is simple and is not of type C.

By concrete computations by the GKM theory, for a simple Lie group G of classical types and of type G_2, Fukukawa, Ishida, and Masuda [FIM], [F] determined the cohomology ring of the GKM graph of G/T. Hence they determined the equivariant integral cohomology ring $H_T^*(G/T)$

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for a Lie group G of type A, B, D, and G_2. In this paper we determine the T-equivariant integral cohomology ring of F_4/T by the GKM theory.

For $x = (x_1, \dots, x_n)$, let $e_i(x)$ denote the i^{th} elementary symmetric polynomial in $x_1, \dots, x_n$. Put $x^k = (x_1^k, \dots, x_n^k)$. For a linear transformation α of $\mathbb{R}x_1 \oplus \cdots \oplus \mathbb{R}x_n$, let $\alpha x = (\alpha x_1, \dots, \alpha x_n)$. Then $e_i(x^k)$ and $e_i(\alpha x)$ denote the i^{th} elementary symmetric polynomial in $x_1^k, \dots, x_n^k$ and $\alpha x_1, \dots, \alpha x_n$, respectively. The following theorem is the main result of this paper. In this theorem $t = (t_1, t_2, t_3, t_4)$, $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$, and ρ is the linear transformation of $\mathbb{R}t_1 \oplus \cdots \oplus \mathbb{R}t_4$ defined as (3.2).

Theorem 1.1. *Let T be a maximal torus of F_4 which acts on F_4/T by the left multiplication, then the T-equivariant integral cohomology ring of F_4/T is given as:*

$$H_T^*(F_4/T) \cong \mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]/(r_1', R_i, r_{2i}, r_{12} \mid 1 \le i \le 4),$$

where $|t_i| = |\gamma| = |\tau_i| = 2$, $|\gamma_i| = 2i$, $|\omega| = 8$,

$$r_1' = e_1(t) - 2\gamma, \qquad R_i = e_i(\tau) - e_i(t) - 2\gamma_i \ \ (i = 1, 2, 3),$$

$$r_{12} = \omega(\omega - e_4(\rho t))(\omega + e_4(\rho^2 t)), \qquad R_4 = e_4(\tau) - e_4(t) - 2\gamma_4 - \omega,$$

$$r_2 = \sum_{j=1}^{2} (-1)^j \gamma_j(\gamma_{2-j} + e_{2-j}(t)), \qquad r_4 = \sum_{j=1}^{4} (-1)^j \gamma_j(\gamma_{4-j} + e_{4-j}(t)) - \omega,$$

$$r_6 = \sum_{j=2}^{4} (-1)^j \gamma_j(\gamma_{6-j} + e_{6-j}(t)) + (\gamma_2 + \gamma^2)\omega, \quad r_8 = \gamma_4(\gamma_4 + e_4(t)) + \omega^2 + (\gamma_4 - e_4(\rho t))\omega.$$

The ordinary integral cohomology ring $H^*(F_4/T)$ was determined by Toda and Watanabe [TW]. We can obtain the integral cohomology ring of F_4/T as a corollary of Theorem 1.1 as follows. There is a fibration sequence

$$F_4/T \longrightarrow ET \times_T F_4/T \xrightarrow{\ p\ } BT.$$

Since the projection $p\colon ET \times_T F_4/T \to BT$ restricts to $p \circ i\colon ET \times_T (F_4/T)^T \to BT$, where i is the inclusion $ET \times_T (F_4/T)^T \to ET \times_T F_4/T$, the induced map $(p \circ i)^*\colon H^*(BT) \to H^*(ET \times_T (F_4/T)^T) = \mathrm{Map}(W(F_4), H^*(BT))$ sends elements of $H^*(BT)$ to constant functions. In Theorem 1.1, t_1, t_2, t_3, t_4, and γ correspond to constant functions (see Section 4). Since the cohomology of F_4/T and BT have vanishing odd parts, the Serre spectral sequence of the fibration p collapses at the E_2-term. Hence $H^*(F_4/T) \cong H_T^*(F_4/T)/(t_1, t_2, t_3, t_4, \gamma)$.

Corollary 1.1 ([TW, Theorem A])**.** *The integral cohomology ring of F_4/T is given as:*

$$H^*(F_4/T) \cong \mathbb{Z}[\tau_i, \gamma_1, \gamma_3, \omega \mid 1 \le i \le 4]/(\overline{r}_1, \overline{r}_2, \overline{r}_3, \overline{r}_4, \overline{r}_6, \overline{r}_8, \overline{r}_{12}),$$

where

$$\overline{r}_1 = 2\gamma_1 - e_1(\tau), \qquad \overline{r}_2 = 2\gamma_1^2 - e_2(\tau),$$

$$\overline{r}_3 = 2\gamma_3 - e_3(\tau), \qquad \overline{r}_4 = e_4(\tau) - 2\gamma_1 e_3(\tau) + 2\gamma_1^4 - 3\omega,$$

$$\overline{r}_6 = -\gamma_1^2 e_4(\tau) + \gamma_3^2, \qquad \overline{r}_8 = 3e_4(\tau)\gamma_1^4 - \gamma_1^8 + 3\omega(\omega + e_3(\tau)\gamma_1),$$

$$\overline{r}_{12} = \omega^3.$$

Corollary 1.1 will be proved in Section 8. Throughout this paper, all cohomology groups and rings will be taken with integer coefficient.

2 GKM graph and its cohomology

Let G be a compact connected Lie group and let T be its maximal torus. Specializing and abstracting the work of Goresky, Kottwitz, and MacPherson [GKM], Guillemin and Zara [GZ] introduced a certain graph to each of whose edge an element of $H^2(BT)$ is given and showed the T-equivariant cohomology of G/T with complex coefficient is recovered from this graph. Let us introduce this special graph. Recall that there is a natural identification

$$\mathrm{Hom}(T, S^1) \cong H^2(BT),$$

where the left hand side is the set of weights of G. Let $W(G)$ and $\Phi(G)$ denote the Weyl group and the root system of G, respectively. Since every root is a weight, we regard $\Phi(G) \subset H^2(BT)$. There is a canonical action of the Weyl group $W(G)$ on $\mathrm{Hom}(T, S^1)$ and it restricts to $\Phi(G)$. We denote this action as $w\alpha$ for $w \in W(G)$ and $\alpha \in H^2(BT)$. Recall that to each $\alpha \in \Phi(G)$, one can assign a reflection σ_α which is an element of the Weyl group $W(G)$.

Definition 2.1. The GKM graph of G/T is the Cayley graph of $W(G)$ with respect to a generating set $\{\sigma_\alpha \in W(G) \mid \alpha \in \Phi(G)\}$ which is equipped with the cohomology classes $\pm w\alpha \in H^2(BT)$ to the edge ww' satisfying $w' = w\sigma_\alpha$. We call $\pm w\alpha$ the label of the edge ww'.

The ambiguity of the sign of the label $\pm w\alpha$ occurs from the equation $w'\alpha = w\sigma_\alpha\alpha = -w\alpha$. Let us introduce the cohomology of the GKM graph. Consider a function $f : W(G) \to H^*(BT)$ between sets. We say that f satisfies the GKM condition or f is a GKM function if for any $w \in W(G)$ and $\alpha \in \Phi(G)$,

$$f(w) - f(w\sigma_\alpha) \in (w\alpha) \subset H^*(BT),$$

where $(x_1, \dots, x_n)$ means the ideal generated by $x_1, \dots, x_n$. It is easy to see that all GKM functions form a subring of $\prod_{W(G)} H^*(BT)$, where we identify the set of all functions $W(G) \to H^*(BT)$ with $\prod_{W(G)} H^*(BT)$. Since the GKM graph of G/T has $W(G)$ as its vertex set, a GKM function assigns an element of $H^*(BT)$ to each vertex of the GKM graph.

Definition 2.2. Let $\mathcal{G}$ be the GKM graph of G/T. The cohomology ring $H^*(\mathcal{G})$ is defined as the subring of $\prod_{W(G)} H^*(BT)$ consisting of all GKM functions.

Guillemin and Zara [GZ, Theorem 1.7.3] restated an important theorem of the GKM theory as

$$H^*_T(G/T; \mathbb{C}) \cong H^*(\mathcal{G}) \otimes \mathbb{C}.$$

Harada, Henriques, and Holm refined this result to the integral cohomology. More precisely, we have:

Theorem 2.1 ([HHH, Theorem 3.1 and Lemma 5.2])**.** *Suppose the Lie group G is simple and let $\mathcal{G}$ be the GKM graph of G/T. If G is not of type C, then there is an isomorphism*

$$H^*_T(G/T) \cong H^*(\mathcal{G}).$$

3 The GKM graph of F_4/T

In this section we describe and analyze the GKM graph of F_4/T. First of all let us choose a maximal torus of F_4. Let T^4 be the standard maximal torus of SO(9) and let $\bar{t}_1$, $\bar{t}_2$, $\bar{t}_3$, $\bar{t}_4 \in H^2(BT^4)$ be the canonical basis. For the universal covering $\mu : \mathrm{Spin}(9) \to \mathrm{SO}(9)$ let $T = \mu^{-1}(T^4)$. Then T is a maximal torus of Spin(9). Since Spin(9) is a Lie subgroup of F_4 (cf. [A, Chapter 8,9,14]), T is also a maximal torus of F_4. We fix a maximal torus of F_4 to T. Let t_i denote $\mu^*(\bar{t}_i) \in H^2(BT)$. By definition we have

$$H^*(BT) = \mathbb{Z}[t_1, t_2, t_3, t_4, \gamma]/(2\gamma - e_1(t)).$$

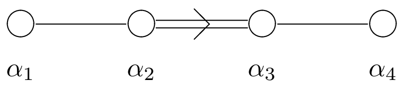
To describe the Weyl group $W(F_4)$ we start with the root system of F_4. The root system $\Phi(F_4)$ is given as:

$$\Phi(F_4) = \{\pm(t_i + t_j), \pm(t_i - t_j), \pm t_k, \frac{1}{2}(\pm t_1 \pm t_2 \pm t_3 \pm t_4) \mid 1 \leq i < j \leq 4,\ 1 \leq k \leq 4\}$$

The roots $\pm(t_i + t_j)$ and $\pm(t_i - t_j)$ are called long roots, and $\pm t_k$ and $\frac{1}{2}(\pm t_1 \pm t_2 \pm t_3 \pm t_4)$ are called short roots. Put

$$\alpha_1 = t_2 - t_3, \qquad \alpha_2 = t_3 - t_4,$$
$$\alpha_3 = t_4, \qquad \alpha_4 = \frac{1}{2}(t_1 - t_2 - t_3 - t_4).$$

Then the Dynkin diagram of F_4 is as:



Then $W(F_4)$ is generated by the reflections σ_{α_i} for $i = 1, 2, 3, 4$. Since Spin(8) is a Lie subgroup of F_4, the root system of Spin(8) is contained in $\Phi(F_4)$, which is given as:

$$\Phi(\mathrm{Spin}(8)) = \{\pm(t_i + t_j), \pm(t_i - t_j) \mid 1 \leq i < j \leq 4\}$$

It consists of all the long roots of the root system $\Phi(F_4)$. Then the Weyl group $W(\mathrm{Spin}(8))$ is generated by the reflections associated with the long roots, and $W(\mathrm{Spin}(8))$ is a subgroup of $W(F_4)$.

Put $W = W(\mathrm{Spin}(8))$. The vertex set $W(F_4)$ of the GKM graph of F_4/T is decomposed into 6 cosets by the next theorem.

Theorem 3.1 ([A, Theorem 14.2]). *The Weyl group W of* Spin(8) *is a normal subgroup of $W(F_4)$ and there is an isomorphism $W(F_4)/W \cong \mathfrak{S}_3$, where $\mathfrak{S}_n$ is the symmetric group on n-letters. Moreover $W(F_4)/W$ permutes the three root pairs*

$$\pm\frac{1}{2}(t_1 + t_2 + t_3 - t_4), \quad \pm\frac{1}{2}(t_1 + t_2 + t_3 + t_4), \quad \pm t_4. \tag{3.1}$$

Let us describe the representatives of $W(F_4)/W$. First we define an element ρ of $W(F_4)$ as

$$\rho = \sigma_{\alpha_3}\sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_0}\sigma_{\alpha_3}\sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_3}\sigma_{\alpha_2}\sigma_{\alpha_4}, \tag{3.2}$$

where α_0 denotes the root $t_1 - t_2$ of Spin(8). By a straightforward calculation, we have

$$\rho t_i = \begin{cases} -\gamma + t_i & (i = 1, 2, 3) \\ \gamma - t_4 & (i = 4), \end{cases} \tag{3.3}$$

$$\rho^2 t_i = \begin{cases} -\gamma + t_4 + t_i & (i = 1, 2, 3) \\ -\gamma & (i = 4), \end{cases}$$

and

$$\rho^3 = \mathrm{id}.$$

By the above equations the root system $\Phi(F_4)$ can be rewritten as:

$$\Phi(F_4) = \{\pm(t_i + t_j), \pm(t_i - t_j), \pm\rho^\varepsilon t_k \mid 1 \le i < j \le 4,\ 1 \le k \le 4,\ 0 \le \varepsilon \le 2\}$$

Note that ρ permutes the three root pairs (3.1) cyclically and $\kappa = \sigma_{t_4}$ interchanges $\pm\frac{1}{2}(t_1 + t_2 + t_3 - t_4) = \pm\rho t_4$ and $\pm\frac{1}{2}(t_1 + t_2 + t_3 + t_4) = \pm\rho^2 t_4$. Hence $W(F_4)/W \cong \mathfrak{S}_3$ is generated by ρ and κ. Since the equation

$$\kappa\rho = \rho^2\kappa \tag{3.4}$$

holds, we have a coset decomposition

$$W(F_4) = \coprod_{\substack{\varepsilon=0,1,2\\ \delta=0,1}} \rho^\varepsilon\kappa^\delta W.$$

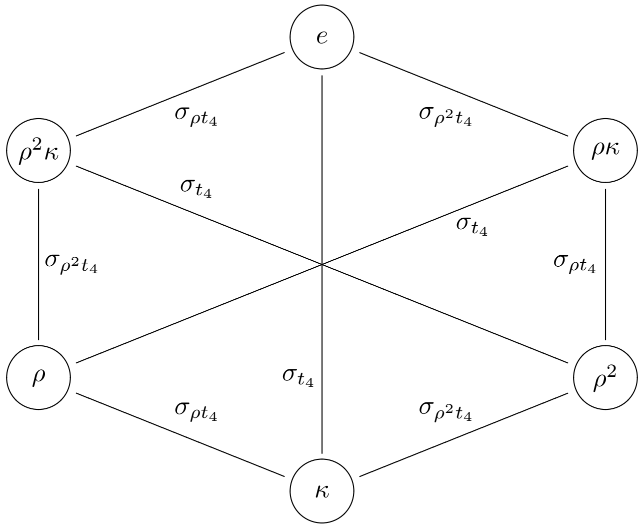
We will describe the GKM graph $\mathcal{F}_4$ of F_4/T. There are $24(= \#\Phi(F_4)/2)$ edges out of each vertex of $\mathcal{F}_4$. The half of these edges correspond to the long roots $\pm(t_i \pm t_j)$ and the other half correspond to the short roots $\pm\rho^\varepsilon t_i$.

The subgraph induced by W is the GKM graph $\mathcal{G}$ of $\mathrm{Spin}(8)/T$ and it is well understood in [FIM]. Let $\rho^\varepsilon\kappa^\delta\mathcal{G}$ be the GKM subgraph induced by $\rho^\varepsilon\kappa^\delta W$ for $\varepsilon = 0, 1, 2$ and $\delta = 0, 1$. For any ε and δ the induced subgraph $\rho^\varepsilon\kappa^\delta\mathcal{G}$ is isomorphic to $\mathcal{G}$ as graphs. Indeed if an edge ww' in $\mathcal{G}$ satisfies $w' = w\sigma_\alpha$ for a root α of Spin(8), then $\rho^\varepsilon\kappa^\delta w$ and $\rho^\varepsilon\kappa^\delta w'$ satisfy $\rho^\varepsilon\kappa^\delta w' = \rho^\varepsilon\kappa^\delta w\sigma_\alpha$, and vice versa. Moreover, labels of edges of $\rho^\varepsilon\kappa^\delta\mathcal{G}$ are also determined by $\mathcal{G}$ as follows. When an edge ww' has a root $\pm\beta$ as its label, the label of the edge connecting $\rho^\varepsilon\kappa^\delta w$ and $\rho^\varepsilon\kappa^\delta w'$ is $\pm\rho^\varepsilon\kappa^\delta\beta$. Remark that if an edge ww' in $\rho^\varepsilon\kappa^\delta\mathcal{G}$ satisfies $w' = w\sigma_\alpha$, then α is one of the long roots.

From the above argument, it is sufficient to consider the edges connecting two of $\rho^\varepsilon\kappa^\delta\mathcal{G}$'s, which correspond to the short roots. Easy calculations show that

$$\sigma_{t_4} = \kappa, \quad \sigma_{\rho t_4} = \rho^2\kappa, \quad \sigma_{\rho^2 t_4} = \rho\kappa.$$

Then the GKM graph $\mathcal{F}_4$ has an induced subgraph below, where e denotes the unit element of $W(F_4)$ and an element of $W(F_4)$ in each circle denotes a vertex of $\mathcal{F}_4$. The labels are calculated later.



(3.5)

We will calculate the reflection σ_α for a short root α to describe $\mathcal{F}_4$. For example let us consider the short root ρt_1 and the reflection $\sigma_{\rho t_1}$. By (3.3) we have $\rho t_1 = \frac{1}{2}(t_1 - t_2 - t_3 - t_4) = \sigma_{t_2}\sigma_{t_3}(\rho t_4)$. Then $\sigma_{\rho t_1} = \sigma_{t_2}\sigma_{t_3}\sigma_{\rho t_4}\sigma_{t_3}\sigma_{t_2}$ and $\sigma_{t_2}\sigma_{t_3} \in W$. Since W is a normal subgroup of $W(F_4)$, we have $W \cdot \rho^2\kappa W = \rho^2\kappa W$ in $W(F_4)/W$. Hence $\sigma_{\rho t_1}$ is also contained in $\rho^2\kappa W$. For any i, it is shown similarly that

$$\sigma_{\rho t_i} \in \rho^2\kappa W, \quad \sigma_{\rho^2 t_i} \in \rho\kappa W$$

and obviously we have

$$\sigma_{t_i} \in \kappa W.$$

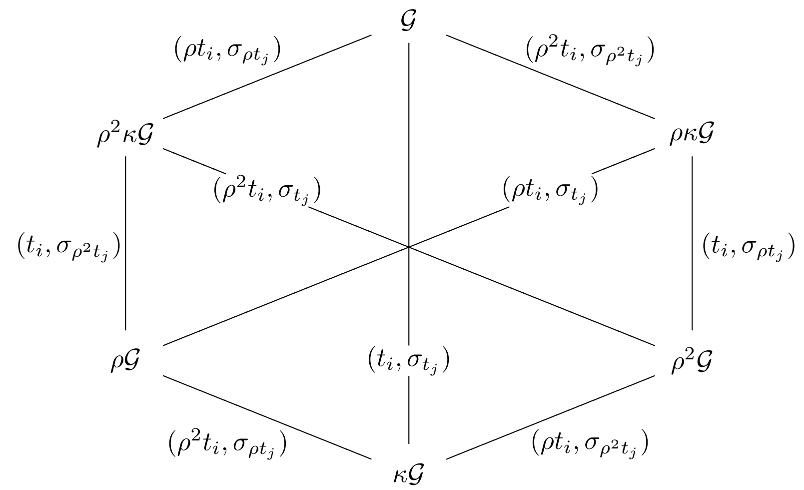
Hence, for any $0 \leq \varepsilon, \varepsilon' \leq 2$ and $\delta = 0, 1$, it is independent from the choice of i and $w \in \rho^{\epsilon'}\kappa^\delta W$ which coset contains $w\sigma_{\rho^\varepsilon t_i}$.

Let us calculate the label of the edge connecting the vertices κ and ρ in the GKM subgraph (3.5), which corresponds to a short root ρt_4. The label of the edge turns out to be $\pm\kappa(\rho t_4)$. It follows from the relation (3.4) that

$$\pm\kappa(\rho t_4) = \pm\rho^2\kappa t_4 = \pm\rho^2 t_4.$$

One can make similar calculations of the labels of other edges in the GKM subgraph (3.5). For any $w \in W$, w fixes three sets of short roots $\{\pm t_i\}_{i=1}^4$, $\{\pm\rho t_i\}_{i=1}^4$ and $\{\pm\rho^2 t_i\}_{i=1}^4$ since w permutes t_i's and changes the signs of even number of t_i's. Hence the label $\pm\rho^\varepsilon\kappa^\delta w(\alpha)$ is calculated similarly for any short root α.

We can now describe a schematic diagram of $\mathcal{F}_4$ as below.



(3.6)

This diagram means the followings. For example, $\mathcal{G}$ and $\rho\mathcal{G}$ are not adjacent in this diagram. It means that for any vertices $w \in W$ and $w' \in \rho W$, they are not adjacent. On the other hand, $\rho\mathcal{G}$ and $\rho\kappa\mathcal{G}$ are adjacent in this diagram, and a pair $(\rho t_i, \sigma_{t_j})$ is assigned to the edge. The first entry ρt_i is a root and the second entry σ_{t_j} is a reflection. If two vertices $w \in \rho W$ and $w' \in \rho\kappa W$ are adjacent in $\mathcal{F}_4$, then they satisfy $w' = w\sigma_{t_j}$ for some j, and the edge ww' is labeled by ρt_i for some i. The label $\pm\rho t_i$ equals to $\pm w t_j$. Especially each vertex of $\rho\mathcal{G}$ is connected to 4 vertices of $\rho\kappa\mathcal{G}$ by the edges correspond to the short roots t_j's ($1 \le j \le 4$), and vice versa. The labels of these edges are $\pm\rho t_i$'s ($1 \le i \le 4$). Every ρt_i's appear as the labels of the edges out of each vertex of $\rho\mathcal{G}$. The situation is the same for any connected two subgraphs in the schematic diagram (3.6).

4 Proof of the main theorem

There is a fibration sequence

$$F_4/T \longrightarrow ET \times_T F_4/T \longrightarrow BT. \tag{4.1}$$

The cohomology rings of F_4/T and BT are free as $\mathbb{Z}$-modules and have vanishing odd parts. As shown in Section 3, $H^*(BT)$ has five generators t_1, t_2, t_3, t_4, and γ of degree 2 with one relation of degree 2. According to [TW], $H^*(F_4/T)$ has τ_1, τ_2, τ_3, τ_4, and γ_1 of degree 2, γ_3 of degree 6, and ω of degree 8 as its generators, and $H^*(F_4/T)$ has seven relations of degree 2, 4, 6, 8, 12, 16, and 24. We can expect $H_T^*(F_4/T)$ has corresponding generators and relations. It is easy to see the Poincaré series of F_4/T and BT are

$$(1 + x^8 + x^{16}) \prod_{i=1}^{4} \frac{1 - x^{4i}}{1 - x^2} \quad \text{and} \quad \frac{1}{(1 - x^2)^4},$$

respectively. Hence we obtain the following proposition by the Serre spectral sequence for (4.1).

Proposition 4.1. *$H_T^*(F_4/T)$ is free as a $\mathbb{Z}$-module and its Poincaré series is*

$$P(H^*(ET\times_T F_4/T),x)=\frac{1}{(1-x^2)^4}(1+x^8+x^{16})\prod_{i=1}^{4}\frac{1-x^{4i}}{1-x^2}.$$

By the Serre spectral sequence for the fibration sequence (4.1), we see that generators of $H_T^*(F_4/T)$ come from the cohomology of F_4/T or BT. Let us define the corresponding GKM functions t_i, γ, τ_i, γ_1 and $\gamma_3 \in \mathrm{Map}(W(F_4), H^*(BT))$ for $1\le i\le 4$ and GKM functions γ_2 and γ_4 to state our results simpler. For any $w\in W(F_4)$

$$\begin{aligned}t_i(w)&=t_i\ \ (i=1,\dots,4)\\ \gamma(w)&=\gamma\\ \tau_i(w)&=w(t_i)\ \ (i=1,\dots,4)\\ \gamma_j&=\frac{1}{2}(e_j(\tau)-e_j(t))\ \ (j=1,2,3),\end{aligned}$$

and

$$\gamma_4(w)=\begin{cases}0 & w\in W\sqcup\rho^2\kappa W,\\ e_4(\rho^2 t) & w\in\rho^2 W\sqcup\rho\kappa W,\\ -e_4(t) & w\in\rho W\sqcup\kappa W.\end{cases}$$

Moreover we define $\omega=e_4(\tau)-e_4(t)-2\gamma_4$. Then

$$\omega(w)=\begin{cases}0 & w\in W\sqcup\kappa W,\\ -e_4(\rho^2 t) & w\in\rho W\sqcup\rho\kappa W,\\ e_4(\rho t) & w\in\rho^2 W\sqcup\rho^2\kappa W.\end{cases}\tag{4.2}$$

Since t_i's and γ are constant functions, they are GKM functions. A straightforward calculation shows that the following relation holds.

$$e_4(t)+e_4(\rho t)+e_4(\rho^2 t)=0\tag{4.3}$$

By the schematic diagram (3.6) of $\mathcal{F}_4$, one can see that γ_4 is a GKM function since $e_4(\rho^\varepsilon t)$ is the product of all $\rho^\varepsilon t_1$, $\rho^\varepsilon t_2$, $\rho^\varepsilon t_3$, and $\rho^\varepsilon t_4$ for $\varepsilon=0,1,2$. The following calculation shows τ_i's satisfy the GKM condition. For any edge ww' which satisfies $w'=w\sigma_\alpha$, we have

$$\begin{aligned}\tau_i(w)-\tau_i(w')&=w(t_i)-w'(t_i)\\ &=w\Big(t_i-\Big(t_i-2\frac{(t_i,\alpha)}{(\alpha,\alpha)}\alpha\Big)\Big)\\ &=2\frac{(t_i,\alpha)}{(\alpha,\alpha)}w\alpha.\end{aligned}$$

Since GKM functions form a ring, for $j=1,2,3$, we see that γ_j's are functions from $W(F_4)$ to $H^*(BT)\otimes\mathbb{Z}[\frac{1}{2}]$ which satisfy the GKM condition tensoring with $\mathbb{Z}[\frac{1}{2}]$. The following calculations show that γ_j's are actually $H^*(BT)$-valued functions. Let us extend ρ to an automorphism of $H^*(BT)$ naturally. For $w \in W \sqcup \kappa W = W(\mathrm{Spin}(9))$ and $\varepsilon = 0, 1, 2$,

$$\begin{aligned}\gamma_j(\rho^\varepsilon w) &= \frac{1}{2}(e_j(\tau) - e_j(t))(\rho^\varepsilon w) \\ &= \frac{1}{2}(\rho^\varepsilon e_j(w(t)) - e_j(t)) \\ &= \rho^\varepsilon\Big(\frac{1}{2}(e_j(w(t)) - e_j(t))\Big) + \frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t)).\end{aligned}$$

Since w only permutes t_i's and changes their signs, it is obvious that $\frac{1}{2}(e_j(w(t)) - e_j(t)) \in H^*(BT)$. Then $\rho^\varepsilon(\frac{1}{2}(e_j(w(t)) - e_j(t))) \in H^*(BT)$. On the other hand one can see that $\frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t)) \in H^*(BT)$ for $\varepsilon = 0, 1, 2$ as follows. When $\varepsilon = 0$, $\frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t)) = 0$ and it is contained in $H^*(BT)$. When $\varepsilon = 1, 2$, Table 1 shows the value of $\frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t))$ for $j = 1, 2, 3$. Then γ_j is a $H^*(BT)$-valued function and then a GKM function.

Table 1: the value of $\frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t))$

	$j = 1$	$j = 2$	$j = 3$
$\varepsilon = 1$	$-\gamma - t_4$	$-\gamma^2 + t_4^2$	$t_4\gamma(\gamma - t_4) - t_4(t_1t_2 + t_2t_3 + t_3t_1)$
$\varepsilon = 2$	$-2\gamma + t_4$	$(-2\gamma + t_4)t_4$	$\gamma^3 - t_4\gamma^2 - \gamma(t_1t_2 + t_2t_3 + t_3t_1)$

The following lemma will be proved in Section 5.

Lemma 4.1 (see [FIM, Lemma 5.4]). *Let $\mathcal{F}_4$ be the GKM graph of F_4/T, then $H^*(\mathcal{F}_4)$ is generated by the GKM functions t_i, γ, τ_i, γ_i, ω $(i = 1, 2, 3, 4)$ as a ring.*

By the fibration sequence (4.1), we can expect some relations hold in $H^*(\mathcal{F}_4)$, which come from the relations of $H^*(BT)$ and $H^*(F_4/T)$. Proposition 4.2 claims the corresponding relations hold in $H^*(\mathcal{F}_4)$.

Proposition 4.2. *The following relations hold in* $H^*(\mathcal{F}_4) \subset \mathrm{Map}(W(F_4), H^*(BT))$*;*

$$r_1' = e_1(t) - 2\gamma = 0, \tag{4.4}$$
$$R_1 = e_1(\tau) - e_1(t) - 2\gamma_1 = 0, \tag{4.5}$$
$$R_2 = e_2(\tau) - e_2(t) - 2\gamma_2 = 0, \tag{4.6}$$
$$R_3 = e_3(\tau) - e_3(t) - 2\gamma_3 = 0, \tag{4.7}$$
$$R_4 = e_4(\tau) - e_4(t) - 2\gamma_4 - \omega = 0, \tag{4.8}$$
$$r_2 = \sum_{j=1}^{2} (-1)^j \gamma_j (\gamma_{2-j} + e_{2-j}(t)) = 0, \tag{4.9}$$
$$r_4 = \sum_{j=1}^{4} (-1)^j \gamma_j (\gamma_{4-j} + e_{4-j}(t)) - \omega = 0, \tag{4.10}$$
$$r_6 = \sum_{j=2}^{4} (-1)^j \gamma_j (\gamma_{6-j} + e_{6-j}(t)) + (\gamma_2 + \gamma^2)\omega = 0, \tag{4.11}$$
$$r_8 = \gamma_4(\gamma_4 + e_4(t)) + \omega^2 + (\gamma_4 - e_4(\rho t))\omega = 0, \tag{4.12}$$
$$r_{12} = \omega(\omega - e_4(\rho t))(\omega + e_4(\rho^2 t)) = 0. \tag{4.13}$$

Proposition 4.2 is proved in Section 6. The following lemma is proved in Section 7.

Lemma 4.2. $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]/(r_1', R_i, r_{2i}, r_{12} \mid 1 \le i \le 4)$ *is free as a* $\mathbb{Z}$*-module, and its Poincaré series coincides with that of* $H_T^*(F_4/T)$*.*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let I denote the ideal $(r_1', R_i, r_{2i}, r_{12} \mid 1 \le i \le 4)$ in the polynomial ring $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]$. We have a surjective ring homomorphism

$$\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4] \to H^*(\mathcal{F}_4)$$

by Lemma 4.1, and it factors through $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]/I \to H^*(\mathcal{F}_4)$ by Proposition 4.2. It follows from Proposition 4.1 and Lemma 4.2 that $H^*(\mathcal{F}_4)$ and $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]/I$ are free as $\mathbb{Z}$-modules. Moreover Lemma 4.2 claims that $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]/I$ and $H^*(\mathcal{F}_4)$ have the same rank in each degree. Therefore the ring homomorphism $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]/I \to H^*(\mathcal{F}_4)$ is an isomorphism and Theorem 1.1 is proved by Theorem 2.1. □

5 Proof of Lemma 4.1

First we introduce some notation for the proof of Lemma 4.1. For a positive integer n, let $[n]$ and $\pm[n]$ be $\{i \in \mathbb{Z} \mid 1 \le i \le n\}$ and $\{\pm i \in \mathbb{Z} \mid 1 \le i \le n\}$, respectively. For $1 \le n \le 4$, let I_n denote an ordered n-tuple $(i_1, \ldots, i_n)$ of elements of $[4]$ which does not include the same entries, and I_n' denote an ordered n-tuple $(i_1', \ldots, i_n')$ of elements of $\pm[4]$ such that $|i_k'| \ne |i_l'|$ for $k \ne l$. We often regard I_n, I_n' as the n-subsets of $[4]$ by the following maps.

$$(i_1, \ldots, i_n) \mapsto \{i_1, \ldots, i_n\}, \quad (i_1', \ldots, i_n') \mapsto \{|i_1'|, \ldots, |i_n'|\}$$

Let $t_{i'} = \mathrm{sgn}(i')t_{|i'|}$. For $\varepsilon = 0, 1, 2$, we define a subset $\rho^\varepsilon W^{I_n}_{I'_n}$ of $W(F_4)$ as:

$$\rho^\varepsilon W^{I_n}_{I'_n} = \{w \in W(F_4) \mid w \in \rho^\varepsilon W(\mathrm{Spin}(9)),\ w(t_{i_k}) = \rho^\varepsilon t_{i'_k}\ (1 \le k \le n)\}$$

We define I_0 and I'_0 to be the empty set. Note that $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$ includes $\rho^\varepsilon W^{I_n}_{I'_n}$ and decomposes as follows.

$$\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}} = \coprod_{i_n \in [4]\setminus I_{n-1}} \rho^\varepsilon W^{(I_{n-1}, i_n)}_{(I'_{n-1}, i'_n)} \sqcup \coprod_{i_n \in [4]\setminus I_{n-1}} \rho^\varepsilon W^{(I_{n-1}, i_n)}_{(I'_{n-1}, -i'_n)} \tag{5.1}$$

For a set $S = \{j_1, \ldots, j_k\}$ of natural numbers with $j_1 < \cdots < j_k$, let x_S denote a sequence $(x_{j_1}, \ldots, x_{j_k})$ for $x = t,\ \rho t,\ \rho^2 t,\ \tau$. For $n \ge 0$, $j \le 4$, and $\varepsilon = 0, 1, 2$, let $\gamma^{(\varepsilon)}_j{}^{I_n}_{I'_n}$ be a function from $\rho^\varepsilon W^{I_n}_{I'_n}$ to $\mathbb{Z}[\frac{1}{2}][t_1, t_2, t_3, t_4]$ defined as

$$\gamma^{(\varepsilon)}_j{}^{I_n}_{I'_n} = \frac{1}{2}(e_j(\tau_{[4]\setminus I_n}) - e_j(\rho^\varepsilon t_{[4]\setminus I'_n})),$$

where I_n and I'_n in the right hand side are regarded as subsets of $[4]$. When $n = 0$ we abbreviate $\gamma^{(\varepsilon)}_j{}^{\emptyset}_{\emptyset}$ by $\gamma^{(\varepsilon)}_j$. If $j \le 0$ or $j > 4 - n$, we define $\gamma^{(\varepsilon)}_j{}^{I_n}_{I'_n} = 0$.

We define a function $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ which is useful in the proof of Lemma 4.1 as:

$$f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n} = \frac{1}{2} \prod_{k \in [4]\setminus I_{n-1}} (\tau_k - \rho^\varepsilon t_{i'_n}).$$

This function is $H^*(BT)$-valued on $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$, since for any $w \in \rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$ there exists $k \in [4]\setminus I_{n-1}$ such that $w \in \rho^\varepsilon W^{(I_{n-1},k)}_{(I'_{n-1},i'_n)} \sqcup \rho^\varepsilon W^{(I_{n-1},k)}_{(I'_{n-1},-i'_n)}$ by the decomposition (5.1), and then $w(t_k) - \rho^\varepsilon t_{i'_n}$ equals to 0 or $-2\rho^\varepsilon t_{i'_n}$. Especially we have

$$f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}(w) = \begin{cases} 0 & w \in \coprod_{k \in [4]\setminus I_{n-1}} \rho^\varepsilon W^{(I_{n-1},k)}_{(I'_{n-1},i'_n)}, \\ -\rho^\varepsilon t_{i'_n} \prod_{k \in [4]\setminus I'_n} (\rho^\varepsilon t_k - \rho^\varepsilon t_{i'_n}) & w \in \coprod_{k \in [4]\setminus I_{n-1}} \rho^\varepsilon W^{(I_{n-1},k)}_{(I'_{n-1},-i'_n)}. \end{cases} \tag{5.2}$$

Let R denote the subring of $H^*(\mathcal{F}_4)$ generated by t_i's, γ, τ_i's, and γ_i's $(1 \le i \le 4)$. The following proposition claims that this function $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ can be replaced partly by an element of R.

Proposition 5.1. *For $1 \le n \le 4$, there is a polynomial in γ_1, γ_2, γ_3, γ_4 over $H^*(BT)$, which coincides with the function $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ on $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$.*

Proposition 5.1 is a consequence of Lemma 5.1 and 5.2 below.

Lemma 5.1. *For $1 \le n \le 4$, there is a polynomial in $\gamma^{(\varepsilon)}_j{}^{I_{n-1}}_{I'_{n-1}}$'s $(1 \le j \le 4 - (n-1))$ over $H^*(BT)$, which coincides with $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ on $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$.*

Lemma 5.2 (cf. [FIM, Lemma 5.3]). *For $1 \le n \le 4$ and $1 \le j \le 4-n$, there is a polynomial in $\gamma_1^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}, \dots, \gamma_{4-n}^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}$ over $H^*(BT)$, which coincides with $\gamma_j^{(\varepsilon)}{}^{(I_{n-1},i_n)}_{(I'_{n-1},i'_n)}$ on $\rho^\varepsilon W^{(I_{n-1},i_n)}_{(I'_{n-1},i'_n)}$. More explicitly,*

$$\gamma_j^{(\varepsilon)}{}^{I_n}_{I'_n} = \begin{cases} \sum_{k=0}^{j-1} \gamma_{j-k}^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}(-\rho^\varepsilon t_{i'_n})^k & \operatorname{sgn} i'_n = 1 \\ \sum_{k=0}^{j-1} \gamma_{j-k}^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}(-\rho^\varepsilon t_{i'_n})^k + \sum_{k=1}^{j} e_{j-k}(\rho^\varepsilon t_{[4]\setminus I'_n})(-\rho^\varepsilon t_{i'_n})^k & \operatorname{sgn} i'_n = -1. \end{cases}$$

Proof of Proposition 5.1. By Lemma 5.1, there is a polynomial in $\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}$'s $(1 \le j \le 4-(n-1))$ over $H^*(BT)$, which coincides with $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ on $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$ for $\varepsilon = 0, 1, 2$. Then by Lemma 5.2 $\gamma_j^{(\varepsilon)}{}^{(I_{n-1},i_n)}_{(I'_{n-1},i'_n)}$ can be replaced by some polynomial in $\gamma_1^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}, \dots, \gamma_{4-n}^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}$ over $H^*(BT)$. By a descending induction on n we reached to a polynomial in $\gamma_1^{(\varepsilon)}, \gamma_2^{(\varepsilon)}, \gamma_3^{(\varepsilon)}, \gamma_4^{(\varepsilon)}$ over $H^*(BT)$, which coincides with $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ on $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$ for $\varepsilon = 0, 1, 2$. Next we need to show that $\gamma_j - \gamma_j^{(\varepsilon)} \in H^*(BT)$ on $\rho^\varepsilon W(\mathrm{Spin}(9))$ for $1 \le j \le 4$ and $\varepsilon = 0, 1, 2$ to complete the proof of Proposition 5.1. By definition we have

$$\gamma_j^{(\varepsilon)} = \gamma_j + \frac{1}{2}(e_j(t) - e_j(\rho^\varepsilon t)) \quad (j = 1, 2, 3).$$

For $\varepsilon = 0, 1, 2$ and $j = 1, 2, 3$, Table 1 shows that $(e_j(t) - e_j(\rho^\varepsilon t))/2 \in H^*(BT)$ and then $\gamma_j - \gamma_j^{(\varepsilon)} \in H^*(BT)$ on $\rho^\varepsilon W(\mathrm{Spin}(9))$. By the definition of γ_4 and the equation (4.3), we have

$$\begin{aligned} \gamma_4^{(0)} &= \gamma_4 && \text{on } W(\mathrm{Spin}(9)), \\ \gamma_4^{(1)} &= \gamma_4 + e_4(t) && \text{on } \rho W(\mathrm{Spin}(9)), \\ \gamma_4^{(2)} &= \gamma_4 - e_4(\rho^2 t) && \text{on } \rho^2 W(\mathrm{Spin}(9)). \end{aligned}$$

Therefore there is a polynomial in $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ over $H^*(BT)$, which coincides with the function $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ on $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$. □

Proof of Lemma 5.1. Without loss of generality, we may suppose that $I_{n-1} = (1, \dots, n-1)$. Note that $e_j(x_S) = 0$ for $j > \#S$ or $j < 0$, and that we have

$$e_j(x_1, \dots, x_{m-1}, x_m) = e_j(x_1, \dots, x_{m-1}) + e_{j-1}(x_1, \dots, x_{m-1})x_m. \tag{5.3}$$

By the definition of $\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}$ we can expand the GKM function $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ as follows.

$$\begin{aligned} \frac{1}{2}\prod_{l=0}^{4-n}(\tau_{n+l} - \rho^\varepsilon t_{i'_n}) =& \frac{1}{2}\sum_{j=0}^{5-n} e_j(\tau_{[4]\setminus I_{n-1}})(-\rho^\varepsilon t_{i'_n})^{5-n-j} \\ =& \frac{1}{2}\sum_{j=0}^{5-n}(2\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}} + e_j(\rho^\varepsilon t_{[4]\setminus I'_{n-1}}))(-\rho^\varepsilon t_{i'_n})^{5-n-j} \end{aligned}$$

Pay attention to the sign of i'_n and recall that $[4] \setminus I'_{n-1} = \{i \in [4] \mid \pm i \notin I'_{n-1}\}$. By (5.3), the above equals to

$$\sum_{j=0}^{5-n} \gamma_j^{(\varepsilon) I_{n-1}}{}_{I'_{n-1}} (-\rho^\varepsilon t_{i'_n})^{5-n-j} + \frac{1}{2} \sum_{j=0}^{5-n} (e_j(\rho^\varepsilon t_{[4]\setminus I'_n}) + e_{j-1}(\rho^\varepsilon t_{[4]\setminus I'_n})\rho^\varepsilon t_{|i'_n|})(-\rho^\varepsilon t_{i'_n})^{5-n-j}$$
$$= \begin{cases} \sum_{j=0}^{5-n} \gamma_j^{(\varepsilon) I_{n-1}}{}_{I'_{n-1}} (-\rho^\varepsilon t_{i'_n})^{5-n-j} & \operatorname{sgn} i'_n = 1 \\ \sum_{j=0}^{5-n} \gamma_j^{(\varepsilon) I_{n-1}}{}_{I'_{n-1}} (-\rho^\varepsilon t_{i'_n})^{5-n-j} + \sum_{j=0}^{4-n} e_j(\rho^\varepsilon t_{[4]\setminus I'_n})(-\rho^\varepsilon t_{i'_n})^{5-n-j} & \operatorname{sgn} i'_n = -1. \end{cases}$$

□

Proof of Lemma 5.2. The relation $\tau_{i_n} = \rho^\varepsilon t_{i'_n}$ holds on $\rho^\varepsilon W^{(I_{n-1}, i_n)}_{(I'_{n-1}, i'_n)}$. Then we have

$$\gamma_j^{(\varepsilon) I_{n-1}}{}_{I'_{n-1}} - \gamma_j^{(\varepsilon)(I_{n-1}, i_n)}{}_{(I'_{n-1}, i'_n)}$$
$$= \frac{1}{2}(e_j(\tau_{i\in[4]\setminus I_{n-1}}) - e_j(\rho^\varepsilon t_{i'\in[4]\setminus I'_{n-1}})) - \frac{1}{2}(e_j(\tau_{i\in[4]\setminus I_n}) - e_j(\rho^\varepsilon t_{i'\in[4]\setminus I'_n}))$$
$$= \frac{1}{2}(e_{j-1}(\tau_{i\in[4]\setminus I_n})\tau_{i_n} - e_{j-1}(\rho^\varepsilon t_{i'\in[4]\setminus I'_n})\rho^\varepsilon t_{|i'_n|})$$
$$= \begin{cases} \gamma_{j-1}^{(\varepsilon) I_n}{}_{I'_n} \rho^\varepsilon t_{i'_n} & \operatorname{sgn} i'_n = 1 \\ \gamma_{j-1}^{(\varepsilon) I_n}{}_{I'_n} \rho^\varepsilon t_{i'_n} + e_{j-1}(\rho^\varepsilon t_{i'\in[4]\setminus I'_n})\rho^\varepsilon t_{i'_n} & \operatorname{sgn} i'_n = -1. \end{cases}$$

Iterated use of this equation shows that

$$\gamma_j^{(\varepsilon)(I_{n-1}, i_n)}{}_{(I'_{n-1}, i'_n)} = \begin{cases} \sum_{k=0}^{j-1} \gamma_{j-k}^{(\varepsilon) I_{n-1}}{}_{I'_{n-1}} (-\rho^\varepsilon t_{i'_n})^k & \operatorname{sgn} i'_n = 1 \\ \sum_{k=0}^{j-1} \gamma_{j-k}^{(\varepsilon) I_{n-1}}{}_{I'_{n-1}} (-\rho^\varepsilon t_{i'_n})^k + \sum_{k=1}^{j} e_{j-k}(\rho^\varepsilon t_{[4]\setminus I'_n})(-\rho^\varepsilon t_{i'_n})^k & \operatorname{sgn} i'_n = -1. \end{cases}$$

□

Now we are ready to prove Lemma 4.1.

Proof of Lemma 4.1. We show that any GKM function $h \in H^*(\mathcal{F}_4)$ belongs to the subring R generated by t_i's, γ, τ_i's, γ_i's, and ω $(1 \le i \le 4)$. By the definition of ρ, the set of all vertices $W(F_4)$ of $\mathcal{F}_4$ decomposes as:

$$W(F_4) = W(\mathrm{Spin}(9)) \sqcup \rho W(\mathrm{Spin}(9)) \sqcup \rho^2 W(\mathrm{Spin}(9)).$$

For each $\varepsilon = 0, 1, 2$, $\rho^\varepsilon W(\mathrm{Spin}(9))$ has a filtration

$$\rho^\varepsilon W^{I_4}_{I'_4} \subset \cdots \subset \rho^\varepsilon W^{I_n}_{I'_n} \subset \rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}} \subset \cdots \subset \rho^\varepsilon W^{I_0}_{I'_0} = \rho^\varepsilon W(\mathrm{Spin}(9)).$$

By descending induction on n, we will show that any GKM function h can be modified to be 0 on $\rho^\varepsilon W^{I_n}_{I'_n}$ by subtracting some GKM function in R. Moreover, in the induction step on n, we give an induction to fill the decomposition (5.1) of $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$.

Let $0 \leq n \leq 4$. The following claim in the case where $n = 0$ shows that h can be modified to be 0 on $W(\mathrm{Spin}(9))$.

Claim 1 (n)**.** *For any ordered n-tuples I_n, I'_n and any function h from $W^{I_n}_{I'_n}$ to $H^*(BT)$ which satisfies the GKM condition on $W^{I_n}_{I'_n}$, there is a GKM function $G \in R$ which coincides with h on $W^{I_n}_{I'_n}$.*

We show this claim by descending induction on n. For $n = 4$, since $W^{I_4}_{I'_4}$ is a one point set, the claim holds obviously. Assume Claim 1 (n) holds, and fix $I_n = (i_1, \dots, i_n)$ and $I'_n = (i'_1, \dots, i'_n)$. Then we have a GKM function which coincides with h on $W^{I_n}_{I'_n}$. Subtracting this GKM function from h, we may assume h vanishes on $W^{I_n}_{I'_n}$. We give an induction to fill the decomposition (5.1) of $W^{I_{n-1}}_{I'_{n-1}}$ as follows. For any $k \in [4] \setminus I_n$, let σ_k denote the reflection associated with $t_k - t_{i_n}$. Then σ_k interchanges t_k and t_{i_n}, and for any $w \in W^{(I_{n-1},k)}_{(I'_{n-1},i'_n)}$, $w\sigma_k$ is contained in $W^{(I_{n-1},i_n)}_{(I'_{n-1},i'_n)}$. By the GKM condition, $h(w) - h(w\sigma_k) = h(w)$ belongs to the ideal generated by $w(t_{i_n} - t_k) = w(t_{i_n}) - t_{i'_n} = \tau_{i_n}(w) - t_{i'_n}$. Put $k_0, \dots, k_{4-n} \in [4] \setminus I_{n-1}$ as $k_0 = i_n$, $k_s < k_t$ for $1 \leq s < t$ and $\{k_0, \dots, k_{4-n}\} \cup I_{n-1} = [4]$.

Claim 2 (t)**.** *If h is a GKM function which vanishes on $\coprod_{s<t} W^{(I_{n-1},k_s)}_{(I'_{n-1},i'_n)}$, there is a GKM function $G \in R$ such that h coincides with $\prod_{s<t}(\tau_{k_s} - t_{i'_n})G$ on $W^{(I_{n-1},k_t)}_{(I'_{n-1},i'_n)}$.*

We show this claim by induction on t $(0 \leq t \leq 4 - n)$. Without loss of generality, we may suppose that $I_{n-1} = (1, \dots, n-1)$ and $k_0 = i_n = n$, $k_1 = n+1, \dots, k_{4-n} = 4$. We rephrase Claim 2 (t) as:

Claim 2 (k)**.** *If h vanishes on $\coprod_{0\leq l<k} W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)}$, there is a GKM function $G \in R$ such that h coincides with $\prod_{0\leq l<k}(\tau_{n+l} - t_{i'_n})G$ on $W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$.*

Obviously $\prod_{0\leq l<k}(\tau_{n+l} - t_{i'_n})$ vanishes on $\coprod_{0\leq l<k} W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)}$. For $w \in W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$, by the GKM condition, there is an element $g_w \in H^*(BT)$ such that

$$h(w) = \Big(\prod_{0\leq l<k}(\tau_{n+l} - t_{i'_n})(w)\Big)g_w.$$

One can verify that a function $G' \colon W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)} \to H^*(BT)$ given by

$$G'(w) = g_w$$

satisfies the GKM condition on $W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$ as follows. Assume that two vertices w, $w' \in W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$ of $\mathcal{F}_4$ satisfy $w' = w\sigma_\alpha$ for some positive root α. Then $\alpha = t_i - t_j$, where $i < j$ and

$i,j\in\{m\in\mathbb{Z}\mid n\le m\le n+k-1$ or $n+k+1\le m\le 4\}$. When $i<j<n+k$ or $n+k<i<j$, the GKM condition says

$$\begin{aligned}h(w)-h(w')&=\Big(\prod_{0\le l<k}(w(t_{n+l})-t_{i'_n})\Big)G'(w)-\Big(\prod_{0\le l<k}(w\sigma_{t_i-t_j}(t_{n+l})-t_{i'_n})\Big)G'(w')\\&=\Big(\prod_{0\le l<k}(w(t_{n+l})-t_{i'_n})\Big)(G'(w)-G'(w'))\end{aligned}$$

belongs to the ideal $(w(t_i-t_j))$. Since $w(t_{n+l})-t_{i'_n}$ and $w(t_i-t_j)$ are relatively prime, $G'(w)-G'(w')$ also belongs to the ideal $(w(t_i-t_j))$. When $i<n+k<j$, the GKM condition says

$$\begin{aligned}&h(w)-h(w')\\&=\Big(\prod_{0\le l<k}(w(t_{n+l})-t_{i'_n})\Big)G'(w)-\Big(\prod_{0\le l<k}(w\sigma_{t_i-t_j}(t_{n+l})-t_{i'_n})\Big)G'(w')\\&=\Big(\prod_{0\le l<k,\ l\ne i}(w(t_{n+l})-t_{i'_n})\Big)\Big((w(t_i)-t_{i'_n})(G'(w)-G'(w'))+(w(t_i)-w(t_j))G'(w')\Big)\end{aligned}$$

belongs to the ideal $(w(t_i-t_j))$. Since $w(t_{n+l})-t_{i'_n}$ and $w(t_i-t_j)$ are relatively prime, $G'(w)-G'(w')$ also belongs to the ideal $(w(t_i-t_j))$. Hence the function G' satisfies the GKM condition on $W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$.

By (descending) induction on n there is a GKM function $G\in R$ such that G and G' coincide on $W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$. Then

$$h-\Big(\prod_{0\le l<k}(\tau_{n+l}-t_{i'_n})\Big)G=0\quad\text{on}\ \coprod_{0\le l\le k}W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)}.$$

Therefore the induction on k proceeds.

Next we fill the other half of the decomposition (5.1). Note that when $I_{n-1}=(1,\dots,n-1)$,

$$f^{(0)}{}^{I_{n-1}}_{i'_n}=\frac{1}{2}\prod_{0\le l\le 4-n}(\tau_{n+l}-t_{i'_n}).$$

Let $0\le k'\le 4-n$.

Claim 3 (k')**.** *If h vanishes on $\coprod_{0\le l\le 4-n}W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)}\sqcup\coprod_{0\le l<k'}W^{(I_{n-1},n+l)}_{(I'_{n-1},-i'_n)}$. There is a GKM function $G\in R$ such that h coincides with $f^{(0)}{}^{I_{n-1}}_{i'_n}\prod_{0\le l<k'}(\tau_{n+l}+t_{i'_n})G$ on $W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$.*

We show this claim by induction on k'. For $w\in W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$, by the GKM condition, $h(w)$ belongs to the ideal generated by the following elements of $H^*(\mathcal{F}_4)$.

$$\begin{aligned}w(t_{n+l}-t_{n+k'})&=w(t_{n+l})+t_{i'_n}&&\text{for } 0\le l<k'\\w(t_{n+l}+t_{n+k'})&=w(t_{n+l})-t_{i'_n}&&\text{for } 0\le l\le 4-n,\ l\ne k'\\w(t_{n+k'})&=-t_{i'_n}&&\end{aligned}$$

For $w \in W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$, by the equation (5.2), there is an element $g_w \in H^*(BT)$ such that

$$h(w) = f^{(0)}{}^{I_{n-1}}_{i'_n}(w)\Big(\prod_{0\le l<k'}(\tau_{n+l}+t_{i'_n})(w)\Big)g_w.$$

One can verify that a function G' given by $G'(w) = g_w$ satisfies the GKM condition on $W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$ similarly as above. By (descending) induction on n there is a GKM function $G \in R$ such that G and G' coincide on $W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$. Then

$$h - f^{(0)}{}^{I_{n-1}}_{i'_n}\Big(\prod_{0\le l<k'}(\tau_{n+l}+t_{i'_n})\Big)G = 0$$

$$\text{on} \coprod_{0\le l\le 4-n} W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)} \sqcup \coprod_{0\le l\le k'} W^{(I_{n-1},n+l)}_{(I'_{n-1},-i'_n)}.$$

Therefore the induction on k' proceeds. By Proposition 5.1 the function

$$f^{(0)}{}^{I_{n-1}}_{i'_n} = \frac{1}{2}\prod_{0\le l\le 4-n}(\tau_{n+l}-t_{i'_n})$$

can be replaced by a polynomial in γ_j's ($1 \le j \le 4$) over $H^*(BT)$. Therefore the (descending) induction on n proceeds, and we may assume h vanishes on $W(\mathrm{Spin}(9)) = W \sqcup \kappa W$.

Next we show that for a GKM function h which vanishes on $W(\mathrm{Spin}(9))$, there is a GKM function $G \in R$ such that $h - \omega G = 0$ on $W(\mathrm{Spin}(9)) \sqcup \rho W(\mathrm{Spin}(9))$, where ω vanishes on $W(\mathrm{Spin}(9))$. Recall that the schematic diagram (3.6) says that each $w \in \rho W \sqcup \rho\kappa W$ are adjacent to 4 vertices of $W \sqcup \kappa W$, and the labels of these edges are $\rho^2 t_i$ ($1 \le i \le 4$) and different each other. The GKM condition says that for $w \in \rho W(\mathrm{Spin}(9))$, $h(w)$ belongs to the ideal $(\rho^2 t_i \mid i = 1, 2, 3, 4)$. For $w \in \rho W(\mathrm{Spin}(9))$, there is an element $g_w \in H^*(BT)$ such that

$$h(w) = -e_4(\rho^2 t)g_w = \omega(w)g_w.$$

It is obvious that a function G' given by $G'(w) = g_w$ satisfies the GKM condition on $\rho W(\mathrm{Spin}(9))$, since the edges in the GKM subgraph induced by $\rho W(\mathrm{Spin}(9))$ have the long roots or ρt_i as their labels and the all positive roots of F_4 are relatively prime in $H^*(BT)$. Then we claim that there is a GKM function G such that $G = G'$ on $\rho W(\mathrm{Spin}(9))$. This claim is proved similarly as above, changing $W^{I_n}_{I'_n}$ to $\rho W^{I_n}_{I'_n}$, $\tau_{k_s} - t_{i'_n}$ to $\tau_{k_s} - \rho t_{i'_n}$, and $f^{(0)}{}^{I_{n-1}}_{i'_n}$ to $f^{(1)}{}^{I_{n-1}}_{i'_n}$.

Finally we show that for a GKM function h which vanishes on $W(\mathrm{Spin}(9)) \sqcup \rho W(\mathrm{Spin}(9))$, there is a GKM function $G \in R$ such that $h - \omega(\omega + e_4(\rho^2 t))G = 0$ as a GKM function on the whole $W(F_4)$, where $\omega + e_4(\rho^2 t)$ vanishes on $\rho W(\mathrm{Spin}(9))$. It is proved similarly as above that for $w \in \rho^2 W(\mathrm{Spin}(9))$, $h(w)$ belongs to the ideal $(t_i, \rho t_i \mid i = 1, 2, 3, 4)$. For $w \in \rho^2 W(\mathrm{Spin}(9))$, there is an element $g_w \in H^*(BT)$ such that

$$h(w) = -e_4(\rho t)e_4(t)g_w = \omega(w)(\omega(w) + e_4(\rho^2 t))g_w,$$

where the latter equality is due to the relation (4.3). Then we claim that a function G' given by $G'(w) = g_w$ satisfies the GKM condition on $\rho^2 W(\mathrm{Spin}(9))$, and that there is a GKM function G such that $G = G'$ on $\rho^2 W(\mathrm{Spin}(9))$. This claim is proved similarly as above, changing $W^{I_n}_{I'_n}$ to $\rho^2 W^{I_n}_{I'_n}$, $\tau_{k_s} - t_{i'_n}$ to $\tau_{k_s} - \rho^2 t_{i'_n}$, and $f^{(0)}{}^{I_{n-1}}_{i'_n}$ to $f^{(2)}{}^{I_{n-1}}_{i'_n}$. The proof is completed. □

6 Proof of Proposition 4.2

We prove Proposition 4.2 in the similar way of [FIM, Proof of Lemma 5.5].

Proof of Proposition 4.2. The relations (4.4), (4.5), (4.6), (4.7), and (4.8) hold obviously by definition, and the relation (4.13) holds by (4.2). To show that (4.9), (4.10), (4.11), and (4.12) hold, we claim that the following relations hold in $H_T^*(\mathcal{F}_4)$.

$$e_1(\tau^2) - e_1(t^2) = 0 \tag{6.1}$$
$$e_2(\tau^2) - e_2(t^2) - 6\omega = 0 \tag{6.2}$$
$$e_3(\tau^2) - e_3(t^2) - e_1(t^2)\omega = 0 \tag{6.3}$$
$$e_4(\tau^2) - e_4(t^2) + 3\omega^2 - 2(e_4(\rho t) - e_4(\rho^2 t))\omega = 0 \tag{6.4}$$

The left-hand side functions of these equations are constant on each $\rho^\varepsilon W(\mathrm{Spin}(9))$ for $\varepsilon = 0, 1, 2$. Calculations of each values on $\rho^\varepsilon W(\mathrm{Spin}(9))$ with (4.3) show that the equations (6.1), (6.2), (6.3), and (6.4) hold.

We show that (6.1), (6.2), (6.3), and (6.4) are divisible by 4 to deduce (4.9), (4.10), (4.11), and (4.12). Let x be an indeterminate and put $X = -6\omega x^4 + e_1(t^2)\omega x^6 + (3\omega^2 - 2(e_4(\rho t) - e_4(\rho^2 t))\omega)x^8$. It follows from (6.1), (6.2), (6.3), and (6.4) that

$$\begin{aligned} 0 &= \prod_{i=1}^{4}(1 - \tau_i^2 x^2) - \prod_{i=1}^{4}(1 - t_i^2 x^2) + X \\ &= \sum_{k=0}^{4}(1 + (-1)^k e_k(\tau)x^k)\sum_{k=0}^{4}(1 + e_k(\tau)x^k) - \sum_{k=0}^{4}(1 + (-1)^k e_k(t)x^k)\sum_{k=0}^{4}(1 + e_k(t)x^k) + X. \end{aligned}$$

We can erase $e_k(\tau)$ by the relation (4.5), (4.6), (4.7), and (4.8), and obtain

$$\begin{aligned} &4\sum_{k=1}^{3}(-1)^k\gamma_k^2 x^{2k} - 8\gamma_1\gamma_3 x^4 + 4\sum_{k=1}^{3}\sum_{i=n_k}^{m_k}(-1)^i\gamma_i e_{2k-i}(t)x^{2k} \\ &+ 2(2\gamma_4 + \omega)x^4 + 4\gamma_2(e_4(t) + 2\gamma_4 + \omega)x^6 + 2e_2(t)(2\gamma_4 + \omega)x^6 + (2e_4(t) + 2\gamma_4 + \omega)(2\gamma_4 + \omega)x^8 + X, \end{aligned}$$

where $n_k = \max\{1, 2k-3\}$ and $m_k = \min\{3, 2k\}$. This calculation is similar to the calculation in [FIM, Proof of Lemma 5.5], but note that $\gamma_4 \neq \frac{1}{2}(e_4(\tau) - e_4(t))$. Then comparing the coefficients, we obtain

$$
\begin{aligned}
0 &= -4\gamma_1^2 + 4(-\gamma_1 e_1(t) + \gamma_2) = 4\sum_{j=1}^{2}(-1)^j\gamma_j(\gamma_{2-j} + e_{2-j}(t)),\\
0 &= 4\gamma_2^2 - 8\gamma_1\gamma_3 + 4(-\gamma_1 e_3(t) + \gamma_2 e_2(t) - \gamma_3 e_1(t)) + 4\gamma_4 - 4\omega\\
&= 4\Big(\sum_{j=1}^{4}(-1)^j\gamma_j(\gamma_{4-j} + e_{4-j}(t)) - \omega\Big),\\
0 &= -4\gamma_3^2 - 4\gamma_3 e_3(t) + 4\gamma_2(e_4(t) + 2\gamma_4 + \omega) + 2e_2(t)(2\gamma_4 + \omega) + (e_1(t)^2 - 2e_2(t))\omega\\
&= 4\Big(\sum_{j=2}^{4}(-1)^j\gamma_j(\gamma_{6-j} + e_{6-j}(t)) + (\gamma_2 + \gamma^2)\omega\Big),\\
0 &= (2e_4(t) + 2\gamma_4 + \omega)(2\gamma_4 + \omega) + (3\omega^2 - 2(e_4(\rho t) - e_4(\rho^2 t))\omega)\\
&= 4\Big(\gamma_4(\gamma_4 + e_4(t)) + \omega^2 + (\gamma_4 - e_4(\rho t))\omega\Big).
\end{aligned}
$$

Regarding GKM functions as elements of $\mathrm{Map}(W(G), H^*(BT)\otimes\mathbb{Q})$, we can divide them by 4 to obtain

$$
\begin{aligned}
0 &= \sum_{j=1}^{2}(-1)^j\gamma_j(\gamma_{2-j} + e_{2-j}(t)), & 0 &= \sum_{j=1}^{4}(-1)^j\gamma_j(\gamma_{4-j} + e_{4-j}(t)) - \omega,\\
0 &= \sum_{j=2}^{4}(-1)^j\gamma_j(\gamma_{6-j} + e_{6-j}(t)) + (\gamma_2 + \gamma^2)\omega, & 0 &= \gamma_4(\gamma_4 + e_4(t)) + \omega^2 + (\gamma_4 - e_4(\rho t))\omega.
\end{aligned}
$$

Since the right-hand sides of these equations remain to be polynomials in $H^*(BT)$-valued GKM functions over $\mathbb{Z}$, these equations hold in $H^*(\mathcal{F}_4) \subset \mathrm{Map}(W(G), H^*(BT))$. □

7 Proof of Lemma 4.2

We will prove Lemma 4.2 by the argument of regular sequences.

Definition 7.1. A sequence $a_1, \dots a_n$ of elements of a ring R is called regular if, for any i, a_i is not a zero divisor in $R/(a_1, \dots, a_{i-1})$.

The following theorems and propositions are useful. Proposition 7.1 and 7.2 are obvious by definition.

Proposition 7.1. *If $a_1, \dots, a_n$ is a regular sequence, then so is $a_1, \dots, a_{i-1}, a_i + b, a_{i+1}, \dots, a_n$ for $1 \le i \le n$ and any $b \in (a_1, \dots, a_{i-1})$.*

Proposition 7.2. *If $a_1, \dots, a_n$ is a regular sequence, then so is $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$ for $1 \le i \le n$.*

Theorem 7.1 ([M, Theorem 16.1]). *If $a_1, \dots, a_n$ is a regular sequence, then so is $a_1^{v_1}, \dots, a_n^{v_n}$ for any positive integers $v_1, \dots, v_n$.*

Theorem 7.2 ([M, Corollary of Theorem 16.3]). *Let A be a Noetherian ring and non-negatively graded. If $a_1, \dots, a_n$ is a regular sequence in A and each a_i is homogeneous of positive degree, then any permutation of $a_1, \dots, a_n$ is again a regular sequence.*

Theorem 7.3 (cf. [NS, Theorem 5.5.1]). *Let F be a field and $R = F[g_i \mid 1 \leq i \leq m]$ a non-negatively graded polynomial ring with $|g_i| > 0$ for any $1 \leq i \leq m$. Assume that $a_1, \dots a_n$ is a regular sequence in R, which consists of homogeneous elements of positive degree. Then the Poincaré series of $R/(a_i \mid 1 \leq i \leq n)$ is given as*

$$\frac{\prod_{i=1}^{n}(1 - x^{|a_i|})}{\prod_{i=1}^{m}(1 - x^{|g_i|})}.$$

Proof. For a non-negatively graded F-module M of finite type, let $P(M, x)$ denote the Poincaré series of M, namely

$$P(M, x) = \sum_{n=0}^{\infty}(\dim_F M_n)x^n,$$

where M_n denotes the degree n part of M. Then obviously we have

$$P(R, x) = \frac{1}{\prod_{i=1}^{m}(1 - x^{|g_i|})}.$$

Since $a_1, \dots a_n$ is a regular sequence, the multiplication by a_i induces an injection on a graded F-module $R/(a_1, \dots, a_{i-1})$. Therefore

$$P(R/(a_1, \dots, a_i), x) = (1 - x^{|a_i|})P(R/(a_1, \dots, a_{i-1}), x).$$

The induction on i completes the proof. □

Proof of Lemma 4.2. Let p be a prime number and

$$M = (\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \leq i \leq 4]/\{r_1', R_i, r_{2i}, r_{12} \mid 1 \leq i \leq 4\}),$$

where $|t_i| = 2$, $|\gamma_i| = 2i$, $|\omega| = 4$. We will show that the Poincaré series of $M \otimes (\mathbb{Z}/p\mathbb{Z})$ does not depend on p. Then the graded $\mathbb{Z}$-module M of finite type must be free. The relations (4.9) and (4.10) say that

$$\gamma_2 = \gamma_1(\gamma_1 + e_1(t)), \qquad \gamma_4 = -\left(\sum_{j=1}^{3}(-1)^j\gamma_j(\gamma_{4-j} + e_{4-j}(t)) - \omega\right),$$

and then we can erase γ_2 and γ_4. Let R denote the polynomial ring $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_1, \gamma_3, \omega \mid 1 \leq i \leq 4]$, r_1', R_i's, and r_i's also denote the corresponding elements of R, and I denote the ideal generated by $\{r_1', R_i, r_6, r_8, r_{12} \mid 1 \leq i \leq 4\}$ in R. Since $M \cong R/I$, it is sufficient to compute the Poincaré series of $(R/I) \otimes (\mathbb{Z}/p\mathbb{Z})$.

When $p=2$, we show that that the sequence

$$r_1', r_{12}, R_4, R_3, R_2, R_1, r_6, r_8$$

is regular and compute the Poincaré series from this sequence. In $(R/I)\otimes(\mathbb{Z}/2\mathbb{Z})$, we have

$$\begin{aligned}
&r_1' = e_1(t), && \\
&R_1 = -(e_1(\tau)-e_1(t)), && R_2 = -(e_2(\tau)-e_2(t)) + (e_1(\tau)-e_1(t))e_1(t), \\
&R_3 = -(e_3(\tau)-e_3(t)), && R_4 = e_4(\tau)-e_4(t)-\omega, \\
&r_6 \equiv \gamma_3^2 \pmod{\gamma, e_i(t), \omega \mid 1\le i\le 4}, && r_8 \equiv \gamma_4^2 \equiv \gamma_2^2 \equiv \gamma_1^8 \pmod{\gamma, e_i(t), \omega \mid 1\le i\le 4}.
\end{aligned}$$

It is well known that the sequence of the elementary symmetric polynomials

$$e_1(x), e_2(x), \dots, e_n(x),$$

that is, the sequence of the Chern classes is regular in $(\mathbb{Z}/p\mathbb{Z})[x_i \mid 1\le i\le n]$ for any prime p. Since a polynomial ring over a field is Noetherian, by Theorem 7.2, the sequence

$$\gamma, e_1(t), e_2(t), e_3(t), e_4(t), \omega, e_4(\tau), e_3(\tau), e_2(\tau), e_1(\tau), \gamma_3^2, \gamma_1^8$$

is regular in $R\otimes(\mathbb{Z}/2\mathbb{Z})$. We modify this sequence by Theorem 7.1 and Proposition 7.1 to obtain the following regular sequence.

$$\gamma, r_1', e_2(t), e_3(t), e_4(t), \omega^3, R_4, R_3, R_2, R_1, r_6, r_8$$

Since $\rho^2 t_4 = -\gamma$ and $e_4(\rho t) = -e_4(t) - e_4(\rho^2 t) \equiv 0 \bmod (\gamma, e_4(t))$, by Proposition 7.1

$$\gamma, r_1', e_2(t), e_3(t), e_4(t), r_{12}, R_4, R_3, R_2, R_1, r_6, r_8$$

is a regular sequence. Hence

$$r_1', r_{12}, R_4, R_3, R_2, R_1, r_6, r_8$$

is a regular sequence by Proposition 7.2. Finally, the Poincaré series of $(R/I)\otimes(\mathbb{Z}/2\mathbb{Z})$ is calculated from the degrees of the generators and the relations by Theorem 7.3, and we have

$$P(M\otimes(\mathbb{Z}/2\mathbb{Z}), x) = \frac{1}{(1-x^2)^4}(1+x^8+x^{16})\prod_{i=1}^{4}\frac{1-x^{4i}}{1-x^2}.$$

Next let us consider the case where $p\ge 3$. Let e_1, e_2, e_3, and e_4 be the left-hand sides of (6.1), (6.2), (6.3), and (6.4) respectively, namely

$$\begin{aligned}
&e_1 = e_1(\tau^2) - e_2(t^2), && e_2 = e_2(\tau^2) - e_2(t^2) - 6\omega, \\
&e_3 = e_3(\tau^2) - e_2(t^2) - e_1(t^2)\omega, && e_4 = e_4(\tau^2) - e_4(t^2) + 3\omega^2 - 2(e_4(\rho t) - e_4(\rho^2 t))\omega.
\end{aligned}$$

Recall that e_1, e_2, e_3, and e_4 are divided by 4 to yield r_2, r_4, r_6, and r_8 respectively. We have

$$\begin{aligned}
M\otimes(\mathbb{Z}/p\mathbb{Z}) &\cong \Big(\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1\le i\le 4]/(r_1', R_i, e_{2i}, r_{12} \mid 1\le i\le 4)\Big)\otimes(\mathbb{Z}/p\mathbb{Z}) \\
&\cong \Big(\mathbb{Z}[t_i, \gamma, \tau_i, \omega \mid 1\le i\le 4]/(r_1', e_{2i}, r_{12} \mid 1\le i\le 4)\Big)\otimes(\mathbb{Z}/p\mathbb{Z}).
\end{aligned}$$

since 2 is invertible in $\mathbb{Z}/p\mathbb{Z}$. We will show the sequence

$$r_1', r_{12}, e_8, e_6, e_4, e_2$$

is a regular sequence. It is well known that the sequence of elementary symmetric polynomial in $\{x_i^2\}_{i=1}^n$

$$e_1(x^2), e_2(x^2), \ldots, e_n(x^2),$$

that is, the sequence of the Pontryagin classes is regular in $(\mathbb{Z}/p\mathbb{Z})[x_i \mid 1 \leq i \leq n]$ for any prime p. By Theorem 7.2, the sequence

$$\gamma, e_1(t), e_2(t), e_3(t), e_4(t), \omega, e_4(\tau^2), e_3(\tau^2), e_2(\tau^2), e_1(\tau^2)$$

is regular in $(\mathbb{Z}/p\mathbb{Z})[t_i, \gamma, \tau_i, \omega \mid 1 \leq i \leq 4]$. We modify this sequence by Theorem 7.1 and Proposition 7.1 to obtain the following regular sequence.

$$\gamma, r_1', e_2(t), e_3(t), e_4(t), r_{12}, e_8, e_6, e_4, e_2$$

Hence

$$r_1', r_{12}, e_8, e_6, e_4, e_2$$

is a regular sequence by Proposition 7.2. Therefore, by Theorem 7.3, we have

$$P(M \otimes (\mathbb{Z}/p\mathbb{Z}), x) = \frac{1}{(1-x^2)^4}(1 + x^8 + x^{16}) \prod_{i=1}^{4} \frac{1 - x^{4i}}{1 - x^2}.$$

□

8 Proof of Corollary 1.1

Proof. By the argument in Section 1 we have the isomorphisms

$$\begin{aligned} H^*(F_4/T) &\cong H_T^*(F_4/T)/(t_1, t_2, t_3, t_4, \gamma) \\ &\cong \mathbb{Z}[\tau_i, \gamma_i, \omega \mid 1 \leq i \leq 4]/(Q_i, q_{2i}, q_{12} \mid 1 \leq i \leq 4), \end{aligned}$$

where

$$\begin{aligned} &Q_i = e_i(\tau) - 2\gamma_i \ \ (i = 1, 2, 3), && Q_4 = e_4(\tau) - 2\gamma_4 - \omega, \\ &q_2 = \gamma_2 - \gamma_1^2, && q_4 = \gamma_4 - 2\gamma_1\gamma_3 + \gamma_2^2 - \omega, \\ &q_6 = 2\gamma_2\gamma_4 - \gamma_3^2 + \gamma_2\omega, && q_8 = \gamma_4^2 + \gamma_4\omega + \omega^2, \\ &q_{12} = \omega^3. \end{aligned}$$

We can regard γ_2 and γ_4 as dependent variables by the relations q_2 and q_4. Let R be the polynomial ring $\mathbb{Z}[\tau_i, \gamma_1, \gamma_3, \omega \mid 1 \leq i \leq 4]$. Then

$$H^*(F_4/T) \cong R/(Q_i, q_6, q_8, q_{12} \mid 1 \leq i \leq 4).$$

Obviously we have

$$Q_i = -\overline{r}_i \ \ (i = 1,2,3), \qquad Q_4 \equiv \overline{r}_4 \pmod{Q_3},$$
$$q_6 \equiv \gamma_2 e_4(\tau) - \gamma_3^2 = -\overline{r}_6 \pmod{Q_4}, \qquad q_{12} = \overline{r}_{12}.$$

Moreover we have

$$\begin{aligned} q_8 &= 4\gamma_1^2\gamma_3^2 - 4\gamma_1^5\gamma_3 + \gamma_1^8 + 3\omega(\omega + 2\gamma_1\gamma_3) - 3\gamma_1^4\omega \\ &\equiv 8\gamma_1^4\gamma_4 + 4\gamma_1^4\omega - 4\gamma_1^5\gamma_3 + \gamma_1^8 + 3\omega(\omega + 2\gamma_1\gamma_3) - 3\gamma_1^4\omega \pmod{q_6} \\ &= 12\gamma_1^5\gamma_3 - 7\gamma_1^8 + 9\gamma_1^4\omega + 3\omega(\omega + 2\gamma_1\gamma_3) - 3\gamma_1^4\omega. \end{aligned}$$

On the other hand,

$$\begin{aligned} \overline{r}_8 &= 3e_4(\tau)\gamma_1^4 - \gamma_1^8 + 3\omega(\omega + e_3(\tau)\gamma_1) \\ &\equiv 3(2\gamma_4 + \omega)\gamma_1^4 - \gamma_1^8 + 3\omega(\omega + 2\gamma_1\gamma_3) \pmod{Q_3, Q_4} \\ &= 12\gamma_1^5\gamma_3 - 7\gamma_1^8 + 9\gamma_1^4\omega + 3\omega(\omega + 2\gamma_1\gamma_3) - 3\gamma_1^4\omega. \end{aligned}$$

Hence $q_8 \equiv \overline{r}_8 \pmod{q_6, Q_3, Q_4}$. Therefore

$$H^*(F_4/T) \cong R/(Q_i, q_6, q_8, q_{12} \mid 1 \leq i \leq 4) \cong R/(\overline{r}_1, \overline{r}_2, \overline{r}_3, \overline{r}_4, \overline{r}_6, \overline{r}_8, \overline{r}_{12}).$$

□

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