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<th>Some group-theoretic aspects of outer Galois representations associated to hyperbolic curves</th>
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<tr>
<td>Issue Date</td>
<td>2015-03-23</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.14989/doctor.k18769">https://doi.org/10.14989/doctor.k18769</a></td>
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<tr>
<td>Rights</td>
<td>Degree regulations Article 9 Paragraph 2 applies to summary publication</td>
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<tr>
<td>Type</td>
<td>Thesis or Dissertation</td>
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This article is a summary of the paper [9], entitled “Some group-theoretic aspects of outer Galois representations associated to hyperbolic curves”. In the paper [9], we discuss some group-theoretic properties of outer Galois representations associated to hyperbolic curves. First, recall the notion of outer Galois representations associated to algebraic varieties. Let \( l \) be a prime number, \( \Sigma \) a set of prime numbers which contains \( l \), \((g, r)\) a pair of nonnegative integers such that \( 2g - 2 + r > 0 \), \( k \) a field of characteristic zero, \( \overline{k} \) an algebraic closure of \( k \), and \( X \) a scheme of finite type, separated, and geometrically connected over \( k \). Write \( G_k := \text{Gal}(\overline{k}/k) \), \( \Delta^\Sigma_X \) for the pro-\( \Sigma \) geometric étale fundamental group of \( X \), i.e., the maximal pro-\( \Sigma \) quotient of the étale fundamental group \( \pi_1(X \otimes \overline{k}) \) of \( X \otimes_k \overline{k} \),

\[
\rho_X^\Sigma : G_k \rightarrow \text{Out}(\Delta^\Sigma_X)
\]

for the pro-\( \Sigma \) outer Galois representation associated to \( X \), \((M_{g,r})_k\) for the moduli stack of \( r \)-pointed smooth proper curves of genus \( g \) over \( k \) whose \( r \) marked points are equipped with an ordering, \( \Delta^\Sigma_{g,r} \) for the pro-\( \Sigma \) completion of the (topological) fundamental group of a topological space obtained by removing \( r \) distinct points from a connected orientable compact topological surface of genus \( g \), and

\[
\rho_{g,r/k}^\Sigma : \pi_1((M_{g,r})_k) \rightarrow \text{Out}(\Delta^\Sigma_{g,r})
\]

for the universal pro-\( \Sigma \) outer monodromy representation of \((M_{g,r})_k\). In the paper [9], we consider the following questions in the case where \( X \) is a hyperbolic curve:

(Q1) Which property of \( \rho_X^\Sigma \) has a group-theoretic generalization?
(Q2) Which group-theoretic properties does \( \rho_X^\Sigma \) have?

In Chapter 1 of the paper [9], we discuss question (Q1), in particular, a group-theoretic generalization of a splitting of the natural outer surjection

\[
\pi_1((M_{g,r})_\mathbb{Q}) \twoheadrightarrow G_\mathbb{Q}
\]

determined by the structure morphism \((M_{g,r})_\mathbb{Q} \rightarrow \text{Spec } \mathbb{Q}\). More precisely, let \( n \) be a positive integer. Write \( \Pi_n \) for the maximal pro-\( \Sigma \) quotient of the étale fundamental group of the \( n \)-th configuration space of a hyperbolic curve \( C \) of type \((g, r)\) over \( \mathbb{Q} \), \( \text{Out}^{\text{FC}}(\Pi_n) \) for the (closed) subgroup of \( \text{Out}(\Pi_n) \) consisting of FC-admissible automorphisms of \( \Pi_n \) (i.e., arising from automorphisms of \( \Pi_n \) that preserve the fiber subgroups of \( \Pi_n \) and the cuspidal inertia subgroups of the fiber subgroups (cf. [11, Definition 1.1,
and $T$ for a split tripod over $\mathbb{Q}$, i.e., $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$. Suppose that either $\Sigma$ contains all prime numbers or satisfies $\sharp(\Sigma) = 1$, and that

$$n \geq \begin{cases} 4 & \text{if } C \text{ is proper}, \\ 3 & \text{if } C \text{ is affine}. \end{cases}$$

Write $GT \subseteq \text{Out}(\Delta^r_T)$ for the pro-$\Sigma$ Grothendieck-Teichmüller group. Then it follows from [5, Theorem B] and the definitions involved that we obtain the following commutative diagram of profinite groups

$$\begin{array}{c}
\pi_1((\mathcal{M}_g,r)_\mathbb{Q}) & \longrightarrow & G_\mathbb{Q} \\
\rho^{\Sigma}_{g,r}/\mathbb{Q} & \downarrow & \downarrow \rho^T \\
\text{Out}^{FC}(\Pi_n) & \longrightarrow & GT \\
\text{Out}(\Delta^r_T) & \longrightarrow & \text{Out}(\Delta^T)
\end{array}$$

where the left-hand lower slanting arrow is a natural injection. Note that, by a well-known injectivity result of Belyi, if $\Sigma$ contains all prime numbers, then $\rho^T$ is injective, and, if, moreover, $g \leq 2$, then $\rho^T_{g,r}/\mathbb{Q}$ is also injective (cf. [1, Theorem 3A, and Theorem 5], [2, Theorem 2.5], [5, Corollary 6.5]). Thus, we shall regard $GT$ (respectively, $\text{Out}^{FC}(\Pi_n)$) as a group-theoretic approximation of $G_\mathbb{Q}$ (respectively, $\pi_1((\mathcal{M}_g,r)_\mathbb{Q})$). In [7], Hoshi and Mochizuki constructed the tripod homomorphism

$$\Xi: \text{Out}^{FC}(\Pi_n) \longrightarrow GT,$$

which is compatible with the natural outer surjection

$$\pi_1((\mathcal{M}_g,r)_\mathbb{Q}) \longrightarrow G_\mathbb{Q},$$

and proved that this homomorphism is surjective. This result may be regarded as a group-theoretic generalization of the fact that the natural outer homomorphism

$$\pi_1((\mathcal{M}_g,r)_\mathbb{Q}) \longrightarrow G_\mathbb{Q}$$

is surjective. In Chapter 1 of the paper [9], we prove the following result:

**Theorem A.** The tripod homomorphism

$$\Xi: \text{Out}^{FC}(\Pi_n) \longrightarrow GT$$

is split.

In particular, although $GT$ is a subgroup of $\text{Out}(\Delta^r_T)$, we obtain an outer $GT$-action on $\Pi_n$. Theorem A may be regarded as a group-theoretic generalization of the fact that the natural outer surjection

$$\pi_1((\mathcal{M}_g,r)_\mathbb{Q}) \longrightarrow G_\mathbb{Q}$$

is split.

In order to prove Theorem A, we also consider a variant of Theorem A, as follows: Here, we do not put the assumption that either $\Sigma$ contains all prime numbers or satisfies $\sharp(\Sigma) = 1$. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-$\Sigma$ PSC-type, i.e., roughly speaking, a system of the dual (semi-)graph of a
pointed stable curve $X$ over an algebraically closed field of characteristic zero and “the pro-$\Sigma$ completions” of Galois categories obtained from irreducible components of $X$, marked points of $X$, and nodes of $X$ (cf. [10, Definition 1.1, (i)]). For a vertex $v \in \text{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall denote by $\mathcal{G}|_v$ a certain semi-graph of anabelioids of pro-$\Sigma$ PSC-type with $\text{Vert}(\mathcal{G}|_v) = \{v\}$ obtained as [6, Definition 2.1, (iii)]. Write $\text{Aut}^{\text{graph}}(\mathcal{G})$ for the group of automorphisms of $\mathcal{G}$ which induce the identity automorphism on the underlying semi-graph of $\mathcal{G}$, and $\text{Glu}(\mathcal{G}) \subseteq \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{graph}}(\mathcal{G}|_v)$ for the closed subgroup of “glueable” collections of automorphisms of the direct product $\prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{graph}}(\mathcal{G}|_v)$ consisting of elements $(\alpha_v)_{v \in \text{Vert}(\mathcal{G})}$ such that the image of $\alpha_v$ in $(\hat{\mathbb{Z}}^\Sigma)^\times$ by the cyclotomic character does not depend on $v \in \text{Vert}(\mathcal{G})$ (cf. [6, Definition 4.9]). In [6], Hoshi and Mochizuki proved that the image of the natural homomorphism $\rho_{\text{Vert}}: \text{Aut}^{\text{graph}}(\mathcal{G}) \rightarrow \text{Glu}(\mathcal{G})$ is equal to $\text{Glu}(\mathcal{G})$. In Chapter 1 of the paper [9], we also prove the following result:

**Theorem B.** Let $\mathcal{G}$ be a totally degenerate semi-graph of anabelioids of pro-$\Sigma$ PSC-type. Then the surjection $\rho_{\text{Vert}}^*: \text{Aut}^{\text{graph}}(\mathcal{G}) \rightarrow \text{Glu}(\mathcal{G})$ is split.

The proof of Theorem B is as follows: First, we prove Theorem B in the case where the cardinality of the set of nodes of $\mathcal{G}$ is 1. Then we prove Theorem B for general $\mathcal{G}$ inductively, by resolving $\mathcal{G}$ and reducing to the case where the cardinality of the set of nodes of $\mathcal{G}$ is 1. By regarding Theorem B as a local version of Theorem A, Theorem A follows from Theorem B.

In Chapter 2 of the paper [9], we discuss question (Q2), in particular, group-theoretic properties of the images of pro-$\{l\}$ outer Galois representations associated to hyperbolic curves and universal pro-$\{l\}$ outer monodromy representations. (Although $\prod$ pro-$\{l\}$ $\prod$ is often written $\prod$ pro-$l$ $\prod$, since we also consider $\prod$ pro-$\Sigma$ $\prod$, we use this notation.) More precisely, let $C$ be a hyperbolic curve over $k$. Write $\rho_C^{\Sigma, \text{ab}}: G_k \rightarrow \text{Aut}((\Delta_{\Sigma}^C)^\text{ab})$ (resp. $\rho_{g,r,k}^{\Sigma, \text{ab}}: \pi_1((\mathcal{M}_{g,r})_k) \rightarrow \text{Aut}((\Delta_{g,r}^\Sigma)^\text{ab})$) for the homomorphism obtained from $\rho_C^{\Sigma}$ (respectively, $\rho_{g,r,k}^{\Sigma}$) and the maximal abelian quotient $\Delta_C^\Sigma \rightarrow (\Delta_{\Sigma}^C)^\text{ab}$ (respectively, $\Delta_{g,r}^\Sigma \rightarrow (\Delta_{g,r}^\Sigma)^\text{ab}$) of $\Delta_C^\Sigma$ (respectively, $\Delta_{g,r}^\Sigma$). Since $(\Delta_C^{\{l\}})^\text{ab}$ (respectively, $(\Delta_{g,r}^{\{l\}})^\text{ab}$) is a free $\mathbb{Z}_l$-module of finite rank, we may regard $\rho_C^{\{l\}, \text{ab}}$ (respectively, $\rho_{g,r,k}^{\{l\}, \text{ab}}$) as one of the most natural $l$-adic representation obtained from $\rho_C^{\{l\}}$ (respectively,
there exists a pair of an injection

\[ \text{im}(\rho_{\ell}) \to \text{im}(\rho_{\ell}) \]  

and \( \text{im}(\rho_{g,r/k}) \to \text{im}(\rho_{g,r/k}) \) injective? More generally, are profinite groups \( \text{im}(\rho_{\ell}) \) and \( \text{im}(\rho_{g,r/k}) \) \( \ell \)-adic Lie groups?

Note that Hoshi proved that, if \( k \) is a number field and \( C \) is proper, then the natural homomorphism \( \text{im}(\rho_{\ell}) \to \text{im}(\rho_{\ell}) \) is not an isomorphism (cf. [3, Corollary 1.3]). In Chapter 2 of the paper [9], we prove the following result:

**Theorem C.** Suppose that \( k \) is \( \ell \)-cyclotomically inertially full, i.e., there exists a pair of an injection \( \overline{\mathbb{Q}} \to \overline{k} \) and a prime \( \ell \) of \( \overline{\mathbb{Q}} \) over \( \ell \) such that the intersection of \( \text{im}(G_{k(\mu_{\infty})} \to G_{\mathbb{Q}(\mu_{\infty})}) \) and the inertia subgroup \( I_{1} \subseteq G_{\mathbb{Q}} \)
of \( \ell \) is an open subgroup of \( I_{1} \cap G_{\mathbb{Q}(\mu_{\infty})} \), where \( \mu_{\infty} \subseteq \overline{\mathbb{Q}} \) is the group of roots of \( \ell \)-power order of unity. Then \( \text{im}(\rho_{\ell}) \) is not an \( \ell \)-adic Lie group.

In particular, in this case, the natural homomorphism \( \text{im}(\rho_{\ell}) \to \text{im}(\rho_{\ell}) \) is not injective.

Since a number field is \( \ell \)-cyclotomically inertially full, Theorem C is a generalization of Hoshi’s result. Theorem C follows from analysis of the pro-\( \{l\} \) outer Galois representation associated to a split tripod, i.e., \( \mathbb{P}^{1} \setminus \{0, 1, \infty\} \), initiated by Ihara. Also, we prove a generalization of Theorem C for pro-\( \Sigma \) outer Galois representations.

As for \( \text{im}(\rho_{g,r/k}) \), we prove the following result:

**Theorem D.** Suppose that \( 3g - 3 + r > 0 \). Then the natural surjection

\[ \text{im}(\rho_{g,r/k}) \twoheadrightarrow \text{im}(\rho_{g,r/k}) \]

is not injective.

Suppose, moreover, that either \( (g, r) \neq (1, 1) \) or \( l = 2 \). Then \( \text{im}(\rho_{g,r/k}) \) is not an \( \ell \)-adic Lie group.

Note that, for Theorem D, \( k \) is not necessarily “arithmetic”. Nevertheless, our proof of Theorem D is based on a deep arithmetic phenomenon concerning the outer Galois representations associated to hyperbolic curves. For example, in order to prove Theorem D, we use a result concerning an outer Galois action on \( \text{im}(\rho_{g,r/k}) \) (cf. [8, Theorem 3.4]) and a result concerning the pro-\( \{l\} \) version of the Grothendieck conjecture. Also, note that, for a positive integer \( r \), Hoshi and the author proved that, if \( \ell \geq 11 \), then a pro-\( \{l\} \) version of the congruence subgroup problem of \( \pi_{1}(\mathcal{M}_{1,r}) \) has a negative answer, i.e., roughly speaking, for any open subgroup \( U \subseteq \pi_{1}(\mathcal{M}_{1,r}) \) of \( \pi_{1}(\mathcal{M}_{1,r}) \), the natural surjection \( U \to \rho_{1,r/k}(U) \) is not the maximal pro-\( \{l\} \) quotient of \( U \) (cf. [4, Corollary 4.10]). In particular, if \( r \geq 2 \) and \( \ell \geq 11 \), then, for any open subgroup \( U \subseteq \pi_{1}(\mathcal{M}_{1,r}) \) of \( \pi_{1}(\mathcal{M}_{1,r}) \), \( \rho_{1,r/k}(U) \) is neither an \( \ell \)-adic Lie group nor the maximal pro-\( \{l\} \) quotient of \( U \). Finally, we prove a partial generalization of Theorem D for universal pro-\( \Sigma \)
outer monodromy representations, and a corollary to Theorem D, which is a partial strengthening of Theorem C.

REFERENCES


