<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>外部群論的面の外的ガロア表現と双曲線関連の研読</td>
</tr>
<tr>
<td>著者</td>
<td>Iijima, Yu</td>
</tr>
<tr>
<td>引用</td>
<td>京都大学</td>
</tr>
<tr>
<td>発行日</td>
<td>2015-03-23</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.14989/doctor.k18769">https://doi.org/10.14989/doctor.k18769</a></td>
</tr>
<tr>
<td>資格</td>
<td>学位規則第9条第2項により要約公開</td>
</tr>
</tbody>
</table>

学位規則第9条第2項により要約公開
A SUMMARY OF “SOME GROUP-THEORETIC ASPECTS OF OUTER GALOIS REPRESENTATIONS ASSOCIATED TO HYPERBOLIC CURVES”

YU IIJIMA

This article is a summary of the paper [9], entitled “Some group-theoretic aspects of outer Galois representations associated to hyperbolic curves”. In the paper [9], we discuss some group-theoretic properties of outer Galois representations associated to hyperbolic curves. First, recall the notion of outer Galois representations associated to algebraic varieties. Let \( p \) be a prime number, \( \Sigma \) a set of prime numbers which contains \( p \), \((g, r)\) a pair of nonnegative integers such that \( 2g - 2 + r > 0 \), \( k \) a field of characteristic zero, \( \bar{k} \) an algebraic closure of \( k \), and \( X \) a scheme of finite type, separated, and geometrically connected over \( k \). Write \( G_k := \text{Gal}(k/k) \), \( \Delta_X^\Sigma \) for the pro-\( \Sigma \) geometric étale fundamental group of \( X \), i.e., the maximal pro-\( \Sigma \) quotient of the étale fundamental group \( \pi_1(X \otimes \bar{k}) \) of \( X \otimes_k \bar{k} \),

\[ \rho_X^\Sigma : G_k \to \text{Out}(\Delta_X^\Sigma) \]

for the pro-\( \Sigma \) outer Galois representation associated to \( X \), \((M_{g,r})_k \) for the moduli stack of \( r \)-pointed smooth proper curves of genus \( g \) over \( k \) whose \( r \) marked points are equipped with an ordering, \( \Delta_{g,r}^\Sigma \) for the pro-\( \Sigma \) completion of the (topological) fundamental group of a topological space obtained by removing \( r \) distinct points from a connected orientable compact topological surface of genus \( g \), and

\[ \rho_{g,r/k}^\Sigma : \pi_1((M_{g,r})_k) \to \text{Out}(\Delta_{g,r}^\Sigma) \]

for the universal pro-\( \Sigma \) outer monodromy representation of \((M_{g,r})_k \). In the paper [9], we consider the following questions in the case where \( X \) is a hyperbolic curve:

(Q1) Which property of \( \rho_X^\Sigma \) has a group-theoretic generalization?

(Q2) Which group-theoretic properties does \( \rho_X^\Sigma \) have?

In Chapter 1 of the paper [9], we discuss question (Q1), in particular, a group-theoretic generalization of a splitting of the natural outer surjection

\[ \pi_1((M_{g,r})_\mathbb{Q}) \to G_\mathbb{Q} \]

determined by the structure morphism \((M_{g,r})_\mathbb{Q} \to \text{Spec} \mathbb{Q} \). More precisely, let \( n \) be a positive integer. Write \( \Pi_n \) for the maximal pro-\( \Sigma \) quotient of the étale fundamental group of the \( n \)-th configuration space of a hyperbolic curve \( C \) of type \((g, r)\) over \( \mathbb{Q} \), \( \text{Out}^\text{FC}(\Pi_n) \) for the (closed) subgroup of \( \text{Out}(\Pi_n) \) consisting of \( \text{FC} \)-admissible automorphisms of \( \Pi_n \) (i.e., arising from automorphisms of \( \Pi_n \) that preserve the fiber subgroups of \( \Pi_n \) and the cuspidal inertia subgroups of the fiber subgroups (cf. [11, Definition 1.1,
(ii))), and \( T \) for a \textit{split tripod} over \( \mathbb{Q} \), i.e., \( \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} \). Suppose that either \( \Sigma \) contains all prime numbers or satisfies \( \#(\Sigma) = 1 \), and that

\[
   n \geq \begin{cases} 
   4 & \text{if } C \text{ is proper}, \\
   3 & \text{if } C \text{ is affine}.
   \end{cases}
\]

Write \( G_T \subseteq \text{Out}(\Delta^T_{\Sigma}) \) for the pro-\( \Sigma \) Grothendieck-Teichmüller group. Then it follows from [5, Theorem B] and the definitions involved that we obtain the following commutative diagram of profinite groups

\[
\begin{array}{c}
\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \ar[r] & G_{\mathbb{Q}} \\
\ar[u] & & \\
\ar[u] & & \\
\text{Out}^{\mathbb{C}F}(\Pi_n) \ar[r] & G_T \\
\ar[u] & & \\
\ar[u] & & \\
\text{Out}(\Delta^T_{\Sigma}) \ar[u] & & G_T \\
\end{array}
\]

where the left-hand lower slanting arrow is a natural injection. Note that, by a well-known \textit{injectivity result of Belyi}, if \( \Sigma \) contains all prime numbers, then \( \rho^T_{\Sigma} \) is injective, and, if, moreover, \( g \leq 2 \), then \( \rho^T_{g,r/\mathbb{Q}} \) is also injective (cf. [1, Theorem 3A, and Theorem 5], [2, Theorem 2.5], [5, Corollary 6.5]). Thus, we shall regard \( G_T \) (respectively, \( \text{Out}^{\mathbb{C}F}(\Pi_n) \)) as a \textit{group-theoretic approximation} of \( G_{\mathbb{Q}} \) (respectively, \( \pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \)). In [7], Hoshi and Mochizuki constructed the \textit{tripod homomorphism}

\[
\Xi : \text{Out}^{\mathbb{C}F}(\Pi_n) \to G_T,
\]

which is compatible with the natural outer surjection

\[
\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \to G_{\mathbb{Q}},
\]

and proved that this homomorphism is \textit{surjective}. This result may be regarded as a \textit{group-theoretic generalization} of the fact that the natural outer homomorphism

\[
\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \to G_{\mathbb{Q}}
\]

is surjective. In Chapter 1 of the paper [9], we prove the following result:

\textbf{Theorem A.} The \textit{tripod homomorphism}

\[
\Xi : \text{Out}^{\mathbb{C}F}(\Pi_n) \to G_T
\]

is \textit{split}.

In particular, although \( G_T \) is a subgroup of \( \text{Out}(\Delta^T_{\Sigma}) \), we obtain an outer \( G_T \)-action on \( \Pi_n \). Theorem A may be regarded as a \textit{group-theoretic generalization} of the fact that the natural outer surjection

\[
\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \to G_{\mathbb{Q}}
\]

is \textit{split}.

In order to prove Theorem A, we also consider a variant of Theorem A, as follows: Here, we do not put the assumption that either \( \Sigma \) contains all prime numbers or satisfies \( \#(\Sigma) = 1 \). Let \( \mathcal{G} \) be a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type, i.e., roughly speaking, a system of the dual (semi-)graph of a
pointed stable curve $X$ over an algebraically closed field of characteristic zero and “the pro-$\Sigma$ completions” of Galois categories obtained from irreducible components of $X$, marked points of $X$, and nodes of $X$ (cf. [10, Definition 1.1, (i)]). For a vertex $v \in \text{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall denote by $\mathcal{G}|_v$ a certain semi-graph of anabelioids of pro-$\Sigma$ PSC-type with $\text{Vert}(\mathcal{G}|_v) = \{v\}$ obtained as [6, Definition 2.1, (iii)]. Write

$$\text{Aut}^{\text{graph}}(\mathcal{G})$$

for the group of automorphisms of $\mathcal{G}$ which induce the identity automorphism on the underlying semi-graph of $\mathcal{G}$, and

$$\text{Glu}(\mathcal{G}) \subseteq \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{graph}}(\mathcal{G}|_v)$$

for the closed subgroup of “glueable” collections of automorphisms of the direct product $\prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{graph}}(\mathcal{G}|_v)$ consisting of elements $(\alpha_v)_{v \in \text{Vert}(\mathcal{G})}$ such that the image of $\alpha_v$ in $\hat{\mathbb{Z}}^\Sigma \times$ by the cyclotomic character does not depend on $v \in \text{Vert}(\mathcal{G})$ (cf. [6, Definition 4.9]). In [6], Hoshi and Mochizuki proved that the image of the natural homomorphism

$$\rho_{\text{Vert}}^{\mathcal{G}} : \text{Aut}^{\text{graph}}(\mathcal{G}) \rightarrow \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{graph}}(\mathcal{G}|_v)$$

is equal to $\text{Glu}(\mathcal{G})$. In Chapter 1 of the paper [9], we also prove the following result:

**Theorem B.** Let $\mathcal{G}$ be a totally degenerate semi-graph of anabelioids of pro-$\Sigma$ PSC-type. Then the surjection

$$\rho_{\text{Vert}}^{\mathcal{G}} : \text{Aut}^{\text{graph}}(\mathcal{G}) \rightarrow \text{Glu}(\mathcal{G})$$

is split.

The proof of Theorem B is as follows: First, we prove Theorem B in the case where the cardinality of the set of nodes of $\mathcal{G}$ is 1. Then we prove Theorem B for general $\mathcal{G}$ inductively, by resolving $\mathcal{G}$ and reducing to the case where the cardinality of the set of nodes of $\mathcal{G}$ is 1. By regarding Theorem B as a local version of Theorem A, Theorem A follows from Theorem B.

In Chapter 2 of the paper [9], we discuss question (Q2), in particular, group-theoretic properties of the images of pro-$\{l\}$ outer Galois representations associated to hyperbolic curves and universal pro-$\{l\}$ outer monodromy representations. (Although $\Gamma$ pro-$\{l\}$ $\Gamma$ is often written $\Gamma$ pro-$l$ $\Gamma$, since we also consider $\Gamma$ pro-$\Sigma$ $\Gamma$, we use this notation.) More precisely, let $C$ be a hyperbolic curve over $k$. Write

$$\rho_C^{\Sigma,\text{ab}} : G_k \rightarrow \text{Aut}((\Delta_C^\Sigma)^{\text{ab}})$$

(resp. $\rho_{g,r,k}^{\Sigma,\text{ab}} : \pi_1((M_{g,r})_k) \rightarrow \text{Aut}((\Delta_{g,r}^\Sigma)^{\text{ab}})$)

for the homomorphism obtained from $\rho_C^{\Sigma}$ (respectively, $\rho_{g,r,k}^{\Sigma}$) and the maximal abelian quotient $\Delta_C^\Sigma \twoheadrightarrow (\Delta_C^\Sigma)^{\text{ab}}$ (respectively, $\Delta_{g,r}^\Sigma \twoheadrightarrow (\Delta_{g,r}^\Sigma)^{\text{ab}}$) of $\Delta_C^\Sigma$ (respectively, $\Delta_{g,r}^\Sigma$). Since $(\Delta_C^{\{l\}})^{\text{ab}}$ (respectively, $(\Delta_{g,r}^{\{l\}})^{\text{ab}}$) is a free $\mathbb{Z}_l$-module of finite rank, we may regard $\rho_{\Sigma}^{\{l\},\text{ab}}$ (respectively, $\rho_{g,r,k}^{\{l\},\text{ab}}$) as one of the most natural $l$-adic representation obtained from $\rho_C^{\{l\}}$ (respectively,
\(\rho^{(l)}_{g,r/k}\). In Chapter 2 of the paper [9], we discuss the following question concerning question (Q2):

Are the natural homomorphisms \(\text{im}(\rho^{(l)}_C) \to \text{im}(\rho^{(l)}_{\text{ab}})\) and \(\text{im}(\rho^{(l)}_{g,r/k}) \to \text{im}(\rho^{(l)}_{\text{ab}})\) injective? More generally, are profinite groups \(\text{im}(\rho^{(l)}_C)\) and \(\text{im}(\rho^{(l)}_{g,r/k})\) \(l\)-adic Lie groups?

Note that Hoshi proved that, if \(k\) is a number field and \(C\) is proper, then the natural homomorphism \(\text{im}(\rho^{(l)}_C) \to \text{im}(\rho^{(l)}_{\text{ab}})\) is not an isomorphism (cf. [3, Corollary 1.3]). In Chapter 2 of the paper [9], we prove the following result:

**Theorem C.** Suppose that \(k\) is \(l\)-cyclotomically inertially full, i.e., there exists a pair of an injection \(\overline{\mathbb{Q}} \hookrightarrow \overline{k}\) and a prime \(l\) of \(\overline{\mathbb{Q}}\) over \(l\) such that the intersection of \(\text{im}(G_{k(\mu^\infty)} \to G_{\overline{\mathbb{Q}}(\mu^\infty)})\) and the inertia subgroup \(I_l \subseteq G_{\overline{\mathbb{Q}}}\) of \(l\) is an open subgroup of \(I_l \cap G_{\overline{\mathbb{Q}}(\mu^\infty)}\), where \(\mu^\infty \subseteq \overline{\mathbb{Q}}\) is the group of roots of \(l\)-power order of unity. Then \(\text{im}(\rho^{(l)}_C)\) is not a \(l\)-adic Lie group.

In particular, in this case, the natural homomorphism \(\text{im}(\rho^{(l)}_C) \to \text{im}(\rho^{(l)}_{\text{ab}})\) is not injective.

Since a number field is \(l\)-cyclotomically inertially full, Theorem C is a generalization of Hoshi’s result. Theorem C follows from analysis of the pro-\(\{l\}\) outer Galois representation associated to a split tripod, i.e., \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\), initiated by Ihara. Also, we prove a generalization of Theorem C for pro-\(\Sigma\) outer Galois representations.

As for \(\text{im}(\rho^{(l)}_{g,r/k})\), we prove the following result:

**Theorem D.** Suppose that \(3g - 3 + r > 0\). Then the natural surjection

\[\text{im}(\rho^{(l)}_{g,r/k}) \twoheadrightarrow \text{im}(\rho^{(l)}_{\text{ab}})\]

is not injective.

Suppose, moreover, that either \((g,r) \neq (1,1)\) or \(l = 2\). Then \(\text{im}(\rho^{(l)}_{g,r/k})\) is not an \(l\)-adic Lie group.

Note that, for Theorem D, \(k\) is not necessarily “arithmetic”. Nevertheless, our proof of Theorem D is based on a deep arithmetic phenomenon concerning the outer Galois representations associated to hyperbolic curves. For example, in order to prove Theorem D, we use a result concerning an outer Galois action on \(\text{im}(\rho^{(l)}_{g,r/k})\) (cf. [8, Theorem 3.4]) and a result concerning the pro-\(\{l\}\) version of the Grothendieck conjecture. Also, note that, for a positive integer \(r\), Hoshi and the author proved that, if \(l \geq 11\), then a pro-\(\{l\}\) version of the congruence subgroup property of \(\pi_1((\mathcal{M}_{1,r})_{\overline{k}})\) has a negative answer, i.e., roughly speaking, for any open subgroup \(U \subseteq \pi_1((\mathcal{M}_{1,r})_{\overline{k}})\) of \(\pi_1((\mathcal{M}_{1,r})_{\overline{k}})\), the natural surjection \(U \to \rho^{(l)}_{1,r/k}(U)\) is not the maximal pro-\(\{l\}\) quotient of \(U\) (cf. [4, Corollary 4.10]). In particular, if \(r \geq 2\) and \(l \geq 11\), then, for any open subgroup \(U \subseteq \pi_1((\mathcal{M}_{1,r})_{\overline{k}})\) of \(\pi_1((\mathcal{M}_{1,r})_{\overline{k}})\), \(\rho^{(l)}_{1,r/k}(U)\) is neither an \(l\)-adic Lie group nor the maximal pro-\(\{l\}\) quotient of \(U\). Finally, we prove a partial generalization of Theorem D for universal pro-\(\Sigma\)
outer monodromy representations, and a corollary to Theorem D, which is a partial strengthening of Theorem C.

REFERENCES


