<table>
<thead>
<tr>
<th>Title</th>
<th>Some group-theoretic aspects of outer Galois representations associated to hyperbolic curves</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Iijima, Yu</td>
</tr>
<tr>
<td>Citation</td>
<td>Kyoto University (京都大学)</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2015-03-23</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.14989/doctor.k18769">https://doi.org/10.14989/doctor.k18769</a></td>
</tr>
<tr>
<td>Right</td>
<td>学位規則第1条第2項により要約公開</td>
</tr>
<tr>
<td>Type</td>
<td>Thesis or Dissertation</td>
</tr>
<tr>
<td>Textversion</td>
<td>none</td>
</tr>
</tbody>
</table>
A SUMMARY OF “SOME GROUP-THEORETIC ASPECTS
OF OUTER GALOIS REPRESENTATIONS ASSOCIATED
TO HYPERBOLIC CURVES”

YU IIJIMA

This article is a summary of the paper [9], entitled “Some group-theoretic aspects of outer Galois representations associated to hyperbolic curves”. In the paper [9], we discuss some group-theoretic properties of outer Galois representations associated to hyperbolic curves. First, recall the notion of outer Galois representations associated to algebraic varieties. Let $l$ be a prime number, $\Sigma$ a set of prime numbers which contains $l$, $(g, r)$ a pair of nonnegative integers such that $2g - 2 + r > 0$, $k$ a field of characteristic zero, $\bar{k}$ an algebraic closure of $k$, and $X$ a scheme of finite type, separated, and geometrically connected over $k$. Write $G_k := \text{Gal}(\bar{k}/k)$, $\Delta^\Sigma_X$ for the pro-$\Sigma$ geometric étale fundamental group of $X$, i.e., the maximal pro-$\Sigma$ quotient of the étale fundamental group $\pi_1(X \otimes \bar{k})$ of $X \otimes_k \bar{k}$,

$$\rho^\Sigma_X : G_k \to \text{Out}(\Delta^\Sigma_X)$$

for the pro-$\Sigma$ outer Galois representation associated to $X$, $(\mathcal{M}_{g,r})_k$ for the moduli stack of $r$-pointed smooth proper curves of genus $g$ over $k$ whose $r$ marked points are equipped with an ordering, $\Delta^\Sigma_{g,r}$ for the pro-$\Sigma$ completion of the (topological) fundamental group of a topological space obtained by removing $r$ distinct points from a connected orientable compact topological surface of genus $g$, and

$$\rho^\Sigma_{g,r/k} : \pi_1((\mathcal{M}_{g,r})_k) \to \text{Out}(\Delta^\Sigma_{g,r})$$

for the universal pro-$\Sigma$ outer monodromy representation of $(\mathcal{M}_{g,r})_k$. In the paper [9], we consider the following questions in the case where $X$ is a hyperbolic curve:

(Q1) Which property of $\rho^\Sigma_X$ has a group-theoretic generalization?
(Q2) Which group-theoretic properties does $\rho^\Sigma_X$ have?

In Chapter 1 of the paper [9], we discuss question (Q1), in particular, a group-theoretic generalization of a splitting of the natural outer surjection

$$\pi_1((\mathcal{M}_{g,r})_\mathbb{Q}) \twoheadrightarrow G_\mathbb{Q}$$

determined by the structure morphism $(\mathcal{M}_{g,r})_\mathbb{Q} \to \text{Spec} \mathbb{Q}$. More precisely, let $n$ be a positive integer. Write $\Pi_n$ for the maximal pro-$\Sigma$ quotient of the étale fundamental group of the $n$-th configuration space of a hyperbolic curve $C$ of type $(g, r)$ over $\overline{\mathbb{Q}}$, $\text{Out}^{\text{FC}}(\Pi_n)$ for the (closed) subgroup of $\text{Out}(\Pi_n)$ consisting of FC-admissible automorphisms of $\Pi_n$ (i.e., arising from automorphisms of $\Pi_n$ that preserve the fiber subgroups of $\Pi_n$ and the cuspidal inertia subgroups of the fiber subgroups (cf. [11, Definition 1.1,}
(ii))), and $T$ for a \textit{split tripod} over $\mathbb{Q}$, i.e., $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Suppose that either $\Sigma$ contains all prime numbers or satisfies $\sharp(\Sigma) = 1$, and that

\[
 n \geq \begin{cases} 
 4 & \text{if } C \text{ is proper}, \\
 3 & \text{if } C \text{ is affine}.
\end{cases}
\]

Write $G_T \subseteq \text{Out}(\Delta^T_\Sigma)$ for the pro-$\Sigma$ Grothendieck–Teichmüller group. Then it follows from [5, Theorem B] and the definitions involved that we obtain the following commutative diagram of profinite groups

\[
\begin{array}{ccc}
\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) & \longrightarrow & G_{\mathbb{Q}} \\
\rho_{g,r/\mathbb{Q}}^\Sigma & \downarrow & \rho_T^\Sigma \\
\text{Out}^{\text{FC}}(\Pi_n) & \longrightarrow & G_T \\
\text{Out}(\Delta^T_\Sigma) & \longrightarrow & \text{Out}(\Delta^T_\Sigma)
\end{array}
\]

where the left-hand lower slanting arrow is a natural injection. Note that, by a well-known injectivity result of Belyi, if $\Sigma$ contains all prime numbers, then $\rho_T^\Sigma$ is injective, and, if, moreover, $g \leq 2$, then $\rho_{g,r/\mathbb{Q}}^\Sigma$ is also injective (cf. [1, Theorem 3A, and Theorem 5], [2, Theorem 2.5], [5, Corollary 6.5]). Thus, we shall regard $G_T$ (respectively, $\text{Out}^{\text{FC}}(\Pi_n)$) as a \textit{group-theoretic approximation} of $G_{\mathbb{Q}}$ (respectively, $\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}))$. In [7], Hoshi and Mochizuki constructed the \textit{tripod homomorphism}

$T : \text{Out}^{\text{FC}}(\Pi_n) \longrightarrow G_T$,

which is compatible with the natural outer surjection

$\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \longrightarrow G_{\mathbb{Q}}$,

and proved that this homomorphism is \textit{surjective}. This result may be regarded as a \textit{group-theoretic generalization} of the fact that the natural outer homomorphism

$\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \longrightarrow G_{\mathbb{Q}}$

is surjective. In Chapter 1 of the paper [9], we prove the following result:

\textbf{Theorem A.} The \textit{tripod homomorphism}

$T : \text{Out}^{\text{FC}}(\Pi_n) \longrightarrow G_T$

is \textit{split}.

In particular, although $G_T$ is a subgroup of $\text{Out}(\Delta^T_\Sigma)$, we obtain an outer $G_T$-action on $\Pi_n$. Theorem A may be regarded as a \textit{group-theoretic generalization} of the fact that the natural outer surjection

$\pi_1((\mathcal{M}_{g,r})_{\mathbb{Q}}) \longrightarrow G_{\mathbb{Q}}$

is \textit{split}.

In order to prove Theorem A, we also consider a variant of Theorem A, as follows: Here, we do not put the assumption that either $\Sigma$ contains all prime numbers or satisfies $\sharp(\Sigma) = 1$. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-$\Sigma$ PSC-type, i.e., roughly speaking, a system of the dual (semi-)graph of a
pointed stable curve $X$ over an algebraically closed field of characteristic zero and “the pro-$\Sigma$ completions” of Galois categories obtained from irreducible components of $X$, marked points of $X$, and nodes of $X$ (cf. [10, Definition 1.1, (i)]). For a vertex $v \in \text{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall denote by $\mathcal{G}|_v$ a certain semi-graph of anabelioids of pro-$\Sigma$ PSC-type with $\text{Vert}(\mathcal{G}|_v) = \{v\}$ obtained as [6, Definition 2.1, (iii)]. Write
\[ \text{Aut}^{\text{grph}}(\mathcal{G}) \]
for the group of automorphisms of $\mathcal{G}$ which induce the identity automorphism on the underlying semi-graph of $\mathcal{G}$, and
\[ \text{Glu}(\mathcal{G}) \subseteq \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{grph}}(\mathcal{G}|_v) \]
for the closed subgroup of “glueable” collections of automorphisms of the direct product $\prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{grph}}(\mathcal{G}|_v)$ consisting of elements $(\alpha_v)_{v \in \text{Vert}(\mathcal{G})}$ such that the image of $\alpha_v$ in $(\hat{\mathbb{Z}}^\Sigma)^\times$ by the cyclotomic character does not depend on $v \in \text{Vert}(\mathcal{G})$ (cf. [6, Definition 4.9]). In [6], Hoshi and Mochizuki proved that the image of the natural homomorphism $\rho_{\text{Vert}}: \text{Aut}^{\text{grph}}(\mathcal{G}) \rightarrow \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{grph}}(\mathcal{G}|_v)$ is equal to $\text{Glu}(\mathcal{G})$. In Chapter 1 of the paper [9], we also prove the following result:

**Theorem B.** Let $\mathcal{G}$ be a totally degenerate semi-graph of anabelioids of pro-$\Sigma$ PSC-type. Then the surjection $\rho_{\mathcal{G}}^{\text{Vert}}: \text{Aut}^{\text{grph}}(\mathcal{G}) \rightarrow \text{Glu}(\mathcal{G})$ is split.

The proof of Theorem B is as follows: First, we prove Theorem B in the case where the cardinality of the set of nodes of $\mathcal{G}$ is 1. Then we prove Theorem B for general $\mathcal{G}$ inductively, by resolving $\mathcal{G}$ and reducing to the case where the cardinality of the set of nodes of $\mathcal{G}$ is 1. By regarding Theorem B as a local version of Theorem A, Theorem A follows from Theorem B.

In Chapter 2 of the paper [9], we discuss question (Q2), in particular, group-theoretic properties of the images of pro-$\{l\}$ outer Galois representations associated to hyperbolic curves and universal pro-$\{l\}$ outer monodromy representations. (Although $\pro$ pro-$\{l\}$ $\pro$ is often written $\pro$ pro-$l$ $\pro$, since we also consider $\pro$ pro-$\Sigma$ $\pro$, we use this notation.) More precisely, let $C$ be a hyperbolic curve over $k$. Write
\[ \rho_C^{\Sigma,\text{ab}}: G_k \rightarrow \text{Aut}((\Delta_C^{\Sigma})^{\text{ab}}) \]
(respectively, $\rho_{g,r/k}^{\Sigma,\text{ab}}: \pi_1((\mathcal{M}_{g,r})_k) \rightarrow \text{Aut}((\Delta_{g,r}^{\Sigma})^{\text{ab}})$) for the homomorphism obtained from $\rho_C^{\Sigma}$ (respectively, $\rho_{g,r/k}^{\Sigma}$) and the maximal abelian quotient $\Delta_C^{\Sigma} \rightarrow (\Delta_C^{\Sigma})^{\text{ab}}$ (respectively, $\Delta_{g,r}^{\Sigma} \rightarrow (\Delta_{g,r}^{\Sigma})^{\text{ab}}$) of $\Delta_C^{\Sigma}$ (respectively, $\Delta_{g,r}^{\Sigma}$). Since $(\Delta_C^{\{l\}})^{\text{ab}}$ (respectively, $(\Delta_{g,r}^{\{l\}})^{\text{ab}}$) is a free $\mathbb{Z}_l$-module of finite rank, we may regard $\rho_C^{\{l\},\text{ab}}$ (respectively, $\rho_{g,r/k}^{\{l\},\text{ab}}$) as one of the most natural $l$-adic representation obtained from $\rho_C^{\{l\}}$ (respectively, $\rho_{g,r/k}^{\{l\}}$). In Chapter 2 of the paper [9], we discuss question (Q2), in particular, group-theoretic properties of the images of pro-$\{l\}$ outer Galois representations associated to hyperbolic curves and universal pro-$\{l\}$ outer monodromy representations. (Although $\pro$ pro-$\{l\}$ $\pro$ is often written $\pro$ pro-$l$ $\pro$, since we also consider $\pro$ pro-$\Sigma$ $\pro$, we use this notation.) More precisely, let $C$ be a hyperbolic curve over $k$. Write
\[ \rho_C^{\Sigma,\text{ab}}: G_k \rightarrow \text{Aut}((\Delta_C^{\Sigma})^{\text{ab}}) \]
(respectively, $\rho_{g,r/k}^{\Sigma,\text{ab}}: \pi_1((\mathcal{M}_{g,r})_k) \rightarrow \text{Aut}((\Delta_{g,r}^{\Sigma})^{\text{ab}})$) for the homomorphism obtained from $\rho_C^{\Sigma}$ (respectively, $\rho_{g,r/k}^{\Sigma}$) and the maximal abelian quotient $\Delta_C^{\Sigma} \rightarrow (\Delta_C^{\Sigma})^{\text{ab}}$ (respectively, $\Delta_{g,r}^{\Sigma} \rightarrow (\Delta_{g,r}^{\Sigma})^{\text{ab}}$) of $\Delta_C^{\Sigma}$ (respectively, $\Delta_{g,r}^{\Sigma}$). Since $(\Delta_C^{\{l\}})^{\text{ab}}$ (respectively, $(\Delta_{g,r}^{\{l\}})^{\text{ab}}$) is a free $\mathbb{Z}_l$-module of finite rank, we may regard $\rho_C^{\{l\},\text{ab}}$ (respectively, $\rho_{g,r/k}^{\{l\},\text{ab}}$) as one of the most natural $l$-adic representation obtained from $\rho_C^{\{l\}}$ (respectively,
In Chapter 2 of the paper [9], we discuss the following question concerning question (Q2):

Are the natural homomorphisms $\text{im}(\rho^{(l)}_C) \to \text{im}(\rho^{(l)}_{C,\text{ab}})$ and $\text{im}(\rho^{(l)}_{g,r/k}) \to \text{im}(\rho^{(l)}_{g,r/k,\text{ab}})$ injective? More generally, are profinite groups $\text{im}(\rho^{(l)}_C)$ and $\text{im}(\rho^{(l)}_{g,r/k})$ l-adic Lie groups?

Note that Hoshi proved that, if $k$ is a number field and $C$ is proper, then the natural homomorphism $\text{im}(\rho^{(l)}_C) \to \text{im}(\rho^{(l)}_{C,\text{ab}})$ is not an isomorphism (cf. [3, Corollary 1.3]). In Chapter 2 of the paper [9], we prove the following result:

**Theorem C.** Suppose that $k$ is l-cyclotomically inertially full, i.e., there exists a pair of an injection $\overline{\mathbb{Q}} \hookrightarrow \overline{k}$ and a prime $l$ of $\overline{\mathbb{Q}}$ over $l$ such that the intersection of $\text{im}(G_{k(\mu_{\infty})} \to G_{\overline{\mathbb{Q}}(\mu_{\infty})})$ and the inertia subgroup $I_l \subseteq G_{\overline{\mathbb{Q}}}$ of $l$ is an open subgroup of $I_l \cap G_{\overline{\mathbb{Q}}(\mu_{\infty})}$, where $\mu_{\infty} \subseteq \overline{\mathbb{Q}}$ is the group of roots of $l$-power order of unity. Then $\text{im}(\rho^{(l)}_C)$ is not an l-adic Lie group.

In particular, in this case, the natural homomorphism $\text{im}(\rho^{(l)}_C) \to \text{im}(\rho^{(l)}_{C,\text{ab}})$ is not injective.

Since a number field is l-cyclotomically inertially full, Theorem C is a generalization of Hoshi’s result. Theorem C follows from analysis of the pro-$\{l\}$ outer Galois representation associated to a split tripod, i.e., $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, initiated by Ihara. Also, we prove a generalization of Theorem C for pro-$\Sigma$ outer Galois representations.

As for $\text{im}(\rho^{(l)}_{g,r/k})$, we prove the following result:

**Theorem D.** Suppose that $3g - 3 + r > 0$. Then the natural surjection $\text{im}(\rho^{(l)}_{g,r/k}) \twoheadrightarrow \text{im}(\rho^{(l)}_{g,r/k,\text{ab}})$ is not injective.

Suppose, moreover, that either $(g, r) \neq (1, 1)$ or $l = 2$. Then $\text{im}(\rho^{(l)}_{g,r/k})$ is not an l-adic Lie group.

Note that, for Theorem D, $k$ is not necessarily “arithmetic”. Nevertheless, our proof of Theorem D is based on a deep arithmetic phenomenon concerning the outer Galois representations associated to hyperbolic curves. For example, in order to prove Theorem D, we use a result concerning an outer Galois action on $\text{im}(\rho^{(l)}_{g,r/k})$ (cf. [8, Theorem 3.4]) and a result concerning the pro-$\{l\}$ version of the Grothendieck conjecture. Also, note that, for a positive integer $r$, Hoshi and the author proved that, if $l \geq 11$, then a pro-$\{l\}$ version of the congruence subgroup problem of $\pi_1((\mathcal{M}_{1,r})_{\overline{k}})$ has a negative answer, i.e., roughly speaking, for any open subgroup $U \subseteq \pi_1((\mathcal{M}_{1,r})_{\overline{k}})$ of $\pi_1((\mathcal{M}_{1,r})_{\overline{k}})$, the natural surjection $U \to \rho^{(l)}_{1,r/k}(U)$ is not the maximal pro-$\{l\}$ quotient of $U$ (cf. [4, Corollary 4.10]). In particular, if $r \geq 2$ and $l \geq 11$, then, for any open subgroup $U \subseteq \pi_1((\mathcal{M}_{1,r})_{\overline{k}})$ of $\pi_1((\mathcal{M}_{1,r})_{\overline{k}})$, $\rho^{(l)}_{1,r/k}(U)$ is neither an l-adic Lie group nor the maximal pro-$\{l\}$ quotient of $U$. Finally, we prove a partial generalization of Theorem D for universal pro-$\Sigma$.
outer monodromy representations, and a corollary to Theorem D, which is a partial strengthening of Theorem C.

REFERENCES


