THESIS
A recipe for multi-metric gravity

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Abstract

In this thesis, we look for theories of interacting multiple gravitational fields. In general, interaction among gravitational fields generates extra degrees of freedom, which turn out to have negative kinetic energy and are called BD-ghosts. Thus, for a long time, it had been thought that healthy theories are impossible. However, recent progress in non-linear massive gravity has opened a way. A generalization of massive gravity leads to a theory of interacting two gravitational fields. Interaction models which contain more gravitational fields are also proposed, but whether or not they contain BD-ghosts is not completely resolved. Hence, in this thesis, we determine when a system of multiple interacting gravitational fields becomes ghost-free.

Contents

Chapter 1

Introduction

It is widely known that we have four kinds of forces, electromagnetic force, weak and strong nuclear forces, and gravitational force. They are mediated by particles named by photon, W, Z bosons, gluon and graviton. In general, particles are classified by the notion of mass and spin. (See Appendix A for details.) Except for graviton, the above particles have spin-1. Graviton is a massless particle with spin-2. Non-linearly self-interacting theories for massless spin-1 and spin-2 particles are uniquely determined. Respectively, they are Yang-Mills theory and general relativity. These two theories are central concern of modern theoretical physics, but compared to Yang-Mills theory, general relativity contains more complicated interactions and difficult to study. Thus, even now, a lot of properties are hidden in a veil of mystery. One may think that a theory with spin-3 would be more difficult, but it is not the case. A spin-3 particle cannot have self interactions [1] and drops out from our arguments. Then, properties of a spin-2 particle may be one of the most challenging theories to study.

In this thesis, we investigate non-linear interactions among multiple kinds of gravitons, not a single graviton. For spin-1 particles, we can consider interactions among them. However, for spin-2 fields, such an example had not been discovered until very recently [2]. For a long time, it had been thought that non-linear interactions of gravitons are impossible because they generate extra degrees of freedom with negative kinetic energy. These modes are unphysical and must be removed, which had bothered us for long. However, the authors of [2] have succeeded in eliminating such an unphysical state and discovered a healthy theory of non-linearly interacting two gravitational fields. After that, two prominent researches have been published. One of them is an extension to a system containing three gravitational fields [3], but whether or not it retains unphysical states is left unresolved in [3]. The other is a research on multiple interacting spin-2 fields written in terms of vielbeins, not metrics [4]. In general, gravitational fields are expressed by metrics, but theories in [4] are written only by vielbeins. Though these vielbein theories do not contain unphysical states, they do not necessarily overlap with metric theories. For instance, in the case of three spin-2 fields, only a special type of metric interaction can be rewritten in a vielbein formulation. Therefore, the purpose of this thesis is to seek healthy theories for interacting multiple spin-2 fields expressed in terms of metrics, not vielbeins. More precisely, we firstly settle the problem of the theory with three metrics, and then determine when theories of multiple metrics exclude unphysical states.

Prior to directly consider interactions among fields, it is convenient to investigate a massive one. This is because, in general, massless fields become massive when interaction among them is switched on. This fact is related to breaking of symmetries due to introducing interactions. As is well known, a massive field has more physical degrees of freedom than its massless counterpart. In four-dimensional cases, a massive spin-s ($s = 1, 2, ...$etc) field has $2s + 1$ dynamical degrees of freedom corresponding to the helicity 0, ± 1,...,$\pm s$ states while massless one has only two degrees of freedom coming from the helicity $\pm s$ modes. These increased degrees of freedom are supplied via breaking of a symmetry. A theory of a free massless spin-s field contains a gauge symmetry, which reduces physical degrees of freedom to two. On the other hand, when we introduce a mass term, the symmetry is broken and we cannot remove degree of freedom. Thus, we have more physical degrees of freedom. A mass term is a special case of more general interaction terms where we include not only one but also two or more fields. If there are free massless fields with no interaction, we should have symmetries which eliminate right number of degrees of freedom. However, interaction terms generally breaks some of these symmetries. As a result, more physical degrees of freedom are left, which should correspond to massive modes. Hence, we see that interactions generally makes massless fields massive.

This discussion suggests that understanding of massive gravity, where a graviton becomes massive, should help us to construct a theory of interacting spin-2 fields. Thus, our first step is focused on massive gravity. We usually introduce a mass term for a scalar or vector field as a quadratic term of the field. The action for them is quadratic and the equation of motion becomes linear when they are free. However, general relativity is a complicated non-linear theory even in the vacuum, and does not seem similar to these simple cases. Hence, we start with linearized general relativity and make the graviton field massive, which is a topic in Chapter 2. Afterward, we consider its non-linear extension in Chapter 3 and 4. At the same time, a healthy theory of interacting two gravitational fields is introduced. In Chapter 5, we consider interacting multiple spin-2 fields written in terms of vielbeins. We also consider their relationship to a metric formulation. Then, based on the result of these chapters, we investigate conditions to construct a metric theory of non-linearly interacting multiple spin-2 fields in Chapter 6. This chapter is based on our original work [5]. Finally, Chapter 7 is devoted to an application of interacting gravitational fields. Especially, we attempt an application to the AdS/CFT correspondence, which is based on our unpublished work [6].

Chapter 2

Linear massive gravity

We begin our long journey toward a theory of interacting gravitational fields. Our first step is to consider linear massive gravity. In this chapter, we linearize general relativity and attempt to introduce a mass term. Linearized general relativity is very similar to electromagnetic field, and it seems easy to make the gravitational field massive. However, even in the linear level, introducing graviton's mass is not so trivial.

As usual, a mass term is introduced simply as a quadratic term of the field. For example, in the four-dimensional flat space-time, a massless real scalar (spin-0) field ϕ has the action

$$S = \int d^4x \Big\{ -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)\Big\}, \tag{2.1}$$

and the equation of motion is given by

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \;\Rightarrow\; \partial_\mu\partial^\mu\phi = 0. \tag{2.2}$$

We introduce a mass term by adding a quadratic term with a coefficient m^2

$$S = \int d^4x \Big\{ -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\Big\}. \tag{2.3}$$

Then, we obtain the equation of motion

$$\frac{\delta S}{\delta\phi} = 0 \;\Rightarrow\; (\partial_\mu\partial^\mu - m^2)\phi = 0, \tag{2.4}$$

and we interpret m as mass of the scalar field. In the case of a vector (spin-1) field A_μ, the situation is almost the same. We have the action and the equation of motion for a massless vector field

$$S = \int d^4x \; -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.5}$$

$$\frac{\delta S}{\delta A_\nu} = 0 \;\Rightarrow \partial_\mu F^{\mu\nu} = 0. \tag{2.6}$$

As is well known, this system is invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$, where Λ is an arbitrary function. Thus, some modes are unphysical. In order to extract true degrees of freedom, we can impose gauge fixing conditions $\partial_\mu A^\mu = 0$ and $A_0 = 0$. Therefore, the number of remaining degrees of freedom is two, which corresponds to the helicity ± 1 modes. We make the vector field massive by introducing a quadratic term

$$S = \int d^4x \Big\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \Big\}, \tag{2.7}$$

which leads to the equation of motion

$$\frac{\delta S}{\delta A_\nu} = 0 \ \Rightarrow \partial_\mu F^{\mu\nu} - m^2 A^\nu = 0. \tag{2.8}$$

Taking the divergence of the equation of motion, we find $\partial_\nu A^\nu = 0$ to conclude that the missive vector obeys

$$(\partial_\mu \partial^\mu - m^2) A_\nu = 0, \quad \partial_\nu A^\nu = 0. \tag{2.9}$$

Since the mass term breaks the gauge invariance, we cannot impose gauge fixing conditions. Instead, the transverse constraint $\partial_\nu A^\nu = 0$ reduces one degree of freedom. Hence, the total number of degrees of freedom is three, which corresponds to the helicity ± 1 and 0 states.

In the above examples, giving mass to a field is straightforward. There is no difficulty. A naive extension of a free vector (spin-1) field is linearized general relativity where we have a tensor (spin-2) field called graviton. However, the situation becomes rather complicated. A massive spin-2 field should have five degrees of freedom corresponding to the helicity 0, ± 1 and ± 2 states. On the contrary, an extra sixth degree of freedom emerges along with introducing a quadratic term, and what is worse, this sixth mode turns out to have negative kinetic energy. Such an unphysical mode has to be excluded, which leads to the so called Fierz-Pauli mass term [7]. This chapter is devoted to investigating the Fierz-Pauli tuning from several viewpoints: the equation of motion, the Stückelberg trick and the Hamiltonian analysis. These tools play an important role also in later chapters. Prior to more complicated non-linear theories, we demonstrate how they work in a simple linear model. The Stückelberg trick is useful to trace the behavior of the extra unphysical degree of freedom which is mixed with other regular ones. It also gives us a key idea for the non-linear extension of linear massive gravity. However, it is difficult to prove the absence of an extra degree of freedom by only the Stückelberg trick. Thus, we need to rely on the Hamiltonian analysis. It gives us a systematic way to count the total number of degrees of freedom contained in the system. When we succeed in constructing a consistent theory of a massive spin-2 field, the Hamiltonian analysis must show five degrees of freedom are left. For more details or a historical overview, review articles [8, 9] are helpful.

2.1 Linearized general relativity

In this section, we derive the action of a graviton or spin-2 field in the linear level. We start with the action of general relativity which is well-known as the Einstein-Hilbert action

$$S_{EH}[g] = \frac{1}{16\pi G}\int d^D x \sqrt{-\det g}(R[g] - 2\Lambda). \tag{2.10}$$

We set the dimension of the space-time to be D composed of one time and $D-1$ spatial directions. In the action, we have a metric $g_{\mu\nu}$ $(\mu, \nu = 0, 1, .., D-1)$ and the scalar curvature

$R[g]$ for $g_{\mu\nu}$. Λ represents a cosmological constant and G is the gravitational constant. The equation of motion is given by

$$\frac{\delta S_{EH}}{\delta g_{\mu\nu}} = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{2.11}$$

This system is invariant under a diffeomorphism

$$x^\mu \to f^\mu(x), \quad g_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} g_{\alpha\beta}\big(f(x)\big) \tag{2.12}$$

with an arbitrary function $f^\mu(x)$, and the invariance is responsible to extract true degrees of freedom. The discussion of a non-linear case is postponed to Section 3.1.1.

Here, we consider a perturbation around a fixed background metric and expand the action up to the second order. Thus, the metric $g_{\mu\nu}$ is decomposed as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{2.13}$$

where $\bar{g}_{\mu\nu}$ is a background metric obeying (2.11) and $h_{\mu\nu}$ represents a fluctuation. The detailed calculation is found in Appendix B. The result is

$$S_{EH} = \frac{1}{16\pi G} \int d^D x \mathcal{L}^{(2)}_{EH}, \tag{2.14}$$

with the Lagrangian density

$$\begin{aligned}
\frac{\mathcal{L}^{(2)}_{EH}}{\sqrt{-\det \bar{g}}} =& \frac{2}{D}\bar{R} + \bar{\nabla}_\mu\big(\bar{\nabla}_\lambda h^{\mu\lambda} - \bar{\nabla}^\mu h^\lambda{}_\lambda\big) \\
& - \frac{1}{4}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\alpha h^{\mu\nu}) + \frac{1}{2}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\nu h^{\alpha\mu}) - \frac{1}{2}(\bar{\nabla}^\mu h^\alpha{}_\alpha)(\bar{\nabla}^\lambda h_{\mu\lambda}) + \frac{1}{4}(\bar{\nabla}^\mu h^\alpha{}_\alpha)(\bar{\nabla}_\mu h^\beta{}_\beta) \\
& + \frac{\bar{R}}{2D}\Big(h^\alpha{}_\beta h^\beta{}_\alpha - \frac{1}{2} h^\alpha{}_\alpha h^\beta{}_\beta\Big) \\
& + \bar{\nabla}_\mu\Big(h^{\alpha\beta}\bar{\nabla}^\mu h_{\alpha\beta} - h^{\mu\alpha}\bar{\nabla}^\beta h_{\alpha\beta} + h^{\mu\alpha}\bar{\nabla}_\alpha h^\beta{}_\beta - h^{\alpha\beta}\bar{\nabla}_\alpha h^\mu{}_\beta + \frac{1}{2}h^\beta{}_\beta\bar{\nabla}^\alpha h^\mu{}_\alpha - \frac{1}{2}h^\alpha{}_\alpha\bar{\nabla}^\mu h^\beta{}_\beta\Big),
\end{aligned} \tag{2.15}$$

where $\bar{\nabla}_\mu$ stands for the covariant derivative constructed from the background metric $\bar{g}_{\mu\nu}$. The space-time indices are raised or lowered by $\bar{g}_{\mu\nu}$ and its inverse $\bar{g}^{\mu\nu}$. When we neglect total derivatives, the Lagrangian density is simplified to be

$$\begin{aligned}
\frac{\mathcal{L}^{(2)}_{EH}}{\sqrt{-\det \bar{g}}} =& - \frac{1}{4}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\alpha h^{\mu\nu}) + \frac{1}{2}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\nu h^{\alpha\mu}) - \frac{1}{2}(\bar{\nabla}^\mu h^\alpha{}_\alpha)(\bar{\nabla}^\lambda h_{\mu\lambda}) + \frac{1}{4}(\bar{\nabla}^\mu h^\alpha{}_\alpha)(\bar{\nabla}_\mu h^\beta{}_\beta) \\
& + \frac{\bar{R}}{2D}\Big(h^\alpha{}_\beta h^\beta{}_\alpha - \frac{1}{2} h^\alpha{}_\alpha h^\beta{}_\beta\Big) + \frac{2}{D}\bar{R}.
\end{aligned} \tag{2.16}$$

The invariance under (2.12) is translated to a gauge symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu \tag{2.17}$$

with an arbitrary vector ξ_μ, which has a role to extract true degrees of freedom in the linear theory.

2.2 The Fierz-Pauli mass term

In this section, we attempt to introduce graviton's mass to linearized general relativity. We add a quadratic term with no derivative as a mass term. However, the choice of a quadratic term is not unique. We start with a general mass term because we do not have a guiding principle. Then, we see that an extra degree of freedom emerges due to the introduced quadratic term, and what is worse, it has negative kinetic energy. In order to exclude it, we have to tune the mass term, which uniquely picks up the so called Fierz-Pauli mass term [7].

In this chapter, we mainly rely on the flat space-time as a background. We denote the Mankowski metric as $\eta_{\mu\nu} := \mathrm{diag}(-1, 1, 1, ..., 1)$, and thus we set $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. We put $\Lambda = 0$ and $\bar{R} = 0$, and total derivatives are discarded. For notational simplicity, we abbreviate the trace part of $h_{\mu\nu}$ as $h = h^{\mu}{}_{\mu}$, and write down the linearized Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G}\int d^D x \mathcal{L}_{EH}, \tag{2.18}$$

with

$$\mathcal{L}_{EH} = -\frac{1}{4}(\partial_\alpha h_{\mu\nu})(\partial^\alpha h^{\mu\nu}) + \frac{1}{2}(\partial_\alpha h_{\mu\nu})(\partial^\nu h^{\alpha\mu}) - \frac{1}{2}(\partial_\mu h)(\partial_\nu h^{\mu\nu}) + \frac{1}{4}(\partial_\mu h)(\partial^\mu h). \tag{2.19}$$

For the purpose to make a graviton massive, we add a quadratic term to the above action. Since we have two kinds of quadratic terms $h^{\mu\nu}h_{\mu\nu}$ and h^2, we try to add a linear combination of them

$$\mathcal{L}_m = ah^2 + bh^{\mu\nu}h_{\mu\nu}, \tag{2.20}$$

where a and b are some constants. Then, we have the action

$$S = \frac{1}{16\pi G}\int d^D x(\mathcal{L}_{EH} + \mathcal{L}_m), \tag{2.21}$$

and the equation of motion $\delta S/\delta h^{\mu\nu} = 0$ is found to be

$$\begin{aligned}&\frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\partial^\alpha\partial_\nu h_{\alpha\mu} - \frac{1}{2}\partial^\alpha\partial_\mu h_{\nu\alpha} + \frac{1}{2}\partial_\mu\partial_\nu h + \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \frac{1}{2}\eta_{\mu\nu}\Box h\\ &+2a\eta_{\mu\nu}h + 2bh_{\mu\nu} = 0,\end{aligned} \tag{2.22}$$

where we have defined $\Box := \partial^\mu\partial_\mu$. We take divergence and trace on (2.22), and obtain

$$a\partial_\nu h + b\partial^\mu h_{\mu\nu} = 0, \tag{2.23}$$

$$-\Big(\frac{D}{2} - 1\Big)\Box h + \Big(\frac{D}{2} - 1\Big)\partial_\alpha\partial_\beta h^{\alpha\beta} + 2(aD + b)h = 0. \tag{2.24}$$

It is easy to see that two equations (2.23) and (2.24) lead to the equation of motion for the trace part

$$\Big(\frac{D}{2} - 1\Big)\Big(1 + \frac{a}{b}\Big)\Box h - 2(Da + b)h = 0. \tag{2.25}$$

Here, we notice that something interesting happens when we set $a + b = 0$. The dynamics of the trace h is lost, and a constraint

$$h = 0 \tag{2.26}$$

is left. Substituting this constraint (2.26) back into the equation (2.23), we get another constraint

$$\partial^\mu h_{\mu\nu} = 0. \tag{2.27}$$

We insert these traceless transverse conditions (2.26) and (2.27) into the original equation of motion (2.22), and obtain a simple equation

$$(\Box + 4b)h_{\mu\nu} = 0. \tag{2.28}$$

In this case, the total number of degrees of freedom is easy to count. The symmetric tensor $h_{\mu\nu}$ has $\frac{1}{2}D(D+1)$ components, but we have constraints (2.26) and (2.27). Thus, the total number of degrees of freedom is $\frac{1}{2}D(D+1) - 1 - D = \frac{1}{2}(D-2)(D+1)$. In the four dimensional case ($D = 4$), we have five degrees of freedom, which should be nothing but spin (2,1,0,-1,-2) modes for a massive spin-2 field. Therefore, it seems natural to put $a + b = 0$ and set $4b = -m^2$, where m is interpreted as graviton's mass. This mass term is the Fierz-Pauli mass term [7]

$$\mathcal{L}_{FP} := -\frac{1}{4}m^2(h^{\mu\nu}h_{\mu\nu} - h^2). \tag{2.29}$$

In the case of $a + b \neq 0$, the trace part becomes dynamical and an extra degree of freedom is included. In fact, this extra mode turns out to have negative kinetic energy. Such a particle is called ghost, and must be excluded from the system. In order to focus on the trace part, we assume that $h_{\mu\nu}$ is diagonal. We decompose the space-time indices into the time and spatial components. The time component is denoted as 0 while spatial ones are represented as $i, j, k, ...$etc. We set $h_{00} = h_{00}(t)$, $h_{0i} = 0$ and $h_{ij} = \phi(t)\delta_{ij}$, where all non-zero components depend only on t and the time derivative is abbreviated as $\dot{\phi} = \partial_0\phi$. Then, the Lagrangian density is simplified to be

$$\mathcal{L} = -\frac{1}{4}(D-1)(D-2)\dot{\phi}^2 + (D-1)\big(a(D-1)+b\big)\phi^2 + (a+b)h_{00}^2 - 2a(D-1)h_{00}\phi. \tag{2.30}$$

We see that h_{00} is merely an auxiliary field. The equation of motion for h_{00} is

$$h_{00} = \frac{a(D-1)}{a+b}\phi, \tag{2.31}$$

which is substituted into the Lagrangian density (2.30) to give

$$\mathcal{L} = -\frac{1}{4}(D-1)(D-2)\dot{\phi}^2 + \frac{(D+1)b}{a+b}(Da+b)\phi^2. \tag{2.32}$$

Hence, there is a field with negative kinetic energy. On the other hand, if we put $a+b = 0$, the equation of motion for h_{00} reads $\phi = 0$. The ghost field ϕ is eliminated. Thus, we conclude that the extra mode we have encountered from the trace part is a ghost. The Fierz-Pauli tuning is necessary to exclude this ghost mode.

2.3 The Stückelberg trick

In Section 2.2, we have seen how a ghost degree of freedom emerges. From the equation of motion, we suspected that the trace part contains an extra degree of freedom which should be absent in a healthy massive theory. Then, we have extracted diagonal components to show that the trace part actually has negative kinetic energy. However, this method is rather ad hoc and difficult to apply to more complicated theories such as non-linear massive gravity which we discuss in Chapter 3. A more powerful and widely applicable method is known as the Stückelberg trick, where we introduce new degrees of freedom along with additional gauge symmetries. These newly introduced degrees of freedom trace a ghost hidden in the original variables.

In this section, we explain the Stückelberg trick in the context of the ghost problem in linear massive gravity. We show that when the Fierz-Pauli tuning is violated, higher order derivatives with respect to the time coordinate emerge on the newly introduced field, which leads to a ghost degree of freedom.

Even outside this chapter, the Stückelberg trick plays an important role. When we investigate the non-linear extension of the linear Fierz-Pauli theory, we heavily rely on it. The Stückelberg trick is also helpful in considering the vDVZ discontinuity in Section 2.5. More details about the Stückelberg trick is found in [8].

We begin by the Lagrangian density $\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_m$, where $\mathcal{L}_{EH}$ and $\mathcal{L}_m$ are given by (2.19) and (2.20) respectively. We left constants a and b in (2.20) undetermined to know the role of the Fierz-Pauli tuning. The Stückelberg trick starts with introducing a new field A_μ through the following replacement

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu \tag{2.33}$$

which imitates the invariance broken by the mass term $\mathcal{L}_m$. Because the massless part of the Lagrangian $\mathcal{L}_{EH}$ has a gauge symmetry $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, $\mathcal{L}_{EH}$ is invariant under the replacement (2.33). On the other hand, the mass term $\mathcal{L}_m$ breaks this gauge symmetry and picks up a change. Thus, the Lagrangian density after the replacement is found to be

$$\begin{aligned}\mathcal{L}(h, A) =& \mathcal{L}_{EH}(h) + \mathcal{L}_m(h) \\ &+ 4ah(\partial^\mu A_\mu) + 4bh^{\mu\nu}(\partial_\mu A_\nu) + 2b(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (2b + 4a)(\partial_\mu A_\nu)(\partial^\nu A^\mu),\end{aligned} \tag{2.34}$$

where $\mathcal{L}_{EH}(h)$ and $\mathcal{L}_m(h)$ do not contain the field A_μ, and are the same as (2.19) and (2.20). The introduction of the new field A_μ comes along with a gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \delta A_\mu = -\xi_\mu. \tag{2.35}$$

The original Lagrangian before the replacement (2.33) can be recovered by fixing this gauge as $A_\mu = 0$. Hence, the theory itself is not changed before and after the operation (2.33). The next step is to introduce one more extra field ϕ via the replacement

$$A_\mu \to A_\mu + \partial_\mu \phi. \tag{2.36}$$

Then, we obtain

$$\begin{aligned}\mathcal{L}(h, A, \phi) =& \mathcal{L}_{EH}(h) + \mathcal{L}_m(h) \\ &+ 4ah(\partial^\mu A_\mu) + 4bh^{\mu\nu}(\partial_\mu A_\nu) + 2b(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (2b + 4a)(\partial_\mu A_\nu)(\partial^\nu A^\mu) \\ &+ 4ah(\partial^\mu \partial_\mu \phi) + 4bh^{\mu\nu}(\partial_\mu \partial_\nu \phi) \\ &+ 8(a + b)(\partial_\mu A_\nu)(\partial^\mu \partial^\nu \phi) + 4(a + b)(\partial_\mu \partial_\nu \phi)(\partial^\mu \partial^\nu \phi).\end{aligned} \tag{2.37}$$

Along with the introduction of ϕ, one more gauge symmetry comes. In total, we have two kinds of gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \delta A_\mu = -\xi_\mu, \tag{2.38}$$

$$\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\phi = -\Lambda. \tag{2.39}$$

By fixing the gauge as $A_\mu = 0$ and $\phi = 0$, we can restore the original Lagrangian. Therefore, we can investigate $\mathcal{L}(h, A, \phi)$ instead of the original $\mathcal{L}_{EH}(h) + \mathcal{L}_m(h)$.

Now, we see that the Lagrangian density (2.37) contains higher order derivative terms such as $(\partial_\mu\partial_\nu\phi)(\partial^\mu\partial^\nu\phi)$. In general, a higher order derivative with respect to the time coordinate carries an extra degree of freedom which is ghost-like [10, 11]. Here, we do not develop the abstract proof. Instead, we show the emergence of a ghost mode through an explicit calculation. In order to simplify the formula, we partially fix the gauge freedom by the conditions $\partial^\mu h_{\mu\nu} = 0$ and $\partial_\mu A^\mu = 0$.

Actually, our gauge fixing is not complete, and fictitious modes may be contained. However, the role of the Stückelberg trick is to guess whether or not a ghost mode is contained, and it is not expected to completely determine the problem. By using the Stückelberg trick, we speculate a appropriate form of a mass term. Then, its validity is proved in another way, which we discuss in the next section.

Integrating by parts and using the gauge conditions, we simplify the Lagrangian density

$$\begin{aligned}\mathcal{L} =& \mathcal{L}_{MG}(h) + 2b(\partial_\mu A_\nu)(\partial^\mu A^\nu) + 4ah(\Box\phi) + 4(a+b)(\Box\phi)(\Box\phi) + \lambda^\nu(\partial^\mu h_{\mu\nu}) + \lambda(\partial_\mu A^\mu),\\ \mathcal{L}_{MG}(h) :=& -\frac{1}{4}(\partial_\alpha h_{\mu\nu})(\partial^\alpha h^{\mu\nu}) + \frac{1}{4}(\partial_\mu h)(\partial^\mu h) + ah^2 + bh^{\mu\nu}h_{\mu\nu}.\end{aligned} \tag{2.40}$$

The last two terms represent constraints coming from the gauge fixing, and λ^ν and λ are corresponding Lagrange multipliers. For the purpose to eliminate the coupling between h and $\Box\phi$, we introduce a shifted graviton field $l_{\mu\nu}$

$$h_{\mu\nu} = l_{\mu\nu} + \frac{8a}{D-1}\phi\eta_{\mu\nu}. \tag{2.41}$$

Then, we obtain

$$\begin{aligned}\mathcal{L} =& \mathcal{L}_{MG}(l) + 2b(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \lambda^\nu(\partial^\mu l_{\mu\nu} + 8a\partial_\nu\phi/(D-1)) + \lambda(\partial_\mu A^\mu)\\ &+ 4(a+b)(\Box\phi)(\Box\phi) + \frac{16a^2 D}{D-1}\phi(\Box\phi) + \frac{64a^2 D(aD+b)}{(D-1)^2}\phi^2 + \frac{16a(aD+b)}{D-1}l\phi.\end{aligned} \tag{2.42}$$

We extract the property of the ϕ part and define

$$\mathcal{L}_\phi := (\Box\phi)(\Box\phi) + A\phi(\Box\phi) + B\phi^2, \tag{2.43}$$

where A and B are some constants. This Lagrangian is physically equivalent to the following two-field Lagrangian

$$\mathcal{L}_{\phi,\psi} := \Big(\Box\phi + \frac{A}{2}\phi\Big)\psi - \frac{1}{4}\psi^2 + \Big(B - \frac{1}{4}A^2\Big)\phi^2. \tag{2.44}$$

In the Lagrangian (2.44), ψ is merely an auxiliary field and the equation of motion for it can be solved in terms of ϕ

$$\frac{\delta\mathcal{L}_{\phi,\psi}}{\delta\psi} = \Box\phi + \frac{A}{2}\phi - \frac{1}{2}\psi = 0 \;\Rightarrow\; \psi = 2\Big(\Box\phi + \frac{A}{2}\phi\Big). \tag{2.45}$$

The above solution is plugged back into the original Lagrangian (2.44), and we can find the equivalence

$$\mathcal{L}_{\phi,\psi}|_{\psi=\psi(\phi)} = \mathcal{L}_{\phi}. \tag{2.46}$$

Hence, we can investigate $\mathcal{L}_{\phi,\psi}$ instead of $\mathcal{L}_{\phi}$. In order to diagonalize the cross derivative term in (2.44), we rename the fields ϕ and ψ in the following way

$$\phi = \sigma + \chi, \quad \psi = \sigma - \chi, \tag{2.47}$$

and obtain

$$\mathcal{L}_{\phi,\psi} = -(\partial_\mu \sigma)(\partial^\mu \sigma) + (\partial_\mu \chi)(\partial^\mu \chi) + \cdots , \tag{2.48}$$

where the remaining terms represented as "$\cdots$" are composed of no-derivative couplings between σ and χ. Thus, we find that one of σ and χ always behaves as a ghost. This discussion can be applied to the Lagrangian density (2.42). Therefore, we conclude that a ghost degree of freedom is contained in (2.42). To exclude this unphysical mode, we have to set $a + b = 0$ and eliminate the higher order derivative. Here, it should be noticed that we discarded the term $(\partial_\mu A_\nu)(\partial^\mu \partial^\nu \phi)$ in (2.37) by the gauge fixing, but this term also disappears under the tuning $a + b = 0$. The Fierz-Pauli tuning eliminates all of the higher order derivatives in (2.37) and the system becomes ghost-free.

2.4 The Hamiltonian analysis

In Section 2.3, we have used the Stückelberg trick to investigate the existence of a ghost degree of freedom. In general, the Stückelberg trick is useful in detecting a ghost, but it is extremely difficult to directly prove the absence of a ghost. If the extra mode is eliminated, we should have the right number of degrees of freedom. Especially, five degrees of freedom must be left in the case of a four-dimensional massive spin-2 field. Thus, we have only to count the total number of degrees of freedom to finish the proof of the absence of a ghost. In Section 2.2, we have counted the total number of degrees of freedom from the equation of motion. However, this method is possible only when the equation of motion is not so complicated. The most systematic way to count the number of degrees of freedom is the Hamiltonian analysis, which we use as a main tool throughout this thesis. In this section, we reconsider linear massive gravity from the view point of the Hamiltonian analysis.

As in Section 2.3, we start with the Lagrangian density $\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_m$, where $\mathcal{L}_{EH}$ and $\mathcal{L}_m$ are given by (2.19) and (2.20) with the constants a and b undetermined. The first step of the Hamiltonian analysis is to find the dynamical variables and construct the Hamiltonian. Hence, we decompose the space-time indices into the time and spatial ones, which we denote $\mu = (0, i)$. We also abbreviate the time derivative as $\partial_0 \phi = \dot{\phi}$. Then, the Lagrangian density is written down:

$$\begin{aligned}\mathcal{L} =&\frac{1}{4}\dot{h}_{ij}\dot{h}_{ij} - \frac{1}{4}\dot{h}_{ii}\dot{h}_{jj} - \frac{1}{2}\dot{h}_{ij}(\partial_j h_{0i} + \partial_i h_{j0}) + \dot{h}_{ii}(\partial_j h_{0j}) \\ &- \frac{1}{4}(\partial_i h_{jk})(\partial_i h_{jk}) + \frac{1}{2}(\partial_i h_{jk})(\partial_k h_{ij}) - \frac{1}{2}(\partial_i h_{kk})(\partial_j h_{ij}) + \frac{1}{4}(\partial_i h_{jj})(\partial_i h_{kk}) \\ &- \frac{1}{2}(\partial_i h_{0j})(\partial_j h_{i0}) + \frac{1}{2}(\partial_i h_{j0})(\partial_i h_{j0}) + \frac{1}{2}h_{00}(\partial_i\partial_i h_{kk} - \partial_i\partial_j h_{ij}) \\ &+ (a+b)h_{00}h_{00} - 2ah_{00}h_{ii} - 2bh_{0i}h_{0i} + ah_{ii}h_{jj} + bh_{ij}h_{ij},\end{aligned} \tag{2.49}$$

where we have performed integrations by parts on h_{00}, and lowered all of the indices. From this decomposition, we find that only h_{ij} $(i, j = 1, 2, ..., D-1)$ are dynamical, and define the canonical momenta

$$\pi_{ij} := \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \frac{1}{2}\dot{h}_{ij} - \frac{1}{2}\delta_{ij}\dot{h}_{kk} - \frac{1}{2}(\partial_j h_{i0} + \partial_i h_{j0}) + \delta_{ij}\partial_k h_{0k}, \tag{2.50}$$

which can be inverted for $\dot{h}_{ij}$

$$\frac{1}{2}\dot{h}_{ij} = \pi_{ij} - \frac{1}{D-2}\delta_{ij}\pi_{kk} + \frac{1}{2}(\partial_j h_{i0} + \partial_i h_{j0}). \tag{2.51}$$

The Hamiltonian density is constructed through the Legendre transformation

$$\begin{aligned}\mathcal{H} :=&\pi_{ij}\dot{h}_{ij} - \mathcal{L} \\ =&\pi_{ij}\pi_{ij} - \frac{1}{D-2}\pi_{ii}\pi_{jj} \\ &+ \frac{1}{4}(\partial_i h_{jk})(\partial_i h_{jk}) - \frac{1}{2}(\partial_i h_{jk})(\partial_k h_{ij}) + \frac{1}{2}(\partial_i h_{kk})(\partial_j h_{ij}) - \frac{1}{4}(\partial_i h_{jj})(\partial_i h_{kk}) \\ &- ah_{ii}h_{jj} - bh_{ij}h_{ij} \\ &- \frac{1}{2}h_{00}\Big(\partial_i\partial_i h_{jj} - \partial_i\partial_j h_{ij} - 4ah_{ii} + 2(a+b)h_{00}\Big) - 2h_{j0}(\partial_i\pi_{ij} - bh_{j0}).\end{aligned} \tag{2.52}$$

In the following analysis, we often use a short hand notation

$$h = h_{ii}, \quad \pi = \pi_{ii}, \tag{2.53}$$

which we must be careful not to confuse with the trace with respect to the space-time indices $h^\mu{}_\mu = -h_{00} + h_{ii}$. The space-time dimension D is sometimes omitted:

$$\int dx = \int d^{D-1}x, \quad \delta(x) = \delta^{(D-1)}(x), \tag{2.54}$$

and we also abbreviate the derivative and integration symbols:

$$\partial^2 = \partial_i\partial_i, \quad \int f + g = \int dx[f(x) + g(x)]. \tag{2.55}$$

2.4.1 the massless case

Prior to the analysis of massive gravity, we consider the massless case where the constants a and b are set to be zero, $a = b = 0$. The Hamiltonian density is given by

$$\mathcal{H} = \mathcal{K} - \frac{1}{2}h_{00}\mathcal{C} - h_{0i}\mathcal{C}_i, \tag{2.56}$$

where the each element is defined as

$$\begin{aligned}\mathcal{K} :=&\pi_{ij}\pi_{ij} - \frac{1}{D-2}\pi_{ii}\pi_{jj} \\ &+ \frac{1}{4}(\partial_i h_{jk})(\partial_i h_{jk}) - \frac{1}{2}(\partial_i h_{jk})(\partial_k h_{ij}) + \frac{1}{2}(\partial_i h_{kk})(\partial_j h_{ij}) - \frac{1}{4}(\partial_i h_{jj})(\partial_i h_{kk}), \end{aligned}\tag{2.57}$$

$$\mathcal{C} :=\partial_i\partial_i h_{jj} - \partial_i\partial_j h_{ij}, \tag{2.58}$$

$$\mathcal{C}_i :=2\partial_i\pi_{ij}. \tag{2.59}$$

In the above formulae, we notice that h_{00} and h_{i0} appear only linearly and can be interpreted as Lagrange multipliers. Variation of the action with respect to these multipliers leads to constraints

$$\mathcal{C} = 0, \qquad \mathcal{C}_i = 0. \tag{2.60}$$

In general, constraints must be preserved along the time evolution which is represented as the Poisson bracket with the Hamiltonian. Thus, the following consistency conditions must be satisfied

$$\dot{\mathcal{C}} = \{\mathcal{C}, H\}_{PB} \approx 0, \qquad \dot{\mathcal{C}}_i = \{\mathcal{C}_i, H\}_{PB} \approx 0, \tag{2.61}$$

where the Hamiltonian H is given by

$$H := \int d^{D-1}x\, \mathcal{H}(x), \tag{2.62}$$

and the Poisson bracket is determined by

$$\big\{F(x), G(y)\big\}_{PB} = \int d^{D-1}z\bigg(\frac{\delta F(x)}{\delta h_{ij}(z)}\frac{\delta G(y)}{\delta \pi_{ij}(z)} - \frac{\delta F(x)}{\delta \pi_{ij}(z)}\frac{\delta G(y)}{\delta h_{ij}(z)}\bigg). \tag{2.63}$$

In the Poisson bracket, all the time coordinates are set to be equal. Since the action is now written as $S = \frac{1}{16\pi G}\int \dot{h}\pi + \cdots$, we should define $\{F, G\}_{PB} = 16\pi G \int \frac{\delta F}{\delta h}\frac{\delta G}{\delta \pi} + \cdots$. However, the constant $\frac{1}{16\pi G}$ is merely an overall factor, and we omit it. The symbol "$\approx$" represents the equality on the hypersurface determined by the constraints $\mathcal{C} = 0$ and $\mathcal{C}_i = 0$.

To begin with, we investigate the Poisson brackets between $\mathcal{C}$ and $\mathcal{C}_i$. Variation of $\mathcal{C}$ with respect to the variable h_{ij} is calculated as

$$\delta_h\mathcal{C} = \partial_i\partial_i\delta h_{kk} - \partial_i\partial_j\delta h_{ij} = \delta_{ij}\partial_k\partial_k\delta h_{ij} - \partial_i\partial_j\delta h_{ij}. \tag{2.64}$$

This formula suggests that the Poisson bracket contains derivatives on the delta function. Hence, it is convenient to introduce an integrated form

$$\langle\langle f\mathcal{C}\rangle\rangle := \int d^{D-1}x f(x)\mathcal{C}(x) \tag{2.65}$$

with an arbitrary function $f(x)$. Then, discarding total derivatives, the variation is given by

$$\delta_h \langle\langle f\mathcal{C}\rangle\rangle = \int d^{D-1}x \big[\delta_{ij}(\partial_k\partial_k f) - (\partial_i\partial_j f)\big]\delta h_{ij}. \tag{2.66}$$

Because $\mathcal{C}$ does not contain the canonical momenta π_{ij}, the variation with respect to π_{ij} is zero. We also notice that $\mathcal{C}_i$ does not contain h_{ij}. Thus, we find

$$\delta_\pi \mathcal{C} = 0, \qquad \delta_h \mathcal{C}_i = 0. \tag{2.67}$$

On the other hand, variation of $\mathcal{C}_i$ with respect to π_{ij} is found to be

$$\delta_\pi \mathcal{C}_i = 2\partial_j \delta\pi_{ij}. \tag{2.68}$$

We define its integrated form

$$\langle\langle f_i \mathcal{C}_i\rangle\rangle := \int d^{D-1}x f_i(x)\mathcal{C}_i(x) \tag{2.69}$$

with any vector valued function $f_i(x)$, and obtain the variation

$$\delta_\pi \langle\langle f_i\mathcal{C}_i\rangle\rangle = \int d^{D-1}x \big[-\partial_i f_j - \partial_j f_i\big]\delta\pi_{ij}, \tag{2.70}$$

where we have symmetrized the coefficient of $\delta\pi_{ij}$. Using these formulae, we calculate the Poisson brackets between $\mathcal{C}$ and $\mathcal{C}_i$. Following two types are immediate from the definition of the Poisson bracket

$$\big\{\mathcal{C}(x), \mathcal{C}(y)\big\}_{PB} = 0, \qquad \big\{\mathcal{C}_i(x), \mathcal{C}_j(y)\big\}_{PB} = 0. \tag{2.71}$$

The remaining type is calculated as follows

$$\begin{aligned}
&\big\{\langle\langle f\mathcal{C}\rangle\rangle, \langle\langle g_i\mathcal{C}_i\rangle\rangle\big\}_{PB} \\
&= \int dx \int dy \int dz \big[\delta_{ij}(\partial_k\partial_k f) - (\partial_i\partial_j f)\big](x)\delta(x-z)\big[-\partial_i g_j - \partial_j g_i\big](y)\delta(y-z) \\
&= \int dx \big[\delta_{ij}(\partial^2 f) - (\partial_i\partial_j f)\big]\big[-\partial_i g_j - \partial_j g_i\big] \\
&= -2\int (\partial^2 f)(\partial_i g_i) - (\partial_i\partial_j f)(\partial_i g_j) = 0.
\end{aligned} \tag{2.72}$$

In the last line, we have discarded total derivatives. This Poisson bracket can also be expressed in the following form

$$\big\{\langle\langle f\mathcal{C}\rangle\rangle, \langle\langle g_i\mathcal{C}_i\rangle\rangle\big\}_{PB} = \int dx \int dy f(x) g_i(y) \big\{\mathcal{C}(x), \mathcal{C}_i(y)\big\}_{PB}, \tag{2.73}$$

and the functions f and g_i are arbitrary. Therefore, we know that

$$\big\{\mathcal{C}(x), \mathcal{C}_i(y)\big\}_{PB} = 0. \tag{2.74}$$

All of the Poisson brackets between $\mathcal{C}$ and $\mathcal{C}_i$ are zero.

Now, we consider the Poisson bracket with the Hamiltonian. The remaining task is to calculate variation of $\mathcal{K}$. The variation with respect to h_{ij} is given by

$$\begin{aligned}\delta_h \mathcal{K} =& \frac{1}{2}(\partial_k h_{ij})\partial_k \delta h_{ij} - \frac{1}{2}(\partial_i h_{jk})\partial_k \delta h_{ij} - \frac{1}{2}(\partial_j h_{ik})\partial_k \delta h_{ij} \\ &+ \frac{1}{2}\delta_{ij}(\partial_k h_{kl})\partial_l \delta h_{ij} + \frac{1}{2}(\partial_i h_{kk})\partial_j \delta h_{ij} - \frac{1}{2}\delta_{ij}(\partial_l h_{kk})\partial_l \delta h_{ij}. \end{aligned} \tag{2.75}$$

In the Hamiltonian H, $\mathcal{K}$ is contained in the integrand. Thus, we define

$$K := \int d^{D-1}x \mathcal{K}(x), \tag{2.76}$$

and represent the variation in the following form

$$\delta_h K = \frac{1}{2}\int dx \Big(- \partial^2 h_{ij} + \partial_k \partial_i h_{jk} + \partial_k \partial_j h_{ik} - \delta_{ij}\partial_k \partial_l h_{kl} - \partial_i \partial_j h + \delta_{ij}\partial^2 h \Big) \cdot \delta h_{ij}. \tag{2.77}$$

We can also obtain the variation with respect to π_{ij}

$$\delta_\pi \mathcal{K} = 2\Big(\pi_{ij} - \frac{1}{D-2}\pi \delta_{ij}\Big)\delta \pi_{ij}, \tag{2.78}$$

$$\delta_\pi K = \int dx\, 2\Big(\pi_{ij} - \frac{1}{D-2}\pi \delta_{ij}\Big)\delta \pi_{ij}. \tag{2.79}$$

Then, we calculate the Poisson bracket between $\mathcal{C}$ and the Hamiltonian H

$$\begin{aligned}\big\{\langle\langle f\mathcal{C}\rangle\rangle,\, H\big\}_{PB} \approx \big\{\langle\langle f\mathcal{C}\rangle\rangle,\, K\big\}_{PB} &= \int dx \int dy \int dz \big[\delta_{ij}(\partial^2 f) - (\partial_i\partial_j f)\big](x)\delta(x-z) \\ &\qquad \times 2\Big(\pi_{ij} - \frac{1}{D-2}\pi\delta_{ij}\Big)(y)\delta(y-z) \\ &= \int dx 2(\partial_j \pi_{ij})(\partial_i f) \\ &= \int dx \mathcal{C}_i \partial_i f. \end{aligned} \tag{2.80}$$

The symbol "$\approx$" means the equality under the constraints $\mathcal{C} = 0$ and $\mathcal{C}_i = 0$ imposed. In general, we have $\{F, h_{00}\mathcal{C}\} = \{F, h_{00}\}\mathcal{C} + \{F, \mathcal{C}\}h_{00}$. Here, we express $f(x)$ as $f(x) = \int dy f(y)\delta(x-y)$, and rewrite the above formula

$$\int dx \mathcal{C}_i(x)\partial_i f(x) = \int dx \int dy \mathcal{C}_i(x)\partial_i^{(x)} f(y)\delta(x-y) = \int dx \int dy f(x)\mathcal{C}_i(y)\partial_i^{(y)}\delta(x-y), \tag{2.81}$$

where we have exchanged names of the integration variables $x \leftrightarrow y$. Hence, we conclude that

$$\big\{\mathcal{C}(x), H\big\}_{PB} \approx \int dy\, \mathcal{C}_i(y)\frac{\partial}{\partial y^i}\delta(x-y). \tag{2.82}$$

The consistency condition $\dot{\mathcal{C}} \approx 0$ is satisfied when we set $\mathcal{C}_i = 0$. The other Poisson bracket is calculated in the same way

$$\begin{aligned}&\big\{\langle\langle f_i\mathcal{C}_i\rangle\rangle,\, H\big\}_{PB} \approx \big\{\langle\langle f_i\mathcal{C}_i\rangle\rangle,\, K\big\}_{PB} \\ &= - \int dx \frac{1}{2}\Big[- \partial^2 h_{ij} + \partial_k\partial_i h_{jk} + \partial_k\partial_j h_{ik} - \delta_{ij}\partial_k\partial_l h_{kl} - \partial_i\partial_j h + \delta_{ij}\partial^2 h\Big]\big[- \partial_i f_j - \partial_j f_i\big] = 0, \end{aligned} \tag{2.83}$$

and we obtain

$$\{C_i(x), H\}_{PB} \approx 0. \tag{2.84}$$

Therefore, two consistency conditions (2.61) are automatically satisfied on the constraint surface determined by $\mathcal{C} = 0$ and $\mathcal{C}_i = 0$, and all of the Lagrange multipliers h_{00} and h_{0i} are left undetermined. These remaining Lagrange multipliers correspond to the gauge freedom contained in the system. Therefore, we can count the total number of degrees of freedom in the following way. In configuration space, the number of the original dynamical variables is $\frac{1}{2}D(D-1)$ coming from the symmetric tensor h_{ij} $(i, j = 1, 2, ..., D)$. In phase space, the canonical momenta π^{ij} are added, and the number of the dynamical variables is doubled $2 \times \frac{1}{2}D(D-1)$. However, all of these dynamical variables are not independent. We have constraints $\mathcal{C} = 0$ and $\mathcal{C}_i = 0$, which reduces D degrees of freedom. Besides, the remaining Lagrange multipliers h_{00} and h_{i0} mean the existence of the gauge freedom. Thus, D degrees of freedom are further eliminated by gauge fixing. Then, the total number of degrees of freedom is $D(D-1) - D - D = (D-1)(D-2) - 2$ in phase space. Divided by two, we conclude that we have

$$\frac{1}{2}(D-1)(D-2) - 1 \tag{2.85}$$

physical degrees of freedom in the configuration space. In the four dimensional case ($D = 4$), only two degrees of freedom is left which is nothing but the helicity ± 2 modes.

2.4.2 the massive case

We proceed to the Hamiltonian analysis of massive gravity ($a \neq 0$ and $b \neq 0$). The Hamiltonian density is given by (2.52). In the case of $a + b \neq 0$ and $b \neq 0$, h_{00} and h_{j0} become auxiliary fields. We can solve the equation of motion for them by other fields as $h_{00} = h_{00}(h_{ij}, \pi_{ij})$ and $h_{0k} = h_{0k}(h_{ij}, \pi_{ij})$, and substitute these solutions back into the original action. Then, the auxiliary fields disappear while the variables h_{ij} and π_{ij} are left. Because we have no constraint, the total number of degrees of freedom is $\frac{1}{2}D(D-1)$. In the four dimensional case ($D = 4$), we have six degrees of freedom. One extra component remains, which is known to be a ghost. The emergence of this extra degree of freedom comes from quadratic terms of h_{00} and h_{i0}. In the massless case, these quadratic terms disappear and h_{00} and h_{i0} become Lagrange multipliers, which leads to the reduction of the degrees of freedom. When the field becomes massive, the number of physical degrees of freedom must increase. Thus, it seems needed to partially eliminate the quadratic terms of the non-dynamical variables h_{00} and h_{i0}.

Now, we focus on the Fierz-Pauli tuning and set $a = -b$. Then, the quadratic term of h_{00} disappears while h_{i0} remains as an auxiliary field. The equation of motion for h_{i0} reads

$$\frac{\delta \mathcal{L}}{\delta h_{j0}} = 2(\partial_i \pi_{ij} - 2b h_{j0}) = 0. \tag{2.86}$$

We can easily solve the above equation

$$h_{j0} = \frac{1}{2b}(\partial_i \pi_{ij}), \tag{2.87}$$

which is plugged back into the Hamiltonian density (2.52) to give

$$\begin{aligned}\mathcal{H} =&\pi_{ij}\pi_{ij} - \frac{1}{D-2}\pi_{ii}\pi_{jj} - \frac{1}{2b}(\partial_i\pi_{ik})(\partial_j\pi_{jk}) \\ &+ \frac{1}{4}(\partial_i h_{jk})(\partial_i h_{jk}) - \frac{1}{2}(\partial_i h_{jk})(\partial_k h_{ij}) + \frac{1}{2}(\partial_i h_{kk})(\partial_j h_{ij}) - \frac{1}{4}(\partial_i h_{jj})(\partial_i h_{kk}) \\ &+ b h_{ii}h_{jj} - b h_{ij}h_{ij} - \frac{1}{2}h_{00}\mathcal{C}_1. \end{aligned} \tag{2.88}$$

Then, we obtain the Hamiltonian density

$$\mathcal{H} = \mathcal{K} - \frac{1}{2}h_{00}\mathcal{C}, \tag{2.89}$$

where

$$\begin{aligned}\mathcal{K} :=&\pi_{ij}\pi_{ij} - \frac{1}{D-2}\pi_{ii}\pi_{jj} - \frac{1}{2b}(\partial_i\pi_{ik})(\partial_j\pi_{jk}) + b h_{ii}h_{jj} - b h_{ij}h_{ij} \\ &+ \frac{1}{4}(\partial_i h_{jk})(\partial_i h_{jk}) - \frac{1}{2}(\partial_i h_{jk})(\partial_k h_{ij}) + \frac{1}{2}(\partial_i h_{kk})(\partial_j h_{ij}) - \frac{1}{4}(\partial_i h_{jj})(\partial_i h_{kk}), \end{aligned} \tag{2.90}$$

$$\mathcal{C} :=\partial_i\partial_i h_{jj} - \partial_i\partial_j h_{ij} + 4b h_{ii}. \tag{2.91}$$

Variation of the action with respect to h_{00} leads to a primary constraint $\mathcal{C} = 0$. The constraint $\mathcal{C} = 0$ must be preserved along the time evolution. Thus, we must imposed the following consistency condition

$$\dot{\mathcal{C}} = \big\{\mathcal{C}, H\big\}_{PB} \approx 0, \qquad H := \int d^{D-1}x\mathcal{H}(x), \tag{2.92}$$

where the symbol "$\approx$" represents the equality on the hypersurface determined by the constraint $\mathcal{C} \approx 0$. Since $\mathcal{C}$ does not contain the canonical momenta π_{ij}, we immediately find

$$\big\{\mathcal{C}(x), \mathcal{C}(y)\big\}_{PB} = 0. \tag{2.93}$$

As in the massless case, it is convenient to consider integrated formulae

$$\langle\langle f\mathcal{C}\rangle\rangle := \int d^{D-1}x f(x)\mathcal{C}(x), \qquad K := \int d^{D-1}x\mathcal{K}(x), \tag{2.94}$$

and prepare their variation

$$\begin{aligned}\delta_h K = \frac{1}{2}\int dx\Big(&-\partial^2 h_{ij} + \partial_k\partial_i h_{jk} + \partial_k\partial_j h_{ik} \\ &- \delta_{ij}\partial_k\partial_l h_{kl} - \partial_i\partial_j h + \delta_{ij}\partial^2 h + 4b\delta_{ij}h - 4b h_{ij}\Big)\delta h_{ij}, \end{aligned} \tag{2.95}$$

$$\delta_\pi K = \int dx\left[2\pi_{ij} - \frac{2}{D-2}\pi\delta_{ij} + \frac{1}{2b}(\partial_l\partial_j\pi_{il}) + \frac{1}{2b}(\partial_l\partial_i\pi_{jl})\right]\delta\pi_{ij}, \tag{2.96}$$

$$\delta_h\langle\langle f\mathcal{C}\rangle\rangle = \int dx\big[\delta_{ij}(\partial^2 f) - (\partial_i\partial_j f) + 4b\delta_{ij}f\big]\delta h_{ij}, \quad \delta_\pi\langle\langle f\mathcal{C}\rangle\rangle = 0, \tag{2.97}$$

where δ_h and δ_π represent variation with respect to h_{ij} and π_{ij}.
Then, we calculate the Poisson bracket with the Hamiltonian $H = K - \frac{1}{2}\langle\langle h_{00}\mathcal{C}\rangle\rangle$

$$\begin{aligned}\big\{\langle\langle f\mathcal{C}\rangle\rangle,\, H\big\}_{PB} \approx& \big\{\langle\langle f\mathcal{C}\rangle\rangle,\, K\big\}_{PB}\\ =& \int dx \big[\delta_{ij}(\partial^2 f) - (\partial_i\partial_j f) + 4b\delta_{ij} f\big]\Big[2\pi_{ij} - \frac{2}{D-2}\pi\delta_{ij} + \frac{1}{b}\partial_l\partial_j\pi_{il}\Big]\\ =& \int dx\, f\cdot 2\Big[\partial_i\partial_j\pi_{ij} - \frac{4b}{D-2}\pi\Big],\end{aligned} \tag{2.98}$$

from which we read the consistency condition

$$\dot{\mathcal{C}} = \big\{\mathcal{C}, H\big\}_{PB} \approx 2\Big(\partial_i\partial_j\pi_{ij} - \frac{4b}{D-2}\pi\Big) \approx 0. \tag{2.99}$$

Obviously, the consistency condition (2.99) is not satisfied automatically, which leads to a secondary constraint

$$\mathcal{C}_{(2)} := \partial_i\partial_j\pi_{ij} - \frac{4b}{D-2}\pi_{ii} = 0. \tag{2.100}$$

The constraint $\mathcal{C}_{(2)} = 0$ must also be preserved along the time evolution, which means that we need one more consistency condition

$$\dot{\mathcal{C}}_{(2)} = \big\{\mathcal{C}_{(2)}, H\big\}_{PB} \approx 0. \tag{2.101}$$

Here, we have two constraints $\mathcal{C} = 0$ and $\mathcal{C}_{(2)} = 0$. Thus, the symbol "≈" should be reinterpreted as "=" on the constraint surface determined by both of the two constraints.

We immediately find that variation of $\mathcal{C}_{(2)}$ with respect to h_{ij} is zero $\delta_h\mathcal{C}_{(2)} = 0$. We prepare the variation with respect to π_{ij}

$$\delta_\pi\langle\langle f\mathcal{C}_{(2)}\rangle\rangle = \int dx\Big[(\partial_i\partial_j f) - \frac{4b}{D-2}f\delta_{ij}\Big]\delta\pi_{ij}, \tag{2.102}$$

and calculate the Poisson bracket with the Hamiltonian

$$\big\{\langle\langle f\mathcal{C}_{(2)}\rangle\rangle, H\big\}_{PB} = \big\{\langle\langle f\mathcal{C}_{(2)}\rangle\rangle, K\big\}_{PB} - \frac{1}{2}\big\{\langle\langle f\mathcal{C}_{(2)}\rangle\rangle, \langle\langle h_{00}\mathcal{C}\rangle\rangle\big\}_{PB}. \tag{2.103}$$

The first term is

$$\begin{aligned}&\big\{\langle\langle f\mathcal{C}_{(2)}\rangle\rangle, K\big\}_{PB}\\ =& -\int dx\frac{1}{2}\Big[-\partial^2 h_{ij} + \partial_k\partial_i h_{jk} + \partial_k\partial_j h_{ik} - \delta_{ij}\partial_k\partial_l h_{kl} - \partial_i\partial_j h + \delta_{ij}\partial^2 h + 4b\delta_{ij}h - 4bh_{ij}\Big]\\ &\quad\times\Big[(\partial_i\partial_j f) - \frac{4b}{D-2}f\delta_{ij}\Big]\\ =& \int dx\, f\Big[\frac{2b}{D-2}\big(\partial_i\partial_j h_{ij} - \partial^2 h\big) + 8b^2 h\Big] = -\int dx\, f\Big[\frac{2b}{D-2}\mathcal{C} - \frac{D-1}{D-2}(8b^2 h)\Big],\end{aligned} \tag{2.104}$$

and the second term is

$$\begin{aligned}\big\{\langle\langle f\mathcal{C}_{(2)}\rangle\rangle, \langle\langle h_{00}\mathcal{C}\rangle\rangle\big\}_{PB} \approx& -\int dx\big[\delta_{ij}(\partial^2 h_{00}) - (\partial_i\partial_j h_{00}) + 4b\delta_{ij}h_{00}\big]\Big[(\partial_i\partial_j f) - \frac{4b}{D-2}f\delta_{ij}\Big]\\ =& \int dx\frac{D-1}{D-2}(16b^2 f h_{00}).\end{aligned} \tag{2.105}$$

Hence, we obtain

$$\{\langle\langle f\mathcal{C}_{(2)}\rangle\rangle, H\}_{PB} \approx -\int dx\, f\frac{2b}{D-2}\Big[\mathcal{C} + 4b(D-1)(h_{00} - h_{ii})\Big], \tag{2.106}$$

and conclude that

$$\{\mathcal{C}_{(2)}, H\}_{PB} \approx -\frac{2b}{D-2}\Big[\mathcal{C} + 4b(D-1)(h_{00} - h_{ii})\Big] \approx -\frac{8b^2(D-1)}{D-2}(h_{00} - h_{ii}). \tag{2.107}$$

Therefore, we can satisfy the consistency condition (2.101) by determining the Lagrange multiplier h_{00} to be

$$h_{00} = h_{ii}, \tag{2.108}$$

which finishes the Hamiltonian analysis. On the constraint surface $\mathcal{C} = \mathcal{C}_{(2)} = 0$, we have the traceless condition $h^\mu_\mu = -h_{00} + h_{ii} = 0$.

We can count the total number of degrees of freedom in the following way. In phase space, the number of the original dynamical variables is $2 \times \frac{1}{2}D(D-1) = D^2 - D$. We have two constraints $\mathcal{C} = 0$ and $\mathcal{C}_{(2)} = 0$, which reduces the number of the degrees of freedom to $D^2 - D - 2$. In configuration space, we divide it by two and have

$$\frac{1}{2}D(D-1) - 1 \tag{2.109}$$

physical degrees of freedom. In the four-dimensional case ($D = 4$), we have 5 degrees of freedom corresponding to the helicity $(0, \pm 1, \pm 2)$ states. In conclusion, we find that the absence of quadratic terms in h_{00} has eliminated the right number of degrees of freedom. This structure is expected to be generalized to non-linear massive gravity.

2.5 The vDVZ discontinuity

Thus far, we have focused only on the ghost problem and concluded that the Fierz-Pauli tuning is the solution. In this section, we change the subject and consider the massless limit of linear massive gravity with the Fierz-Pauli mass term. Because we know that general relativity has strong predictive power on observations, we expect that the massless limit should coincide with general relativity. However, in the linear level, there is a gap between predictions from general relativity and massive gravity in the massless limit. This phenomenon is called the vDVZ (van Dam, Veltman and Zakharov) discontinuity [12, 13], which we now discuss. Resolution of this issue is one of motivations to extend massive gravity from the linear level to the non-linear level. Non-linear effects are expected to remove this pathology. Thus, in this section, we briefly sketch how this problem emerges.

We consider the four dimensional case including a matter energy momentum tensor

$$S = M_p^2 \int d^4x(\mathcal{L}_{EH} + \mathcal{L}_{FP}) - \frac{1}{2}\int d^4x h_{\mu\nu}T^{\mu\nu}, \tag{2.110}$$

where $\mathcal{L}_{EH}$ and $\mathcal{L}_{FP}$ are given by (2.19) and (2.29), and we have defined the Plank mass $M_p^2 := 1/16\pi G$. If we take directly the massless limit in the above action, we recover pure general relativity, but massive modes are lost. For our purpose, we need to take the massless limit with all of massive modes retained. Thus, we rely on the Stückelberg trick which we have used in Section 2.3. After introducing two new fields A_μ and ϕ as in (2.33) and (2.36), we obtain the action

$$\begin{aligned} S =& M_p^2 \int d^4x\, \mathcal{L}_{EH}(h) + \mathcal{L}_{FP}(h) - \frac{m^2}{4} F^{\mu\nu}F_{\mu\nu} \\ &\qquad - m^2\big(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu\big) - m^2\big(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial_\mu\partial^\mu\phi\big) \\ &- \frac{1}{2}\int d^4x\, \big(h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu\partial_\nu\phi\big)T^{\mu\nu}, \end{aligned} \tag{2.111}$$

where we have defined $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. Then, we assume conservation of the energy momentum tensor $\partial_\mu T^{\mu\nu} = 0$, and normalize A_μ and ϕ as $\hat{A}_\mu := mA_\mu$ and $\hat{\phi} := m^2\phi$, which leads to the action

$$\begin{aligned} S =& M_p^2 \int d^4x\, \mathcal{L}_{EH}(h) + \mathcal{L}_{FP}(h) - \frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} \\ &\qquad - m\big(h_{\mu\nu}\partial^\mu \hat{A}^\nu - h\partial_\mu \hat{A}^\mu\big) - \big(h_{\mu\nu}\partial^\mu\partial^\nu\hat{\phi} - h\partial_\mu\partial^\mu\hat{\phi}\big) \\ &- \frac{1}{2}\int d^4x\, h_{\mu\nu}T^{\mu\nu}. \end{aligned} \tag{2.112}$$

We also note the gauge symmetries corresponding to (2.38) and (2.39)

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \qquad \delta\hat{A}_\mu = -m\xi_\mu, \tag{2.113}$$

$$\delta\hat{A}_\mu = \partial_\mu\Lambda, \qquad \delta\hat{\phi} = -m\Lambda. \tag{2.114}$$

Now, we take the massless limit $m \to 0$, and obtain the action

$$S = M_p^2 \int d^4x\, \mathcal{L}_{EH}(h) - \frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} - \big(h_{\mu\nu}\partial^\mu\partial^\nu\hat{\phi} - h\partial_\mu\partial^\mu\hat{\phi}\big) - \frac{1}{2}\int d^4x\, h_{\mu\nu}T^{\mu\nu}. \tag{2.115}$$

Here, we shift the graviton field $h_{\mu\nu}$ as $h_{\mu\nu} = h'_{\mu\nu} + \frac{2}{D-2}\phi\eta_{\mu\nu}$ to diagonalize the coupling between $h_{\mu\nu}$ and ϕ. The result is

$$S = M_p^2 \int d^4x\, \Big(\mathcal{L}_{EH}(h') - \frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} - \frac{3}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi}\Big) - \frac{1}{2}\int d^4x\, \big(h'_{\mu\nu}T^{\mu\nu} + \phi T\big), \tag{2.116}$$

and the remaining symmetries are

$$\delta h'_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \quad \delta\hat{A}_\mu = \partial_\mu\Lambda. \tag{2.117}$$

Thus, we have one massless spin-2, one massless spin-1 and one spin-0 fields, which contains $2+2+1 = 5$ degrees of freedom. Since no degree of freedom is lost, the action (2.116) describes exactly the massless limit of massive gravity.

The important point is that not only $h'_{\mu\nu}$ but also ϕ couples to matter. Schematically, the equation of motion for them is given by

$$\partial^2 h' = \frac{1}{M_p^2}T, \quad \partial^2\phi = \frac{1}{M_p^2}T. \tag{2.118}$$

We can find that h and ϕ contribute in the same order $h \sim \phi$ to the original graviton field $h_{\mu\nu} = h'_{\mu\nu} + \phi\eta_{\mu\nu}$. Since pure general relativity corresponds to the field $h'_{\mu\nu}$, the massless limit of massive gravity deviates from general relativity. This is the origin of the vDVZ-discontinuity. However, if we introduce some non-linear term $\Xi[\phi]$ to the equation of motion for ϕ

$$\partial^2 h' = \frac{1}{M_p^2}T, \quad \partial^2\phi + \Xi[\phi] = \frac{1}{M_p^2}T, \tag{2.119}$$

there is a possibility that the nonlinear term becomes dominant in some regime, and the contribution from ϕ gets small, $h' \gg \phi$. In this situation, general relativity is restored $h \sim h'$, which is called the Vainshtein mechanism. Therefore, the non-linear extension of linear-massive gravity is expected to resolve the vDVZ discontinuity.

2.6 The Fierz-Pauli mass term on curved space-times

We finish this chapter with a note about the Fierz-Pauli mass term on curved background space-times. Throughout this chapter, we have considered how to exclude an extra degree of freedom which is ghost like, and have not taken care of the possibility that ghosts are contained in the remaining degrees of freedom. On the Minkowski background, the remaining ones are guaranteed to be "regular" particles, which comes from the property of the Poincare-group. However, when a background is curved, the above argument is not applicable. Ghost like modes may be contained even if we exclude the extra degrees of freedom. For example, on the de-Sitter background, we have a ghost when graviton's mass is small compared to the Hubble parameter [14]. In this section, we attempt to detect this type of ghost degrees of freedom. We rely on the Stückelberg trick which we have introduced in section 2.3.

On a curved background, the massless part of the Lagrangian density is given by (2.15). We add the Fierz-Pauli mass term to obtain the action $S = \frac{1}{16\pi G}\int d^D x \mathcal{L}(h)$ and

$$\begin{aligned}\frac{\mathcal{L}(h)}{\sqrt{-\det\bar{g}}} =& -\frac{1}{4}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\alpha h^{\mu\nu}) + \frac{1}{2}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\nu h^{\alpha\mu}) - \frac{1}{2}(\bar{\nabla}_\mu h)(\bar{\nabla}_\nu h^{\mu\nu}) + \frac{1}{4}(\bar{\nabla}_\mu h)(\bar{\nabla}^\mu h)\\ &+ \frac{\bar{R}}{2D}\Big(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\Big) - \frac{1}{4}m^2(h^{\mu\nu}h_{\mu\nu} - h^2),\end{aligned} \tag{2.120}$$

where we have neglected total derivatives and the zeroth order term. We perform index manipulations by the background metric $\bar{g}_{\mu\nu}$ and its inverse $\bar{g}^{\mu\nu}$.

We introduce a Stückelberg field A_μ

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu A_\nu + \nabla_\nu A_\mu, \tag{2.121}$$

and one more Stückelberg field ϕ

$$A_\mu \to A_\mu + \nabla_\mu\phi. \tag{2.122}$$

Then, we have the Lagrangian density

$$\begin{aligned}\frac{\mathcal{L}(h,A,\phi)}{\sqrt{-\det\bar{g}}} =& \frac{\mathcal{L}(h)}{\sqrt{-\det\bar{g}}}\\ &- \frac{1}{4}m^2 F^{\mu\nu}F_{\mu\nu} + \frac{1}{D}m^2\bar{R}A^\mu A_\mu - m^2(h^{\mu\nu}\nabla_\mu A_\nu - h\nabla_\mu A^\mu) + \frac{2m^2\bar{R}}{D}A^\mu\nabla_\mu\phi\\ &+ \frac{m^2\bar{R}}{D}(\nabla^\mu\phi)(\nabla_\mu\phi) - m^2(h^{\mu\nu}\nabla_\mu\nabla_\nu\phi - h\Box\phi),\end{aligned} \tag{2.123}$$

where $F_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and a relation $\nabla_\mu A_\nu \nabla^\nu A^\mu = (\nabla_\mu A^\mu)^2 - \bar{R}_{\mu\nu} A^\mu A^\nu$ has been used. Along with the two Stückelberg fields, two gauge symmetries are introduced

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \qquad \delta A_\mu = -\xi_\mu, \tag{2.124}$$

$$\delta A_\mu = \nabla_\mu \lambda, \qquad \delta\phi = -\lambda. \tag{2.125}$$

Here, we notice that the Fierz-Pauli tuning eliminates higher order derivatives which is the source of the extra ghost like degrees of freedom. Now, we define the shifted graviton field $h'_{\mu\nu}$ as

$$h_{\mu\nu} = h'_{\mu\nu} + \frac{2m^2}{D-2}\phi g_{\mu\nu}. \tag{2.126}$$

The role of this redefinition is to eliminate terms like $h^{\mu\nu}\nabla_\mu\nabla_\nu\phi$ or $h\Box\phi$. Then, the Lagrangian density written by $h'_{\mu\nu}$ is

$$\begin{aligned} \frac{\mathcal{L}(h', A, \phi)}{\sqrt{-\det \bar{g}}} =& \frac{\mathcal{L}(h')}{\sqrt{-\det \bar{g}}} - \frac{1}{4}m^2 F^{\mu\nu}F_{\mu\nu} + \frac{1}{D}m^2 \bar{R} A^\mu A_\mu \\ & - m^2(h'^{\mu\nu}\nabla_\mu A_\nu - h'\nabla_\mu A^\mu) + m^2\Big(\frac{D-1}{D-2}m^2 - \frac{\bar{R}}{D}\Big)(2\phi\nabla_\mu A^\mu + h'\phi) \\ & - m^2\Big(\frac{D-1}{D-2}m^2 - \frac{\bar{R}}{D}\Big)\Big((\nabla_\mu\phi)(\nabla^\mu\phi) - m^2\frac{2D}{D-2}\phi^2\Big). \end{aligned} \tag{2.127}$$

In the above formula, the coefficient of the kinetic term of the scalar field ϕ is $\frac{D-1}{D-2}m^2 - \frac{\bar{R}}{D}$ which can be positive or negative due to the value of the background curvature $\bar{R}$. This fact suggests the possibility that we have ghosts in the remaining degrees of freedom. There is no generic prescription to eliminate this type of ghosts, which must be handled case by case. In general, the term "ghost-free" massive gravity means only the absence of ghost-like extra degrees of freedom. Other types of ghost may be contained, depending on the background, solutions or matter couplings. They are not considered in this thesis.

Chapter 3

Non-linear massive gravity

Chapter 2 has been devoted to the problem of how to construct a consistent theory of massive gravity in the linear level. We have achieved the Fierz-Pauli mass term which contains the right number of degrees of freedom. However, we have encountered a pathology called the vDVZ discontinuity, where predictions from massless limit of massive gravity deviates from those of general relativity. One of motivations for non-linear massive gravity is to resolve this problem by means of non-linear effects. Non-linear effects are also expected to explain observations such as accelerated expansion of the universe or dark energy, but we do not treat these topics in this thesis.

In this chapter, we extend linear massive gravity and construct a theory of non-linear massive gravity. Firstly, we make only the kinetic term (the Einstein-Hilbert term) non-linear while the Fierz-Pauli mass term is left in the linear level. In this setting, we see that an extra degree of freedom is recovered via non-linearity and it behaves as a ghost. Thus, we try to remove it by making the Fierz-Pauli mass term non-linear. For this purpose, the Stückelberg trick is useful. We use the Stückelberg trick in the non-linear context and find the emergence of higher order derivatives which turns out to carry a ghost. Then, we add non-linear terms to the mass term to cancel potentially dangerous higher order derivatives. These newly introduced non-linear terms are combined to become a non-linear mass term. We also attempt further extensions to bi or tri-metric gravity.

As in the linear level, the final step is to count the total number of degrees of freedom, where we rely on the Hamiltonian analysis. The Hamiltonian analysis of fully non-linear massive gravity contains rather lengthy calculations. Hence, the Hamiltonian analysis is postponed to the next chapter.

For more details and related topics, we can consult [8, 9].

3.1 The Fierz-Pauli mass term with the non-linear kinetic term

As a first step toward fully non-linear massive gravity, we attempt to make only the kinetic term non-linear while we leave the Fierz-Pauli mass term in the linear level. Then, we consider the action

$$S = \frac{1}{16\pi G}\int d^D x\left[\sqrt{-\det g}R - \frac{1}{4}m^2\eta^{\mu\alpha}\eta^{\nu\beta}(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta})\right]. \tag{3.1}$$

A full metric $g_{\mu\nu}$ is composed of the background Minkowski metric $\eta_{\mu\nu}$ and a fluctuation $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{3.2}$$

The mass term explicitly contains the background metric. Thus, the diffeomorphism invariance is broken. To begin with, we count the total number of degrees of freedom by the Hamiltonian analysis, and find that there is an extra degree of freedom.

3.1.1 The Hamiltonian analysis

The Hamiltonian analysis containing the non-linear Einstein-Hilbert term is performed based on the ADM (d+1) decomposition [15, 16].

the ADM decomposition

We regard a space-time as a foliation of spatial slices which have the unit normal vector n^μ and the tangent vector $e^\mu{}_i$, where we write indices on the spatial slices as $i, j, k = 1, 2, ..., d$. If we denote general coordinates as X^μ and coordinates on the spatial slices as x^i, the tangent vector can be expressed as $e^\mu_i = \frac{\partial X^\mu}{\partial x^i}$. Since the slicing is spatial, the normal vector is time-like, namely $n^\mu n_\mu = -1$. A time coordinate t is combined with the spatial coordinates x^i to form space-time coordinates. However, there is no natural choice of the time on general space-times. Thus, we define a vector

$$T^\mu := Nn^\mu + N^i e^\mu{}_i, \tag{3.3}$$

and regard it as the tangent vector of the time direction $T^\mu = \frac{\partial X^\mu}{\partial t}$. Coefficients N and N^i determine the direction of the time evolution, and is called the shift and the lapse respectively. Then, the line element is given by

$$\begin{aligned} ds^2 =& g_{\mu\nu} dX^\mu dX^\nu \\ =& g_{\mu\nu}\big(T^\mu dt + e^\mu{}_i dx^i\big)\big(T^\nu dt + e^\nu{}_j dx^j\big) \\ =& g_{\mu\nu}\Big(Nn^\mu dt + \big(N^i dt + dx^i\big)e^\mu{}_i\Big)\Big(Nn^\nu dt + \big(N^j dt + dx^j\big)e^\nu{}_j\Big) \\ =& -N^2 dt^2 + \gamma_{ij}\big(N^i dt + dx^i\big)\big(N^j dt + dx^j\big), \end{aligned} \tag{3.4}$$

where we have introduced the spatial metric $\gamma_{ij} := g_{\mu\nu} e^\mu{}_i e^\nu{}_j$. Equivalently in a matrix form, we can write it as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \gamma_{ij}N^i N^j & \gamma_{jk}N^k \\ \gamma_{ik}N^k & \gamma_{ij} \end{pmatrix}. \tag{3.5}$$

In the following, we denote the inverse of γ_{ij} as γ^{ij}, namely $\gamma^{ik}\gamma_{kj} = \delta^i_j$. We also denote the covariant derivative constructed only from the spatial metric γ_{ij} and γ^{ij} as $\mathcal{D}_i$.

The Gauss-Codazzi relation represents the relation between the curvature of a $(d+1)$-dimensional space-time and that of d-dimensional spatial subspace

$${}^{(d+1)}R = {}^{(d)}R + \big(K_{\mu\nu}K^{\mu\nu} - K^2\big) - 2\nabla_\mu\big(n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu\big), \tag{3.6}$$

where $K_{\mu\nu}$ is the extrinsic curvature and $K = K^{\mu}{}_{\mu}$. We neglect total derivatives and rewrite the Einstein-Hilbert action as

$$\begin{aligned} S_{EH} =& \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-\det g}\, {}^{(d+1)}R \\ =& \frac{1}{16\pi G} \int dt\, d^d x\, N \sqrt{\det \gamma} \left({}^{(d)}R + K_{ij}K^{ij} - K^2 \right). \end{aligned} \tag{3.7}$$

In the above formula, the extrinsic curvature can be expressed as

$$K_{ij} = \frac{1}{2N} \left(\dot{\gamma}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i \right), \tag{3.8}$$

$$K = \gamma^{ij} K_{ij}. \tag{3.9}$$

the massless case

Prior to analyze massive gravity, we consider the case of general relativity. In the ADM formalism, the action is given by

$$S = \frac{1}{16\pi G} \int dt\, d^d x\, \mathcal{L}, \quad \mathcal{L} = N \sqrt{\det \gamma} \left({}^{(d)}R + K_{ij}K^{ij} - K^2 \right). \tag{3.10}$$

It is obvious that the dynamical variables are the spatial metric γ_{ij}. Hence, we define the canonical momenta

$$\pi^{ij} := \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = \sqrt{\det \gamma} \left(K^{ij} - \gamma^{ij} K \right). \tag{3.11}$$

If we invert (3.11), we have

$$\dot{\gamma}_{ij} = \frac{N}{\sqrt{\det \gamma}} \left(2\pi_{ij} - \pi \gamma_{ij} \right) + \mathcal{D}_i N_j + \mathcal{D}_j N_i. \tag{3.12}$$

Instead of (3.12), we use the formula for the extrinsic curvature (3.8)

$$\dot{\gamma}_{ij} = 2N K_{ij} + \mathcal{D}_i N_j + \mathcal{D}_j N_i, \tag{3.13}$$

from which we obtain

$$\dot{\gamma}_{ij} \pi^{ij} = (\mathcal{D}_i N_j + \mathcal{D}_j N_i) \pi^{ij} + 2N \sqrt{\det \gamma} \left(K_{ij}K^{ij} - K^2 \right). \tag{3.14}$$

We also calculate

$$\pi_{ij}\pi^{ij} = (\det \gamma) \left(K_{ij}K^{ij} + (d-2)K^2 \right), \tag{3.15}$$

$$\pi^2 = (\det \gamma)(d-1)^2 K^2, \tag{3.16}$$

and find

$$K^2 = \frac{1}{(d-1)^2 \det \gamma} \pi^2, \tag{3.17}$$

$$K_{ij}K^{ij} = \frac{1}{\det \gamma} \pi_{ij}\pi^{ij} - \frac{d-2}{(d-1)^2 \det \gamma} \pi^2, \tag{3.18}$$

which lead to

$$K_{ij}K^{ij} - K^2 = \frac{1}{\det\gamma}\Big(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2\Big). \tag{3.19}$$

Then, the action can be read as

$$S = \frac{1}{16\pi G}\int dt\, d^dx\,\Big[\dot{\gamma}_{ij}\pi^{ij} + N\mathcal{R}_0 + N^i\mathcal{R}_i\Big], \tag{3.20}$$

where each element is defined by

$$\begin{aligned}\mathcal{R}_0 :=& \sqrt{\det\gamma}\,{}^{(d)}R - \frac{1}{\sqrt{\det\gamma}}\Big(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2\Big)\\ =& \sqrt{\det\gamma}\,{}^{(d)}R - \frac{1}{\sqrt{\det\gamma}}\Big(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\Big)\pi^{ij}\pi^{kl}, \qquad (3.21)\\ \mathcal{R}_i :=& 2\mathcal{D}_j\pi^j{}_i = 2\gamma_{ik}\mathcal{D}_j\pi^{jk}. \qquad (3.22)\end{aligned}$$

Hence, N and N^i are interpreted as Lagrange multipliers and variation with respect to them leads to constraints

$$\mathcal{R}_0 = 0, \qquad \mathcal{R}_i = 0. \tag{3.23}$$

These constraints must be preserved along the time evolution, which is represented as consistency conditions

$$\frac{d}{dt}\mathcal{R}_0 = \big\{\mathcal{R}_0,\, H\big\}_{PB} \approx 0, \quad \frac{d}{dt}\mathcal{R}_i = \big\{\mathcal{R}_i,\, H\big\}_{PB} \approx 0. \tag{3.24}$$

The symbol "$\approx$" stands for the equality on the hypersurface determined by the constraints, and the Hamiltonian H is give by

$$H = \int d^dx\big(-N\mathcal{R}_0 - N^i\mathcal{R}_i\big). \tag{3.25}$$

The Poisson bracket is determined by the following formula

$$\big\{F(x), G(y)\big\}_{PB} = \int d^dz\left[\frac{\delta F(x)}{\delta\gamma_{ij}(z)}\frac{\delta G(y)}{\delta\pi^{ij}(z)} - \frac{\delta F(x)}{\delta\pi^{ij}(z)}\frac{\delta G(y)}{\delta\gamma_{ij}(z)}\right], \tag{3.26}$$

where the integration is only on the spatial coordinates and the time component is set to be equal on each factor. Here, it should be noted that we are neglecting the overall factor $16\pi G$ for notational simplicity.

We explicitly calculate the Poisson brackets between $\mathcal{R}_0$ and $\mathcal{R}_i$ in Appendix C, and the result is

$$\begin{aligned}\big\{\mathcal{R}_0(x), \mathcal{R}_0(y)\big\}_{PB} &= \mathcal{R}_i(y)\mathcal{D}^i_{(y)}\delta^{(d)}(x-y) - \mathcal{R}_i(x)\mathcal{D}^i_{(x)}\delta^{(d)}(x-y) \approx 0, \qquad (3.27)\\ \big\{\mathcal{R}_0(x), \mathcal{R}_i(y)\big\}_{PB} &= -\mathcal{R}_0(y)\mathcal{D}^{(x)}_i\delta(x-y) \approx 0, \qquad (3.28)\\ \big\{\mathcal{R}_i(x), \mathcal{R}_j(y)\big\}_{PB} &= \mathcal{R}_i(y)\mathcal{D}^{(y)}_j\delta^{(d)}(x-y) - \mathcal{R}_j(x)\mathcal{D}^{(x)}_i\delta^{(d)}(x-y) \approx 0. \qquad (3.29)\end{aligned}$$

Therefore, we see that the consistency conditions (3.24) are automatically satisfied, and the Lagrange multipliers N and N^i are left undetermined. These remaining Lagrange multipliers correspond to the diffeomorphism invariance contained in general relativity.

The total number of degrees of freedom can be counted. In phase space, the number of the original dynamical variables is $2 \times \frac{1}{2}d(d+1)$ coming from symmetric tensors h_{ij} and π^{ij} $(i, j = 1, 2, .., d)$. The constraints $\mathcal{R}_0$ and $\mathcal{R}_i$ eliminate $d+1$ degrees of freedom. The gauge freedom corresponding to the remaining Lagrange multipliers reduce further $d+1$ degrees of freedom. Then, divided by two, the total number of degrees of freedom is

$$\frac{1}{2}d(d+1) - (d+1). \tag{3.30}$$

In the case of a four dimensional space-time ($d = 3$), we have two degrees of freedom, which is compatible with the analysis in linearized general relativity.

the massive case

In the massive case $m \neq 0$, the action is given by (3.1). Obviously, it is convenient to regard the full metric $g_{\mu\nu}$ as a basic variable. Thus, $h_{\mu\nu}$ is interpreted as $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Since the mass term contains no derivative, the definition of the canonical momenta is not changed. We apply the ADM decomposition (3.5) for the full metric $g_{\mu\nu}$, and obtain the action

$$\begin{aligned} S =& \frac{1}{16\pi G}\int dt\, d^dx \Big[\dot{\gamma}_{ij}\pi^{ij} + N\mathcal{R}_0 + N^i\mathcal{R}_i \\ &- \frac{1}{4}m^2\Big(\delta^{ik}\delta^{jl}(h_{ij}h_{kl} - h_{ik}h_{jl}) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} + 2N_iN_j(\gamma^{ij} - \delta^{ij})\Big)\Big], \end{aligned} \tag{3.31}$$

where $h_{ij} := \gamma_{ij} - \delta_{ij}$. We notice that the crucial point is the break down of linearity for N and N^i. They become auxiliary fields and the equation of motion for them can be solved as

$$N = \frac{\mathcal{R}_0}{m^2\delta^{ij}h_{ij}}, \quad N_i = \frac{1}{m^2}(\gamma^{ij} - \delta^{ij})^{-1}\mathcal{R}^j. \tag{3.32}$$

These formulae are plugged back into the original action, and we have no constraint and no Lagrange multiplier. Therefore, all the dynamical variables h_{ij} $(i, j = 1, 2, ..., d)$ remain. The total number of degrees of freedom is counted to be

$$\frac{1}{2}d(d+1). \tag{3.33}$$

In the case of a four dimensional space-time ($d = 3$), we have six degrees of freedom. One extra degree of freedom is recovered via the non-linear extension of the kinetic term. This extra one is called the BD (Boulware and Deser) ghost [17].

3.1.2 The Stückelberg trick

We have seen that the combination of the non-linear Einstein-Hilbert term and the linear Fierz-Pauli mass term leads to an extra degree of freedom. Then, we show that it is actually a ghost. For this purpose, the Stückelberg trick is useful. In Section 2.3, we have introduced the Stückelberg trick in the linear level. Hence, we need to extend it to the non-linear level [18].

The essence of the Stückelberg trick is to introduce a new field imitating the broken symmetry. In the action (3.1), the invariance broken by the mass term is the diffeomorphism invariance of the Einstein-Hilbert term. Therefore, we introduce a new field Y^μ and perform the following replacement

$$g_{\mu\nu}(x) \to G_{\mu\nu}(x) := \frac{\partial Y^\alpha}{\partial x^\mu}\frac{\partial Y^\beta}{\partial x^\nu} g_{\alpha\beta}\big(Y(x)\big). \tag{3.34}$$

Since the Einstein-Hilbert term retains the diffeomorphism invariance, it is not changed. Only the mass term picks up a change. Thus, We have only to focus on $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ in the Fierz-Pauli mass term, and replace it by $H_{\mu\nu} := G_{\mu\nu} - \eta_{\mu\nu}$. Then, the action is given by

$$S = \frac{1}{16\pi G}\int d^D x\left[\sqrt{-\det g}R - \frac{1}{4}m^2\eta^{\mu\alpha}\eta^{\nu\beta}(H_{\mu\nu}H_{\alpha\beta} - H_{\mu\alpha}H_{\nu\beta})\right] - \frac{1}{2}\int d^D x h_{\mu\nu}T^{\mu\nu}. \tag{3.35}$$

For later convenience, we have added a matter coupling. $T^{\mu\nu}$ represents its energy momentum tensor. Along with the new field Y^α, a gauge symmetry is introduced

$$g_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu} g_{\alpha\beta}\big(f(x)\big), \quad Y^\mu(x) \to f^{-1}\big(Y(x)\big)^\mu, \tag{3.36}$$

where f is a function corresponding to the gauge freedom. This is because the transformation for Y^α after that for $g_{\mu\nu}$ can be expressed as

$$\begin{aligned}\frac{\partial Y^\alpha}{\partial x^\mu}\frac{\partial Y^\beta}{\partial x^\nu} g_{\alpha\beta}\big(Y(x)\big) \to & \frac{\partial Y^\alpha}{\partial x^\mu}\frac{\partial Y^\beta}{\partial x^\nu}\frac{\partial f^\lambda}{\partial Y^\alpha}(Y)\frac{\partial f^\rho}{\partial Y^\beta}(Y) g_{\lambda\rho}\big(f(Y(x))\big) \\ = & \frac{\partial f^\lambda(Y(x))}{\partial x^\mu}\frac{\partial f^\rho(Y(x))}{\partial x^\nu} g_{\lambda\rho}\big(f(Y(x))\big) \\ \to & \frac{\partial Y^\lambda}{\partial x^\mu}\frac{\partial Y^\rho}{\partial x^\nu} g_{\lambda\rho}\big(Y(x)\big),\end{aligned} \tag{3.37}$$

and $G_{\mu\nu}$ itself is invariant. The gauge fixing $Y^\alpha = x^\alpha$ returns the original action (3.1).

Now, we shift Y^α to define A^α

$$Y^\alpha(x) =: x^\alpha + A^\alpha(x), \tag{3.38}$$

and expand the invariant tensor $G_{\mu\nu}$

$$\begin{aligned}G_{\mu\nu} = & \big(\delta^\alpha_\mu + \partial_\mu A^\alpha\big)\big(\delta^\beta_\nu + \partial_\nu A^\beta\big)\Big(g_{\alpha\beta} + A^\lambda\partial_\lambda g_{\alpha\beta} + \frac{1}{2}A^\lambda A^\rho\partial_\lambda\partial_\rho g_{\alpha\beta}\Big) \\ = & g_{\mu\nu} + (\partial_\mu A^\alpha)g_{\alpha\nu} + (\partial_\nu A^\beta)g_{\beta\mu} + (\partial_\mu A^\alpha)(\partial_\nu A^\beta)g_{\alpha\beta} + (\text{terms containing } \partial^{n\geq 1}g).\end{aligned} \tag{3.39}$$

We perform index manipulations by the background metric $\eta_{\mu\nu}$ and obtain

$$\begin{aligned}H_{\mu\nu} = & h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + (\partial_\mu A^\alpha)(\partial_\nu A_\alpha) \\ & + (\partial_\mu A^\alpha)h_{\alpha\nu} + (\partial_\nu A^\beta)h_{\beta\mu} + (\partial_\mu A^\alpha)(\partial_\nu A^\beta)h_{\alpha\beta} + (\text{terms containing } \partial^{n\geq 1}h).\end{aligned} \tag{3.40}$$

Here, we introduce one more Stückelberg field ϕ through the replacement

$$A_\mu \to A_\mu + \partial_\mu\phi, \tag{3.41}$$

and find

$$
\begin{aligned}
H_{\mu\nu} = h_{\mu\nu} &+ \partial_\mu A_\nu + \partial_\nu A_\mu + (\partial_\mu A^\alpha)(\partial_\nu A_\alpha) \\
&+ 2\partial_\mu \partial_\nu \phi + (\partial_\mu \partial^\alpha \phi)(\partial_\nu \partial_\alpha \phi) \\
&+ (\text{terms containing } h^{n\geq 1}).
\end{aligned} \tag{3.42}
$$

Gauge symmetries are found from the transformation law for Y^α (3.36).

Our present purpose is to expand the action (3.35) by $h_{\mu\nu}$, A_μ and ϕ and see whether a ghost degree of freedom appears. In the following, we treat the four dimensional case $D = 4$ and put $M_p^2 := \frac{1}{16\pi G}$. Firstly, we focus on the Einstein-Hilbert part. The expansion is

$$
\begin{aligned}
S_{EH} =& M_p^2 \int d^4x \sqrt{-\det g} R \\
=& \int d^4x \Big\{ -\frac{M_p^2}{4} (\partial_\lambda h_{\mu\nu})(\partial^\lambda h^{\mu\nu}) + \cdots + M_p^2 \times (\text{terms of } h^{n\geq 3}) \Big\},
\end{aligned} \tag{3.43}
$$

which contains the kinetic part for $h_{\mu\nu}$. In order to eliminate the dimensional coefficient attached to the kinetic term, we normalize $h_{\mu\nu}$ as

$$
\hat{h}_{\mu\nu} := M_p h_{\mu\nu}. \tag{3.44}
$$

Then, the Einstein-Hilbert action is rewritten as

$$
S_{EH} = \int d^4x \Big\{ -\frac{1}{4}(\partial_\lambda \hat{h}_{\mu\nu})(\partial^\lambda \hat{h}^{\mu\nu}) + \cdots + \frac{1}{M_p}(\text{terms of } \hat{h}^3) + \frac{1}{M_p^2}(\text{terms of } \hat{h}^4) + \cdots \Big\}. \tag{3.45}
$$

On the other hand, the Fierz-Pauli mass term is expanded as

$$
\begin{aligned}
S_{FP} :=& - M_p^2 \frac{m^2}{4} \int d^4x \big(H_{\mu\nu}H^{\mu\nu} - H^2\big) \\
=& \int d^4x \Big\{ - M_p^2 \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - h^2) - M_p^2 \frac{m^2}{4} F_{\mu\nu}F^{\mu\nu} \\
& \qquad - M_p^2 m^2 (h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) - M_p^2 m^2 (h_{\mu\nu}\partial^\mu\partial^\nu \phi - h\partial_\mu\partial^\mu \phi) \Big\} \\
& \qquad + M_p^2 m^2 (\text{higher than third order terms of } h, A, \phi) \Big\}.
\end{aligned} \tag{3.46}
$$

We normalize A_μ and ϕ in the same way as $h_{\mu\nu}$

$$
\hat{A}_\mu := m M_p A_\mu, \qquad \hat{\phi} := m^2 M_p \phi, \tag{3.47}
$$

and obtain

$$
\begin{aligned}
S_{FP} = \int d^4x \Big\{ & -\frac{m^2}{4}(\hat{h}_{\mu\nu}\hat{h}^{\mu\nu} - \hat{h}^2) - \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - m(\hat{h}_{\mu\nu}\partial^\mu \hat{A}^\nu - \hat{h}\partial_\mu \hat{A}^\mu) - (\hat{h}_{\mu\nu}\partial^\mu\partial^\nu \hat{\phi} - \hat{h}\partial_\mu\partial^\mu \hat{\phi}) \\
& + (\text{higher than third order terms of } \hat{h}, \hat{A}, \hat{\phi}) \Big\}.
\end{aligned} \tag{3.48}
$$

We also consider the matter coupling term

$$
\begin{aligned}
S_{coup} :=& -\frac{1}{2}\int d^4x H_{\mu\nu}T^{\mu\nu} \\
=& -\int d^4x \Big\{ \frac{1}{2}h_{\mu\nu}T^{\mu\nu} + (\partial_\nu A_\mu)T^{\mu\nu} + (\partial_\mu\partial_\nu\phi)T^{\mu\nu} \\
& \qquad + T\times(\text{higher than second order terms of } h, A, \phi)\Big\}.
\end{aligned} \tag{3.49}
$$

We assume that the energy momentum tensor $T^{\mu\nu}$ is conserved $\partial_\mu T^{\mu\nu} = 0$, and find

$$
S_{coup} = -\int d^4x \Big\{ \frac{1}{2M_p}\hat{h}_{\mu\nu}T^{\mu\nu} + T\times(\text{higher than second order terms of } h, A, \phi)\Big\}. \tag{3.50}
$$

Now, we have various interaction terms, but important ones are coming from the expansion of S_{FP}. They are expressed as

$$
m^2 M_p^2\, h^{n(h)}(\partial A)^{n(A)}(\partial^2\phi)^{n(\phi)} = \Lambda_\lambda^{4-n(h)-2n(A)-3n(\phi)}(\hat{h})^{n(h)}(\partial\hat{A})^{n(A)}(\partial^2\hat{\phi})^{n(\phi)}, \tag{3.51}
$$

where

$$
\Lambda_\lambda := m\left(\frac{M_p}{m}\right)^{\frac{1}{\lambda}}, \quad \lambda := \frac{3n(\phi)+2n(A)+n(h)-4}{n(\phi)+n(A)+n(h)-2} = 2 + \frac{n(\phi)-n(h)}{n(\phi)+n(A)+n(h)-2}. \tag{3.52}
$$

In general, it is natural to think that the Planck mass M_p is larger than graviton's mass m. Thus, we assume $M_p > m$ and find that Λ_λ decreases when λ increases. Focusing on higher than third order interactions ($n(\phi)+n(A)+n(h) \geq 3$), the maximum value of λ is 5 when we have $n(A) = 0, n(h) = 0, n(\phi) = 3$. Hence, the minimum value of Λ_λ is $\Lambda_5 = (M_p m^4)^{\frac{1}{5}}$. On the other hand, a condition $4 - n(h) - 2n(A) - 3n(\phi) < 0$ always holds because we do not have a term with $n(h) = 3, n(A) = 0, n(\phi) = 0$. Therefore, the interaction terms (3.51) have negative mass dimension and non-renormalizable. Λ_5 is the cut-off scale as an effective field theory.

Since the expanded action contains a lot of interaction terms, it seems difficult to detect a ghost degree of freedom. Then, we focus on the cut off scale, and take a limit called Λ_5 decoupling limit

$$
m \to 0, \quad M_p \to \infty, \quad T \to \infty, \quad \Lambda_5, \frac{T}{M_p} : \text{fixed}. \tag{3.53}
$$

In this limit, we have

$$
\Lambda_{\lambda<5} = m\left(\frac{M_p}{m}\right)^{\frac{1}{\lambda}} = \Lambda_5\left(\frac{M_p}{m}\right)^{\frac{1}{\lambda}-\frac{1}{5}} \to \infty. \tag{3.54}
$$

Therefore, among higher than third order interaction terms, only the term with $n(\phi) = 3, n(A) = 0, n(h) = 0$ remains, whose explicit formula is given by

$$
\begin{aligned}
& -M_p^2\frac{m^2}{4}\Big\{4(\partial_\mu\partial_\nu\phi)(\partial_\mu\partial^\alpha\phi)(\partial_\nu\partial_\alpha\phi) - 4(\Box\phi)(\partial_\mu\partial_\alpha\phi)(\partial^\mu\partial^\alpha\phi)\Big\} \\
=& \frac{1}{2\Lambda_5^5}\Big\{(\Box\hat{\phi})^3 - (\Box\hat{\phi})(\partial_\mu\partial_\nu\hat{\phi})(\partial^\mu\partial^\nu\hat{\phi})\Big\}.
\end{aligned} \tag{3.55}
$$

In order to diagonalize the coupling between $h_{\mu\nu}$ and ϕ, we shift the graviton field $\hat{h}_{\mu\nu}$ as $\hat{h}_{\mu\nu} = \hat{h}'_{\mu\nu} + \hat{\phi}\eta_{\mu\nu}$, and obtain the action

$$\begin{aligned} S = \int d^4x \Big\{ &- \frac{1}{4}(\partial_\alpha \hat{h}'_{\mu\nu})(\partial^\alpha \hat{h}'^{\mu\nu}) + \frac{1}{2}(\partial_\alpha \hat{h}'_{\mu\nu})(\partial^\nu \hat{h}'^{\alpha\mu}) - \frac{1}{2}(\partial_\mu \hat{h}')(\partial_\nu \hat{h}'^{\mu\nu}) + \frac{1}{4}(\partial_\mu \hat{h}')(\partial^\mu \hat{h}') \\ &- \frac{1}{2M_p}\hat{h}'_{\mu\nu}T^{\mu\nu} - \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} \\ &- \frac{3}{2}(\partial_\mu\hat{\phi})(\partial^\mu\hat{\phi}) + \frac{1}{2\Lambda_5^5}\Big[(\Box\hat{\phi})^3 - (\Box\hat{\phi})(\partial_\mu\partial_\nu\hat{\phi})(\partial^\mu\partial^\nu\hat{\phi})\Big] - \frac{1}{2M_p}\hat{\phi}T\Big\}. \end{aligned} \tag{3.56}$$

We can see that non-linearity has lead to higher order derivatives with respect to time which operate on ϕ. This fact suggests the emergence of a ghost degree of freedom [10, 11]. For simplicity, we consider such a Lagrangian density below (More details are found in [19].):

$$\mathcal{L}_{\hat{\phi}} := -(\partial_\mu\hat{\phi})(\partial^\mu\hat{\phi}) + \frac{1}{\Lambda_5^5}(\Box\hat{\phi})^3 - \frac{1}{M_p}\hat{\phi}T. \tag{3.57}$$

This is a simplified version of the scalar part in (3.56). The point is that an non-linear term $(\Box\phi)^3$ is not a total derivative combination, and it cannot exclude a higher order derivative with respect to the time coordinate. In addition, we introduce one more Lagrangian density

$$\mathcal{L}_{\hat{\phi},\psi} := -(\partial_\mu\hat{\phi})(\partial^\mu\hat{\phi}) - 2(\partial_\mu\hat{\phi})(\partial^\mu\psi) + \frac{4}{3}\sqrt{\frac{2}{3}}\Lambda_5^{5/2}\psi^{\frac{3}{2}} - \frac{1}{M_p}\hat{\phi}T \tag{3.58}$$

The field ψ is an auxiliary field. The equation of motion for ψ is given by

$$\frac{\delta\mathcal{L}_{\hat{\phi},\psi}}{\delta\psi} = 2\Box\hat{\phi} + 2\sqrt{\frac{2}{3}}\Lambda_5^{5/2}\psi^{\frac{1}{2}} = 0, \tag{3.59}$$

which is solved as a function of ϕ

$$\psi^{\frac{1}{2}} = -\sqrt{\frac{2}{3}}\Lambda_5^{-5/2}\Box\hat{\phi}. \tag{3.60}$$

Substituting this solution back into $\mathcal{L}_{\hat{\phi},\psi}$, we find

$$\mathcal{L}_{\hat{\phi},\psi}\Big|_{\psi^{\frac{1}{2}}=-\sqrt{\frac{2}{3}}\Lambda_5^{-5/2}\Box\hat{\phi}} = -(\partial_\mu\hat{\phi})(\partial^\mu\hat{\phi}) + \frac{1}{\Lambda_5^5}(\Box\hat{\phi})^3 + \frac{1}{M_p}\hat{\phi}T = \mathcal{L}_{\hat{\phi}}. \tag{3.61}$$

Thus, two Lagrangian densities $\mathcal{L}_{\hat{\phi}}$ and $\mathcal{L}_{\hat{\phi},\psi}$ are physically equivalent. We can investigate $\mathcal{L}_{\hat{\phi},\psi}$ instead of $\mathcal{L}_{\hat{\phi}}$. Here, we put

$$\hat{\phi} = \varphi - \psi, \tag{3.62}$$

and obtain

$$\mathcal{L}_{\chi,\psi} = \mathcal{L}_{\hat{\phi}=\varphi-\psi,\psi} = -(\partial_\mu\varphi)(\partial^\mu\varphi) + (\partial_\mu\psi)(\partial^\mu\psi) + \frac{4}{3}\sqrt{\frac{2}{3}}\Lambda_5^{5/2}\psi^{\frac{3}{2}} - \frac{1}{M_p}\varphi T + \frac{1}{M_p}\psi T. \tag{3.63}$$

In the above formula, ψ behaves as a ghost. Therefore, we conclude that the 6th extra degree of freedom is a ghost mode. This is called the BD (Boulware and Deser) ghost [17].

3.2 How to eliminate the BD-ghost

In Section 3.1, we have considered the half non-linearization, where the non-linear Einstein-Hilbert term and the linear Fierz-Pauli mass term are combined. Effects of the half non-linearization has been investigated by the Stückelberg trick. Then, we have found that the half non-linearity leads to higher order derivatives which carry a ghost degree of freedom called the BD-ghost. At first sight, this fact seems catastrophic. However, we should notice that this theory is not fully non-linear. There is room to cancel these dangerous higher order derivatives by non-linearly extending the Fierz-Pauli mass term. Thus, we add non-linear terms constructed from a fluctuation $h_{\mu\nu}$ such as $h^3, h^4, ...$ with no derivative. We adjust them to cancel higher order derivatives when we expand the action by the Stückelberg fields. In this section, we try to describe how this program works. The purpose of this section is not to explicitly construct a theory of non-linear massive gravity. However, the calculation below gives us an important clue to determine a non-linear and ghost-free mass term. More details are found in [20, 21].

Following [20, 21], we start with the action

$$S = M_p^2 \int d^4x \left[\sqrt{-\det g} R - \frac{m^2}{4} \sqrt{-\det g}\, g^{\mu\alpha} g^{\nu\beta} (h_{\mu\nu} h_{\alpha\beta} - h_{\mu\alpha} h_{\nu\beta}) \right], \tag{3.64}$$

where $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is understood. Compared to the action (3.1), the above action (3.64) already contains some non-linearity in the mass term . This is merely for our convenience.

Our main tool is the Stückelberg trick, but we introduce it in a rather different way from that in Section 3.1.2. We bring in a field Y^μ through the following replacement

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \to H_{\mu\nu} = g_{\mu\nu} - \bar{G}_{\mu\nu}, \tag{3.65}$$

where

$$\bar{G}_{\mu\nu}(x) := \frac{\partial Y^\alpha}{\partial x^\mu}(x) \frac{\partial Y^\beta}{\partial x^\nu}(x) \eta_{\alpha\beta}. \tag{3.66}$$

The replacement is done only on $h_{\mu\nu}$ in the mass term. Then, the action is given by

$$S = M_p^2 \int d^4x \left[\sqrt{-\det g} R - \frac{m^2}{4} \sqrt{-\det g}\, g^{\mu\alpha} g^{\nu\beta} (H_{\mu\nu} H_{\alpha\beta} - H_{\mu\alpha} H_{\nu\beta}) \right]. \tag{3.67}$$

Along with Y^α, a gauge symmetry is introduced

$$g_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} g_{\alpha\beta}\big(f(x)\big), \quad Y^\mu(x) \to Y^\mu(f(x))), \tag{3.68}$$

where f is a function corresponding to the gauge freedom. This is because $\bar{G}_{\mu\nu}$ transforms as a tensor field

$$\begin{aligned} \frac{\partial Y^\alpha}{\partial x^\mu}(x) \frac{\partial Y^\beta}{\partial x^\nu}(x) \eta_{\alpha\beta} \to & \frac{\partial Y^\alpha}{\partial x^\mu}(f(x)) \frac{\partial Y^\beta}{\partial x^\nu}(f(x)) \eta_{\alpha\beta} \\ = & \frac{\partial f^\lambda}{\partial x^\mu} \frac{\partial Y^\alpha}{\partial f^\lambda}(f(x)) \frac{\partial f^\rho}{\partial x^\nu} \frac{\partial Y^\beta}{\partial f^\rho}(f(x)) \eta_{\alpha\beta} = \frac{\partial f^\lambda}{\partial x^\mu} \frac{\partial f^\rho}{\partial x^\nu} \frac{\partial Y^\alpha}{\partial f^\lambda}(f(x)) \frac{\partial Y^\beta}{\partial f^\rho}(f(x)) \eta_{\alpha\beta}. \end{aligned} \tag{3.69}$$

Thus, $H_{\mu\nu}$ also obeys the transformation law for a tensor field. The gauge fixing $Y^\alpha = x^\alpha$ restores the original action (3.64).

Now, we shift Y^α to define A^α

$$Y^\alpha = x^\alpha - A^\alpha. \tag{3.70}$$

Then, $H_{\mu\nu}$ is expanded as

$$\begin{aligned}H_{\mu\nu} =& g_{\mu\nu} - (\delta^\alpha_\mu - \partial_\mu A^\alpha)(\delta^\beta_\nu - \partial_\nu A^\beta)\eta_{\alpha\beta} \\ =& h_{\mu\nu} + (\partial_\mu A^\alpha)\eta_{\alpha\nu} + (\partial_\nu A^\beta)\eta_{\beta\mu} - (\partial_\mu A^\alpha)(\partial_\nu A^\beta)\eta_{\alpha\beta} \\ =& h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu - (\partial_\mu A^\alpha)(\partial_\nu A_\alpha),\end{aligned} \tag{3.71}$$

where index manipulations are done by the background metric $\eta_{\mu\nu}$. We introduce one more Stückelberg field ϕ through the replacement

$$A^\alpha \to A^\alpha + \partial^\alpha \phi, \tag{3.72}$$

and the expansion of $H_{\mu\nu}$ reads

$$\begin{aligned}H_{\mu\nu} =& h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu - (\partial_\mu A^\alpha)(\partial_\nu A_\alpha) \\ &+ 2\partial_\mu\partial_\nu\phi - (\partial_\mu\partial^\alpha\phi)(\partial_\nu\partial_\alpha\phi) \\ &- (\partial_\mu A^\alpha)(\partial_\nu\partial_\alpha\phi) - (\partial_\nu A^\beta)(\partial_\mu\partial_\beta\phi).\end{aligned} \tag{3.73}$$

As in Section 3.1.2, we expand the action (3.67) by $h_{\mu\nu}$, A_μ and ϕ. We remember (3.51) and write down the general form of interaction terms coming from the mass term

$$m^2 M_p^2\, h^{n(h)}(\partial A)^{n(A)}(\partial^2\phi)^{n(\phi)} = \Lambda_\lambda^{4-n(h)-2n(A)-3n(\phi)}(\hat{h})^{n(h)}(\partial\hat{A})^{n(A)}(\partial^2\hat{\phi})^{n(\phi)}, \tag{3.74}$$

with

$$\Lambda_\lambda = m\left(\frac{M_p}{m}\right)^{\frac{1}{\lambda}}, \quad \lambda = \frac{3n(\phi)+2n(A)+n(h)-4}{n(\phi)+n(A)+n(h)-2} = 2 + \frac{n(\phi)-n(h)}{n(\phi)+n(A)+n(h)-2}. \tag{3.75}$$

Hat symbols represent normalized fields $\hat{h}_{\mu\nu} = M_p h_{\mu\nu}$, $\hat{A}_\mu = m M_p A_\mu$ and $\hat{\phi} = m^2 M_p \phi$. Since our main interest is in higher order derivatives, we focus on higher than third order $n(\phi) + n(A) + n(h) \geq 3$ terms. We can classify them according to the value of λ

$\cdot\, \lambda > 5$:

$2n(\phi) + 3n(A) + 4n(h) < 6 \Rightarrow$ no solution

$\cdot\, \lambda = 5$:

$2n(\phi) + 3n(A) + 4n(h) = 6 \Rightarrow (n(\phi),\, n(A),\, n(h)) = (3, 0, 0)$

$\cdot\, 4 < \lambda < 5$:

$n(\phi) + 2n(A) + 3n(h) < 4,\ 2n(\phi) + 3n(A) + 4n(h) > 6 \Rightarrow$ no solution

$\cdot\, \lambda = 4$:

$n(\phi) + 2n(A) + 3n(h) = 4 \Rightarrow (n(\phi),\, n(A),\, n(h)) = (2, 1, 0), (4, 0, 0)$

$\cdot\, 3 < \lambda < 4$:

$n(A) + 2n(h) < 2,\ n(\phi) + 2n(A) + 3n(h) > 4,$

$\Rightarrow (n(\phi),\, n(A),\, n(h)) = (n \geq 5, 0, 0)_{\lambda=3+\frac{2}{n-2}}, (n \geq 3, 1, 0)_{\lambda=3+\frac{1}{n-1}}$

$\cdot\, \lambda = 3$:

$$n(A) + 2n(h) = 2 \Rightarrow (n(\phi),\, n(A),\, n(h)) = (n \geq 2, 0, 1), (n \geq 1, 2, 0), \tag{3.76}$$

which is summarized in the following table.

scale	Λ_5	Λ_4	$\Lambda_{3+\frac{2}{n-2}}$	$\Lambda_{3+\frac{1}{n-1}}$	Λ_3	$\Lambda_{n>3}$
interaction	$(\partial^2\hat{\phi})^3$	$(\partial^2\hat{\phi})^4$ $(\partial\hat{A})(\partial^2\hat{\phi})^2$	$(\partial^2\hat{\phi})^{n\geq 5}$	$(\partial\hat{A})(\partial^2\hat{\phi})^{n\geq 3}$	$\hat{h}(\partial^2\hat{\phi})^{n\geq 2}$ $(\partial\hat{A})^2(\partial^2\hat{\phi})^{n\geq 1}$	$\cdots$

All of these interaction terms can have potentially dangerous higher order derivatives. Our present purpose is to cancel out them by adding non-linear terms of $H_{\mu\nu}$. In the following calculations, we abbreviate $\phi_{\mu\nu} = \partial_\mu\partial_\nu\phi$ and $A_{\mu\nu} = \partial_\mu A_\nu$ for notational simplicity. Besides, we denote the trace with respect to $g_{\mu\nu}$ as $\langle \cdot \rangle$ and the trace with respect to $\eta_{\mu\nu}$ as $[\cdot]$. For example, we write

$$\langle\phi\rangle = g^{\mu\nu}\phi_{\mu\nu}, \qquad \langle h\phi\rangle = g^{\nu\alpha}g^{\mu\beta}h_{\mu\nu}\phi_{\alpha\beta}, \tag{3.77}$$

$$[\phi] = \eta^{\mu\nu}\phi_{\mu\nu}, \qquad [h\phi] = \eta^{\nu\alpha}\eta^{\mu\beta}h_{\mu\nu}\phi_{\alpha\beta}. \tag{3.78}$$

Now, we start the program with the action (3.67)

$$S = M_p^2\int d^4x\Big[\sqrt{-g}R - \frac{m^2}{4}\mathcal{L}_2\Big], \quad \mathcal{L}_2 := \sqrt{-g}\Big(\langle H^2\rangle - \langle H\rangle^2\Big). \tag{3.79}$$

We know that $\mathcal{L}_2$ contains higher order derivative terms which lead to the appearance of a extra ghost degree of freedom. One of them is $(\partial^2\phi)^3$. We add a third order term of $H_{\mu\nu}$, which we denote $\mathcal{L}_3$, to cancel this $(\partial^2\phi)^3$ term. The explicit form of $\mathcal{L}_3$ is completely fixed to cancel the potentially dangerous $(\partial^2\phi)^3$ term coming from $\mathcal{L}_2$. Here, we have another freedom. We can also add a third order term $\mathcal{L}_3^{TD}$ which returns a total derivative about the $(\partial^2\phi)^3$ part. Thus, we get a mass term $\mathcal{L}_2 + \mathcal{L}_3 + \alpha_3\mathcal{L}_3^{TD}$, where α_3 is a constant. The coefficient α_3 is adjusted to cancel other third order dangerous terms such as $h(\partial^2\phi)^2$. From the third order mass term $\mathcal{L}_2+\mathcal{L}_3+\alpha_3\mathcal{L}_3^{TD}$, we encounter further higher order derivative terms such as $(\partial^2\phi)^4$. Thus, we add a fourth order term $\mathcal{L}_4$ to cancel the $(\partial^2\phi)^4$ term. Also, we can add another fourth order term $\mathcal{L}_4^{TD}$ which returns a total derivative combination about $(\partial^2\phi)^4$. We make use of $\mathcal{L}_4^{TD}$ to eliminate other fourth order ghost like terms. Whether this program works or not is a non-trivial problem. We consider up to the fourth order cancellation for $(\partial^2\phi)^2$, $(\partial^2\phi)^3$, $(\partial^2\phi)^4$, $(\partial A)(\partial^2\phi)^2$, $(\partial A)(\partial^2\phi)^3$, $h(\partial^2\phi)^2$ and $h(\partial^2\phi)^3$. Such a term $(\partial A)^2(\partial^2\phi)$ is difficult to investigate. The details are found in [22], but we do not consider it here.

From the expansion of $\mathcal{L}_2$, we find

$$(\partial^2\phi)^2 \;: \text{total derivative} \tag{3.80}$$

$$(\partial^2\phi)^3 \;: -4[\phi^3] + 4[\phi][\phi^2] \tag{3.81}$$

$$(\partial^2\phi)^4 \;: [\phi^4] - [\phi^2]^2 \tag{3.82}$$

$$h(\partial^2\phi)^2 \;: 4[h][\phi^2] - 2[h][\phi]^2 + 8[h\phi][\phi] - 10[h\phi^2] \tag{3.83}$$

$$h(\partial^2\phi)^3 \;: -2[h][\phi^3] + 2[h][\phi][\phi^2] + 8[h\phi^3] - 4[h\phi][\phi^2] - 4[h\phi^2][\phi] \tag{3.84}$$

$$(\partial A)(\partial^2\phi)^2 \;: -12[A\phi^2] + 4[A][\phi^2] + 8[A\phi][\phi] \tag{3.85}$$

$$(\partial A)(\partial^2\phi)^3 \;: 4[A\phi^3] - 4[A\phi][\phi^2]. \tag{3.86}$$

The next order term $\mathcal{L}_3$ is determined to cancel the $(\partial^2\phi)^3$ term (3.81)

$$\mathcal{L}_3 := -\frac{1}{2}\sqrt{-g}\Big(\langle H\rangle\langle H^2\rangle - \langle H^3\rangle\Big), \tag{3.87}$$

and the expansion is

$$
\begin{aligned}
(\partial^2\phi)^3 &: -4[\phi][\phi^2]+4[\phi^3] &(3.88)\\
(\partial^2\phi)^4 &: 2[\phi^2]^2+4[\phi][\phi^3]-6[\phi^4] &(3.89)\\
h(\partial^2\phi)^2 &: -2[h][\phi^2]-4[h\phi][\phi]+6[h\phi^2] &(3.90)\\
h(\partial^2\phi)^3 &: 4[h][\phi^3]-2[h][\phi][\phi^2]+6[h\phi][\phi^2]+10[h\phi^2][\phi]-18[h\phi^3] &(3.91)\\
(\partial A)(\partial^2\phi)^2 &: -4[A][\phi^2]-8[A\phi][\phi]+12[A\phi^2] &(3.92)\\
(\partial A)(\partial^2\phi)^3 &: 4[A][\phi^3]+12[A\phi^2][\phi]+8[A\phi][\phi^2]-24[A\phi^3]. &(3.93)
\end{aligned}
$$

Also, we add a total derivative term

$$
\mathcal{L}_3^{TD} := -\sqrt{-g}\Big(3\langle H\rangle\langle H^2\rangle - 2\langle H^3\rangle - \langle H\rangle^3\Big), \qquad (3.94)
$$

which contains

$$
\begin{aligned}
(\partial^2\phi)^3 &: \text{total derivative} &(3.95)\\
(\partial^2\phi)^4 &: 12[\phi^2]^2+24[\phi][\phi^3]-24[\phi^4]-12[\phi]^2[\phi^2] &(3.96)\\
h(\partial^2\phi)^2 &: -12[h][\phi^2]+12[h][\phi]^2-24[h\phi][\phi]+24[h\phi^2] &(3.97)\\
h(\partial^2\phi)^3 &: -24[h][\phi][\phi^2]+20[h][\phi^3]+4[h][\phi]^3\\
&\quad +36[h\phi][\phi^2]-72[h\phi^3]+60[h\phi^2][\phi]-24[h\phi][\phi]^2 &(3.98)\\
(\partial A)(\partial^2\phi)^2 &: 24\big([A][\phi]^2-[A][\phi^2]+2[A\phi^2]-2[A\phi][\phi]\big) = \text{total derivative} &(3.99)\\
(\partial A)(\partial^2\phi)^3 &: 24[A][\phi^3]-24[A][\phi][\phi^2]-96[A\phi^3]+48[A\phi][\phi^2]+72[A\phi^2][\phi]-24[A\phi][\phi]^2. &(3.100)
\end{aligned}
$$

How to construct total derivatives is a topic in Appendix D. Then, we obtain the mass term $\mathcal{L}_2+\mathcal{L}_3+\alpha_3\mathcal{L}_3^{TD}$, whose third order terms are

$$
\begin{aligned}
(\partial^2\phi)^2 &: \text{total derivative} &(3.101)\\
(\partial^2\phi)^3 &: \text{total derivative} &(3.102)\\
h(\partial^2\phi)^2 &: (2-12\alpha_3)\big([h][\phi^2]-[h][\phi]^2\big)+(4-24\alpha_3)([h\phi][\phi]-[h\phi^2]) &(3.103)\\
(\partial A)(\partial^2\phi)^2 &: \text{total derivative.} &(3.104)
\end{aligned}
$$

The $h(\partial^2\phi)^2$ term (3.103) seems to contain higher order derivatives. However, if we decompose the coordinates into the time and the spatial components $\mu=(0,i)$, we can see that higher order derivatives with respect to the time coordinate is not contained

$$
\begin{aligned}
[h][\phi^2]-[h][\phi]^2 =& h\big(2\partial_0^2\phi\triangle\phi-2(\partial_0\partial_i\phi)^2-(\triangle\phi)^2+(\partial_i\partial_j\phi)^2\big) &(3.105)\\
[h\phi][\phi]-[h\phi^2] =& h_{00}\big(\partial_0^2\phi\triangle\phi-(\partial_0\partial_i\phi)^2\big)+2h_{0i}\big(\partial_0\partial_i\phi\triangle\phi-\partial_0\partial_j\phi\partial_i\partial_j\phi\big)\\
&+h_{ij}\big(\partial_i\partial_j\phi\Box\phi-\partial_i\partial_\mu\phi\partial_j\partial^\mu\phi\big), &(3.106)
\end{aligned}
$$

where we have defined $\triangle := \partial_i\partial_i$. More simply, we can find $[\phi^2]-[\phi]^2 \propto \mathcal{L}_2^{TD}(\partial\partial\phi)$ and $[h\phi][\phi]-[h\phi^2] \propto h^{\mu\nu}X^{(2)}_{\mu\nu}(\partial\partial\phi)$ from Appendix D to conclude there is no higher order derivative. Anyway, no ghost emerges from (3.103). The fourth order $(\partial^2\phi)^4$ term coming from $\mathcal{L}_2+\mathcal{L}_3+\alpha_3\mathcal{L}_3^{TD}$ is

$$
(\partial^2\phi)^4 \;:\; -(5+24\alpha_3)[\phi^4]+(1+12\alpha_3)[\phi^2]^2+(4+24\alpha_3)[\phi][\phi^3]-12\alpha_3[\phi]^2[\phi^2]. \qquad (3.107)
$$

This should be canceled by $\mathcal{L}_4$. Hence, we determine

$$\mathcal{L}_4 := \frac{\sqrt{-g}}{16}\Big((5+24\alpha_3)\big\langle H^4\big\rangle - (1+12\alpha_3)\big\langle H^2\big\rangle^2 - (4+24\alpha_3)\big\langle H\big\rangle\big\langle H^3\big\rangle + 12\alpha_3\big\langle H^2\big\rangle\big\langle H\big\rangle^2\Big), \tag{3.108}$$

which contains

$$\begin{aligned}
(\partial^2\phi)^4 &: (5+24\alpha_3)[\phi^4] - (1+12\alpha_3)[\phi^2]^2 - (4+24\alpha_3)[\phi][\phi^3] + 12\alpha_3[\phi]^2[\phi^2] && (3.109)\\
h(\partial^2\phi)^3 &: -(2+12\alpha_3)[h][\phi^3] + 12\alpha_3[h][\phi][\phi^2] + (10+48\alpha_3)[h\phi^3]\\
&\quad -(2+24\alpha_3)[h\phi][\phi^2] - (6+36\alpha_3)[h\phi^2][\phi] + 12\alpha_3[h\phi][\phi]^2 && (3.110)\\
(\partial A)(\partial^2\phi)^3 &: (20+96\alpha_3)[A\phi^3] - (4+48\alpha_3)[A\phi][\phi^2] - (4+24\alpha_3)[A][\phi^3]\\
&\quad -(12+72\alpha_3)[A\phi^2][\phi] + 24\alpha_3[A\phi][\phi]^2 + 24\alpha_3[A][\phi^2][\phi]. && (3.111)
\end{aligned}$$

We also add a total derivative combination

$$\mathcal{L}_4^{TD} = \sqrt{-g}\Big(\big\langle H\big\rangle^4 - 6\big\langle H^2\big\rangle\big\langle H\big\rangle^2 + 8\big\langle H^3\big\rangle\big\langle H\big\rangle + 3\big\langle H^2\big\rangle^2 - 6\big\langle H^4\big\rangle\Big), \tag{3.112}$$

which is expanded to give

$$\begin{aligned}
(\partial^2\phi)^4 &: \text{total derivative} && (3.113)\\
h(\partial^2\phi)^3 &: 32[h][\phi]^3 - 96[h][\phi][\phi^2] + 64[h][\phi^3]\\
&\quad -192[h\phi^3] + 96[h\phi][\phi^2] + 192[h\phi^2][\phi] - 96[h\phi][\phi]^2 && (3.114)\\
(\partial A)(\partial^2\phi)^3 &: \text{total derivative.} && (3.115)
\end{aligned}$$

Then, we obtain the fourth order mass term $\mathcal{L}_2 + \mathcal{L}_3 + \alpha_3\mathcal{L}_3^{TD} + \mathcal{L}_4 + \alpha_4\mathcal{L}_4^{TD}$, from which we have

$$\begin{aligned}
(\partial^2\phi)^4 &: \text{total derivative} && (3.116)\\
(\partial A)(\partial^2\phi)^3 &: \text{total derivative} && (3.117)\\
h(\partial^2\phi)^3 &: (4\alpha_3 + 32\alpha_4)\Big(2[h][\phi^3] - 3[h][\phi][\phi^2] + [h][\phi]^3\\
&\quad -6[h\phi^3] + 3[h\phi][\phi^2] + 6[h\phi^2][\phi] - 3[h\phi][\phi]^2\Big). && (3.118)
\end{aligned}$$

The $h(\partial^2\phi)^3$ term (3.118) seems to contain a ghost, but we can show the absence of higher order derivatives with respect to time. It is merely a combination $h^{\mu\nu}X^{(3)}_{\mu\nu}$ defined in Appendix D. Hence, we do not encounter a ghost degree of freedom.

Thus far, our program works well. Here, we should notice that the condition to eliminate higher order derivatives on ϕ seems to be enough. In the above calculation, we have determined non-linear terms $\mathcal{L}_3$ and $\mathcal{L}_4$ in order to cancel the $(\partial^2\phi)^3$ and $(\partial^2\phi)^4$ terms. Other conditions have not been used. We have also added total derivative combinations $\mathcal{L}_3^{TD}$ and $\mathcal{L}_4^{TD}$ to cancel remaining higher order derivatives, but they have not played any role in this purpose. Other ghost like terms in $h(\partial^2\phi)^2$ or $h(\partial^2\phi)^3$ have been canceled automatically, and the constants α_3 and α_4 have been left undetermined. Therefore, it seems that the condition which makes the ϕ part total derivative eliminates all of dangerous higher order derivatives. Actually, the explicit construction of non-linear massive gravity has been succeeded in this line [23].

3.3 dRGT massive gravity

Now, we are in the position to construct a theory of fully non-linear massive gravity, which is called dRGT (de-Rham, Gabadadze and Tolley) massive gravity [23]. In Section 3.2, we have obtained a suggestion that total derivative combinations for the scalar field ϕ introduced via the Stückelberg trick may lead to a ghost-free and non-linear mass term. In this section, we carry out this strategy. We remember that the Stückelberg fields A_μ and ϕ have been introduced via $H_{\mu\nu}$ in (3.73). In order to focus only on the ϕ part, we put $h_{\mu\nu} = 0$ and $A_\mu = 0$, and see

$$H_{\mu\nu}\big|_{h=0,A=0} = 2\partial_\mu\partial_\nu\phi - (\partial_\mu\partial_\alpha\phi)(\partial^\alpha\partial_\nu\phi), \tag{3.119}$$

which we write $H = 2\partial\partial\phi - (\partial\partial\phi)^2$ in a matrix form.

Here, we define

$$\mathcal{K}^\mu{}_\nu := \delta^\mu{}_\nu - \sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu} = \sum_{n=1}^{\infty} d_n (H^n)^\mu{}_\nu, \qquad d_n := \frac{(2n)!}{(2n-1)(n!)^2 4^n}. \tag{3.120}$$

The upper index on $H^\mu{}_\nu$ is raised by $g^{\mu\nu}$, namely $H^\mu{}_\nu := g^{\mu\lambda}H_{\lambda\nu}$. The square root of a matrix X is denoted as $\sqrt{X}$, and defined as $(\sqrt{X})^\mu{}_\lambda(\sqrt{X})^\lambda{}_\nu = X^\mu{}_\nu$. The purpose to bring in the tensor $\mathcal{K}_{\mu\nu}$ is to make the ϕ dependence in (3.119) simple. Setting $h_{\mu\nu} = 0$ and $A_\mu = 0$, we find

$$\begin{aligned}\mathcal{K}\big|_{h=0,A=0} =& 1 - \sqrt{1-H}\big|_{h=0,A=0} \\ &= 1 - \sqrt{1 - 2\partial\partial\phi + (\partial\partial\phi)^2} \\ &= 1 - \sqrt{|1-\partial\partial\phi|^2} \\ &= 1 - |1-\partial\partial\phi| \\ &= \partial\partial\phi,\end{aligned} \tag{3.121}$$

which is a matrix form of $\mathcal{K}^\mu{}_\nu\big|_{h=0,A=0} = \partial^\mu\partial_\nu\phi$. In the last step, we have used a condition $1 - \partial\partial\phi > 0$. We can read it as $1 - \frac{\hat{\phi}}{m^2 M_p r^2} > 0$. In the next section, we consider some solutions for $\hat{\phi}$ (3.140) or (3.143). Under the cut off scale $r < \Lambda_3^{-1}$, these solutions are estimated to be $\hat{\phi} < \frac{1}{r}$ and $1 - \frac{\hat{\phi}}{m^2 M_p r^2} > 0$ is satisfied.

Then, we propose the following action

$$S = \frac{1}{16\pi G}\int d^D x\sqrt{-\det g}\Big[R - \frac{m^2}{4}W(g,\mathcal{K})\Big], \quad W(g,\mathcal{K}) := \sum_{n=0}^{D}\alpha_n \mathcal{L}_n^{TD}(\mathcal{K}), \tag{3.122}$$

where $\mathcal{L}_n^{TD}$ is the total derivative combination defined in (D.2) at appendix D, and α_n is a constant. If we insist that constant and linear terms of $h_{\mu\nu}$ should not appear, we have to set $\alpha_0 = \alpha_1 = 0$. Besides, $\alpha_2 = -4$ restores the Fierz-Pauli mass term. When we set $h_{\mu\nu} = 0$ and $A_\mu = 0$, the mass term becomes

$$W(\eta, \partial\partial\phi) = \text{total derivative}. \tag{3.123}$$

Therefore, we expect that we have succeeded in constructing a theory of non-linear and ghost-free massive gravity. This theory is called dRGT massive gravity. We postpone the proof of the absence of a ghost to Section 4, and proceed related topics.

The action (3.122) is often used in a rather different formulation. We remember (3.66) in which the Stückelberg field Y^μ has been originally introduced

$$H_{\mu\nu} = g_{\mu\nu} - \frac{\partial Y^\alpha}{\partial x^\mu}\frac{\partial Y^\beta}{\partial x^\nu}\eta_{\alpha\beta} \quad \text{and} \quad H^\mu{}_\nu = \delta^\mu{}_\nu - g^{\mu\lambda}\frac{\partial Y^\alpha}{\partial x^\lambda}\frac{\partial Y^\beta}{\partial x^\nu}\eta_{\alpha\beta}, \tag{3.124}$$

and express the tensor $\mathcal{K}^\mu{}_\nu$ as

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{g^{\mu\lambda}f_{\lambda\nu}}, \qquad f_{\mu\nu} := \frac{\partial Y^\alpha}{\partial x^\mu}\frac{\partial Y^\beta}{\partial x^\nu}\eta_{\alpha\beta}. \tag{3.125}$$

Then, the mass term is represented as

$$\sum_{n=0}^{D} \alpha_n \mathcal{L}_n^{TD}(\mathcal{K}) = \sum_{n=0}^{D} \alpha'_n \mathcal{L}_n^{TD}\big(\sqrt{g^{-1}f}\big), \tag{3.126}$$

where α'_n is a constant written by a linear combination of α_n $(n = 0, 1, ..., D)$. Thus far, we assumed that the background metric is the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, .., 1)$. However, we can remove this assumption. The absence of a ghost can be proved on general background metrics. Hence, we forget the origin of $f_{\mu\nu}$ and regard $f_{\mu\nu}$ as another non-dynamical metric.

In conclusion, the action of non-linear massive gravity is generally given in the form [24]

$$S_{MG} = \frac{1}{16\pi G}\int d^D x\sqrt{-\det g}\Big[R + 2m^2\sum_{n=0}^{D}\beta_n e_n\big(\sqrt{g^{-1}f}\big)\Big], \tag{3.127}$$

where β_n is a constant and e_n is defined in Appendix D.

3.4 The Vainshtein mechanism

In this section, we consider the Vainshtein mechanism, namely a resolution of the vDVZ discontinuity discussed in Section 2.5. Here, we consider only the simplest setting. More detailed or general discussions are found in [25] and references therein.

We focus on the four dimensional case ($D = 4$) with the Minkowski background, and return to the expansion by the Stückelberg fields. We remember the action (3.122) and add a matter coupling

$$S = M_p^2\int d^D x\sqrt{-\det g}\Big[R - \frac{m^2}{4}W(g, \mathcal{K})\Big] - \frac{1}{2}\int d^4x\, H_{\mu\nu}T^{\mu\nu}. \tag{3.128}$$

Here, conservation of the energy momentum tensor $\partial_\mu T^{\mu\nu} = 0$ is assumed. We expand the action by the Stückelberg fields through $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and (3.73)

$$\begin{aligned} H_{\mu\nu} =& h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu - (\partial_\mu A^\alpha)(\partial_\nu A_\alpha) \\ &+ 2\partial_\mu\partial_\nu\phi - (\partial_\mu\partial^\alpha\phi)(\partial_\nu\partial_\alpha\phi) \\ &- (\partial_\mu A^\alpha)(\partial_\nu\partial_\alpha\phi) - (\partial_\nu A^\beta)(\partial_\mu\partial_\beta\phi). \end{aligned} \tag{3.129}$$

We also remember the table

scale	Λ_5	Λ_4	$\Lambda_{3+\frac{2}{n-2}}$	$\Lambda_{3+\frac{1}{n-1}}$	Λ_3	$\Lambda_{n>3}$
interaction	$(\partial^2\hat{\phi})^3$	$(\partial^2\hat{\phi})^4$ $(\partial\hat{A})(\partial^2\hat{\phi})^2$	$(\partial^2\hat{\phi})^{n\geq 5}$	$(\partial\hat{A})(\partial^2\hat{\phi})^{n\geq 3}$	$\hat{h}(\partial^2\hat{\phi})^{n\geq 2}$ $(\partial\hat{A})^2(\partial^2\hat{\phi})^{n\geq 1}$	$\cdots$

where the normalization $\hat{h}_{\mu\nu} = M_p h_{\mu\nu}$, $\hat{A}_\mu = mM_pA_\mu$ and $\hat{\phi} = m^2M_p\phi$ is understood. We already know that $(\partial^2\hat{\phi})^{n\geq 3}$ terms are combined to become total derivatives. Now, we assume that we encounter an interaction term such as $(\partial A)(\cdots\partial^2\phi\cdots)$, where the second term $(\cdots\partial^2\phi\cdots)$ does not contain $h_{\mu\nu}$ and A_μ. Originally, A_μ has been introduced via the replacement $A_\mu \to A_\mu + \partial_\mu\phi$. Thus, we should also have a term $(\partial^2\phi)(\cdots\partial^2\phi\cdots)$, which is a total derivative. Then, from the argument in Appendix D, we notice that $(\partial_\mu A_\nu)(\cdots\partial^2\phi\cdots)$ must be a total derivative. Therefore, interaction terms between the scale Λ_5 and $\Lambda_{3+\frac{1}{n-1}}$ are all total derivatives. This fact means that the cut off scale of dRGT massive gravity is raised to Λ_3. We focus on this scale to investigate the Vainshtein mechanism.

We take a limit called Λ_3 decoupling limit

$$m \to 0, \quad M_p \to \infty, \quad T \to \infty, \quad \Lambda_3,\ \frac{T}{M_p} : \text{fixed}. \tag{3.130}$$

Interaction terms with scale $\Lambda_{n>3}$ disappear, and only those with Λ_3 remain. They are $\hat{h}(\partial^2\hat{\phi})^{n\geq 2}$ and $(\partial\hat{A})^2(\partial^2\hat{\phi})^{n\geq 1}$. Here, we notice that A_μ is quadratic in the action. Hence, the equation of motion for A_μ reads $(\cdots)A_\mu = 0$, which means that $A_\mu = 0$ is a solution. In the following, we consistently set $A_\mu = 0$. Therefore, in this situation, the mass term in the action (3.128) becomes

$$-M_p^2\frac{m^2}{4}\int d^4x h^{\mu\nu}\bar{X}_{\mu\nu}, \tag{3.131}$$

where we have defined

$$\bar{X}_{\mu\nu} := \frac{\delta}{\delta h_{\mu\nu}}\Big(\sqrt{-\det g}\, W(g,\mathcal{K})\Big)\Big|_{A=0,h=0}. \tag{3.132}$$

We recall the definition of $W(g,\mathcal{K})$ in (3.122)

$$W(g,\mathcal{K}) = \sum_{n=0}^{4}\alpha_n\mathcal{L}_n^{TD}(\mathcal{K}), \tag{3.133}$$

and apply the formula (D.29). In order to restore the Fierz-Pauli mass term at the linear level, we set $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 = -4$. Then, we find

$$\bar{X}_{\mu\nu} = \frac{1}{2}\sum_{n=2}^{4}\alpha_n\big(X_{\mu\nu}^{(n)} + nX_{\mu\nu}^{(n-1)}\big) = \frac{1}{2}\big(2\alpha_2X_{\mu\nu}^{(1)} + (\alpha_2+3\alpha_3)X_{\mu\nu}^{(2)} + (\alpha_3+4\alpha_4)X_{\mu\nu}^{(3)}\big). \tag{3.134}$$

As we mention in Appendix D, $X_{\mu\nu}^{(n)}$ is a linear combination of $(\partial\partial\phi)^n$ terms, and note that $h^{\mu\nu}\bar{X}_{\mu\nu}$ contains no higher order derivative with respect to the time coordinate. Taking the normalization $\hat{h_{\mu\nu}}$, $\hat{A}_\mu$ and $\hat{\phi}$ into account, we obtain the action

$$S = \int d^4x\, \hat{h}_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}\hat{h}_{\mu\nu} - \frac{1}{8}\hat{h}^{\mu\nu}\Big(-8\hat{X}_{\mu\nu}^{(1)} + \frac{3\alpha_3-4}{\Lambda_3^3}\hat{X}_{\mu\nu}^{(2)} + \frac{\alpha_3+4\alpha_4}{\Lambda_3^6}\hat{X}_{\mu\nu}^{(3)}\Big) - \frac{1}{2M_p}\hat{h}_{\mu\nu}T^{\mu\nu}, \tag{3.135}$$

where $h_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}h_{\lambda\rho}$ represents the kinetic term coming from the Einstein-Hilbert action found in (2.19). In order to diagonalize a coupling $\hat{h}^{\mu\nu}\hat{X}^{(1)}_{\mu\nu}$, we shift $\hat{h}_{\mu\nu}$ as $\hat{h}_{\mu\nu} = h'_{\mu\nu} + \hat{\phi}\eta_{\mu\nu}$. Using a relation $\Upsilon_{\mu\nu}{}^{\lambda\rho}(\phi\eta_{\lambda\rho}) = -\frac{(D-2)}{4}X^{(1)}_{\mu\nu}$, the action can be written as

$$S = \int d^4x\, h'_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}h'_{\lambda\rho} - h'^{\mu\nu}\Big(\frac{3\alpha_3-4}{8\Lambda_3^3}\hat{X}^{(2)}_{\mu\nu} + \frac{\alpha_3+4\alpha_4}{8\Lambda_3^6}\hat{X}^{(3)}_{\mu\nu}\Big) - \frac{1}{2M_p}h'_{\mu\nu}T^{\mu\nu}$$
$$+ \frac{1}{2}\hat{\phi}\hat{X}^{(1)\mu}_{\mu} - \frac{3\alpha_3-4}{8\Lambda_3^3}\hat{\phi}\hat{X}^{(2)\mu}_{\mu} - \frac{\alpha_3+4\alpha_4}{8\Lambda_3^6}\hat{\phi}\hat{X}^{(3)\mu}_{\mu} - \frac{1}{2M_p}\hat{\phi}T. \quad (3.136)$$

We can diagonalize one more coupling $h'^{\mu\nu}\hat{X}^{(2)}_{\mu\nu}$ by shifting $h'_{\mu\nu}$ as

$$h'_{\mu\nu} = h''_{\mu\nu} + \frac{3\alpha_3-4}{4\Lambda_3^3}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}). \quad (3.137)$$

We employ relations (D.31), (D.32) and $\Upsilon_{\mu\nu}{}^{\lambda\rho}(\partial_\lambda\phi\partial_\rho\phi) = \frac{1}{4}X^{(2)}_{\mu\nu}$ to obtain the action

$$S = \int d^4x\, h''_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}h''_{\lambda\rho} - \frac{\alpha_3+4\alpha_4}{8\Lambda_3^6}h''^{\mu\nu}\hat{X}^{(3)}_{\mu\nu} - \frac{1}{2M_p}h''_{\mu\nu}T^{\mu\nu}$$
$$- \frac{3}{2}(\partial\hat{\phi})^2 + \frac{3(3\alpha_3-4)}{8\Lambda_3^3}(\partial\hat{\phi})^2[\hat{\phi}] - \frac{(3\alpha_3-4)^2 - 8(\alpha_3+4\alpha_4)}{32\Lambda_3^6}(\partial\hat{\phi})^2\big([\hat{\phi}]^2 - [\hat{\phi}^2]\big)$$
$$- \frac{5(3\alpha_3-4)(\alpha_3+4\alpha_4)}{64\Lambda_3^9}(\partial\hat{\phi})^2\big([\hat{\phi}]^3 - 3[\hat{\phi}][\hat{\phi}^2] + 2[\hat{\phi}^3]\big) - \frac{1}{2M_p}\hat{\phi}T, \quad (3.138)$$

where we have abbreviated $(\partial\phi)^2 = (\partial_\mu\phi)(\partial^\mu\phi)$, and the notation $[\cdot]$ is found in (3.78). At first glance, the action (3.138) seems to contain higher order derivatives with respect to time. However, from (D.32), we see that such terms can be discarded via total derivative combinations. This fact means that we cannot apply the argument in Section 3.1.2, with which we have detected a ghost.

Now, we set $\alpha_3 + 4\alpha_4 = 0$ to eliminate the remaining coupling between $h''_{\mu\nu}$ and $\hat{\phi}$. Notice that the tensor field $h''_{\mu\nu}$ corresponds to the gravitational field in pure general relativity while that in massive gravity is $\hat{h} \sim h'' + \phi$. Schematically, the equation of motion for $h''_{\mu\nu}$ and $\hat{\phi}$ is given by

$$\partial^2 h'' = \frac{1}{M_p}T, \quad \partial^2\hat{\phi} \pm \frac{1}{\Lambda_3^3}\partial^4\hat{\phi}^2 + \frac{1}{\Lambda_3^6}\partial^6\hat{\phi}^3 = \frac{1}{M_p}T. \quad (3.139)$$

Here, we consider a point source $T \sim M\delta^{(3)}(x)$ with mass M, and assume static solutions. At large distances, non-linear terms are negligible. Then, we have solutions

$$h'' \sim \frac{M}{M_p}\frac{1}{r}, \quad \hat{\phi} \sim \frac{M}{M_p}\frac{1}{r}. \quad (3.140)$$

The scale where non-linear terms start to contribute is estimated from the above solution and

$$\frac{\hat{\phi}}{r^2} \sim \frac{\hat{\phi}^2}{\Lambda_3^3 r^4}. \quad (3.141)$$

The result is called the Vainshtein radius

$$r_V := \frac{1}{\Lambda_3}\Big(\frac{M}{M_p}\Big)^{\frac{1}{3}}. \tag{3.142}$$

Inside the Vainshtein radius, the solution for $\hat{\phi}$ can be estimated to be

$$\hat{\phi} \sim \Big(\frac{M}{M_p}\Big)^{\frac{1}{2}}\Lambda_3^{\frac{3}{2}}r^{\frac{1}{2}}, \quad \hat{\phi} \sim \Big(\frac{M}{M_p}\Big)^{\frac{1}{3}}\Lambda_3^{2}r. \tag{3.143}$$

The second one comes when the cubic term in (3.139) dominates. Thus, we can find

$$\frac{\hat{\phi}}{h''} \sim \Big(\frac{r}{r_V}\Big)^{\frac{1}{2}}, \Big(\frac{r}{r_V}\Big)^{2} \ll 1. \tag{3.144}$$

Therefore, in small scales, the gravitational field in massive gravity $\hat{h} \sim h'' + \phi$ does not deviate from that in general relativity $\hat{h} \sim h''$. The vDVZ discontinuity is resolved at least in this simple setting.

3.5 Extension to bimetric gravity

dRGT massive gravity has one peculiar feature. It contains a non-dynamical metric, which breaks the diffeomorphism invariance. Thus, it seems natural to promote the non-dynamical metric to dynamical one. This theory contains interacting two metrics and is called bimetric gravity [2]. We give the dynamics to the non-dynamical metric in dRGT massive gravity, and obtain the action

$$\begin{aligned} S_{BG} =& \frac{1}{16\pi G_g}\int d^Dx\sqrt{-\det g}R[g] + \frac{1}{16\pi G_f}\int d^Dx\sqrt{-\det g}R[f] \\ &+ 2\Big(16\pi G_g + 16\pi G_f\Big)^{-1}m^2\int d^Dx\sqrt{-\det g}\sum_{n=0}^{D}\beta_n e_n\big(\sqrt{g^{-1}f}\big), \end{aligned} \tag{3.145}$$

where $R[g]$ and $R[f]$ represents the curvatures for two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ respectively. G_g and G_f are two kinds of gravitational constants. We can also include cosmological constants, but we do not consider them here. In the action (3.145), we can rewrite the interaction term in a rather different form. We recall a property (D.12) to find

$$e_n\big(\sqrt{g^{-1}f}\big) = \det\sqrt{g^{-1}f}\,e_{D-n}\big((\sqrt{g^{-1}f})^{-1}\big). \tag{3.146}$$

For a matrix X with the inverse X^{-1}, we know that $\det\sqrt{X} = \sqrt{\det X}$ and $(\sqrt{X})^{-1} = \sqrt{(X^{-1})}$ hold. Then, we notice a relation

$$\sqrt{-\det g}\,e_n\big(\sqrt{g^{-1}f}\big) = \sqrt{-\det f}\,e_{D-n}\big(\sqrt{f^{-1}g}\big), \tag{3.147}$$

and see that the metric $g_{\mu\nu}$ has no special meaning compared to the other metric $f_{\mu\nu}$.

In general relativity, we have a diffeomorphism invariance which guarantees the massless degrees of freedom. When the invariance is broken, degrees of freedom increases and the graviton becomes massive. If the interaction in bimetric gravity (3.145) is switched off ($m = 0$), we have two independent general relativity and two diffeomorphism invariances are contained. These symmetries extract a couple of true massless degrees of freedom. However, the interaction term breaks one of them, and only one overall diffeomorphism invariance is left. We expect that this remaining invariance reduces degrees of freedom corresponding to one massless graviton. Thus, if the theory is consistent, namely we have no extra ghost mode, bimetric gravity should have one massless and one massive graviton. In this section, we see this fact in the linear level. Non-linear analysis is a topic in Chapter 4.

We consider a perturbation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad f_{\mu\nu} = \bar{g}_{\mu\nu} + l_{\mu\nu}, \tag{3.148}$$

where $\bar{g}_{\mu\nu}$ is a background metric for both two metrics, and $h_{\mu\nu}$ and $l_{\mu\nu}$ stand for fluctuations. We expand the action (3.145) up to the second order in $h_{\mu\nu}$ and $l_{\mu\nu}$. For notational simplicity, we write the trace with respect to the background metric as

$$[h] = \bar{g}^{\mu\lambda} h_{\lambda\mu}, \qquad [l] = \bar{g}^{\mu\lambda} l_{\lambda\mu}. \tag{3.149}$$

In the following, index manipulations are performed by the background metric $\bar{g}_{\mu\nu}$ and its inverse $\bar{g}^{\mu\nu}$.

Using expansion formulae (B.18) and (B.20), we find the result in Appendix E. In the four dimensional case ($D = 4$), we find

$$\begin{aligned}
&\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\big(\sqrt{g^{-1}f}\big)\\
=&(\beta_0+4\beta_1+6\beta_2+4\beta_3+\beta_4)+\Big(\frac{1}{2}\beta_0+\frac{3}{2}\beta_1+\frac{3}{2}\beta_2+\frac{1}{2}\beta_3\Big)[h]+\Big(\frac{1}{2}\beta_1+\frac{3}{2}\beta_2+\frac{3}{2}\beta_3+\frac{1}{2}\beta_4\Big)[l]\\
&+\Big(-\frac{1}{4}\beta_0-\frac{5}{8}\beta_1-\frac{1}{2}\beta_2-\frac{1}{8}\beta_3\Big)[h^2]+\Big(-\frac{1}{4}\beta_1-\frac{1}{2}\beta_2-\frac{1}{4}\beta_3\Big)[hl]+\Big(-\frac{1}{8}\beta_1-\frac{1}{2}\beta_2-\frac{5}{8}\beta_3-\frac{1}{4}\beta_4\Big)[l^2]\\
&+\Big(\frac{1}{8}\beta_0+\frac{1}{4}\beta_1+\frac{1}{8}\beta_2\Big)[h]^2+\Big(\frac{1}{4}\beta_1+\frac{1}{2}\beta_2+\frac{1}{4}\beta_3\Big)[h][l]+\Big(\frac{1}{8}\beta_2+\frac{1}{4}\beta_3+\frac{1}{8}\beta_4\Big)[l]^2.
\end{aligned} \tag{3.150}$$

If we imposing the condition that linear terms should disappear

$$\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3 = 0, \qquad \beta_1 + 3\beta_2 + 3\beta_3 + \beta_4 = 0, \tag{3.151}$$

we obtain

$$\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\big(\sqrt{g^{-1}f}\big) = \frac{1}{8}(\beta_1+2\beta_2+\beta_3)\Big(\big[(h-l)^2\big]-\big[h-l\big]^2\Big), \tag{3.152}$$

where we have eliminated β_0 and β_4 via (3.151). Therefore, the Fierz-Pauli tuning has been restored. Notice that the condition (3.151) also removes the zeroth order term, which means that $\sum_n e_n\big(\sqrt{g^{-1}g}\big) = 0$ holds even in the non-linear level. The simplest setting is said to be minimal, which is defined as

$$\beta_0 = 3, \quad \beta_1 = -1, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = 1. \tag{3.153}$$

Anyway, when we adjust the parameters to recover the Fierz-Pauli tuning, the linearized action of bimetric gravity becomes

$$\begin{aligned} S_{BG} =& \frac{1}{16\pi G_g}\int d^D x\, h_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}\, h_{\lambda\rho} + \frac{1}{16\pi G_f}\int d^D x\, l_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}\, l_{\lambda\rho} \\ &+ \frac{m^2}{4}\frac{1}{16\pi G_g + 16\pi G_f}\int d^D x\Big([(h-l)^2] - [h-l]^2\Big), \end{aligned} \tag{3.154}$$

where $h_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}h_{\lambda\rho}$ represents the kinetic term coming from the Einstein-Hilbert action found in (2.16). In order to simplify the kinetic terms, we define normalized variables $h_{\mu\nu} =: \sqrt{16\pi G_g}\,\hat{h}_{\mu\nu}$ and $l_{\mu\nu} =: \sqrt{16\pi G_f}\,\hat{l}_{\mu\nu}$. Then, the action (3.154) can be read as

$$\begin{aligned} S_{BG} =& \int d^D x\, \hat{h}_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}\, \hat{h}_{\lambda\rho} + \int d^D x\, \hat{l}_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}\, \hat{l}_{\lambda\rho} \\ &+ \frac{m^2}{4}\frac{1}{16\pi G_g + 16\pi G_f}\int d^D x\Big([(\sqrt{16\pi G_g}\,\hat{h} - \sqrt{16\pi G_f}\,\hat{l}\,)^2] - [\sqrt{16\pi G_g}\,\hat{h} - \sqrt{16\pi G_f}\,\hat{l}\,]^2\Big). \end{aligned} \tag{3.155}$$

Here, we introduce new variables

$$u_{\mu\nu} := \frac{\sqrt{16\pi G_g}\,\hat{h}_{\mu\nu} - \sqrt{16\pi G_f}\,\hat{l}_{\mu\nu}}{16\pi G_g + 16\pi G_f}, \quad v_{\mu\nu} := \frac{\sqrt{16\pi G_f}\,\hat{h}_{\mu\nu} + \sqrt{16\pi G_g}\,\hat{l}_{\mu\nu}}{16\pi G_g + 16\pi G_f} \tag{3.156}$$

and rewrite the action. The result is

$$S_{BG} = \int d^D x\, v_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}v_{\lambda\rho} + \int d^D x\, u_{\mu\nu}\Upsilon^{\mu\nu\lambda\rho}u_{\lambda\rho} + \frac{m^2}{4}\int d^D x\big([u^2] - [u]^2\big). \tag{3.157}$$

Thus, we have one massless and one massive graviton, and no ghost degree of freedom is contained at least in the linear level.

3.6 Further extension to trimetric gravity ?

A naive extension of bimetric gravity has been originally proposed in [3], where we have three metrics $g_{\mu\nu}$, $f_{\mu\nu}$ and $h_{\mu\nu}$, and the action is given by

$$\begin{aligned} S_{TG} =& M_g^2\int d^4x\sqrt{-\det g}R[g] + M_f^2\int d^4x\sqrt{-\det f}R[f] + M_h^2\int d^4x\sqrt{-\det h}R[h] \\ &+ 2m_1^2 M_{gf}^2\int d^4x\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\big(\sqrt{g^{-1}f}\big) \\ &+ 2m_2^2 M_{fh}^2\int d^4x\sqrt{-\det f}\sum_{n=0}^{4}\beta'_n e_n\big(\sqrt{f^{-1}h}\big) \\ &+ 2m_3^2 M_{hg}^2\int d^4x\sqrt{-\det h}\sum_{n=0}^{4}\beta''_n e_n\big(\sqrt{h^{-1}g}\big). \end{aligned} \tag{3.158}$$

A lot of constants are introduced to adjust dimensions.

Bimetric gravity has been actually proved to be ghost-free [26, 27], which is a topic in the next chapter, but the same method is not applicable to trimetric gravity. Thus, the ghost problem in non-linear trimetric gravity has not been solved in [3]. We resolve this problem in Chapter 6.

Chapter 4

The Hamiltonian analysis of dRGT massive/bimetric gravity

In this chapter we prove the absence of the BD-ghost in dRGT massive gravity. At the same time, we also prove the fact that bimetric gravity contains the right number of degrees of freedom, namely that of one massless and one massive graviton [26, 27]. Our tool is the Hamiltonian analysis, with which we directly count the total number of degrees of freedom. Actually, no-ghost proof in dRGT massive gravity is carried out by the Stückelberg trick [28, 29, 30, 31], but it is extremely difficult and we do not know how to extend it to bimetric gravity. Thus, we employ the ADM decomposition and rely on the Hamiltonian analysis. The essential point of the proof is common for dRGT massive/bimetric gravity though it is not applicable to trimetric gravity. We focus on the four dimensional case $D = 4$, but the dimensionality is expected not to be crucial in this proof.

The action is now given by

$$S_{MG} = M_p^2 \int d^4x \sqrt{-\det g}\Big[R + 2m^2 \sum_{n=0}^{4} \beta_n\big(\sqrt{g^{-1}f}\big)\Big] \tag{4.1}$$

for dRGT massive gravity, and

$$\begin{aligned} S_{BG} =& M_g^2 \int d^4x \sqrt{-\det g} R[g] + M_f^2 \int d^4x \sqrt{-\det g} R[f] \\ &+ 2M_{eff}^2 m^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n e_n\big(\sqrt{g^{-1}f}\big) \end{aligned} \tag{4.2}$$

for bimetric gravity. We have several kinds of the Planck mass M_p, M_g and M_f, and effective mass M_{eff} is defined as

$$M_{eff}^2 := \Big(\frac{1}{M_f^2} + \frac{1}{M_g^2}\Big)^{-1}. \tag{4.3}$$

Prior to the actual proof, we remember the discussion in Section 2.4, where we have performed the Hamiltonian analysis on the linear massive gravity with the Fierz-Pauli mass term. There, we have found that the difference of degrees of freedom between massless and massive gravitons comes from the break down of linearity for non-dynamical variables. In the massless case, all of the non-dynamical variables appear linearly in the action and are interpreted as Lagrange multipliers. Their existence reduces the number of physical degrees of freedom to two. On the other hand, in the massive case, some of these variables become quadratic. If we do not have the Fierz-Pauli tuning, all of them get quadratic to become auxiliary fields. These auxiliary fields are removed from the action via the equation of motion for them. Thus, we cannot eliminate any dynamical degrees of freedom, and an extra sixth mode remains which is ghost like. When we apply the Fierz-Pauli tuning, one of the non-dynamical variables recovers linearity while the others are left quadratic. This structure reduces the extra ghost mode and gives the right number of degrees of freedom to a massive graviton, which we know five.

Therefore, it seems natural to think that the same situation should occur also in non-linear massive gravity. In Section 3.1.1, we have attempted the non-linear extension of linear massive gravity with the Fierz-Pauli mass term. Then, we have seen the break down of linearity for all non-dynamical variables, which fails to reduce the number of degrees of freedom. The result is the emergence of the BD-ghost. Hence, we expect that, in dRGT massive gravity, only one of the non-dynamical variables appear linearly and the others remain non-linear. Our expectation is actually correct, but this fact is not so easy to see. Non-dynamical variables in the ADM formalism is the lapse function N and the shift vector N^i. By comparing to the analysis in linear massive gravity, it seems reasonable to expect that N becomes linear while N^i remains non-linear. However, we cannot see it even if we directly write down the action in terms of the ADM variables. The non-linear mass term is composed of the square root $\sqrt{g^{-1}f}$, which carries non-linear combinations of $N, N^2, N^3, ...$ and $N^i, (N^i)^2, (N^i)^3, ...$ Hence, we need to mix the lapse N and the shift N^i to define a new variable n^i through a transformation

$$(N, N^i) \to (N, n^i = n^i(N, N^k)). \tag{4.4}$$

The role of the new function $n^i(N, N^i)$ is to absorb higher order terms of N and to make N appear linearly. This is the essential point of the following no-ghost proof.

4.1 The ADM decomposition

4.1.1 The decomposition of metrics

The Hamiltonian analysis is based on the ADM decomposition. Thus, we start with decomposing the mass/interaction term. The (3+1) decomposition of the Einstein-Hilbert part is found in Section 3.1.1. Notations for the ADM decomposition is the same as those in Section 3.1.1. One metric $g_{\mu\nu}$ is decomposed as

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -N^2 + \gamma_{lk} N^l N^k & \gamma_{ij} N^i \\ \gamma_{ij} N^j & \gamma_{ij} \end{pmatrix}, \tag{4.5}$$

and its inverse is

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \frac{1}{N^2} \begin{pmatrix} -1 & N^j \\ N^i & N^2\gamma^{ij} - N^i N^j \end{pmatrix}, \tag{4.6}$$

where γ^{ij} is the inverse of the spatial metric γ_{ij}. In the original definition, the shift vector N^i is upper indexed. For notational simplicity, we also use the lower indexed shift defined as

$N_i := \gamma_{ij} N^j$.
Along with $g_{\mu\nu}$, we decompose the other metric $f_{\mu\nu}$

$$f_{\mu\nu} = \begin{pmatrix} f_{00} & f_{0j} \\ f_{i0} & f_{ij} \end{pmatrix} = \begin{pmatrix} -L^2 + \omega_{lk} N^l N^k & \omega_{ij} L^i \\ \omega_{ij} L^j & \omega_{ij} \end{pmatrix}, \tag{4.7}$$

and its inverse

$$f^{\mu\nu} = \begin{pmatrix} f^{00} & f^{0j} \\ f^{i0} & f^{ij} \end{pmatrix} = \frac{1}{L^2} \begin{pmatrix} -1 & L^j \\ L^i & L^2 \omega^{ij} - L^i L^j \end{pmatrix}. \tag{4.8}$$

We denote the spatial part f_{ij} as ω_{ij}, and ω^{ij} means the inverse of ω_{ij}. L and L^i are interpreted as the lapse and the shift for $f_{\mu\nu}$, where L^i is defined with an upper index. In the following, we often use the lower indexed shift defined as $L_i := \omega_{ij} L^j$. Here, it should be noted that we never lower an index on L^i by γ_{ij} and N^i by ω_{ij}.

4.1.2 The new shift vector

Under the decomposition in Section 4.1.1, we write down the basic element $g^{-1} f$

$$N^2 (g^{-1} f)^\mu{}_\nu = \begin{pmatrix} (^0{}_0) & (^0{}_j) \\ (^i{}_0) & (^i{}_j) \end{pmatrix}$$

$$= \begin{pmatrix} -f_{00} + N^l f_{l0} & -f_{0j} + N^l f_{lj} \\ N^2 \gamma^{il} f_{l0} - N^i(-f_{00} + N^l f_{l0}) & N^2 \gamma^{il} f_{lj} - N^i(-f_{0j} + N^l f_{lj}) \end{pmatrix}$$

$$= \begin{pmatrix} -L^2 + L_k L^k + N^l L_l & -L_j + N^l \omega_{lj} \\ N^2 \gamma^{il} L_l - N^i(L^2 - L_k L^k + N^l L_l) & N^2 \gamma^{il} \omega_{lj} - N^i(-L_j + N^l \omega_{lj}) \end{pmatrix}. \tag{4.9}$$

We can see the complicated dependence on the Lapse and the shift. Thus, the mass/interaction term necessarily contains higher order terms for them, which makes the absence of a ghost degree of freedom extremely unclear. Our strategy is now to ensure the linearity of the lapse N through a variable change $(N, N^i) \to (N, n^i(N, N^k))$. If we express N^i by the transformed variables as $N^i = N^i(N, n^k)$, the function $N^i(N, n^k)$ should be linear in N. Otherwise, we encounter an non-linear term of N from $N^i \mathcal{R}_i$ which is contained in the Einstein-Hilbert action. Hence, we set

$$N^i = c^i + N d^i, \tag{4.10}$$

where c^i and d^i are some functions of n^i and independent of N. We can determine a transformation $(N, N^i) \to (N, n^i(N, N^k))$ by fixing these functions c^i and d^i. We realize the linearity of N through a proper choice of them. Then, we write down (4.9) via (4.10) as

$$N^2 g^{-1} f = E_0 + N E_1 + N^2 E_2, \tag{4.11}$$

where $E_{0,1,2}$ are matrices independent of N and given by

$$E_0 = \begin{pmatrix} \underbrace{-f_{00} + c^l f_{l0}} & \underbrace{-f_{0j} + c^l f_{lj}} \\ -(\underbrace{-f_{00} + c^l f_{l0}})c^i & -c^i(\underbrace{-f_{0j} + c^l f_{lj}}) \end{pmatrix} \tag{4.12}$$

$$= \begin{pmatrix} L^2 - L_k L^k + c^l L_l & -L_j + c^l \omega_{lj} \\ -(L^2 - L_k L^k + c^l L_l)c^i & -c^i(-L_j + c^l \omega_{lj}) \end{pmatrix}, \tag{4.13}$$

$$E_1 = \begin{pmatrix} d^l f_{l0} & d^l f_{lj} \\ -d^l f_{l0} c^i - (\underset{\sim}{-f_{00} + c^l f_{l0}}) d^i & -c^i d^l f_{lj} - d^i (\underset{\sim}{-f_{0j} + c^l f_{lj}}) \end{pmatrix} \tag{4.14}$$

$$= \begin{pmatrix} d^l L_l & d^l \omega_{lj} \\ -d^l L_l c^i - (L^2 - L_k L^k + c^l L_l) d^i & -c^i d^l \omega_{lj} - d^i(-L_j + c^l \omega_{lj}) \end{pmatrix}, \tag{4.15}$$

$$E_2 = \begin{pmatrix} 0 & 0 \\ (\gamma^{il} - d^i d^l) f_{l0} & (\gamma^{il} - d^i d^l) f_{lj} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (\gamma^{il} - d^i d^l) L_l & (\gamma^{il} - d^i d^l) \omega_{lj} \end{pmatrix}. \tag{4.16}$$

In the wavy underlines, we set

$$a_\mu := -f_{0\mu} + c^l f_{l\mu} \tag{4.17}$$

for notational simplicity.

Since N is expected to appear linearly after a variable change $(N, N^i) \to (N, n^i(N, N^i))$, we have to seek the condition that all of $N e_n\big(\sqrt{g^{-1} f}\big)$ become linear in N. In fact, this condition is satisfied when we have

$$N \sqrt{g^{-1} f} = A + NB, \tag{4.18}$$

where A and B are some matrices independent of N. We take the square

$$N^2 g^{-1} f = A^2 + N(AB + BA) + N^2 B^2, \tag{4.19}$$

and compare to (4.11), from which we obtain

$$A^2 = E_0, \quad B^2 = E_2, \quad AB + BA = E_1. \tag{4.20}$$

We can easily find the matrices A and B obeying (4.20). Because the square of E_0 satisfies a relation

$$\begin{aligned} E_0^2 =& \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -c^i a_j \end{pmatrix}^2 \\ =& \begin{pmatrix} a_0^2 - a_0 c^l a_l & a_0 a_j - a_l c^l a_j \\ -a_0^2 c^i + a_0 c^i a_l c^l & -a_0 c^i a_j + c^i a_l c^l a_j \end{pmatrix} \\ =& (a_0 - c^l a_l) \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -c^i a_j \end{pmatrix} \\ =& (a_0 - c^l a_l) E_0, \end{aligned} \tag{4.21}$$

the solution for A is given by

$$A = \frac{1}{\sqrt{x}} \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -c^i a_j \end{pmatrix}, \qquad x := a_0 - c^l a_l = L^2 - L_k L^k + 2 c^l L_l - c^l c^k \omega_{lk}. \tag{4.22}$$

On the other hand, $B^2 = E_2$ can be solved as

$$B = \sqrt{x} \begin{pmatrix} 0 & 0 \\ D^i{}_k \omega^{kl} f_{l0} & D^i{}_j \end{pmatrix}, \qquad \sqrt{x} D^i{}_j := \sqrt{(\gamma^{il} - d^i d^l) f_{lj}}, \tag{4.23}$$

where we have factored out $\sqrt{x}$ for convenience. We have introduced a matrix $D^i{}_j$, and it has some properties we use in later calculations. Firstly, we note that D can be expressed as $D = \sqrt{S\omega}$ with a symmetric matrix S. Then, we expand it as $D = \sqrt{1 + (S\omega - 1)} = \sum_n d_n (S\omega - 1)^n = \sum_n \tilde{d}_n (S\omega)^n$ with some coefficients d_n and $\tilde{d}_n$. From this expanded formula and symmetric properties of S and ω, we can find a relation

$$\omega_{ik} D^k{}_j = \omega_{jk} D^k{}_i. \tag{4.24}$$

Next, we contract the above formula with ω^{li}. We get $\omega^{li}\omega_{ik} D^k{}_j = \omega^{li}\omega_{jk} D^k{}_i$ which leads to $D^l{}_j = \omega^{li}\omega_{jk} D^k{}_i$. Once more contracted with ω^{mj}, we obtain $\omega^{mj} D^l{}_j = \omega^{mj}\omega_{jk}\omega^{li} D^k{}_i = \omega^{li} D^m{}_i$. Thus, we can find another relation

$$\omega^{ik} D^j{}_k = \omega^{jk} D^i{}_k. \tag{4.25}$$

These two relations are used repeatedly.

Though we have determined A and B, they must also satisfy the remaining relation $AB + BA = E_1$. We write down the left hand side

$$\begin{aligned} AB + BA = &\begin{pmatrix} a_i D^i{}_k \omega^{kl} f_{l0} & a_i D^i{}_j \\ -c^i a_j D^j{}_k \omega^{kl} f_{l0} & -c^i a_k D^k{}_j \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ a_0 D^i{}_k \omega^{kl} f_{l0} - a_0 D^i{}_k c^k & D^i{}_k \omega^{kl} f_{l0} a_j - D^i{}_k c^k a_j \end{pmatrix}, \end{aligned} \tag{4.26}$$

and read equations

$$a_i D^i{}_k \omega^{kl} f_{l0} = d^l f_{l0}, \tag{4.27}$$

$$a_i D^i{}_j = d^l f_{lj}, \tag{4.28}$$

$$c^i a_j D^j{}_k \omega^{kl} f_{l0} - a_0 D^i{}_k \omega^{kl} f_{l0} + a_0 D^i{}_k c^k = d^l f_{l0} c^i + a_0 d^i, \tag{4.29}$$

$$c^i a_k D^k{}_j - D^i{}_k \omega^{kl} f_{l0} a_j + D^i{}_k c^k a_j = c^i d^l f_{lj} + d^i a_j. \tag{4.30}$$

From Eq.(4.27), we have

$$(d^l - a_i D^i{}_k \omega^{kl}) f_{l0} = 0. \tag{4.31}$$

Using the definition of a_i (4.17), $f_{ij} = \omega_{ij}$ and relations (4.24) and (4.25), Eq.(4.28) leads to

$$d^l = \omega^{lj} a_i D^i{}_j \tag{4.32}$$

$$= \omega^{lj}(-f_{0i} + c^k f_{ki}) D^i{}_j = -f_{0i}\omega^{ij} D^l{}_j + \omega^{lj} c^k f_{ji} D^i{}_k = -f_{0i}\omega^{ij} D^l{}_j + D^l{}_k c^k. \tag{4.33}$$

We also obtain from Eq.(4.29)

$$a_0(d^i - D^i{}_k c^k + D^i{}_k \omega^{kl} f_{l0}) + (d^l - a_j D^j{}_k \omega^{kl}) c^i f_{l0} = 0, \tag{4.34}$$

and from Eq.(4.30)

$$a_j(d^i - D^i{}_k c^k + D^i{}_k \omega^{kl} f_{l0}) + f_{lj}(d^l - \omega^{lm} a_k D^k{}_m) c^i = 0. \tag{4.35}$$

These equations (4.31), (4.33), (4.34) and (4.35) lead to one condition

$$d^i = D^i{}_k (c^k - \omega^{kl} f_{l0}), \tag{4.36}$$

which connects two functions c^i and d^i introduced in changing variables (4.10). Note that $D^i{}_k$ depends on c^i and d^i. Thus far, we have not specified how c^i and d^i actually depends on n^i. Therefore, we can determine c^i and d^i as functions of n^i by introducing a relation among c^i, d^i and n^i. For our purpose, the following relation works well

$$n^i =: c^i - \omega^{ik} f_{k0}, \tag{4.37}$$

from which we have

$$c^i = n^i + L^i, \tag{4.38}$$
$$d^i = D^i{}_k n^k. \tag{4.39}$$

We can determine $D^i{}_k$ by n^i in the following way. We substitute (4.38) and (4.39) into the definition of $D^i{}_j$ which says

$$\sqrt{x} D^i{}_j := \sqrt{(\gamma^{il} - d^i d^l)\omega_{lj}}, \quad x := a_0 - c^l a_l = L^2 - L_k L^k + 2c^l L_l - c^l c^k \omega_{lk}, \tag{4.40}$$

and obtain

$$\sqrt{x} D^i{}_j = \sqrt{(\gamma^{il} - D^i{}_m n^m D^l{}_k n^k)\omega_{lj}} = \sqrt{(\gamma^{il} - D^i{}_m n^m n^k D^l{}_k)\omega_{lj}}, \tag{4.41}$$
$$x = L^2 - n^i n^k \omega_{ki}. \tag{4.42}$$

In a matrix form, it can be expressed as

$$\sqrt{x} D = \sqrt{(\gamma^{-1} - D n n^T D^T)\omega}, \quad x = L^2 - n^T \omega n, \tag{4.43}$$

where the subscript "T" stand for the transposed matrix. This equation determines $D^i{}_j$ as a function of n^i. We take the square of both sides of (4.43)

$$x D^i{}_l D^l{}_j = (\gamma^{il} - D^i{}_m n^m n^k D^l{}_k)\omega_{lj} = \gamma^{il}\omega_{lj} - D^i{}_l n^l n^m D^k{}_j \omega_{km}, \tag{4.44}$$

which can be read as

$$D^i{}_l Q^l{}_k D^k{}_j = \gamma^{il}\omega_{lj}, \quad Q^l{}_k := x\delta^l{}_k + n^l n^m \omega_{mk} \tag{4.45}$$
$$\Leftrightarrow DQD = \gamma^{-1}\omega, \quad Q := x\mathbf{1} + n^T \omega n. \tag{4.46}$$

Then, we get $DQDQ = \gamma^{-1}\omega Q$ and read $DQ = \sqrt{\gamma^{-1}\omega Q}$. Hence, we find the solution

$$D = \big(\sqrt{\gamma^{-1}\omega Q}\big) Q^{-1}, \tag{4.47}$$

where the inverse of Q is given by

$$Q^{-1} = \frac{1}{x}(1 - L^{-2} n n^T \omega). \tag{4.48}$$

Therefore, we have the solution $D = D(\gamma, \omega, n^i, L)$, and obtain $d^i = d^i(\gamma, \omega, n^k, L)$ via (4.39).

Here, we have one point to notice. In the Hamiltonian analysis of dRGT massive gravity, we do not need to take care of the lapse L and the shift L^i for the non-dynamical metric $f_{\mu\nu}$. However, in analyzing bimetric gravity, they must also be traced. Hence, we have to simplify the complicated dependence on L in $D^i{}_j$. We recall that D is determined by the equation (4.43) and define normalized quantities $\hat{n}^i$ and $\hat{x}$ as

$$n^i =: L\hat{n}^i, \qquad x = L^2(1 - \hat{n}^T\omega\hat{n}) =: L^2\hat{x}. \tag{4.49}$$

Then, (4.43) can be written as

$$L\sqrt{\hat{x}}D = \sqrt{(\gamma^{-1} - L^2 D\hat{n}\hat{n}^T D^T)\omega}. \tag{4.50}$$

In order to eliminate the L dependence, we normalize $D^i{}_j$ as

$$\hat{D}^i{}_j := LD^i{}_j \tag{4.51}$$

and obtain the equation

$$\sqrt{\hat{x}}\hat{D} = \sqrt{(\gamma^{-1} - \hat{D}\hat{n}\hat{n}^T\hat{D}^T)\omega}, \qquad \hat{x} = 1 - \hat{n}^T\omega\hat{n}. \tag{4.52}$$

The solution for $\hat{D}$ can be found as before

$$\hat{D} = \left(\sqrt{\gamma^{-1}\omega\hat{Q}}\right)\hat{Q}^{-1}, \tag{4.53}$$

where

$$\hat{Q} = \hat{x} + \hat{n}^T\omega\hat{n}, \qquad \hat{Q}^{-1} = \frac{1}{\hat{x}}(1 - \hat{n}\hat{n}^T\omega). \tag{4.54}$$

Thus, we have the solution $\hat{D} = \hat{D}(\gamma, \omega, \hat{n}^i)$, and the complicated dependence on L is removed. In this "hat" quantities, (4.38) and (4.39) read $c^i = L\hat{n}^i + L^i$ and $d^i = \hat{D}^i{}_k\hat{n}^k$, and we obtain $d^i = d^i(\gamma, \omega, \hat{n}^k)$. The transformation law is give by

$$N^i = c^i + Nd^i = L\hat{n}^i + L^i + N\hat{D}^i_k(\gamma, \omega, \hat{n})\hat{n}^k. \tag{4.55}$$

In the following, we use this "hat" formulae. Sometimes, a relation $\sqrt{x}D = \sqrt{\hat{x}}\hat{D}$ is useful.

4.1.3 Linearity of the lapse

Now, we show the linearity of N. In fact, the mass/interaction term is linear in not only N but also L and L^i. To begin with, we consider $e_1(\sqrt{g^{-1}f}) = \mathrm{Tr}\sqrt{g^{-1}f}$. From (4.18), we immediately find

$$Ne_1(\sqrt{g^{-1}f}) = \mathrm{Tr}A + N\mathrm{Tr}B. \tag{4.56}$$

Matrices A and B are explicitly given by (4.22) and (4.23). Hence, we can calculate their trace

$$\mathrm{Tr}A = \mathrm{Tr}\frac{1}{\sqrt{x}}\begin{pmatrix} a_0 & a_j \\ -a_0c^i & -c^ia_j \end{pmatrix} = \frac{a_0 - c^ia_i}{\sqrt{x}} = \sqrt{x} = L\sqrt{\hat{x}}, \tag{4.57}$$

$$\mathrm{Tr}B = \sqrt{x}\mathrm{Tr}D = \sqrt{\hat{x}}\mathrm{Tr}\hat{D}, \tag{4.58}$$

and obtain

$$Ne_1(\sqrt{g^{-1}f}) = L\sqrt{\hat{x}} + N\sqrt{\hat{x}}\mathrm{Tr}\hat{D} = L\sqrt{\hat{x}}(\omega,\hat{n}) + N(\sqrt{\hat{x}}\mathrm{Tr}\hat{D})(\gamma,\omega,\hat{n}). \tag{4.59}$$

Therefore, we find the linearity of N and L. This linearity is retained even for e_2, e_3 and e_4.

For a preparation, we notice that $E_0^2 = xE_0$ leads to

$$A^n = x^{\frac{1}{2}(n-2)}E_0, \tag{4.60}$$

and $\mathrm{Tr}A^n = x^{n/2}$ holds because we know $\mathrm{Tr}E_0 = x$. On the other hand, we have $(\mathrm{Tr}A)^n = x^{n/2}$. Thus, we obtain an important relation

$$\mathrm{Tr}A^n = (\mathrm{Tr}A)^n. \tag{4.61}$$

We have one more important relation containing both A and B. We can see $\mathrm{Tr}(A^2B) = \mathrm{Tr}(E_0B)$ and $\mathrm{Tr}(A)\mathrm{Tr}(AB) = \sqrt{x}\mathrm{Tr}(E_0B/\sqrt{x}) = \mathrm{Tr}(E_0B)$, which leads to

$$\mathrm{Tr}(A^2B) = \mathrm{Tr}(A)\mathrm{Tr}(AB). \tag{4.62}$$

In the following, we use a short hand notation $\mathrm{Tr}^nX := (\mathrm{Tr}X)^n$ for a matrix X.

Using relations (4.61), (4.62) and (4.18), we obtain

$$\begin{aligned} Ne_2(\sqrt{g^{-1}f}) =& \frac{1}{2}N\Big(\mathrm{Tr}^2(\frac{1}{N}A+B) - \mathrm{Tr}(\frac{1}{N}A+B)^2\Big) \\ =& (\mathrm{Tr}A)(\mathrm{Tr}B) - \mathrm{Tr}(AB) + \frac{1}{2}N\Big(\mathrm{Tr}^2B - \mathrm{Tr}B^2\Big), \end{aligned} \tag{4.63}$$

$$\begin{aligned} Ne_3(\sqrt{g^{-1}f}) =& \frac{N}{6}\Big(\mathrm{Tr}^3(\frac{1}{N}A+B) - 3\mathrm{Tr}(\frac{1}{N}A+B)\mathrm{Tr}(\frac{1}{N}A+B)^2 + 2\mathrm{Tr}(\frac{1}{N}A+B)^3\Big) \\ =& \mathrm{Tr}(AB^2) - \mathrm{Tr}(AB)\mathrm{Tr}B + \frac{1}{2}(\mathrm{Tr}A)\big(\mathrm{Tr}^2B - \mathrm{Tr}B^2\big) \\ &+ \frac{1}{6}N\big(\mathrm{Tr}^3B - 3(\mathrm{Tr}B)(\mathrm{Tr}B^2) + 2\mathrm{Tr}B^3\big), \end{aligned} \tag{4.64}$$

and

$$\sqrt{\det\gamma}Ne_4(\sqrt{g^{-1}f}) = \sqrt{-\det g}\det\sqrt{g^{-1}f} = \sqrt{-\det f} = L\sqrt{\det\omega}. \tag{4.65}$$

We can calculate these traces. Using $a_i = n^k\omega_{ki}$, $c^i = n^i + L^i$, and formulae for A and B (4.22) and (4.23), we find

$$\mathrm{Tr}A = \sqrt{x} = L\sqrt{\hat{x}}, \tag{4.66}$$

$$\mathrm{Tr}(AB) = a_iD^i{}_k\omega^{kl}f_{l0} - c^ia_kD^k{}_i = -n^i\omega_{ij}D^j{}_kn^k = -n^T\omega Dn = -L\hat{n}^T\omega\hat{D}\hat{n}. \tag{4.67}$$

We apply (4.40) squared $xD^i{}_kD^k{}_j = (\gamma^{ik} - d^id^k)\omega_{kj}$ to the trace of AB^2, and find

$$\begin{aligned} \mathrm{Tr}(AB^2) =& \frac{1}{\sqrt{x}}\Big[a_i(\gamma^{il} - d^id^l)f_{l0} - c^ia_k(\gamma^{kl} - d^kd^l)\omega_{li}\Big] \\ =& -\sqrt{x}n^i\omega_{ij}D^j{}_kD^k{}_ln^l = -\sqrt{x}n^T\omega D^2n = -L\sqrt{\hat{x}}\hat{n}^T\omega\hat{D}^2\hat{n}. \end{aligned} \tag{4.68}$$

If we employ (4.41) squared $xD^2 = (\gamma^{-1} - Dnn^T D^T)\omega$, the trace of AB^2 is also expressed as

$$\mathrm{Tr}(AB^2) = -\frac{1}{\sqrt{x}} n^T \omega(\gamma^{-1} - Dnn^T D^T)\omega n. \tag{4.69}$$

We proceed to calculate the trace containing only B

$$\mathrm{Tr}B = \sqrt{x}\mathrm{Tr}D = \sqrt{\hat{x}}\mathrm{Tr}\hat{D}, \tag{4.70}$$

$$\mathrm{Tr}B^2 = x\mathrm{Tr}D^2 = \hat{x}\mathrm{Tr}\hat{D}^2, \tag{4.71}$$

$$\mathrm{Tr}B^3 = \sqrt{x}^3\mathrm{Tr}D^3 = \sqrt{\hat{x}}^3\mathrm{Tr}\hat{D}^3. \tag{4.72}$$

Collecting these trace formulae, we write down the elements of the mass/interaction term

$$Ne_0\big(\sqrt{g^{-1}f}\big) = N, \tag{4.73}$$

$$Ne_1\big(\sqrt{g^{-1}f}\big) = L\sqrt{\hat{x}} + N\sqrt{\hat{x}}\mathrm{Tr}\hat{D}, \tag{4.74}$$

$$Ne_2\big(\sqrt{g^{-1}f}\big) = L\big(\hat{x}\mathrm{Tr}\hat{D} + \hat{n}^T\omega\hat{D}\hat{n}\big) + \frac{1}{2}N\big(\hat{x}\mathrm{Tr}^2\hat{D} - \hat{x}\mathrm{Tr}\hat{D}^2\big), \tag{4.75}$$

$$\begin{aligned} Ne_3\big(\sqrt{g^{-1}f}\big) =& L\Big[-\sqrt{\hat{x}}\hat{n}^T\omega\hat{D}^2\hat{n} + \big(\hat{n}^T\omega\hat{D}\hat{n}\big)\sqrt{\hat{x}}\mathrm{Tr}\hat{D} + \frac{1}{2}\sqrt{\hat{x}}\big(\hat{x}\mathrm{Tr}^2\hat{D} - \hat{x}\mathrm{Tr}\hat{D}^2\big)\Big] \\ &+ \frac{1}{6}N\Big[\hat{x}^{\frac{3}{2}}\mathrm{Tr}^3\hat{D} - 3\hat{x}^{\frac{3}{2}}(\mathrm{Tr}\hat{D})(\mathrm{Tr}\hat{D}^2) + 2\hat{x}^{\frac{3}{2}}\mathrm{Tr}\hat{D}^3\Big], \end{aligned} \tag{4.76}$$

$$Ne_4\big(\sqrt{g^{-1}f}\big) = L\frac{\sqrt{\det\omega}}{\sqrt{\det\gamma}}. \tag{4.77}$$

Therefore, we obtain

$$N\sum_{n=0}^{4}\beta_n e_n\big(\sqrt{g^{-1}f}\big) = LU(\gamma,\omega,\hat{n}) + NV(\gamma,\omega,\hat{n}), \tag{4.78}$$

where U and V are defined by

$$\begin{aligned} U(\gamma,\omega,\hat{n}) :=& \beta_4\frac{\sqrt{\det\omega}}{\sqrt{\det\gamma}} + \beta_1\sqrt{\hat{x}} + \beta_2\big[\hat{x}\mathrm{Tr}\hat{D} + \hat{n}^T\omega\hat{D}\hat{n}\big] \\ &+ \beta_3\Big[\sqrt{\hat{x}}(\mathrm{Tr}\hat{D})\big(\hat{n}^T\omega\hat{D}\hat{n}\big) - \sqrt{\hat{x}}\hat{n}^T\omega\hat{D}^2\hat{n} + \frac{1}{2}\hat{x}^{\frac{3}{2}}\big(\mathrm{Tr}^2\hat{D} - \mathrm{Tr}\hat{D}^2\big)\Big], \end{aligned} \tag{4.79}$$

$$\begin{aligned} V(\gamma,\omega,\hat{n}) :=& \beta_0 + \beta_1\sqrt{\hat{x}}\mathrm{Tr}\hat{D} + \frac{1}{2}\beta_2\hat{x}\big(\mathrm{Tr}^2\hat{D} - \mathrm{Tr}\hat{D}^2\big) \\ &+ \frac{1}{6}\beta_3\hat{x}^{\frac{3}{2}}\Big[\mathrm{Tr}^3\hat{D} - 3(\mathrm{Tr}\hat{D})(\mathrm{Tr}\hat{D}^2) + 2\mathrm{Tr}\hat{D}^3\Big]. \end{aligned} \tag{4.80}$$

The linearity for N and L, and also that for L^i is confirmed.

4.1.4 The Hamiltonian formulation of the action

Since the mass term or the interaction term in (4.1) and (4.2) do not contain derivatives, the definition of the canonical momenta is not changed from that in Section 3.1.1. We denote the canonical momenta for γ_{ij} and ω_{ij} as π^{ij} and p^{ij} respectively. Then, the action $S = \int d^4x\mathcal{L}$ has the Lagrangian density

$$\mathcal{L}_{MG}/M_p^2 = \pi^{ij}\dot{\gamma}_{ij} + N\mathcal{R}_0 + \mathcal{R}_i\big(L\hat{n}^i + L^i + N\hat{D}^i{}_k\hat{n}^k\big) + 2m^2\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\big(\sqrt{g^{-1}f}\big) \tag{4.81}$$

$$\begin{aligned} =&\pi^{ij}\dot{\gamma}_{ij} + (L\hat{n}^i + L^i)\mathcal{R}_i + 2m^2 L\sqrt{\det\gamma}U(\gamma,\omega,\hat{n}) \\ &+ N\big[\mathcal{R}_0 + R_i\hat{D}^i{}_k\hat{n}^k + 2m^2\sqrt{\det\gamma}V(\gamma,\omega,\hat{n})\big] \end{aligned} \tag{4.82}$$

for dRGT massive gravity, and

$$\begin{aligned} \mathcal{L}_{BG} =& M_g^2\pi^{ij}\dot{\gamma}_{ij} + \underset{\sim\sim\sim}{M_f^2 p^{ij}\dot{\omega}_{ij}} \\ &+ L^i\big[M_g^2\mathcal{R}_i^{(g)} + \underset{\sim\sim\sim}{M_f^2\mathcal{R}_i^{(f)}}\big] + L\big[\underset{\sim\sim\sim}{M_f^2\mathcal{R}_0^{(f)}} + M_g^2\hat{n}^i\mathcal{R}_i^{(g)} + 2m^2M_{eff}^2\sqrt{\det\gamma}U(\gamma,\omega,\hat{n})\big] \\ &+ N\big[M_g^2\mathcal{R}_0^{(g)} + M_g^2\mathcal{R}_i^{(g)}\hat{D}^i{}_k\hat{n}^k + 2m^2M_{eff}^2\sqrt{\det\gamma}V(\gamma,\omega,\hat{n})\big] \end{aligned} \tag{4.83}$$

for bimetric gravity. $\mathcal{R}_0$ and $\mathcal{R}_i$ are defined as (3.21) and (3.22). The subscript (g) and (f) represent "for $g_{\mu\nu}$" and "for $f_{\mu\nu}$". The difference comes from terms with the wavy underline, which do not contain $\hat{n}^i$. Hence, variation with respect to $\hat{n}^i$ leads to the same formula.

Now, we see the linearity for N, L and L^i, and the non-linear dependence on n^i. We expect that the variation with respect to n^i leads to an equation to determine the auxiliary field n^i, and the solution does not break the linearity for N, L and L^i when it is substituted back into the action.

4.2 Variation with respect to the new shift

In this section, we calculate variation with respect $\hat{n}^i$. We often use a formula

$$\delta\mathrm{Tr}\big(\sqrt{X}\big) = \frac{n}{2}\mathrm{Tr}\big(\sqrt{X}^{\,n-2}\delta X\big), \tag{4.84}$$

where X is a matrix. We also rely on properties $\omega_{ik}D^k{}_j = \omega_{jk}D^k{}_i$ and $\omega^{ik}D^j{}_k = \omega^{jk}D^i{}_k$.

Since the mass/interaction term is constructed from the traces of matrices A and B, we calculate their variation. We start with the simplest one

$$\delta\mathrm{Tr}A = \delta\sqrt{x} = -x^{-\frac{1}{2}}n^i\omega_{ij}\delta n^j = -\frac{1}{\sqrt{x}}n^T\omega\delta n. \tag{4.85}$$

Variation of the trace of B needs a little trick. Using (4.41) or its matrix form, we calculate it in such a way

$$\begin{aligned}
\delta\mathrm{Tr}B =&\delta\mathrm{Tr}\sqrt{(\gamma^{-1}-Dnn^TD^T)\omega}\\
=&\frac{1}{2}\mathrm{Tr}\Big(\sqrt{(\gamma^{-1}-Dnn^TD^T)\omega}^{\,-1}\delta(\gamma^{-1}-Dnn^TD^T)\omega\Big)\\
=&-\frac{1}{2}\frac{1}{\sqrt{x}}(D^{-1})^i{}_j\delta(D^j{}_ln^ln^kD^m{}_k)\omega_{mi}\\
=&-\frac{1}{2\sqrt{x}}(D^{-1})^i{}_j\Big(\delta(D^j{}_ln^l)D^m{}_kn^k\omega_{mi}+D^j{}_ln^l\delta(D^m{}_kn^k)\omega_{mi}\Big)\\
=&-\frac{1}{2\sqrt{x}}(D^{-1})^i{}_j\Big(\delta(D^j{}_ln^l)D^m{}_in^k\omega_{mk}+D^j{}_ln^l\delta(D^m{}_kn^k)\omega_{mi}\Big)\\
=&-\frac{1}{2\sqrt{x}}\Big(\delta(D^j{}_ln^l)n^k\omega_{jk}+n^i\delta(D^m{}_kn^k)\omega_{mi}\Big)\\
=&-\frac{1}{\sqrt{x}}n^T\omega\delta(Dn). \qquad (4.86)
\end{aligned}$$

Similarly, we also obtain

$$\delta\mathrm{Tr}AB=-\delta(n^T\omega Dn)=-(\delta n^i)\omega_{ij}D^j{}_kn^k-n^i\omega_{ij}\delta(D^j{}_kn^k)=-n^T\omega D\delta n-n^T\omega\delta(Dn), \quad (4.87)$$

and

$$\begin{aligned}
\delta\mathrm{Tr}AB^2 =&-\delta\big(\sqrt{x}n^T\omega D^2n\big)\\
=&-\delta\big(\sqrt{x}\omega_{kj}D^j{}_in^iD^k{}_ln^l\big)\\
=&\frac{1}{\sqrt{x}}\big(n^T\omega D^2n\big)n^T\omega\delta n-2\sqrt{x}n^T\omega D\delta(Dn). \qquad (4.88)
\end{aligned}$$

Here, we can find an important relation from (4.88). We remember that $\mathrm{Tr}(AB^2)$ can be expressed in a different way (4.69)

$$\mathrm{Tr}(AB^2)=-\frac{1}{\sqrt{x}}n^T\omega(\gamma^{-1}-Dnn^TD^T)\omega n=-\frac{1}{\sqrt{x}}n^i\omega_{ij}(\gamma^{jl}-D^j{}_mn^mn^pD^l{}_p)\omega_{lk}n^k. \quad (4.89)$$

In this formula (4.89), variation acting on $1/\sqrt{x}$ leads to

$$-\delta\Big(\frac{1}{\sqrt{x}}\Big)n^T\omega(\gamma^{-1}-Dnn^TD^T)\omega n=-x^{-\frac{3}{2}}(n^T\omega\delta n)n^T\omega xD^2n=-\frac{1}{\sqrt{x}}(n^T\omega D^2n)n^T\omega\delta n, \quad (4.90)$$

where we have used $xD^2=(\gamma^{-1}-Dnn^TD^T)\omega$ after the variation . In the same way, variation of (4.89) with respect to n^i which exists outside the bracket is

$$-2\frac{1}{\sqrt{x}}n^T\omega xD^2\delta n=-2\sqrt{x}n^T\omega D^2\delta n, \quad (4.91)$$

and that within the bracket is

$$\frac{1}{\sqrt{x}}n^i\omega_{ij}\Big(\delta(D^j{}_mn^m)D^l{}_pn^p+D^j{}_mn^m\delta(D^l{}_pn^p)\Big)\omega_{lk}n^k=\frac{2}{\sqrt{x}}(n^T\omega Dn)n^T\omega\delta(Dn). \quad (4.92)$$

Hence, we have

$$\delta \mathrm{Tr} AB^2 = -\frac{1}{\sqrt{x}}(n^T\omega D^2 n)n^T\omega\delta n - 2\sqrt{x}n^T\omega D^2\delta n + \frac{2}{\sqrt{x}}(n^T\omega D n)n^T\omega\delta(Dn). \tag{4.93}$$

Comparing (4.88) and (4.93), we obtain

$$\Big[\sqrt{x}n^T\omega D + \frac{1}{\sqrt{x}}(n^T\omega Dn)n^T\omega\Big]\delta(Dn) = \Big[\sqrt{x}n^T\omega D^2 + \frac{1}{\sqrt{x}}(n^T\omega D^2 n)n^T\omega\Big]\delta n. \tag{4.94}$$

Equivalently, we can express it as

$$\Big[\sqrt{x}D + \frac{1}{\sqrt{x}}Dnn^T\omega\Big]\delta(Dn) = \Big[\sqrt{x}D^2 + \frac{1}{\sqrt{x}}(n^T\omega D^2 n)\mathbf{1}\Big]\delta n \tag{4.95}$$

with unit matrix $\mathbf{1}$. Remaining ones are calculated to be

$$\delta \mathrm{Tr} B^2 = \delta \mathrm{Tr}(\gamma^{-1} - Dnn^TD^T)\omega = -2n^T\omega D\delta(Dn), \tag{4.96}$$

$$\delta \mathrm{Tr} B^3 = \delta \mathrm{Tr}\big[(\gamma^{-1} - Dnn^TD^T)\omega\big]^{\frac{3}{2}} = -3\sqrt{x}n^T\omega D^2\delta(Dn). \tag{4.97}$$

We collect these formulae for variation and obtain

$$\delta N e_1\big(\sqrt{g^{-1}f}\big) = \delta \mathrm{Tr} A + N\delta \mathrm{Tr} B = -\frac{1}{\sqrt{x}}n^T\omega\delta(n + NDn), \tag{4.98}$$

$$\begin{aligned}\delta N e_2\big(\sqrt{g^{-1}f}\big) =& \delta \mathrm{Tr} A\cdot \mathrm{Tr} B + \mathrm{Tr} A\cdot\delta \mathrm{Tr} B - \delta \mathrm{Tr}(AB) + \frac{N}{2}\Big(2\delta \mathrm{Tr} B\cdot \mathrm{Tr} B - \delta \mathrm{Tr} B^2\Big)\\ =& n^T\omega\big[D - \mathbf{1}\mathrm{Tr} D\big]\delta(n + NDn).\end{aligned} \tag{4.99}$$

e_3 is rather complicated

$$\begin{aligned}\delta N e_3\big(\sqrt{g^{-1}f}\big) =& \delta \mathrm{Tr}(AB^2) - \delta \mathrm{Tr}(AB)\mathrm{Tr} B - \mathrm{Tr}(AB)\delta \mathrm{Tr} B\\ &+ \frac{1}{2}\delta \mathrm{Tr} A\cdot\big(\mathrm{Tr}^2 B - \mathrm{Tr} B^2\big) + \frac{1}{2}\mathrm{Tr} A\cdot\big(2\delta \mathrm{Tr} B\cdot \mathrm{Tr} B - \delta \mathrm{Tr} B^2\big)\\ &+ \frac{N}{6}\Big(3\mathrm{Tr}^2 B\cdot\delta \mathrm{Tr} B - 3\delta \mathrm{Tr} B\cdot \mathrm{Tr} B^2 - 3\mathrm{Tr} B\cdot\delta \mathrm{Tr} B^2 + 2\delta \mathrm{Tr} B^3\Big)\\ =& \Big[\frac{1}{\sqrt{x}}(n^T\omega D^2 n)n^T\omega + \sqrt{x}(\mathrm{Tr} D)n^T\omega D - \frac{\sqrt{x}}{2}\big(\mathrm{Tr}^2 D - \mathrm{Tr} D^2\big)n^T\omega\Big]\delta n\\ &- \Big[\sqrt{x}n^T\omega D + \frac{1}{\sqrt{x}}(n^T\omega Dn)n^T\omega\Big]\delta(Dn)\\ &+ \frac{\sqrt{x}}{2}\Big[-\big(\mathrm{Tr}^2 D - \mathrm{Tr} D^2\big)n^T\omega + 2(\mathrm{Tr} D)n^T\omega D - 2n^T\omega D^2\Big],\end{aligned} \tag{4.100}$$

but we use (4.94) and conclude

$$\delta N e_3\big(\sqrt{g^{-1}f}\big) = -\sqrt{x}n^T\omega\Big[D^2 - D\mathrm{Tr} D + \frac{1}{2}\mathbf{1}\big(\mathrm{Tr}^2 D - \mathrm{Tr} D^2\big)\Big]\delta(n + NDn). \tag{4.101}$$

Thus, variation of the Lagrangian density with respect to n^i is given by

$$\begin{aligned}\frac{1}{M_p^2}\delta\mathcal{L} =& \mathcal{R}_i\delta(n^i + ND^i{}_k n^k) + 2m^2\sqrt{\det\gamma}\Big\{-\frac{\beta_1}{\sqrt{x}}n^T\omega + \beta_2 n^T\omega\big(D - \mathbf{1}\mathrm{TrD}\big)\\ &- \beta_3\sqrt{x}n^T\omega\Big[D^2 - D\mathrm{Tr} D + \frac{1}{2}\mathbf{1}\big(\mathrm{Tr}^2 D - \mathrm{Tr} D^2\big)\Big]\Big\}\delta(n + NDn).\end{aligned} \tag{4.102}$$

In the case of bimetric gravity, we have only to replace m^2 by $m^2 M_{eff}^2/M_g^2$. The last factor in (4.102) can be interpreted as

$$\frac{\partial}{\partial n^j}(n^i + N D^i{}_k) = \frac{\partial N^i}{\partial n^j}. \tag{4.103}$$

Since the variable change should be invertible, the transformation matrix $\partial N^i/\partial n^j$ must have the inverse. Therefore, the variational principle leads to an equation

$$\mathcal{R}_i^{(g)} - 2m^2\sqrt{\det\gamma}\frac{1}{\sqrt{x}}n^T\omega\Big\{\beta_1\mathbf{1} + \beta_2\sqrt{x}\big(\mathbf{1}\mathrm{Tr}D - D\big) + \beta_3 x\Big[D^2 - D\mathrm{Tr}D + \frac{1}{2}\mathbf{1}\big(\mathrm{Tr}^2 D - \mathrm{Tr}D^2\big)\Big]\Big\}_i = 0, \tag{4.104}$$

which is equivalently in the hat formula

$$\mathcal{R}_i^{(g)} - 2m^2\sqrt{\det\gamma}\frac{1}{\sqrt{\hat{x}}}\hat{n}^T\omega\Big\{\beta_1\mathbf{1} + \beta_2\sqrt{\hat{x}}\big(\mathbf{1}\mathrm{Tr}\hat{D} - \hat{D}\big) + \beta_3 \hat{x}\Big[\hat{D}^2 - \hat{D}\mathrm{Tr}\hat{D} + \frac{1}{2}\mathbf{1}\big(\mathrm{Tr}^2 \hat{D} - \mathrm{Tr}\hat{D}^2\big)\Big]\Big\}_i = 0. \tag{4.105}$$

For later convenience, we name the left hand side of the above equation as $\mathcal{C}_i$. We solve this equation as $\hat{n}^i = \hat{n}^i(\gamma, \pi, \omega)$ and substitute it back into the original action. Then, we start the Hamiltonian analysis. Here, the important point is that the solution $\hat{n}^i(\gamma, \pi, \omega)$ is independent of L, L^i and N. We recall that $\hat{D} = \hat{D}\big(\gamma, \omega, \hat{n}\big)$ also does not depend on L, L^i and N. Thus, non-linear terms of N, L and L^i never appear in the action.

In the minimal model where $\beta_1 = -1$, $\beta_2 = 0$ and $\beta_3 = 0$, we can explicitly solve the equation (4.105). The minimal model drastically simplifies (4.105)

$$\sqrt{\hat{x}}\mathcal{R}_i + 2m^2\sqrt{\det\gamma}\,\omega_{ij}\hat{n}^j = 0. \tag{4.106}$$

Squaring this equation with the metric ω, we have

$$\big(\sqrt{\hat{x}}\mathcal{R}_i + 2m^2\sqrt{\det\gamma}\,\omega_{ij}\hat{n}^j\big)\omega^{il}\big(\sqrt{\hat{x}}\mathcal{R}_l - 2m^2\sqrt{\det\gamma}\,\omega_{lk}\hat{n}^k\big) = 0, \tag{4.107}$$

which is calculated to be

$$\hat{x}\mathcal{R}_i\omega^{il}\mathcal{R}_l - 4m^4(\det\gamma)\omega_{jk}\hat{n}^j\hat{n}^k = 0. \tag{4.108}$$

We substitute $\hat{x} = 1 - \hat{n}^T\omega\hat{n}$, and obtain

$$\hat{n}^i\omega_{ij}\hat{n}^j = \frac{(\mathcal{R}_k\omega^{ki})\omega_{ij}(\omega^{jl}\mathcal{R}_l)}{4m^4(\det\gamma) + \mathcal{R}_i\omega^{ij}\mathcal{R}_j}. \tag{4.109}$$

The above equation can be easily solved as

$$\hat{n}^i = \frac{\mathcal{R}_j\omega^{ji}}{\sqrt{4m^4(\det\gamma) + \mathcal{R}_k\omega^{kl}\mathcal{R}_l}}, \tag{4.110}$$

and

$$\hat{x} = \frac{4m^4(\det\gamma)}{4m^4(\det\gamma) + \mathcal{R}^T\omega^{-1}\mathcal{R}}. \tag{4.111}$$

4.3 The Hamiltonian analysis

In Section 4.2, we have found the equation of motion (4.105) for the auxiliary field $\hat{n}^i$, and obtained the solution $\hat{n}^i = \hat{n}^i(\gamma, \pi, \omega)$. We substitute the solution into the original action, and eliminate the auxiliary field n^i. The action is linear in L, L^i and N because $\hat{n}^i(\gamma, \pi, \omega)$ and $\hat{D}\big(\gamma, \omega, \hat{n}\big)$ does not depend on L, L^i and N. Then, we proceed to the main part of the Hamiltonian analysis. We recall the action $S = \int d^4x \mathcal{L}$ with the Lagrangian density

$$\mathcal{L}_{MG}/M_p^2 = \pi^{ij}\dot{\gamma}_{ij} - \mathcal{H}_f + N\mathcal{C}, \tag{4.112}$$

$$\mathcal{H}_f := -(L\hat{n}^i + L^i)\mathcal{R}_i - 2m^2 L\sqrt{\det\gamma}\,U(\gamma, \omega, \hat{n}), \tag{4.113}$$

$$\mathcal{C} := \mathcal{R}_0 + \mathcal{R}_i \hat{D}^i{}_k \hat{n}^k + 2m^2\sqrt{\det\gamma}\,V(\gamma, \omega, \hat{n}), \tag{4.114}$$

for dRGT massive gravity, and

$$\mathcal{L}_{BG} = M_g^2\pi^{ij}\dot{\gamma}_{ij} + \underset{\sim\sim\sim}{M_f^2 p^{ij}\dot{\omega}_{ij}} - \mathcal{H}_f + N\mathcal{C}, \tag{4.115}$$

$$\mathcal{H}_f := -L^i\big[M_g^2\mathcal{R}_i^{(g)} + \underset{\sim\sim\sim}{M_f^2\mathcal{R}_i^{(f)}}\big] - L\big[\underset{\sim\sim\sim}{M_f^2\mathcal{R}_0^{(f)}} + M_g^2\hat{n}^i\mathcal{R}_i^{(g)} + 2m^2M_{eff}^2\sqrt{\det\gamma}\,U(\gamma, \omega, \hat{n})\big], \tag{4.116}$$

$$\mathcal{C} := M_g^2\mathcal{R}_0^{(g)} + M_g^2\mathcal{R}_i^{(g)}\hat{D}^i{}_k\hat{n}^k + 2m^2M_{eff}^2\sqrt{\det\gamma}\,V(\gamma, \omega, \hat{n}), \tag{4.117}$$

for bimetric gravity. The difference is only in the wavy under lines. We also remember the explicit formulae of U and V

$$U(\gamma, \omega, \hat{n}) = \beta_4\frac{\sqrt{\det\omega}}{\sqrt{\det\gamma}} + \beta_1\sqrt{\hat{x}} + \beta_2\big[\hat{x}\mathrm{Tr}\hat{D} + \hat{n}^T\omega\hat{D}\hat{n}\big]$$
$$+ \beta_3\Big[\sqrt{\hat{x}}(\mathrm{Tr}\hat{D})\big(\hat{n}^T\omega\hat{D}\hat{n}\big) - \sqrt{\hat{x}}\hat{n}^T\omega\hat{D}^2\hat{n} + \frac{1}{2}\hat{x}^{\frac{3}{2}}\big(\mathrm{Tr}^2\hat{D} - \mathrm{Tr}\hat{D}^2\big)\Big], \tag{4.118}$$

$$V(\gamma, \omega, \hat{n}) = \beta_0 + \beta_1\sqrt{\hat{x}}\mathrm{Tr}\hat{D} + \frac{1}{2}\beta_2\hat{x}\big(\mathrm{Tr}^2\hat{D} - \mathrm{Tr}\hat{D}^2\big)$$
$$+ \frac{1}{6}\beta_3\hat{x}^{\frac{3}{2}}\Big[\mathrm{Tr}^3\hat{D} - 3(\mathrm{Tr}\hat{D})(\mathrm{Tr}\hat{D}^2) + 2\mathrm{Tr}\hat{D}^3\Big]. \tag{4.119}$$

In these formulae, $\hat{n}^i$ is interpreted as the solution of the equation (4.105), which we denote $\hat{n}^i = \hat{n}^i(\gamma, \pi, \omega)$.

Now, we take variation with respect to N and obtain a primary constraint

$$\mathcal{C}(\gamma, \pi, \omega, \hat{n}(\gamma, \pi, \omega)) = 0. \tag{4.120}$$

This constraint must be preserved along the time evolution, which is generated by the Hamiltonian

$$H := \int d^3x(\mathcal{H}_f - N\mathcal{C}). \tag{4.121}$$

Hence, the Poisson bracket with the Hamiltonian H must be zero

$$\frac{d}{dt}\mathcal{C}(x) = \big\{\mathcal{C}(x), H\big\}_{PB} \approx \int d^3y\big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\}_{PB} - \int d^3y N(y)\big\{\mathcal{C}(x), \mathcal{C}(y)\big\}_{PB} \approx 0, \tag{4.122}$$

where the symbol "≈" means the equality on the hypersurface determined by constraints. The Poisson bracket is given by

$$\big\{F(x),G(y)\big\}_{PB}=\int d^3z\Big[\frac{\delta F(x)}{\delta\gamma_{mn}(z)}\frac{\delta G(y)}{\delta\pi^{mn}(z)}-\frac{\delta F(x)}{\delta\pi^{mn}(z)}\frac{\delta G(y)}{\delta\gamma_{mn}(z)}+(\omega\text{ and }p\text{ derivatives})\Big], \tag{4.123}$$

and that between $\mathcal{C}(x)$ and $\mathcal{C}(y)$ is derived in Appendix F. The result is

$$\big\{\mathcal{C}(x),\mathcal{C}(y)\big\}_{PB}=-\Big[P^i(x)\frac{\partial}{\partial x^i}\delta^{(3)}(x-y)-P^i(y)\frac{\partial}{\partial y^i}\delta^{(3)}(x-y)\Big],\quad P^i:=\mathcal{C}\hat{D}^i{}_j n^j. \tag{4.124}$$

Thus, we conclude

$$\big\{\mathcal{C}(x),\mathcal{C}(y)\big\}_{PB}\approx 0. \tag{4.125}$$

Here, notice that this result holds for both of dRGT massive gravity and bimetric gravity.

In the case of dRGT massive gravity, the coefficient of N at (4.122) disappears, and the Lagrange multiplier N cannot be determined. Therefore, we have a secondary constraint $\mathcal{C}_{(2)}=0$, which we derive in Appendix F,

$$\begin{aligned}\mathcal{C}_{(2)}:=&\int d^3y\big\{\mathcal{C}(x),\mathcal{H}_f(y)\big\}\\=&m^2L\big(\gamma_{mn}\pi^k{}_k-2\pi_{mn}\big)U^{mn}+2m^2\sqrt{\det\gamma}\,\gamma_{im}\mathcal{D}_n(LU^{mn})\hat{D}^i{}_k\hat{n}^k+\mathcal{C}\,\mathcal{D}_i(L\hat{n}^i+L^i)\\&+\big(\hat{D}^i{}_k\hat{n}^k\mathcal{R}_j-2m^2\sqrt{\det\gamma}\,\bar{V}^{il}\gamma_{lj}\big)\mathcal{D}_i(L\hat{n}^j+L^j)+\big(\mathcal{D}_i\mathcal{R}_0+\hat{D}^j{}_k\hat{n}^k\mathcal{D}_i\mathcal{R}_j\big)(L\hat{n}^i+L^i),\end{aligned} \tag{4.126}$$

where U and $\bar{V}$ are defined in (F.57) and (F.46). The first term in (4.126) contains π^{ij} linearly without derivatives. Since such a term cannot be found in $\mathcal{C}$ and $\mathcal{C}_i$, secondary constraint $\mathcal{C}_{(2)}$ does not automatically vanish even on the constraint surface. If we consider bimetric gravity, additional terms contribute to the above formula. The consistency condition on $\mathcal{C}_{(2)}$ for the time evolution

$$\frac{d\mathcal{C}_{(2)}}{dt}=\big\{\mathcal{C}_{(2)}(x),H\big\}_{PB}\approx\int d^3y\big\{\mathcal{C}_{(2)}(x),\mathcal{H}_f(y)\big\}_{PB}-\int d^3yN(y)\big\{\mathcal{C}_{(2)}(x),\mathcal{C}(y)\big\}_{PB}\approx 0 \tag{4.127}$$

have to be further imposed. The explicit calculation seems extremely difficult, but if we are able to confirm

$$\big\{\mathcal{C}_{(2)}(x),\mathcal{H}_f(y)\big\}_{PB}\not\approx 0,\quad \big\{\mathcal{C}_{(2)}(x),\mathcal{C}(y)\big\}_{PB}\not\approx 0, \tag{4.128}$$

we can find that the consistency condition on $\mathcal{C}_{(2)}$ determines the Lagrange multiplier N, and no additional constraint emerges. At least in the linear level, we know that (4.128) is satisfied, which we have seen in Section 2.4.2. We also know that dRGT massive gravity reduces to the Fierz-Pauli mass term in the linear level. Taking into account these facts, we believe that (4.128) is actually satisfied. Therefore, we have two constraints and no remaining Lagrange multipliers. The total number of degrees of freedom is $(6\times 2-2)/2=5$, which means the absence of the BD-ghost.

In the case of bimetric gravity, we have the Hamiltonian density

$$\mathcal{H} = \mathcal{H}_f - N\mathcal{C} = -L^i\mathcal{C}_i^{(L)} - L\mathcal{C}^{(L)} - N\mathcal{C}, \tag{4.129}$$

$$\mathcal{C}_i^{(L)} := M_g^2\mathcal{R}_i^{(g)} + M_f^2\mathcal{R}_i^{(f)}, \tag{4.130}$$

$$\mathcal{C}^{(L)} := M_f^2\mathcal{R}_0^{(f)} + M_g^2\hat{n}^i\mathcal{R}_i^{(g)} + 2m^2M_{eff}^2\sqrt{\det\gamma}\,U(\gamma,\omega,\hat{n}). \tag{4.131}$$

In addition to N, we have other Lagrange multipliers L and L^i. Variation with respect to them leads to primary constraints

$$\mathcal{C} = 0, \quad \mathcal{C}^{(L)} = 0, \quad \mathcal{C}_i^{(L)} = 0. \tag{4.132}$$

We can speculate the structure of their Poisson brackets as follows. Firstly, we recall the fact that bimetric gravity retains one overall diffeomorphism invariance. Thus, it seems reasonable to think that $\mathcal{C}_i^{(L)}$ and $\mathcal{C}^{(L)}$ should generate corresponding transformation and $\{\mathcal{C}_i^{(L)}, \mathcal{C}_j^{(L)}\}_{PB} \approx 0$, $\{\mathcal{C}^{(L)}, \mathcal{C}_i^{(L)}\}_{PB} \approx 0$ and $\{\mathcal{C}^{(L)}, \mathcal{C}^{(L)}\}_{PB} \approx 0$ hold. In general, one of generators may be a combination of $\mathcal{C}$ and $\mathcal{C}^{(L)}$, but the essence of the following argument does not change. Among these generators, $\mathcal{C}_i^{(L)}$ should generate spatial transformation, and on the (spatial) constraint surface, $\mathcal{C} = 0$ is satisfied on every point. Then, we find $\{\mathcal{C}_i^{(L)}, \mathcal{C}\}_{PB} \approx 0$. On the other hand, the result (4.126) suggests $\{\mathcal{C}, \mathcal{C}^L\}_{PB} \not\approx 0$. This is because all of $\mathcal{C}$, $\mathcal{C}^{(L)}$ and $\mathcal{C}_i^{(L)}$ do not have a linear and no derivative term of π^{ij}. Actually, our speculation is correct, which is explicitly shown in [32]. Therefore, the consistency condition for the primary constraints can be read as

$$\frac{d}{dt}\mathcal{C} = \int d^3y \big\{\mathcal{C}(x),\, \mathcal{H}(y)\big\}_{PB} \approx -\int d^3y L(y) \big\{\mathcal{C}(x),\, \mathcal{C}^{(L)}(y)\big\}_{PB} \not\approx 0, \tag{4.133}$$

$$\frac{d}{dt}\mathcal{C}_i^{(L)} = \int d^3y \big\{\mathcal{C}_i^{(L)}(x),\, \mathcal{H}(y)\big\}_{PB} \approx 0, \tag{4.134}$$

$$\frac{d}{dt}\mathcal{C}^{(L)} = \int d^3y \big\{\mathcal{C}^{(L)}(x),\, \mathcal{H}(y)\big\}_{PB} \approx -\int d^3y N(y) \big\{\mathcal{C}^{(L)}(x),\, \mathcal{C}(y)\big\}_{PB} \not\approx 0. \tag{4.135}$$

From (4.133) and (4.135), we have one secondary constraint. The consistency condition on this secondary constraint determines one of the Lagrange multipliers, and the Hamiltonian analysis ends. Eventually, we have five primary constraints and one secondary constraint. The number of the undetermined Lagrange multipliers is four. The total number of degrees of freedom is counted to be $(12 \times 2 - 5 - 1 - 4)/2 = 7$ which means one massless and one massive gravitons. We have no ghost-like extra degree of freedom.

The analysis in this chapter has focused only on the four dimensional case. However, generalization to higher dimensions would be straightforward though the amount of calculations increases. This expectation is supported by the analysis in chapter 5.

Chapter 5

Multi-vielbein gravity

In this chapter, we consider a theory introduced in [4]. We have devoted all of Chapter 4 to the Hamiltonian analysis of dRGT massive/bimetric gravity, where a large amount of calculations has been carried out. The origin of this complexity is clear. It comes from the fact that the mass/interaction term is non-linearly dependent on the lapse N as well as the shift N^i. We see N^2, N^3,..., and $(N^i)^2$, $(N^i)^3$,..., through the expansion of the square root $\sqrt{g^{-1}f}$. This non-linearity obscures the absence of the BD-ghost. If the BD-ghost should be eliminated, we must have a constraint to reduce degrees of freedom. In general, we get a constraint from variation with respect to a variable appearing linearly in the action. Thus, such a linear variable needs to be found. We have had a lot of effort to deal with this problem.

Therefore, it seems that the removal of the square root drastically simplifies the analysis. Then, we focus on a variable called vielbein which is interpreted as a square root of a metric. We expect that a vielbein opens up the square root in the interaction term and makes the constraint structure extremely clear.

We begin this chapter with revisiting general relativity from a perspective of a vielbein. Then, we introduce interactions among multiple kinds of vielbeins. We construct an interaction term which carries no additional ghost-like degree of freedom. At the end of this chapter, we consider a relationship between a metric theory and a vielbein theory.

5.1 General relativity in a vielbein formulation

In this section, we bring in a new variable called vielbein, which can be interpreted as a square root of a metric. We revisit general relativity in a vielbein formulation, and perform the Hamiltonian analysis. This is a preparation for the next sections, where we introduce interactions among vielbeins and count the number of degrees of freedom by the Hamiltonian analysis.

5.1.1 The ADM decomposition

We introduce a vielbein E_μ^A as a square root of a metric $g_{\mu\nu}$

$$g_{\mu\nu} =: E_\mu^A E_\nu^B \eta_{AB}, \tag{5.1}$$

where η_{AB} is the Minkowski metric. Subscripts $A, B, ..$ run on indices of the Minkowski space-time. We also introduce the inverse of the vielbein E_μ^A, which we denote by E_A^μ. The inverse vielbein E^μ_A is determined by the following relations

$$E^A_\mu E^\mu_B = \delta^A_B\,, \qquad E^A_\mu E^\nu_A = \delta^\nu_\mu. \tag{5.2}$$

The action of general relativity (2.10) can be rewritten in a vielbein formulation via (5.1)

$$S_{EH} = \frac{1}{16\pi G}\int d^Dx(\det E)\,R[E], \tag{5.3}$$

where $R[E]$ is the scalar curvature written by the vielbein. In order to proceed to the Hamiltonian analysis, we need to apply the ADM decomposition to the vielbein. Here, we fix our notation about space-time indices. We use Greek letters $\mu, \nu, ...$ to represent general space-time indices, and their spatial components are denoted as $i, j, ...$ Index manipulations for them are performed by the general metric $g_{\mu\nu}$ and its spatial metric γ_{ij}. On the other hand, capital letters A,B,... run on indices of the Minkowski space-time, and we write their spatial components as $a, b, ...$ They are manipulated by the Minkowski metric $\eta_{\mu\nu}$ and its spatial component δ_{ab}. We have two kinds of time components, which seems confusing. Hence, we assign the index 0 to the time component of the general space-time, and assign the index t to that of the Minkowski space-time.

If we regard E^A_μ as the A component of the vector E_μ, vectors $E_0, E_{i=1,2,...}$ are linearly independent because the determinant of the vielbein should not be zero: $\det E^A_\mu \neq 0$. Then, we rename the vector E_i as V_i, and E_0 can be written as a linear combination of V_i plus one more independent vector M

$$E^A_i =: V^A_i, \qquad E^A_0 =: N^i V^A_i + NM^A, \tag{5.4}$$

where N and N^i represent coefficients. We substitute (5.4) into the definition (5.1), and regard the coefficients N and N^i as the lapse and the shift. We expect that the metric $g_{\mu\nu}$ returns the ADM decomposed form (3.5)

$$E^A_\mu E^B_\nu \eta_{AB} = \begin{pmatrix} -N^2 + N^i N_i & N_j \\ N_i & \gamma_{ij} \end{pmatrix}, \tag{5.5}$$

from which we obtain

$$M^A M^B \eta_{AB} = -1, \quad M^A V^B_i \eta_{AB} = 0, \quad V^A_i V^B_j \eta_{AB} = \gamma_{ij}. \tag{5.6}$$

This is the general (d+1)-decomposition in a vielbein formulation. If we put $M^A = (1, 0, 0, 0, ...)$, we have $V^0_i = 0$ and the formula dramatically simplifies. In this case, we denote the remaining V^a_i as e^a_i, and the decomposed vielbein reads

$$E^A_\mu = \begin{pmatrix} E^t_0 & E^a_0 \\ E^t_i & E^a_i \end{pmatrix} = \begin{pmatrix} N & N^j e^a_j \\ 0 & e^a_i \end{pmatrix}. \tag{5.7}$$

A general vielbein can be recovered through a Lorentz boost

$$\Lambda(\mathbf{p})^A{}_B := \begin{pmatrix} \tilde\gamma & p^a \\ p_b & \delta^a_b + \frac{1}{\tilde\gamma+1}p^a p_b \end{pmatrix}, \quad \tilde\gamma := \sqrt{1 + p^a p_a}, \tag{5.8}$$

which takes $M^A = (1, \mathbf{0})$ to $(\tilde{\gamma}, \mathbf{p})$ with $\mathbf{p} := (p^1, p^2, ...)$. Then, a general vielbein is parametrized by $\mathbf{p}$

$$E_\mu^A = \Lambda(\mathbf{p})^A{}_B \begin{pmatrix} N & N^j e_j^a \\ 0 & e_i^a \end{pmatrix}_\mu^B = \begin{pmatrix} N\tilde{\gamma} + N^j e_j^b p_b & N p^a + N^j e_j^b \left(\delta_b^a + \frac{1}{\tilde{\gamma}+1} p_b p^a\right) \\ e_i^b p_b & e_i^b \left(\delta_b^a + \frac{1}{\tilde{\gamma}+1} p_b p^a\right) \end{pmatrix}. \tag{5.9}$$

However, the Einstein-Hilbert action is invariant under local Lorentz transformations because vielbein dependence comes through the relation to the metric (5.1). Hence, $\mathbf{p}$ dependence disappears from the action (5.3), and we have only to consider the upper triangular form (5.7). Therefore, the spatial component e_i^a plays an important role, which constructs the spatial metric

$$\gamma_{ij} = e_i^a e_j^b \delta_{ab}. \tag{5.10}$$

The inverse is denoted as e_a^i and satisfies

$$e_i^a e_b^i = \delta_b^a, \qquad e_i^a e_a^j = \delta_i^j, \qquad \gamma^{ij} = e_a^i e_b^j \delta^{ab}. \tag{5.11}$$

In the following, indices are raised and lowered by the spatial metrics γ_{ij}, γ^{ij}, δ_{ab} and δ^{ab}. We can construct a spatial vielbein with two lower or two upper indices, for example $e_i^b \delta_{ab}$. We can write it both e_{ai} and e_{ia}. The order of the indices a and i is arbitrary, which will never confuse us.

5.1.2 The Hamiltonian analysis

We proceed to perform the Hamiltonian analysis in a vielbein formulation. We remember calculations in section 3.1.1. We have written the Einstein-Hilbert action with the phase space variables (3.20), where the canonical momenta π_{ij} has been defined by (3.11). Here, we think that the action (3.20) is written by the spatial vielbein e_i^a through π^{ij} and γ_{ij}. This is a short cut treatment, more detailed discussion is found in [33].

We focus on the first term $\dot{\gamma}_{ij}\pi^{ij}$ in (3.20), and rewrite it with the spatial vielbein

$$\dot{\gamma}_{ij}\pi^{ij} = (\dot{e}_i^a e_j^b \delta_{ab} + e_i^a \dot{e}_j^b \delta_{ab})\pi^{ij} = \dot{e}_i^a (\pi^{ij} + \pi^{ji}) e_{ja} = 2\dot{e}_i^a \pi^{ij} e_{ja}, \tag{5.12}$$

which means that the canonical momentum conjugate to e_i^a is

$$\Pi_a^i := 2\pi^{ij} e_{ja}. \tag{5.13}$$

Since we regard π^{ij} as a function determined by (3.11), (5.13) defines Π_a^i as a function of e_i^a. From the definition (5.13), we can obtain

$$\pi^{ij} = \frac{1}{2}\Pi_a^i e^{aj}. \tag{5.14}$$

Because π^{ij} defined by (3.11) is symmetric, we should have the condition

$$\Pi_a^{[i} e^{j]a} = 0 \Leftrightarrow \Pi_{[a}^i e_{b]i} = 0. \tag{5.15}$$

This condition comes from the fact that we have doubled the number of dynamical variables. The spatial metric γ_{ij} is symmetric and contains $\frac{1}{2}d(d+1)$ components, while the spatial vielbein has no symmetric property and has d^2 components. The above condition reduces $\frac{1}{2}d(d-1)$

variables from the spatial vielbein and no extra degree of freedom is introduced. Thus, we must impose a constraint

$$\mathcal{P}_{ab} := \Pi^i_{[a}e_{b]i} = 0. \tag{5.16}$$

Then, we go to phase space. We have the action

$$S_{EH} = \frac{1}{16\pi G}\int dtd^dx\Big[\dot{e}^a_i\Pi^i_a + N\mathcal{R}_0 + N^i\mathcal{R}_i + \frac{1}{2}\lambda^{ab}\mathcal{P}_{ab}\Big], \tag{5.17}$$

where λ_{ab} is a Lagrange multiplier and antisymmetric in a, b. The canonical variables are e^a_i and Π^i_a. $\mathcal{R}_0$ and $\mathcal{R}_i$ are given by e^a_i and Π^i_a through (5.10) and (5.14). Variation with respect to N, N^i and λ^{ab} leads to three kinds of constraints

$$\mathcal{R}_0 = 0, \quad \mathcal{R}_i = 0, \quad \mathcal{P}_{ab} = 0. \tag{5.18}$$

Now, we calculate the Poisson bracket

$$\{F, G\}_{PB} = \int d^dz\Big[\frac{\delta F}{\delta e^a_i(z)}\frac{\delta G}{\delta \Pi^i_a(z)} - \frac{\delta G}{\delta e^a_i(z)}\frac{\delta F}{\delta \Pi^i_a(z)}\Big] \tag{5.19}$$

among constraints. In $\mathcal{R}_0$ and $\mathcal{R}_i$, the dependence on e^a_i and Π^i_a comes only through γ_{ij} and π^{ij}. In such a case, we have

$$\frac{\delta F}{\delta e^a_i}\frac{\delta G}{\delta \Pi^i_a} = \Big(\frac{\delta F}{\delta\gamma_{kl}}\frac{\delta\gamma_{kl}}{\delta e^a_i} + \frac{\delta F}{\delta\pi^{kl}}\frac{\delta\pi^{kl}}{\delta e^a_i}\Big)\Big(\frac{\delta G}{\delta\gamma_{mn}}\frac{\delta\gamma_{mn}}{\delta\Pi^i_a} + \frac{\delta G}{\delta\pi^{mn}}\frac{\delta\pi^{mn}}{\delta\Pi^i_a}\Big). \tag{5.20}$$

Derivatives operating on γ_{ij} are calculated from (5.10) and derivatives on π^{ij} are obtained from (5.14). They are given by

$$\frac{\delta\gamma_{kl}}{\delta e^a_i} = \delta^i_k e_{la} + \delta^i_l e_{ka}, \qquad \frac{\delta\gamma_{mn}}{\delta\Pi^i_a} = 0, \tag{5.21}$$

$$\frac{\delta\pi^{kl}}{\delta e^a_i} = -\frac{1}{2}\Pi^k_c e^l_a e^{ic}, \qquad \frac{\delta\pi^{kl}}{\delta\Pi^i_a} = \frac{1}{2}\delta^k_i e^{la}. \tag{5.22}$$

In calculating $\delta\pi/\delta e$, we have used a property $e^i_a e^a_j = \delta^i_j$ which leads to a useful relation $\delta e^i_a = -e^i_b(\delta e^b_j)e^j_a$. From these formulae, we find

$$\frac{\delta\gamma_{kl}}{\delta e^a_i}\frac{\delta\pi^{mn}}{\delta\Pi^i_a} = \frac{1}{2}(\delta^k_m\delta^l_n + \delta^k_n\delta^l_m), \qquad \frac{\delta\pi^{kl}}{\delta e^a_i}\frac{\delta\pi^{mn}}{\delta\Pi^i_a} = -\frac{1}{4}\Pi^k_a\gamma^{ln}e^{ma}, \tag{5.23}$$

and conclude

$$\{F, G\}^{(e,\Pi)}_{PB} = \{F, G\}^{(\gamma,\pi)}_{PB} - \frac{1}{4}\int d^dz\frac{\delta F}{\delta\pi^{kl}(z)}\frac{\delta G}{\delta\pi^{mn}(z)}\big(\Pi^k_a e^{ma} - \Pi^m_a e^{ka}\big)\gamma^{ln}(z). \tag{5.24}$$

The subscript (e, Π) stands for the Poisson bracket constructed from functional derivatives with respect to e^a_i and Π^i_a, namely (5.19). Similarly, the subscript (γ, π) represents that of γ_{ij} and π^{ij}. Their difference vanishes when we impose the constraint

$$\mathcal{P}_{ab} = 0 \Leftrightarrow \Pi^{[i}_a e^{j]a} = 0. \tag{5.25}$$

Thus, we think that the Poisson brackets among $\mathcal{R}_0$ and $\mathcal{R}_i$ are the same as those in the metric formulation.

Next, we calculate the Poisson brackets containing $\mathcal{P}_{ab}$. Derivatives of $\mathcal{P}_{ab}$ are easy to calculate

$$\frac{\delta \mathcal{P}_{ab}}{\delta e_i^c} = \frac{1}{2}\big(\Pi_a^i \delta_{cb} - \Pi_b^i \delta_{ca}\big), \qquad \frac{\delta \mathcal{P}_{ab}}{\delta \Pi_c^i} = \frac{1}{2}\big(\delta_a^c e_{bi} - \delta_b^c e_{ai}\big). \tag{5.26}$$

It is convenient to use an integrated version for $\mathcal{P}_{ab}$. We introduce $\mathcal{P}(f) := \int d^d x f^{ab}(x)\mathcal{P}_{ab}(x)$ with some antisymmetric tensor f^{ab}. When a functional F is dependent on e_i^a and Π_a^i only through (5.10) and (5.14), we can calculate as

$$\begin{aligned}\big\{\mathcal{P}(f),\, F\big\}_{PB} &= \int \frac{\delta \mathcal{P}(f)}{\delta e_i^a}\frac{\delta F}{\delta \Pi_a^i} - \frac{\delta F}{\delta e_i^a}\frac{\delta \mathcal{P}(f)}{\delta \Pi_a^i} \\ &= \int \frac{\delta \mathcal{P}(f)}{\delta e_i^a}\left(\frac{\delta \gamma_{kl}}{\delta \Pi_a^i}\frac{\delta F}{\delta \gamma_{kl}} + \frac{\delta \pi^{kl}}{\delta \Pi_a^i}\frac{\delta F}{\delta \pi^{kl}}\right) - \left(\frac{\delta \gamma_{kl}}{\delta e_i^a}\frac{\delta F}{\delta \gamma_{kl}} + \frac{\delta \pi^{kl}}{\delta e_i^a}\frac{\delta F}{\delta \pi^{kl}}\right)\frac{\delta \mathcal{P}(f)}{\delta \Pi_a^i}.\end{aligned} \tag{5.27}$$

We neglect delta functions because we always encounter two delta functions and three integrations. They leave overall one integration. Then, recalling $\delta\gamma/\delta\Pi = 0$ and noticing

$$\frac{\delta \gamma_{kl}}{\delta e_i^a}\frac{\delta \mathcal{P}(f)}{\delta \Pi_a^i} = f^{ab}\big(e_{bk}e_{al} + e_{bl}e_{ak}\big) = 0, \tag{5.28}$$

we find

$$\big\{\mathcal{P}(f),\, F\big\}_{PB} = \int \left(\frac{\delta \mathcal{P}(f)}{\delta e_i^a}\frac{\delta \pi^{kl}}{\delta \Pi_a^i} - \frac{\delta \mathcal{P}(f)}{\delta \Pi_a^i}\frac{\delta \pi^{kl}}{\delta e_i^a}\right)\frac{\delta F}{\delta \pi^{kl}}. \tag{5.29}$$

In fact, the integrand is zero since we can see

$$\frac{\delta \mathcal{P}(f)}{\delta e_i^a}\frac{\delta \pi^{kl}}{\delta \Pi_a^i} - \frac{\delta \mathcal{P}(f)}{\delta \Pi_a^i}\frac{\delta \pi^{kl}}{\delta e_i^a} = \frac{1}{2}f^{ab}\big(\Pi_a^k e_b^l + \Pi_b^k e_a^l\big) = 0. \tag{5.30}$$

Hence, we obtain

$$\big\{\mathcal{P}(f),\, F\big\}_{PB} = 0, \tag{5.31}$$

and conclude that the Poisson brackets between $\mathcal{P}_{ab}$ and $\mathcal{R}_0$ or $\mathcal{R}_i$ vanish.

The remaining one is the Poisson bracket between $\mathcal{P}_{ab}$. Straightforwardly, we calculate to find

$$\big\{\mathcal{P}(f),\, \mathcal{P}(q)\big\}_{PB} = \int \big(f^{ab}q^{cd} - f^{cd}q^{ab}\big)\delta_{cb}\mathcal{P}_{ad}. \tag{5.32}$$

Therefore, we know that the consistency conditions are automatically satisfied on the constraint surface

$$\frac{d\mathcal{R}_0}{dt} = \big\{\mathcal{R}_0, H\big\}_{PB} \approx 0, \quad \frac{d\mathcal{R}_i}{dt} = \big\{\mathcal{R}_i, H\big\}_{PB} \approx 0, \quad \frac{d\mathcal{P}_{ab}}{dt} = \big\{\mathcal{P}_{ab}, H\big\}_{PB} \approx 0. \tag{5.33}$$

We need no additional constraint. Besides, all of the Lagrange multipliers are left undetermined. The total number of degrees of freedom is counted as follows. In phase space, the number of the original variables is $2d^2$. Then, constraints reduce $d+1+\frac{1}{2}d(d-1)$ degrees of freedom. In addition, the remaining Lagrange multipliers mean existence of gauge freedoms. By fixing them, we further reduce $d+1+\frac{1}{2}d(d-1)$ degrees of freedom. Then, the total number of degrees of freedom turns out to be

$$\frac{1}{2}\Big[2d^2-2\Big(d+1+\frac{1}{2}d(d-1)\Big)\Big]=\frac{1}{2}d(d-1)-1. \tag{5.34}$$

In a four-dimensional case ($d=3$), we have two degrees of freedom, which is compatible with the case of general relativity written by the metric language.

5.2 Ghost-free multi-vielbein gravity

In this section, we consider interacting vielbeins and construct a ghost-free model. We assume that we have $\mathcal{N}$ metrics $g(I)_{\mu\nu}$ $(I=1,2,..,\mathcal{N})$ and define corresponding vielbeins $E(I)_\mu^A$ as

$$g(I)_{\mu\nu}=:E(I)_\mu^A E(I)_\nu^B \eta_{AB}. \tag{5.35}$$

Now, we introduce an interaction term given by

$$U\big(E(1),..,E(\mathcal{N})\big):=\sum_{I_1,\ldots,I_D}^{\mathcal{N}} T(I_1 I_2\cdots I_D)\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D}E(I_1)_{\mu_1}^{A_1}E(I_2)_{\mu_2}^{A_2}\cdots E(I_D)_{\mu_D}^{A_D}, \tag{5.36}$$

where $\epsilon^{\mu_1\mu_2\cdots\mu_D}$ and $\epsilon_{A_1A_2\cdots A_D}$ represent antisymmetrization symbols determined by $\epsilon^{012\cdots d}=1$ and $\epsilon_{012\cdots d}=1$. The coefficient $T(I_1I_2\cdots I_D)$ is set to be symmetric. The concept of this interaction term is to linearize all the lapse and the shift. In the formula for a general vielbein (5.9), the lapse and the shift appear linearly and are contained only in time components. Thus, antisymmetrization (5.36) returns only linear terms of them. The above interaction term clearly breaks each diffeomorphism and local Lorentz invariance, but the overall ones are retained. Under the overall local Lorentz transformation

$$E(I)_\mu^A\to\Lambda^A{}_B E(I)_\mu^B \quad (I=1,2,...,\mathcal{N}), \tag{5.37}$$

interaction term U transforms as

$$\begin{aligned}
&\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D}E(I_1)_{\mu_1}^{A_1}E(I_2)_{\mu_2}^{A_2}\cdots E(I_D)_{\mu_D}^{A_D}\\
\to&\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D}\Lambda^{A_1}{}_{B_1}E(I_1)_{\mu_1}^{B_1}\Lambda^{A_2}{}_{B_2}E(I_2)_{\mu_2}^{B_2}\cdots\Lambda^{A_D}{}_{B_D}E(I_D)_{\mu_D}^{B_D}\\
=&\frac{1}{D!}\big(\epsilon_{A_1A_2\cdots A_D}\epsilon^{B_1B_2\cdots B_D}\Lambda^{A_1}{}_{B_1}\Lambda^{A_2}{}_{B_2}\cdots\Lambda^{A_D}{}_{B_D}\big)\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{C_1C_2\cdots C_D}E(I_1)_{\mu_1}^{C_1}E(I_2)_{\mu_2}^{C_2}\cdots E(I_D)_{\mu_D}^{C_D}\\
=&(\det\Lambda)\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D}E(I_1)_{\mu_1}^{A_1}E(I_2)_{\mu_2}^{A_2}\cdots E(I_D)_{\mu_D}^{A_D},
\end{aligned} \tag{5.38}$$

where we have used a property

$$\epsilon^{A_1A_2\cdots A_D}\epsilon_{B_1B_2\cdots B_D}=D!\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_D}_{B_D]}. \tag{5.39}$$

Under the overall general coordinate transformation

$$E(I)_\mu^A(x)\to\frac{\partial f^\nu}{\partial x^\mu}E(I)_\nu^A\big(f(x)\big) \quad (I=1,2,...,\mathcal{N}), \tag{5.40}$$

interaction term U becomes

$$\begin{aligned}&\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D}E(I_1)^{A_1}_{\mu_1}E(I_2)^{A_2}_{\mu_2}\cdots E(I_D)^{A_D}_{\mu_D}\\ \to&\Big(\det\frac{\partial f}{\partial x}\Big)\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D}E(I_1)^{A_1}_{\mu_1}E(I_2)^{A_2}_{\mu_2}\cdots E(I_D)^{A_D}_{\mu_D}.\end{aligned}\tag{5.41}$$

Thus, integrated interaction term $\int d^DxU$ is invariant under the overall local Lorentz and general coordinate diffeomorphism.

Then, we have the action

$$S=\sum_{I=1}^{\mathcal{N}}\int d^Dx\mathcal{L}_{EH}(E(I))+\int d^DxU\big(E(1),...,E(\mathcal{N})\big).\tag{5.42}$$

$\mathcal{L}_{EH}(E(I))$ is the Einstein-Hilbert term for each vielbein $E(I)$ with $1/16\pi G_{(I)}$ included. In order to perform the Hamiltonian analysis, we rely on the ADM decomposition and assume that vielbeins are parametrized by vectors $\mathbf{p}(I)$ as in (5.9) for each $I=1,2,...,\mathcal{N}$:

$$E(I)^A_\mu=\begin{pmatrix}N(I)\tilde{\gamma}(I)+N(I)^je(I)^b_jp(I)_b & N(I)p(I)^a+N(I)^je(I)^b_j\big[\delta^a_b+\frac{1}{\tilde{\gamma}(I)+1}p(I)_bp(I)^a\big]\\ e(I)^b_ip(I)_b & e(I)^b_i\big[\delta^a_b+\frac{1}{\tilde{\gamma}(I)+1}p(I)_bp(I)^a\big]\end{pmatrix}.\tag{5.43}$$

Each of the Einstein-Hilbert term is invariant under a Local Lorentz transformation $E(I)^A_\mu\to\Lambda(I)^A_BE(I)^B_\mu$. Hence, all of $\mathbf{p}(I)$ drop from the Einstein-Hilbert terms. They remain only in the interaction term U. Besides, we decompose spatial vielbeins as

$$e(I)^a_i=[e^{q(I)}]^a{}_b\hat{e}(I)^b_i\quad(I=1,2,...,\mathcal{N}),\tag{5.44}$$

where $q(I)$ is an antisymmetric matrix $q(I)^a{}_c\delta^{cb}=-q(I)^b{}_c\delta^{ca}$ and it generates a spatial rotation on the Minkowski indices. This matrix $q(I)$ has $d(d-1)/2$ components. Thus, remaining $d(d+1)/2$ degrees of freedom are left in $\hat{e}(I)$. Since the Einstein Hilbert term is invariant under a spatial rotation (5.44), $q(I)$ drops out from the kinetic term and becomes an auxiliary field. For instance, we can suppose that $\hat{e}(I)$ is chosen to satisfy the following condition

$$\frac{\partial U}{\partial\hat{e}(I)^a_i}\hat{e}(I)^b_i\delta_{bc}=\frac{\partial U}{\partial\hat{e}(I)^c_i}\hat{e}(I)^b_i\delta_{ba},\tag{5.45}$$

which extracts $d(d+1)/2$ degrees of freedom. On the other hand, the equation of motion for $q(I)$ is given by

$$\frac{\partial U}{\partial q(I)^a{}_b}\delta_{bc}=\frac{\partial U}{\partial q(I)^c{}_b}\delta_{ba}.\tag{5.46}$$

Using (5.44), we can rewrite it as

$$\frac{\partial U}{\partial\hat{e}(I)^b_i}[e^{-q(I)}]^b{}_a\hat{e}^e_i[e^{q(I)}]^d{}_e\delta_{dc}=\frac{\partial U}{\partial\hat{e}(I)^b_i}[e^{-q(I)}]^b{}_c\hat{e}^e_i[e^{q(I)}]^d{}_e\delta_{da}.\tag{5.47}$$

Then, we expand both sides about $q(I)$ and find a solution

$$q(I)^a{}_b=0.\tag{5.48}$$

This is because the zeroth order expansion is nothing but (5.46). Hence, $q(I)$ disappears from the action. In the following argument, we use $\hat{e}_i^a$ instead of e_i^a, and assume that auxiliary fields $q(I)$ $(I = 1, 2, ..., \mathcal{N})$ are solved as above.

In vielbeins parametrized by vectors $\mathbf{p}(I)$, the lapse $N(I)$ and the shift $N(I)^i$ appear only linearly. In addition, the interaction term U is constructed by antisymmetrizing vielbeins. Hence, the interaction term U contains only linear terms of $N(I)$ and $N(I)^i$. Therefore, schematically, the action can be decomposed as

$$S = \sum_{I=1}^{\mathcal{N}} \int d^D x \Big(\hat{\Pi}(I)_a^i \dot{\hat{e}}(I)_i^a + N(I) \big[\mathcal{R}(I)_0(\hat{e}, \hat{\Pi}) + \mathcal{Q}(I)_0(\hat{e}, p) \big] + N(I)^i \big[\mathcal{R}(I)_i(\hat{e}, \hat{\Pi}) + \mathcal{Q}(I)_i(\hat{e}, p) \big] \Big), \tag{5.49}$$

where $\mathcal{Q}(I)_0$ and $\mathcal{Q}(I)_i$ represent contributions from the interaction term U. We do not have constraints $\mathcal{P}(I)_{ab}$ $(I = 1, 2, ..., \mathcal{N})$ because we have already fixed spatial rotations on the Minkowski indices before going to phase space.

Now, we count the total number of degrees of freedom. To begin with, we remember that we have the overall local Lorentz invariance, especially overall boost invariance with which we set $\mathbf{p}(1) = 0$. Then, using constraints obtained from variation with respect to $N(I)_i$ for $I = 2, 3, .., \mathcal{N}$, we solve the remaining $\mathbf{p}(I)$ as

$$\mathcal{R}(I)_i(\hat{e}, \hat{\Pi}) + \mathcal{Q}(I)_i(\hat{e}, p) = 0 \;\Rightarrow\; p(I)^a = p(I)^a(\hat{e}, \hat{\Pi}) \quad (I = 2, 3, ..., \mathcal{N}). \tag{5.50}$$

We substitute the above solution $\mathbf{p}(I)(\hat{e}, \hat{\Pi})$ into the action, and obtain

$$S = \int d^D x \Bigg\{ \sum_{I=1}^{\mathcal{N}} \bigg(\hat{\Pi}(I)_a^i \dot{\hat{e}}(I)_i^a + N(I) \Big[\mathcal{R}(I)_0(\hat{e}, \hat{\Pi}) + \mathcal{Q}(I)_0\big(\hat{e}, p(\hat{e}, \hat{\Pi})\big) \Big] \bigg) + N(1)^i \Big[\mathcal{R}(1)_i(\hat{e}, \hat{\Pi}) + \mathcal{Q}(1)_i\big(\hat{e}, p(\hat{e}, \hat{\Pi})\big) \Big] \Bigg\}. \tag{5.51}$$

Variation with respect to $N(I)$ and $N(1)^i$ leads to constraints $\mathcal{R}_0(I) + \mathcal{Q}_0(I) = 0$ and $\mathcal{R}_i(1) + \mathcal{Q}_i(1) = 0$. Among $\mathcal{R}_0(1) + \mathcal{Q}_0(1)$,..., $\mathcal{R}_0(\mathcal{N}) + \mathcal{Q}_0(\mathcal{N})$, one of them is combined with $\mathcal{R}_i(1) + \mathcal{Q}_i(1)$ to generate the overall general coordinate transformation. The remaining $\mathcal{N} - 1$ constraints are related to broken invariances of coordinate changes.

Actually, we have to calculate the Poisson bracket with the Hamiltonian, but it seems extremely troublesome to complete this task. Only the bi-vielbein case is completed [34, 35]. For more general cases, the following rule is expected to hold. A constraint generating a transformation under which the invariance is retained gives rise to no additional constraint, and the corresponding Lagrange multiplier is left undetermined. On the other hand, each constraint generating a transformation under which the invariance is broken leads to a secondary constraint. Then, the consistency condition on this secondary constraint determines one Lagrange multiplier. Here, we apply the above rule though we may need detailed investigations about this point. Some related studies are found in [36, 37].

Constraints and gauge fixing conditions relating to the overall general coordinate diffeomorphism reduce $2(d + 1)$ degrees of freedom. Remaining constraints responsible for broken symmetries of coordinate changes generate secondary constraints, and these primary and secondary ones remove further $2(\mathcal{N}-1)$ degrees of freedom. Therefore, the total number of degrees of freedom is counted to be

$$\begin{aligned}&\frac{1}{2}\Big(2\times\frac{1}{2}d(d+1)\mathcal{N}-2(d+1)-2(\mathcal{N}-1)\Big)\\=&\frac{1}{2}d(d-1)-1+(\mathcal{N}-1)\left[\frac{1}{2}d(d+1)-1\right].\end{aligned}\tag{5.52}$$

In four-dimensional space-times ($d=3$), we have

$$2+5(\mathcal{N}-1)\tag{5.53}$$

degrees of freedom, which is nothing but one massless and $(\mathcal{N}-1)$ massive gravitons.

5.3 Toward metric formulations

As a final work in this chapter, we consider whether or not multi-vielbein theories can be translated into some metric theories. A vielbeins comes from doubling the number of variables, which introduces an extra Lorentz invariance. Thus, we expect that a constraint related to this Lorentz transformation turns out to be a clue to a metric formulation.

5.3.1 The constraint from the Lorentz transformation

We recall the action given by

$$S=S_{EH}+\int d^Dx U\big(E(1),...,E(\mathcal{N})\big),\quad S_{EH}=\int d^Dx\mathcal{L}_{EH}=\sum_{I=1}^{\mathcal{N}}\int d^Dx\mathcal{L}_{EH}(E(I)).\tag{5.54}$$

We know that the Einstein-Hilbert term S_{EH} is invariant under each local Lorentz transformation

$$E(I)^A_\mu\to\Lambda(I)^A{}_B E(I)^B_\mu\quad(I=1,2,...,\mathcal{N}).\tag{5.55}$$

For our purpose, it is convenient to consider an infinitesimal transformation given by

$$\Lambda(I)^A{}_B\simeq\delta^A_B+\omega(I)^A{}_B,\tag{5.56}$$

where $\omega(I)^A{}_B$ is antisymmetric

$$\omega(I)^A{}_B\eta^{BC}=-\omega(I)^C{}_B\eta^{BA}.\tag{5.57}$$

The invariance of S_{EH} is represented as

$$\delta_\omega S_{EH}=\int d^Dx\frac{\delta\mathcal{L}_{EH}}{\delta E(I)^A_\mu}\omega(I)^A{}_B E(I)^B_\mu=0.\tag{5.58}$$

We can take arbitrary $\omega(I)^A{}_B$ which is antisymmetric. Hence, we obtain

$$\frac{\delta\mathcal{L}_{EH}}{\delta E(I)^A_\mu}E(I)^C_\mu\eta_{CB}-\frac{\delta\mathcal{L}_{EH}}{\delta E(I)^B_\mu}E(I)^C_\mu\eta_{CA}=0.\tag{5.59}$$

We combine (5.59) with the equation of motion for $E(I)^A_\mu$

$$\frac{\delta \mathcal{L}_{EH}}{\delta E(I)^A_\mu} + \frac{\delta U}{\delta E(I)^A_\mu} = 0, \tag{5.60}$$

and obtain a constraint written by only the interaction term U

$$\frac{\delta U}{\delta E(I)^A_\mu} E(I)^C_\mu \eta_{CB} = \frac{\delta U}{\delta E(I)^B_\mu} E(I)^C_\mu \eta_{CA}. \tag{5.61}$$

Here, we rewrite the interaction term (5.36) in a suitable form to consider a relationship to a metric formulation. Using inverse vielbeins $E(J)^\mu_A$, we can write $E(I_i)^{A_i}_{\mu_i} = E(J)^{A_i}_{\nu_i} E(J)^{\nu_i}_{B_i} E(I_i)^{B_i}_{\mu_i}$. Then, the interaction term is written as

$$\begin{aligned}
&\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D} E(I_1)^{A_1}_{\mu_1} E(I_2)^{A_2}_{\mu_2}\cdots E(I_D)^{A_D}_{\mu_D}\\
=&\epsilon^{\mu_1\cdots\mu_D}\epsilon_{A_1\cdots A_D} E(J)^{A_1}_{\nu_1}\cdots E(J)^{A_D}_{\nu_D}\big(E(J)^{\nu_1}_{B_1}E(I_1)^{B_1}_{\mu_1}\big)\cdots\big(E(J)^{\nu_D}_{B_D}E(I_D)^{B_D}_{\mu_D}\big)\\
=&\frac{1}{D!}\epsilon_{A_1\cdots A_D}\epsilon^{\lambda_1\cdots\lambda_D}E(J)^{A_1}_{\lambda_1}\cdots E(J)^{A_D}_{\lambda_D}\epsilon^{\mu_1\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_D}\big(E(J)^{\nu_1}_{B_1}E(I_1)^{B_1}_{\mu_1}\big)\cdots\big(E(J)^{\nu_D}_{B_D}E(I_D)^{B_D}_{\mu_D}\big)\\
=&\big(\det E(J)\big)\epsilon^{\mu_1\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_D}\big(E(J)^{\nu_1}_{B_1}E(I_1)^{B_1}_{\mu_1}\big)\cdots\big(E(J)^{\nu_D}_{B_D}E(I_D)^{B_D}_{\mu_D}\big),
\end{aligned} \tag{5.62}$$

where we have used

$$\epsilon^{\mu_1\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_D} = D!\delta^{\mu_1}_{[\nu_1}\cdots\delta^{\mu_D}_{\nu_D]}. \tag{5.63}$$

We can also write $E(I_i)^{A_i}_{\mu_i} = E(J)^{B_i}_{\mu_i} E(J)^{\nu_i}_{B_i} E(I_i)^{A_i}_{\nu_i}$, and using $\epsilon^{A_1\cdots A_D}\epsilon_{B_1\cdots B_D} = D!\delta^{A_1}_{[B_1}\cdots\delta^{A_D}_{B_D]}$, we obtain

$$\begin{aligned}
&\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D} E(I_1)^{A_1}_{\mu_1} E(I_2)^{A_2}_{\mu_2}\cdots E(I_D)^{A_D}_{\mu_D}\\
=&\big(\det E(J)\big)\epsilon_{A_1\cdots A_D}\epsilon^{B_1\cdots B_D}\big(E(J)^{\nu_1}_{B_1}E(I_1)^{A_1}_{\nu_1}\big)\cdots\big(E(J)^{\nu_D}_{B_D}E(I_D)^{A_D}_{\nu_D}\big).
\end{aligned} \tag{5.64}$$

5.3.2 The case of bi-vielbein

Figure 5.1: Each box describes a vielbein. The link represents interaction.

We consider the case where only two kinds of vielbeins interact, which we denote by $E(1)$ and $E(2)$. It is convenient to represent this setting by a diagram in Fig.5.1. Notice that vielbeins construct metrics

$$g(1)_{\mu\nu} = E(1)^A_\mu E(1)^B_\nu \eta_{AB}, \qquad g(2)_{\mu\nu} = E(2)^A_\mu E(2)^B_\nu \eta_{AB}. \tag{5.65}$$

We extract a basic element of the interaction term

$$\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D} E(1)^{A_1}_{\mu_1}\cdots E(1)^{A_n}_{\mu_n} E(2)^{A_{n+1}}_{\mu_{n+1}}\cdots E(2)^{A_D}_{\mu_D}, \tag{5.66}$$

where n copies of $E(1)$ and $(D-n)$ copies of $E(2)$ are contained. We can write (5.66) in the form (5.62)

$$
\begin{aligned}
&\big(\det E(1)\big)\epsilon_{\mu_1\cdots\mu_D}\epsilon^{\nu_1\cdots\nu_D}\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_n}_{\nu_n}\big(E(1)^{\mu_{n+1}}_{A_{n+1}}E(2)^{A_{n+1}}_{\nu_{n+1}}\big)\cdots\big(E(1)^{\mu_D}_{A_D}E(2)^{A_D}_{\nu_D}\big)\\
=&\big(\det E(1)\big)n!\epsilon_{\mu_{n+1}\cdots\mu_D}\epsilon^{\nu_{n+1}\cdots\nu_D}\big(E(1)^{\mu_{n+1}}_{A_{n+1}}E(2)^{A_{n+1}}_{\nu_{n+1}}\big)\cdots\big(E(1)^{\mu_D}_{A_D}E(2)^{A_D}_{\nu_D}\big) \qquad (5.67)\\
\propto&\sqrt{\det g(1)}e_{D-n}(E(1)^{-1}E(2)), \qquad (5.68)
\end{aligned}
$$

where we have used (D.5), (D.11) and (D.2). Therefore, if the replacement

$$
E(1)^{\mu}_{A}E(2)^{A}_{\nu} \leftrightarrow \sqrt{g(1)^{-1}g(2)}^{\mu}{}_{\nu} \qquad (5.69)
$$

is possible, we can recover bimetric gravity. In fact, this replacement is allowed owing to the constraint (5.61) which is related to the Lorentz transformation. In order to make sure this fact, we write the interaction term (5.66) in the form (5.64)

$$
\begin{aligned}
&\big(\det E(1)\big)\epsilon_{A_1\cdots A_D}\epsilon^{B_1\cdots B_D}\delta^{A_1}_{B_1}\cdots\delta^{A_n}_{B_n}\big(E(1)^{\mu_{n+1}}_{B_{n+1}}E(2)^{A_{n+1}}_{\mu_{n+1}}\big)\cdots\big(E(1)^{\mu_D}_{B_D}E(2)^{A_D}_{\mu_D}\big)\\
=&\big(\det E(1)\big)n!\epsilon_{A_{n+1}\cdots A_D}\epsilon^{B_{n+1}\cdots B_D}\big(E(1)^{\mu_{n+1}}_{B_{n+1}}E(2)^{A_{n+1}}_{\mu_{n+1}}\big)\cdots\big(E(1)^{\mu_D}_{B_D}E(2)^{A_D}_{\mu_D}\big). \qquad (5.70)
\end{aligned}
$$

In Appendix D, we can find explicit formulae for antisymmetrization such as (5.70), from which we know that (5.70) can be expressed as a combination of traces

$$
\big(E(1)^{\lambda_m}_{C_1}E(2)^{C_m}_{\lambda_m}\big)\big(E(1)^{\lambda_{m-1}}_{C_m}E(2)^{C_{m-1}}_{\lambda_{m-1}}\big)\cdots\big(E(1)^{\lambda_2}_{C_3}E(2)^{C_2}_{\lambda_2}\big)\big(E(1)^{\lambda_1}_{C_2}E(2)^{C_1}_{\lambda_1}\big). \qquad (5.71)
$$

For notational simplicity, we define

$$
X_A{}^B := E(1)^{\lambda}_{A}E(2)^{B}_{\lambda}, \qquad (5.72)
$$

and read (5.71) as

$$
X_{C_1}{}^{C_m}X_{C_m}{}^{C_{m-1}}\cdots X_{C_3}{}^{C_2}X_{C_2}{}^{C_1}. \qquad (5.73)
$$

In the simplest case where the interaction term U is given by

$$
U \propto X_C{}^C, \qquad (5.74)
$$

the constraint (5.61) leads to

$$
E(1)^{\mu}_{A}E(2)^{C}_{\mu}\eta_{CB} = E(1)^{\mu}_{B}E(2)^{C}_{\mu}\eta_{CA} \Leftrightarrow X_A{}^C\eta_{CB} = X_B{}^C\eta_{CA}. \qquad (5.75)
$$

Actually, this relation always solves the Lorentz constraint even if the interaction term is not restricted to $U \propto X_C{}^C$. We apply the constraint (5.61) to a more general element (5.73), and find that

$$
X_A{}^{C_m}X_{C_m}{}^{C_{m-1}}\cdots X_{C_3}{}^{C_2}X_{C_2}{}^{C}\eta_{CB} = X_B{}^{C_m}X_{C_m}{}^{C_{m-1}}\cdots X_{C_3}{}^{C_2}X_{C_2}{}^{C}\eta_{CA} \qquad (5.76)
$$

must be satisfied. However, we can easily prove by induction that (5.76) is automatically satisfied when (5.75) holds.

If we use the relation (5.75), we can rewrite the element $g(1)^{-1}g(2)$ as

$$\begin{aligned} g(1)^{\mu\lambda}g(2)_{\lambda\nu} =& E(1)^{\mu}_{A}E(1)^{\lambda}_{B}\eta^{AB}E(2)^{C}_{\lambda}E(2)^{D}_{\nu}\eta_{CD} \\ &= E(1)^{\mu}_{A}E(1)^{\lambda}_{D}\eta^{AB}E(2)^{C}_{\lambda}E(2)^{D}_{\nu}\eta_{CB} \\ &= E(1)^{\mu}_{A}E(2)^{A}_{\lambda}E(1)^{\lambda}_{D}E(2)^{D}_{\nu}. \end{aligned} \tag{5.77}$$

Therefore, we find

$$E(1)^{\mu}_{A}E(2)^{A}_{\nu} = \left(\sqrt{g(1)^{-1}g(2)}\,\right)^{\mu}{}_{\nu}, \tag{5.78}$$

and restore bimetric gravity.

Here, it should be noted, taking the inverse of both sides of (5.75), we find another relation

$$E(2)^{\mu}_{A}E(1)^{C}_{\mu}\eta_{CB} = E(2)^{\mu}_{B}E(1)^{C}_{\mu}\eta_{CA}. \tag{5.79}$$

Thus, the Lorentz constraint (5.61) on $E(1)$ is automatically satisfied when that on $E(2)$ holds. This is because we have one symmetry coming from the overall Lorentz transformation. In a more general multi-vielbein case, one of the set (5.61) is automatically satisfied.

5.3.3 More general cases

We continue to the case with three vielbeins $E(1)$, $E(2)$ and $E(3)$. However, we soon recognize that interacting three vielbeins are not always translated into a metric formulation.

To begin with, we consider the chain type interaction represented by a diagram in Fig.5.2.

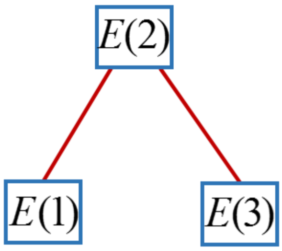

Figure 5.2: The diagram represents the chain type interaction.

In this type, the interaction term can be expressed as $U = U_{12}+U_{23}$, where U_{12} and U_{23} mean interactions between $(E(1), E(2))$ and $(E(2), E(3))$ respectively. The point is that no interaction between $(E(3), E(1))$ or among $((E(1), E(2), E(3)))$ are included. Focusing on the coupling between $(E(1), E(2))$, we express the interaction term U_{12} as $U_{12} \sim (\det E(2))\mathcal{L}^{TD}\big(E(2)^{-1}E(1)\big)$, where $\mathcal{L}^{TD}$ is a linear combination of $\mathcal{L}^{TD}_{n}$ defined in Appendix D. We apply the constraint (5.61) to $E(1)$, and find

$$E(2)^{\mu}_{A}E(1)^{C}_{\mu}\eta_{CB} = E(2)^{\mu}_{B}E(1)^{C}_{\mu}\eta_{CA}. \tag{5.80}$$

Taking the inverse of both sides, we also obtain

$$E(1)^{\mu}_{A}E(2)^{C}_{\mu}\eta_{CB} = E(1)^{\mu}_{B}E(2)^{C}_{\mu}\eta_{CA}. \tag{5.81}$$

We proceed to the constraint on $E(2)$. We express the interaction terms containing $E(2)$ as $U_{12} \sim (\det E(1))\mathcal{L}^{TD}(E(1)^{-1}E(2))$ and $U_{23} \sim (\det E(3))\mathcal{L}^{TD}(E(3)^{-1}E(2))$. Then, we apply the constraint (5.61) to $E(2)$. Here, contributions coming from U_{12} cancel each other because we already have a relation (5.81). Thus, the constraint on $E(2)$ is calculated from only U_{23}, and we obtain

$$E(3)^\mu_A E(2)^C_\mu \eta_{CB} = E(3)^\mu_B E(2)^C_\mu \eta_{CA}. \tag{5.82}$$

Taking the inverse of both sides, we also obtain

$$E(2)^\mu_A E(3)^C_\mu \eta_{CB} = E(2)^\mu_B E(3)^C_\mu \eta_{CA}, \tag{5.83}$$

which is nothing but the constraint on $E(3)$. Therefore, using these three relations, we can restore trimetric gravity with one interaction cut.

When the interaction becomes loop type, namely $U = U_{12}+U_{23}+U_{31}$, the situation becomes complicated and we cannot recover trimetric gravity. The corresponding diagram is in Fig.5.3.

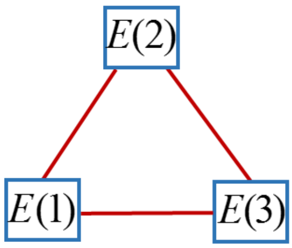

Figure 5.3: The diagram represents the loop type interaction.

As an example, we consider an interaction given by

$$U = \alpha_{12}(\det E(1))E(1)^\mu_A E(2)^A_\mu + \alpha_{23}(\det E(2))E(2)^\mu_A E(3)^A_\mu + \alpha_{31}(\det E(3))E(3)^\mu_A E(1)^A_\mu, \tag{5.84}$$

where α_{12}, α_{23} and α_{31} are constants. Using variation $\delta \det E = (\det E)E^\mu_A \delta E^A_\mu$ and $\delta E^\nu_B = -E^\nu_A E^\mu_B \delta E^A_\mu$, we calculate the constraint (5.61). The result is found to be

$$\alpha_{12}(\det E(1))E(1)^\mu_A E(2)^C_\mu \eta_{CB} - \alpha_{23}(\det E(2))E(2)^\mu_A E(3)^C_\mu \eta_{CB} = (A \leftrightarrow B), \tag{5.85}$$
$$\alpha_{23}(\det E(2))E(2)^\mu_A E(3)^C_\mu \eta_{CB} - \alpha_{31}(\det E(3))E(3)^\mu_A E(1)^C_\mu \eta_{CB} = (A \leftrightarrow B), \tag{5.86}$$
$$\alpha_{31}(\det E(3))E(3)^\mu_A E(1)^C_\mu \eta_{CB} - \alpha_{12}(\det E(1))E(1)^\mu_A E(2)^C_\mu \eta_{CB} = (A \leftrightarrow B), \tag{5.87}$$

from which we cannot obtain the key relations like (5.81) or (5.83). Therefore, trimetric gravity is not recovered from the loop type interaction.

In the case of an interaction among $(E(1), E(2), E(3))$, the situation is the same as that in the loop type. We cannot in general recover trimetric gravity. We call this pattern as branching link type interaction and represent it by a diagram in Fig.5.4. For example, we consider

$$\begin{aligned} U =& \epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{A_1A_2\cdots A_D}E(1)^{A_1}_{\mu_1}\cdots E(1)^{A_{D-2}}_{\mu_{D-2}}E(2)^{A_{D-1}}_{\mu_{D-1}}E(3)^{A_D}_{\mu_D} \\ =& \propto \det E(1)\big[(E(1)^\mu_A E(2)^A_\mu)(E(1)^\nu_B E(3)^B_\nu) - E(1)^\mu_A E(2)^B_\mu E(1)^\nu_B E(3)^A_\nu\big]. \end{aligned} \tag{5.88}$$

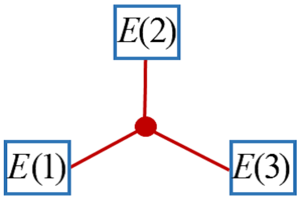

Figure 5.4: The diagram represents the branching link type interaction.

The constraint (5.61) leads to

$$(E(1)^{\nu}_{C'}E(2)^{C'}_{\nu})E(1)^{\mu}_{A}E(3)^{C}_{\mu}\eta_{CB} - E(1)^{\mu}_{A}E(2)^{C'}_{\mu}E(1)^{\nu}_{C'}E(3)^{C}_{\nu}\eta_{CB} = (A \leftrightarrow B), \tag{5.89}$$
$$(E(1)^{\nu}_{C'}E(3)^{C'}_{\nu})E(1)^{\mu}_{A}E(2)^{C}_{\mu}\eta_{CB} - E(1)^{\mu}_{A}E(3)^{C'}_{\mu}E(1)^{\nu}_{C'}E(2)^{C}_{\nu}\eta_{CB} = (A \leftrightarrow B), \tag{5.90}$$

and independent one obtained from variation with respect to $E(1)$. Hence, we cannot obtain the key relations necessary for a metric interpretation.

We can easily generalise this discussion to more general cases where interacting multiple vielbeins are contained. Only the tree type interaction such as Fig.5.5 can be translated into a metric formulation. If loop or branching link structures are included, interaction terms cannot be reformulated in the language of metrics.

In a different viewpoint, how to translate a vielbein theory is discussed also in [38].

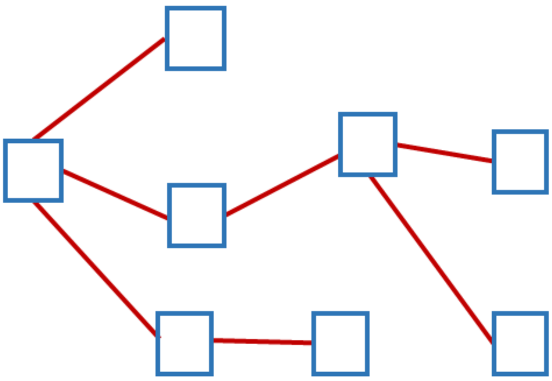

Figure 5.5: The diagram represents an example of the tree type interaction. Each box describes a vielbein.

Chapter 6

Multi-metric gravity

In Chapter 4, we have proved the absence of the BD-ghost in dRGT massive/bimetric gravity via the Hamiltonian analysis. We have seen that the essential point of the proof is to linearize the lapse by the transformation of the lapse and the shift variables. However, this transformation cannot be applied to the case of trimetric or more general multimetric gravity, and whether BD-ghosts exist or not has been left unanswered. On the other hand, in Chapter 5, we have considered interacting multiple gravitational fields in a vielbein formulation. Actually, we have constructed a ghost-free interaction written by vielbeins, but they cannot always be translated into a metric formulation. Only the tree type interaction can be translated into the language of metrics. Thus, the ghost problem of multimetric gravity containing loop or branching link interaction has not been resolved.

In this chapter, we analyze them and prove that they suffer from BD-ghosts, and clarify when multimetric gravity becomes ghost-free. The difficulty of the Hamiltonian analysis in Chapter 4 has come from the non-linear dependence of the lapse, and we have paid a lot of trouble to linearize it. This strategy has been retained even in Chapter 5. We have introduced vielbeins along with antisymmetrization to linearize all the lapse and shift variables. The linearity of these variables has made the structure of the constraints extremely clear.

Now, we take the same strategy, namely linearization of the lapse. We focus on the homogeneous space-times, where the shift vector disappears. We see that interaction terms of multimetric gravity becomes linear in the lapse, and the constraint structure is drastically simplified. Therefore, we can perform the Hamiltonian analysis explicitly, and count the total number of degrees of freedom to decide whether or not BD-ghosts are contained.

This chapter is based on our original work [5]. We focus on four-dimensional space-times.

6.1 Bimetric gravity revisited

Figure 6.1: Each blob describes a metric. The link represents interaction.

In this section, we revisit bimetric gravity and demonstrate our method to probe a BD-ghost. Since, in Chapter 4 and Chapter 5, we have already proved that bimetric gravity is ghost-free, we make sure that the same conclusion can be obtained by using a simple mini-superspace approximation.

We recall the action of ghost-free bimetric gravity (3.145)

$$S = M_g^2 \int d^4x \sqrt{-\det g} R[g] + M_f^2 \int d^4x \sqrt{-\det f} R[f] + 2m^2 M_{gf}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n e_n\left(\sqrt{g^{-1}f}\right), \tag{6.1}$$

where the first and the second terms are the Einstein-Hilbert action for the metric g and that for f, and $R[g]$ and $R[f]$ represent scalar curvatures for these two metrics. M_g and M_f stand for Planck masses $M_g := \frac{1}{16\pi G_g}$ and $M_f := \frac{1}{16\pi G_f}$. The last term describes the interaction between two metrics, and β_n are dimensionless coupling constants. The other constants m and M_{gf} are introduced to adjust the mass dimension, the details of which are not relevant in this chapter. The interaction term is constructed from antisymmetrization functions e_n $(n = 0, 1, 2, 3, 4)$ defined in Appendix D. For convenience, we represent the existence of the interaction by a diagram in Fig.6.1, and we abbreviate trace operations as $\mathrm{tr}^n\mathbb{X} = \left(\mathrm{tr}\mathbb{X}\right)^n$ and $\mathrm{tr}\mathbb{X}^n = \mathrm{tr}(\mathbb{X}^n)$.

Now, we perform the Hamiltonian analysis based on the ADM formalism. In particular, to make the analysis tractable, we employ the mini-superspace approach. Namely, we assume spatial homogeneity and express metrics in terms of ADM variables as

$$g_{\mu\nu}dx^\mu dx^\nu = -N(t)^2 dt^2 + \gamma_{ij}(t) dx^i dx^j \ , \tag{6.2}$$

where N is a lapse function and γ_{ij} is a spatial metric. Similarly, we can take the following ansatz

$$f_{\mu\nu}dx^\mu dx^\nu = -L(t)^2 dt^2 + \omega_{ij}(t) dx^i dx^j \ , \tag{6.3}$$

where L is a lapse function and ω_{ij} is a spatial metric. It is convenient to write them in a matrix form,

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} -N^2 & 0 \\ 0 & \gamma_{ij} \end{pmatrix} , \quad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & 0 \\ 0 & \gamma^{ij} \end{pmatrix} , \\ f_{\mu\nu} &= \begin{pmatrix} -L^2 & 0 \\ 0 & \omega_{ij} \end{pmatrix} , \quad f^{\mu\nu} = \begin{pmatrix} -1/L^2 & 0 \\ 0 & \omega^{ij} \end{pmatrix} , \end{aligned} \tag{6.4}$$

where γ^{ij} and ω^{ij} are inverse matrices of corresponding spatial metrics γ_{ij} and ω_{ij}. Then, a basic element of the interaction term can be calculated to be

$$\left(g^{-1}f\right)^\mu{}_\nu = \begin{pmatrix} L^2/N^2 & 0 \\ 0 & \gamma^{il}\omega_{lj} \end{pmatrix} \Longrightarrow \sqrt{g^{-1}f} = \begin{pmatrix} L/N & 0 \\ 0 & \sqrt{\gamma^{-1}\omega} \end{pmatrix} . \tag{6.5}$$

In the right hand side of (6.5), we see that the lapse functions N and L appear only through the combination L/N. Besides, higher order terms such as $(L/N)^2$, $(L/N)^3$,...etc do not appear. Because the function e_n contains antisymmetrization, we have only up to linear terms of L/N in $\sum_{n=0}^4 \beta_n e_n\left(\sqrt{g^{-1}f}\right)$. Thus, the interaction term $\sqrt{-\det g}\sum_{n=0}^4 \beta_n e_n\left(\sqrt{g^{-1}f}\right)$ becomes linear in N and L. This feature is an advantage of our homogeneous ansatz and is not restricted to the four-dimensional case. The explicit calculation is as follows. The first one is given by

$$\begin{aligned}\sqrt{-\det g}\, e_1\big(\sqrt{g^{-1}f}\big) &= \sqrt{-\det g}\,\mathrm{tr}\sqrt{g^{-1}f} \\ &= \sqrt{\det\gamma}\, N\Big(L/N + \mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big) \\ &= \sqrt{\det\gamma}\,\Big(L + N\mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big)\ . \end{aligned} \tag{6.6}$$

The second one becomes

$$\begin{aligned}\sqrt{-\det g}\, e_2\big(\sqrt{g^{-1}f}\big) &= \sqrt{-\det g}\,\frac{1}{2}\Big(\mathrm{tr}^2\sqrt{g^{-1}f} - \mathrm{tr}\big(g^{-1}f\big)\Big) \\ &= \sqrt{\det\gamma}\, N\,\frac{1}{2}\Big\{\Big(L/N + \mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big)^2 - L^2/N^2 - \mathrm{tr}\big(\gamma^{-1}\omega\big)\Big\} \\ &= \sqrt{\det\gamma}\,\Big\{L\,\mathrm{tr}\sqrt{\gamma^{-1}\omega} + \frac{1}{2}N\Big(\mathrm{tr}^2\sqrt{\gamma^{-1}\omega} - \mathrm{tr}\big(\gamma^{-1}\omega\big)\Big)\Big\}\ . \end{aligned} \tag{6.7}$$

The third one can be calculated as

$$\begin{aligned}\sqrt{-\det g}\, e_3\big(\sqrt{g^{-1}f}\big) &= \sqrt{-\det g}\,\frac{1}{6}\Big(\mathrm{tr}^3\sqrt{g^{-1}f} - 3\mathrm{tr}\sqrt{g^{-1}f}\,\mathrm{tr}\big(g^{-1}f\big) + 2\mathrm{tr}\big(g^{-1}f\big)^{3/2}\Big) \\ &= \sqrt{\det\gamma}\, N\,\frac{1}{6}\Big\{\Big(L/N + \mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big)^3 \\ &\quad -3\Big(L/N + \mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big)\Big(L^2/N^2 + \mathrm{tr}\big(\gamma^{-1}\omega\big)\Big) + 2\Big(L^3/N^3 + \mathrm{tr}\big(\gamma^{-1}\omega\big)^{3/2}\Big)\Big\} \\ &= \sqrt{\det\gamma}\,\Big\{\frac{1}{2}L\Big(\mathrm{tr}^2\sqrt{\gamma^{-1}\omega} - \mathrm{tr}\big(\gamma^{-1}\omega\big)\Big) \\ &\quad +\frac{1}{6}N\Big(\mathrm{tr}^3\sqrt{\gamma^{-1}\omega} - 3\mathrm{tr}\sqrt{\gamma^{-1}\omega}\,\mathrm{tr}\big(\gamma^{-1}\omega\big) + 2\mathrm{tr}\big(\gamma^{-1}\omega\big)^{3/2}\Big)\Big\}\ . \end{aligned} \tag{6.8}$$

The last one is

$$\sqrt{-\det g}\, e_4\big(\sqrt{g^{-1}f}\big) = \sqrt{-\det g}\ \det\sqrt{g^{-1}f} = \sqrt{-\det f} = L\sqrt{\det\omega}\ . \tag{6.9}$$

To sum up, the interaction term reads

$$\begin{aligned}&\sqrt{-\det g}\,\sum_{n=0}^{4}\beta_n e_n\big(\sqrt{g^{-1}f}\big) \\ =&N\,\sqrt{\det\gamma}\Big[\beta_0 + \beta_1\mathrm{tr}\sqrt{\gamma^{-1}\omega} + \frac{1}{2}\beta_2\Big(\mathrm{tr}^2\sqrt{\gamma^{-1}\omega} - \mathrm{tr}\big(\gamma^{-1}\omega\big)\Big) \\ &\qquad \frac{1}{6}\beta_3\Big(\mathrm{tr}^3\sqrt{\gamma^{-1}\omega} - 3\mathrm{tr}\sqrt{\gamma^{-1}\omega}\,\mathrm{tr}\big(\gamma^{-1}\omega\big) + 2\mathrm{tr}\big(\gamma^{-1}\omega\big)^{3/2}\Big)\Big] \\ &+L\,\Big[\sqrt{\det\gamma}\Big\{\beta_1 + \beta_2\mathrm{tr}\sqrt{\gamma^{-1}\omega} + \frac{1}{2}\beta_3\Big(\mathrm{tr}^2\sqrt{\gamma^{-1}\omega} - \mathrm{tr}\big(\gamma^{-1}\omega\big)\Big)\Big\} + \beta_4\sqrt{\det\omega}\Big]. \end{aligned} \tag{6.10}$$

Here, we should notice an important point. When we count the number of physical degrees of freedom, the following fact must be taken into account. In the vacuum cases, we can diagonalize one of two spatial metrics using overall spatial coordinate transformations. Performing a spatial coordinate transformation $x^i \to \Lambda(t_0)^i{}_j\, x^j$, we can set one spatial metric at the time $t = t_0$, $\gamma_{ij}(t_0)$, a unit matrix δ_{ij}. Moreover, since the orthogonal transformation dose not change

$\gamma_{ij}(t_0) = \delta_{ij}$, we can diagonalize $\dot{\gamma}_{ij}(t_0)$ simultaneously by using this freedom. At this stage, homogeneous spatial coordinates are completely fixed. Now, γ_{ij} and $\dot{\gamma}_{ij}$ is diagonal at the time $t = t_0$ as an initial condition. Then we assume diagonal form of $\gamma_{ij}(t)$ at all time, and insert it into the equation of motion. Any contradiction never occurs in vacuum. Thus, we conclude that one spacial metric $\gamma_{ij}(t)$ can be diagonalized because of the uniqueness of the solution. Hence, the number of component of one of two spatial metrics reduces from 6 to 3. This fact will be used later.

For simplicity, we assume that interactions are minimal, namely

$$\beta_0 = 3, \quad \beta_1 = -1, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = 1. \tag{6.11}$$

Clearly, this simplification does not lose any generality concerning with the ghost analysis.

Then, setting $\int d^3x = 1$, the Lagrangian defined by $S = \int dt\mathcal{L}$ reads

$$\mathcal{L} = M_g^2 \pi^{ij}\dot{\gamma}_{ij} + M_f^2 p^{ij}\dot{\omega}_{ij} - NC_N - LC_L, \tag{6.12}$$

and

$$C_N := \frac{M_g^2}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^i{}_i\pi^j{}_j\Big) - M_g^2\sqrt{\det\gamma}\,{}^{(3)}R[\gamma] + a_1\sqrt{\det\gamma}\big(\mathrm{tr}\sqrt{\gamma^{-1}\omega} - 3\big), \tag{6.13}$$

$$C_L := \frac{M_f^2}{\sqrt{\det\omega}}\Big(p^{ij}p_{ij} - \frac{1}{2}p^i{}_ip^j{}_j\Big) - M_f^2\sqrt{\det\omega}\,{}^{(3)}R[\omega] + a_1\big(\sqrt{\det\gamma} - \sqrt{\det\omega}\big), \tag{6.14}$$

where π^{ij} and p^{ij} are canonical conjugate momenta of γ_{ij} and ω_{ij}. The first two terms of C_N and C_L come from the Einstein-Hilbert term in the action. Thus, ${}^{(3)}R[\gamma]$ and ${}^{(3)}R[\omega]$ are spatial scalar curvatures composed from γ and ω, respectively. The last term comes from the interaction, and we use $a_1 := 2m^2M_{gf}^2$. We can see that the lapse functions N and L behave as Lagrange multipliers, and variation with respect to them leads to constraints

$$C_N = 0, \qquad C_L = 0. \tag{6.15}$$

These two constraints must be preserved along the time evolution. Hence, we must impose consistency conditions on them

$$\dot{C}_N = \big\{C_N\,,\,H\big\}_{PB} \approx 0, \qquad \dot{C}_L = \big\{C_L\,,\,H\big\}_{PB} \approx 0, \tag{6.16}$$

where H is the Hamiltonian given by

$$H = NC_N + LC_L, \tag{6.17}$$

and the Poisson bracket $\{F, G\}_{PB}$ is now defined by

$$\big\{F\,,\,G\big\}_{PB} = \frac{1}{M_g^2}\bigg(\frac{\partial F}{\partial\gamma_{mn}}\frac{\partial G}{\partial\pi^{mn}} - \frac{\partial F}{\partial\pi^{mn}}\frac{\partial G}{\partial\gamma_{mn}}\bigg) + \frac{1}{M_f^2}\bigg(\frac{\partial F}{\partial\omega_{mn}}\frac{\partial G}{\partial p^{mn}} - \frac{\partial F}{\partial p^{mn}}\frac{\partial G}{\partial\omega_{mn}}\bigg). \tag{6.18}$$

Here, "≈ 0" means "$= 0$" on the hypersurface determined by constraints. Notice that $\{F, F\}_{PB} = 0$ because of spatial homogeneity, and we find

$$\big\{C_N\,,\,H\big\}_{PB} \approx L\big\{C_N\,,\,C_L\big\}_{PB} =: LC_{NL}, \tag{6.19}$$

$$\big\{C_L\,,\,H\big\}_{PB} \approx N\big\{C_L\,,\,C_N\big\}_{PB} =: NC_{LN}. \tag{6.20}$$

In order to check if additional constraints arise or not, we have to calculate the Poisson bracket C_{NL}. From the calculation presented in Appendix G, we obtain

$$C_{NL} = \{C_N \,,\, C_L\}_{PB} = a_1 \left[\frac{1}{2}\pi^i{}_i - \sqrt{\frac{\det\gamma}{\det\omega}} \left(\frac{1}{2} p^i{}_i \operatorname{tr}\sqrt{\gamma^{-1}\omega} - \operatorname{tr}\left(\sqrt{\gamma^{-1}\omega}\, p\,\omega\right)\right)\right]. \tag{6.21}$$

This leads to one secondary constraint $C_{NL} \approx 0$. We must also impose the consistency condition on this secondary constraint

$$\dot{C}_{NL} = N\{C_{NL} \,,\, C_N\}_{PB} + L\{C_{NL} \,,\, C_L\}_{PB} \approx 0. \tag{6.22}$$

The explicit calculation can be found in Appendix G, and the result is $\{C_{NL} \,,\, C_N\}_{PB} \not\approx 0$ and $\{C_{NL} \,,\, C_J\}_{PB} \not\approx 0$. Therefore, this condition determines one of two Lagrange multipliers N and L. The remaining multiplier describes the overall time reparametrization invariance in bimetric gravity.

The number of components of two spatial metrics and their canonical conjugates is 24. Since we can diagonalize one of the two metrics, we should subtract 6 from this number. Recall that there are two primary constraints and one secondary constraint. Furthermore, as we have one undetermined Lagrange multiplier, we have to put one gauge condition. Thus, the total number of degrees of freedom should be $(24-6-2-1-1)/2 = 7$ in configuration space, which matches degrees of freedom of one massless graviton and one massive graviton. This proves that BD-ghost is absent in bimetric gravity described by the action (6.1).

6.2 Ghost in trimetric gravity

We apply the method explained in the previous section to trimetric gravity. In contrast to the case of bimetric gravity, there are three kinds of interaction, namely, the chain type, the loop type and the branching link type interaction. In this section, we investigate the chain type and the loop type interaction. Interaction with a branching pattern is considered in Section 6.4.

We remember the action for trimetric gravity given in Section 3.6

$$\begin{aligned} S =& M_g^2 \int d^4x \sqrt{-\det g} R[g] + M_f^2 \int d^4x \sqrt{-\det f} R[f] + M_h^2 \int d^4x \sqrt{-\det h} R[h] \\ &+ 2m_1^2 M_{gf}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n e_n\left(\sqrt{g^{-1}f}\right) \\ &+ 2m_2^2 M_{fh}^2 \int d^4x \sqrt{-\det f} \sum_{n=0}^{4} \beta'_n e_n\left(\sqrt{f^{-1}h}\right) \\ &+ 2m_3^2 M_{hg}^2 \int d^4x \sqrt{-\det h} \sum_{n=0}^{4} \beta''_n e_n\left(\sqrt{h^{-1}g}\right), \end{aligned} \tag{6.23}$$

where we have three metrics $g_{\mu\nu}$, $f_{\mu\nu}$ and $h_{\mu\nu}$, from which we construct scalar curvatures $R[g]$, $R[f]$ and $R[h]$. In previous chapters, we have denoted a perturbation as $h_{\mu\nu}$, but in this chapter it does not represent a perturbation. The interaction terms contain free parameters β_n, β'_n, β''_n, and mass parameters $m_1, m_2, m_3, M_{gf}, M_{fh}, M_{hg}$. A Planck mass M_h is also introduced. It should be noted that there exists one overall diffeomorphism invariance in this trimetric theory which makes one of gravitons massless. If this trimetric gravity contains no extra degree of freedom, the total number of degrees of freedom should be $2 + 5 + 5 = 12$, which comes from one massless graviton and two massive gravitons. From now on, we use

$$a_1 := 2m_1^2 M_{gf}^2\,, \quad a_2 := 2m_2^2 M_{fh}^2\,, \quad a_3 := 2m_3^2 M_{hg}^2 \tag{6.24}$$

for notational simplicity. If we have $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$, all pairs (g, f), (f, h) and (h, g) interact and we call it the loop type interaction. When one of a_i $(i = 1, 2, 3)$ is set to zero, two of three pairs of interactions remain, which we call the chain type interaction. The chain type interaction is already concluded to be ghost-free via the vielbein formalism in Chapter 5. On the contrary, whether or not the loop type interaction is ghost-free has been left unanswered. Now, we settle this problem.

Apparently, the full Hamiltonian constraint analysis is almost impossible. To circumvent this difficulty, we take the method used in the previous section. We assume spatial homogeneity and express metrics in terms of ADM variables as

$$g_{\mu\nu}dx^\mu dx^\nu = -N(t)^2 dt^2 + \gamma_{ij}(t)dx^i dx^j, \tag{6.25}$$

where N is a lapse function and γ_{ij} is a spatial metric. Similarly, we can take the following ansatz

$$f_{\mu\nu}dx^\mu dx^\nu = -L(t)^2 dt^2 + \omega_{ij}(t)dx^i dx^j, \tag{6.26}$$

and

$$h_{\mu\nu}dx^\mu dx^\nu = -Q(t)^2 dt^2 + \rho_{ij}(t)dx^i dx^j, \tag{6.27}$$

where L and Q are lapse functions and ω_{ij} and ρ_{ij} are spatial metrics.

To perform the Hamiltonian analysis, we need the Lagrangian in the ADM variables. Similar to the case of bimetric gravity, we assume that interactions are minimal

$$(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = (\beta_0', \beta_1', \beta_2', \beta_3', \beta_4') = (\beta_0'', \beta_1'', \beta_2'', \beta_3'', \beta_4'') = (3, -1, 0, 0, 1), \tag{6.28}$$

and set $\int d^3x = 1$. Then, we obtain the Lagrangian

$$\mathcal{L} = M_g^2 \pi^{ij}\dot{\gamma}_{ij} + M_f^2 p^{ij}\dot{\omega}_{ij} + M_h^2 \phi^{ij}\dot{\rho}_{ij} - NC_N - LC_L - QC_Q, \tag{6.29}$$

where π^{ij}, p^{ij} and ϕ^{ij} are canonical conjugate momenta of γ_{ij}, ω_{ij} and ρ_{ij}, and we have defined

$$\begin{aligned} C_N :=& \frac{M_g^2}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^i{}_i\pi^j{}_j\Big) - M_g^2\sqrt{\det\gamma}\,{}^{(3)}R[\gamma] \\ &+ a_1\sqrt{\det\gamma}\big(\mathrm{tr}\sqrt{\gamma^{-1}\omega} - 3\big) + a_3\big(\sqrt{\det\rho} - \sqrt{\det\gamma}\big), \end{aligned} \tag{6.30}$$

$$\begin{aligned} C_L :=& \frac{M_f^2}{\sqrt{\det\omega}}\Big(p^{ij}p_{ij} - \frac{1}{2}p^i{}_ip^j{}_j\Big) - M_f^2\sqrt{\det\omega}\,{}^{(3)}R[\omega] \\ &+ a_2\sqrt{\det\omega}\big(\mathrm{tr}\sqrt{\omega^{-1}\rho} - 3\big) + a_1\big(\sqrt{\det\gamma} - \sqrt{\det\omega}\big), \end{aligned} \tag{6.31}$$

$$\begin{aligned} C_Q :=& \frac{M_h^2}{\sqrt{\det\rho}}\Big(\phi^{ij}\phi_{ij} - \frac{1}{2}\phi^i{}_i\phi^j{}_j\Big) - M_h^2\sqrt{\det\rho}\,{}^{(3)}R[\rho] \\ &+ a_3\sqrt{\det\rho}\big(\mathrm{tr}\sqrt{\rho^{-1}\gamma} - 3\big) + a_2\big(\sqrt{\det\omega} - \sqrt{\det\rho}\big). \end{aligned} \tag{6.32}$$

The first line of each C_N, C_L and C_Q comes from the Einstein-Hilbert term in the action, so ${}^{(3)}R[\gamma]$, ${}^{(3)}R[\omega]$ and ${}^{(3)}R[\rho]$ are spatial scalar curvatures calculated from γ, ω and ρ, respectively. The other terms can be derived from the explicit formula of the interaction term (6.10). Obviously, we can see that the lapse functions N, L and Q behave as Lagrange multipliers, and variation with respect to them leads to three constraints

$$C_N = 0, \qquad C_L = 0, \qquad C_Q = 0. \tag{6.33}$$

These constraints must be preserved along the time evolution. Thus, we need consistency conditions for them

$$\dot{C}_N = \{C_N\,,\, H\}_{PB} \approx 0, \quad \dot{C}_L = \{C_L\,,\, H\}_{PB} \approx 0, \quad \dot{C}_Q = \{C_Q\,,\, H\}_{PB} \approx 0. \tag{6.34}$$

Here, the Hamiltonian H is given by

$$H = NC_N + LC_L + QC_Q, \tag{6.35}$$

and we find $\{F, F\}_{PB} = 0$ due to spatial homogeneity. Therefore, we have following three consistency conditions

$$\begin{aligned}
\{C_N\,,\, H\}_{PB} &\approx L\{C_N\,,\, C_L\}_{PB} + Q\{C_N\,,\, C_Q\}_{PB} \approx 0,\\
\{C_L\,,\, H\}_{PB} &\approx N\{C_L\,,\, C_N\}_{PB} + Q\{C_L\,,\, C_Q\}_{PB} \approx 0,\\
\{C_Q\,,\, H\}_{PB} &\approx N\{C_Q\,,\, C_N\}_{PB} + L\{C_Q\,.\, C_L\}_{PB} \approx 0.
\end{aligned} \tag{6.36}$$

Whether or not additional secondary constraints arise depends on the Poisson brackets,

$$C_{NL} := \{C_N\,,\, C_L\}_{PB}, \quad C_{LQ} := \{C_L\,,\, C_Q\}_{PB}, \quad C_{QN} := \{C_Q\,,\, C_N\}_{PB}. \tag{6.37}$$

The explicit calculation of the first Poisson bracket is found in Appendix G, and the result is

$$C_{NL} = a_1\left[\frac{1}{2}\pi^i{}_i - \sqrt{\frac{\det\gamma}{\det\omega}}\left(\frac{1}{2}p^i{}_i\,\mathrm{tr}\sqrt{\gamma^{-1}\omega} - \mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big)\right)\right]. \tag{6.38}$$

By performing permutations among $g = (N, \gamma)$, $f = (L, \omega)$ and $h = (Q, \rho)$, we also obtain

$$C_{LQ} = a_2\left[\frac{1}{2}p^i{}_i - \sqrt{\frac{\det\omega}{\det\rho}}\left(\frac{1}{2}\phi^i{}_i\,\mathrm{tr}\sqrt{\omega^{-1}\rho} - \mathrm{tr}\big(\sqrt{\omega^{-1}\rho}\,\phi\,\rho\big)\right)\right] \tag{6.39}$$

and

$$C_{QN} = a_3\left[\frac{1}{2}\phi^i{}_i - \sqrt{\frac{\det\rho}{\det\gamma}}\left(\frac{1}{2}\pi^i{}_i\,\mathrm{tr}\sqrt{\rho^{-1}\gamma} - \mathrm{tr}\big(\sqrt{\rho^{-1}\gamma}\,\pi\,\gamma\big)\right)\right]. \tag{6.40}$$

In general, quantities inside the bracket does not vanish.

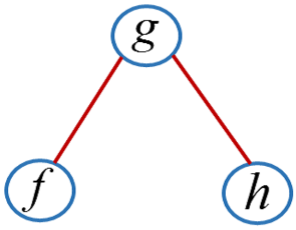

Figure 6.2: The diagram represents the chain type interaction.

6.2.1 Chain type interaction

In this subsection, we consider the chain type interaction

$$a_1 \neq 0, \quad a_2 = 0, \quad a_3 \neq 0, \tag{6.41}$$

which cut interaction between f and h as in Fig.6.2.

In any case, there are primary constraints (6.33). Since $C_{QL} = C_{LQ} = 0$ trivially holds, consistency conditions (6.36) lead to equations

$$LC_{NL} + QC_{NQ} \approx 0, \quad NC_{LN} \approx 0, \quad NC_{QN} \approx 0. \tag{6.42}$$

Hence, we have two secondary constraints

$$C_{NL} \approx 0, \quad C_{NQ} \approx 0. \tag{6.43}$$

Needless to say, these secondary constraints have to be preserved along the time evolution, which requires the following consistency conditions

$$\dot{C}_{NL} = \{C_{NL}\,,\, H\}_{PB} = N\{C_{NL}\,,\, C_N\}_{PB} + L\{C_{NL}\,,\, C_L\}_{PB} + Q\{C_{NL}\,,\, C_Q\}_{PB} \approx 0, \tag{6.44}$$

$$\dot{C}_{QL} = \{C_{QL}\,,\, H\}_{PB} = N\{C_{QL}\,,\, C_N\}_{PB} + L\{C_{QL}\,,\, C_L\}_{PB} + Q\{C_{QL}\,,\, C_Q\}_{PB} \approx 0. \tag{6.45}$$

In Appendix G, we calculate the Poisson brackets in the above equations, and see they do not vanish. Then, we know that conditions (6.44) and (6.45) determine two of three Lagrange multipliers N, L and Q. The remaining multiplier is related to the overall gauge transformation. Eventually, we have five constraints and one gauge freedom. We remember that dynamical modes are spatial metrics, and each of them has six components. However, as is already explained, we can diagonalize one of them. Hence, trimetric gravity has $3+6+6 = 15$ degrees of freedom in configuration space and $15 \times 2 = 30$ in phase apace. Thus, the total number of degrees of freedom is $(30-5-1)/2 = 12$ which matches the physical degrees of one massless and two massive gravitons. Therefore, no BD-ghost exists in the spectrum. This conclusion is consistent with that obtained by the vielbein method in Chapter 5.

6.2.2 Loop type interaction

Now, we consider the loop type interaction represented by a diagram in Fig.6.3

$$a_1 \neq 0, \quad a_2 \neq 0, \quad a_3 \neq 0. \tag{6.46}$$

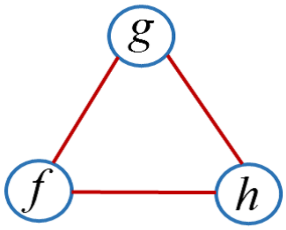

Figure 6.3: The diagram represents the loop type interaction.

In this case, we know that

$$\{C_N \,,\, C_L\}_{PB} \neq 0, \quad \{C_L \,,\, C_Q\}_{PB} \neq 0, \quad \{C_Q \,,\, C_N\}_{PB} \neq 0 \tag{6.47}$$

holds even on the constraint surface. Hence, consistency conditions (6.36) do not generate any secondary constraint. Instead, it determines Lagrange multipliers N, L and Q. However, due to the antisymmetric property of the Poisson brackets

$$C_{NL} = \{C_N \,,\, C_L\}_{PB} = -\{C_L \,,\, C_N\}_{PB} = -C_{LN} \tag{6.48}$$

$$C_{LQ} = \{C_L \,,\, C_Q\}_{PB} = -C_{QL} \tag{6.49}$$

$$C_{QN} = \{C_Q \,,\, C_N\}_{PB} = -C_{NQ}, \tag{6.50}$$

only two of them are determined. For example, if we choose

$$L = -\frac{C_{NQ}}{C_{NL}}\, Q, \quad N = -\frac{C_{LQ}}{C_{LN}}\, Q, \tag{6.51}$$

all of consistency conditions (6.36) are satisfied.

To conclude, we have three primary constraints and one undetermined Lagrange multiplier. Hence, we need one gauge condition to fix it, which is associated with the overall time reparametrization invariance. In trimetric gravity, as is already counted, there are $3+6+6 = 15$ degrees of freedom in configuration space and $15 \times 2 = 30$ in phase apace. In phase space, we have three constraints and one gauge condition, so the total number of degrees of freedom is $(30 - 3 - 1)/2 = 13$. If no BD-ghost is present, there must be $2 + 5 + 5 = 12$ degrees of freedom which comes from one massless graviton and two massive gravitons. Therefore, one extra degree of freedom exists and it should be a BD-ghost. Here, we have just counted the total number of degrees of freedom, but we can actually confirm that this extra one behaves as a ghost [36]. Thus, we have proved the existence of a BD-ghost in trimetric gravity with the loop type interaction.

6.3 General multimetric gravity without branching interaction pattern

In Section 6.2, we have discussed about trimetric gravity with the chain type or loop type interaction, and found that the loop type interaction cannot exclude a BD-ghost. However, the conclusion is not restricted to the case of trimetric gravity. Even if we introduce more interacting metrics, almost the same conclusion holds. The chain type interaction remains ghost-free while the loop type interaction suffers from BD-ghosts. In this section, we consider $\mathcal{N}$ dynamical metrics g_k ($k = 1, 2, .., \mathcal{N}$), and explicitly calculate the number of BD-ghosts if they exist.

Now, we consider the following interaction term

$$\sum_{k=1}^{\mathcal{N}} a_k \sqrt{-\det g_k} \sum_{n=0}^{4} \beta_{k,n} e_n\Big(\sqrt{g_k^{-1} g_{k+1}}\Big), \tag{6.52}$$

where we define $g_{\mathcal{N}+1} := g_1$. For later purpose, we also use $g_0 := g_{\mathcal{N}}$. Let us describe the interaction between two metrics g_k and g_{k+1} in terms of ADM variables

$$ds_k^2 = -N_k^2(t)dt^2 + \gamma_{k,ij}(t)dx^i dx^j. \tag{6.53}$$

Schematically, the interaction can be written as

$$\sqrt{-\det g_k} \sum_{n=0}^{4} \beta_{k,n} e_n\Big(\sqrt{g_k^{-1} g_{k+1}}\Big) = N_k F_k\big(\gamma_k : \gamma_{k+1}\big) + N_{k+1} G_k\big(\gamma_k : \gamma_{k+1}\big), \tag{6.54}$$

where F_k and G_k are some functions determined by parameters $\beta_{k,n}$. Thus, the total interaction term is given by

$$\sum_{k=1}^{\mathcal{N}} a_k \sqrt{-\det g_k} \sum_{n=0}^{4} \beta_{k,n} e_n\Big(\sqrt{g_k^{-1} g_{k+1}}\Big) = \sum_{k=1}^{\mathcal{N}} N_k \Big\{a_k F_k\big(\gamma_k : \gamma_{k+1}\big) + a_{k-1} G_{k-1}\big(\gamma_{k-1} : \gamma_k\big)\Big\}. \tag{6.55}$$

Here, we introduce canonical conjugate momentum π_k for γ_k, and obtain the Lagrangian

$$L = \sum_{k=1}^{\mathcal{N}} \pi_k \dot{\gamma}_k - H, \tag{6.56}$$

where the Hamiltonian H is given by

$$H = \sum_{k=1}^{\mathcal{N}} N_k C_k, \qquad C_k := C_k^0\big(\gamma_k, \pi_k\big) - a_k F_k\big(\gamma_k : \gamma_{k+1}\big) - a_{k-1} G_{k-1}\big(\gamma_{k-1} : \gamma_k\big). \tag{6.57}$$

In the Hamiltonian, C_k^0 comes from the Einstein Hilbert term for g_k, so it contains γ_k and its canonical conjugate momentum π_k. The lapse functions N_k ($k = 1, 2, .., \mathcal{N}$) behave as Lagrange multipliers, and variation with respect to them leads to $\mathcal{N}$ primary constraints

$$C_k = 0 \ , \qquad (k = 1, 2, .., \mathcal{N}) \ . \tag{6.58}$$

As is usual, the consistency condition about the time evolution for them must be imposed

$$\dot{C}_k \approx C_{k,k-1} N_{k-1} + C_{k,k+1} N_{k+1} \approx 0, \qquad (k = 1, 2, .., \mathcal{N}) \ , \tag{6.59}$$

where we have defined $C_{k,l} := \big\{C_k \, , C_l\big\}_{PB}$. This Poisson bracket $C_{k,l}$ satisfies $C_{k,l} = 0$ for $|k - l| \geqq 2$ because C_k contains only γ_{k-1}, γ_k, γ_{k+1} and π_k. When we encounter N_0 and $N_{\mathcal{N}+1}$, they are understood as $N_0 := N_{\mathcal{N}}$ and $N_{\mathcal{N}+1} := N_1$.

Note that the explicit calculation gives rise to an important information

$$C_{k,k+1} \propto a_k \ . \tag{6.60}$$

Besides, the structure of this matrix depends on whether the number of cites is odd or even. For example, in the case $\mathcal{N} = 4$, we have

$$C_{k,l} = \begin{pmatrix} 0 & C_{1,2} & 0 & C_{1,4} \\ -C_{1,2} & 0 & C_{2,3} & 0 \\ 0 & -C_{2,3} & 0 & C_{3,4} \\ -C_{1,4} & 0 & -C_{3,4} & 0 \end{pmatrix} , \tag{6.61}$$

while, in the case of $\mathcal{N} = 5$, we obtain

$$C_{k,l} = \begin{pmatrix} 0 & C_{1,2} & 0 & 0 & C_{1,5} \\ -C_{1,2} & 0 & C_{2,3} & 0 & 0 \\ 0 & -C_{2,3} & 0 & C_{3,4} & 0 \\ 0 & 0 & -C_{3,4} & 0 & C_{4,5} \\ -C_{1,5} & 0 & 0 & -C_{4,5} & 0 \end{pmatrix} . \tag{6.62}$$

In the case of odd number of metrics, we cannot split the equations into two independent sets. On the other hand, in the case of even number of metrics, we can split a set of equations into independent two groups of equations. Hence, we have to discuss two cases, separately.

6.3.1 Chain type interaction

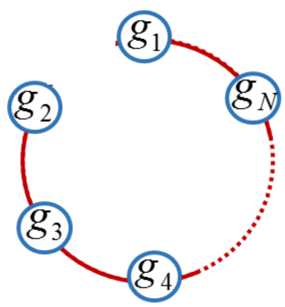

Figure 6.4: The diagram represents the chain type interaction.

Prior to the loop type, we consider the chain type interaction, where one of $(g_k, g_{k+1})_{k=1,2,..,\mathcal{N}}$ interactions is cut as in Fig. 6.4. Here, we set $a_1 = 0$. Then, the consistency conditions for the primary constraints (6.59) lead to $\mathcal{N} - 1$ secondary constraints

$$C_{k,k+1} \approx 0, \ (k = 2, 3, .., \mathcal{N}). \tag{6.63}$$

These secondary constraints must be preserved along the time evolution. Thus, we further impose

$$\dot{C}_{k,k+1} \approx \sum_{l=1}^{\mathcal{N}} \{C_{k,k+1} \, , \, C_l\}_{PB} N_l \approx 0 \quad (k = 2, 3, .., \mathcal{N}), \tag{6.64}$$

which determine $\mathcal{N}-1$ of N_k $(k=1,2,..,\mathcal{N})$, and only one Lagrange multiplier remains undetermined. Therefore, the total number of degrees of freedom can be deduced as

$$\frac{1}{2}\Big(2\big(3+6(\mathcal{N}-1)\big)-\mathcal{N}-(\mathcal{N}-1)-1\Big)=2+5(\mathcal{N}-1). \tag{6.65}$$

It is clear that this result corresponds to one massless and $\mathcal{N}-1$ massive gravitons. Therefore, there exists no BD-ghost. This conclusion is also consistent with the result obtained by the vielbein method in Chapter 5.

6.3.2 Loop type interaction

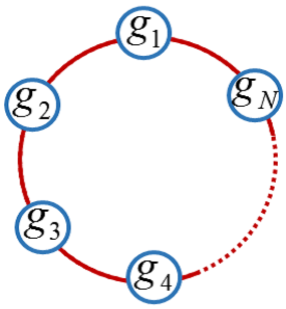

Figure 6.5: The diagram represents the loop type interaction.

If all of $(g_k, g_{k+1})_{k=1,2,..,\mathcal{N}}$ interactions exist as in Fig.6.5, the analysis gets a little complicated. We have to discuss odd and even numbers, separately.

Odd number of metrics

Firstly, we consider the case where $\mathcal{N}=2m+1$ with $(m=1,2,3,4,...)$. In this case, we can classify the equations (6.59) into the following four parts

$$C_{2k,2k-1}N_{2k-1}+C_{2k,2k+1}N_{2k+1}=0 \qquad (k=1,2,3,...,m), \tag{6.66}$$
$$C_{2k-1,2k-2}N_{2k-2}+C_{2k-1,2k}N_{2k}=0 \qquad (k=2,3,...,m), \tag{6.67}$$
$$C_{1,2m+1}N_{2m+1}+C_{1,2}N_2=0, \tag{6.68}$$
$$C_{2m+1,2m}N_{2m}+C_{2m+1,1}N_1=0. \tag{6.69}$$

Solving (6.66), we see that all of N_{2k+1} $(k=1,2,..,m)$ can be expressed by N_1. Similarly, (6.67) can be used to express N_{2k} $(k=2,3,..,m)$ in terms of N_2. Substituting these results into (6.68) and (6.69), we obtain a single equation which determines N_2 by N_1. Thus, (6.59) determines $\mathcal{N}-1$ Lagrange multipliers, and one multiplier is left undetermined, which reflects the existence of the overall gauge symmetry. In the case of odd number of metrics, there is no secondary constraint, while we need one gauge condition to fix the gauge degree of freedom. In conclusion, the total number of degrees of freedom can be calculated as

$$\frac{1}{2}\Big(2\big(3+6(\mathcal{N}-1)\big)-\mathcal{N}-1\Big)=2+5(\mathcal{N}-1)+\frac{\mathcal{N}-1}{2}\,. \tag{6.70}$$

Here, the first two terms correspond to massless and massive gravitons, respectively. The last one should be BD-ghosts, and the number of BD-ghosts is given by $(\mathcal{N}-1)/2$.

Even number of metrics

Next, we consider the case $\mathcal{N} = 2m+2$, where m is a natural number. In this case, we can split the consistency conditions (6.59) into two independent sets of equations,

$$C_{k,k-1}N_{k-1} + C_{k,k+1}N_{k+1} = 0 \qquad (k = 1,3,5,..,2m+1), \tag{6.71}$$
$$C_{k,k-1}N_{k-1} + C_{k,k+1}N_{k+1} = 0 \qquad (k = 2,4,6,..,2m+2). \tag{6.72}$$

The first set (6.71) contains only N_k $(k = 2,4,6,..,2m+2)$, and the second set (6.72) contains only N_k $(k = 1,3,5,..,2m+1)$. Here, if a component $C_{k,k\pm1}$ is found in the set (6.71), $C_{k\pm1,k} = -C_{k,k\pm1}$ must be in the other set (6.72) and vice versa. Therefore, in each set, every component $C_{k,k\pm1}$ appears only once. Now, we define

$$D_{i,j} := C_{2i-1,2j}\,, \qquad M_j := N_{2j} \qquad (i,j = 1,2,3,..,m+1). \tag{6.73}$$

Note that $D_{ij} \neq 0$ only for $i - j = 0, 1$. Then, the equations in (6.71) can be written as

$$\sum_j D_{i,j}M_j = 0 \qquad (i = 1,2,3,..,m+1), \tag{6.74}$$

which we can divide into two parts

$$D_{1,1}M_1 + D_{1,m+1}M_{m+1} = 0, \tag{6.75}$$
$$D_{i,i-1}M_{i-1} + D_{i,i}M_i = 0 \qquad (i = 2,3,..,m+1). \tag{6.76}$$

Using the series (6.76), we can solve all of M_i $(i = 2,3,..,m+1)$ in terms of M_1. However, the relation between M_1 and M_{m+1} obtained from (6.76) cannot be coincide with the formula (6.75) because the equations in (6.76) does not contain $D_{1,1}$ and $D_{1,m+1}$. Thus, we have to impose an additional constraint in order to get non-trivial Lagrange multipliers. This is a secondary constraint expressed by

$$\det D_{ij} = 0. \tag{6.77}$$

If the determinant is not vanish, all of M_j must be zero. Under this condition (6.77), m of M_j $(j = 1,2,..,m+1)$ are determined, and one is left undetermined.

Now, we take the latter set (6.72) and define

$$E_{i,j} := C_{2i,2j-1}\,, \qquad W_j := N_{2j-1} \qquad (i,j = 1,2,3,..,m+1). \tag{6.78}$$

The same argument applies, so we get a secondary constraint $\det E_{ij} = 0$, and one of W_j $(j = 1,2,..,m+1)$ is left undetermined. However, matrix E_{ij} satisfies $E_{ij} = -D_{ji}$. Hence, $\det E_{ij} = 0$ is not a new constraint. Therefore, from (6.59), we get one secondary constraint $\det D_{ij} = 0$ and two undetermined Lagrange multipliers . Then, we must impose a consistency condition for the secondary constraint

$$\frac{d}{dt}\det D_{ij} \approx \sum_{k=1}^{\mathcal{N}} \big\{ \det D_{ij}\,,\, C_k \big\}_{PB} N_k \approx 0, \tag{6.79}$$

which reduces the number of undetermined Lagrange multipliers from two to one.

To summarize, there are $\mathcal{N}$ primary constraints and one secondary constraint, and we need one gauge condition. Thus, we come to the conclusion that the total number of degrees of freedom is

$$\frac{1}{2}\Big(2\big(3+6(\mathcal{N}-1)\big)-\mathcal{N}-1-1\Big)=2+5(\mathcal{N}-1)+\frac{\mathcal{N}-2}{2}\,. \tag{6.80}$$

Here, again, the first two terms correspond to massless and massive gravitons, respectively. Hence, the number of BD-ghosts should be $(\mathcal{N}-2)/2$.

6.4 General multimetric gravity with branching interaction patterns

In this section, we study branching interaction patterns. In an interaction diagram, branching can occur at a node which represents a metric or a midpoint of a link. We call these two patterns as branching node type and branching link type respectively. The former overlaps with a vielbein theory, but the latter does not.

6.4.1 Branching node type interaction

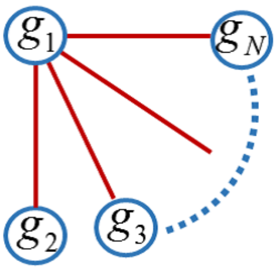

Figure 6.6: The diagram represents the branching node type interaction.

Branching node type interaction can be expressed as a diagram in Fig.6.6, and the interaction term is given by

$$\sum_{k=2}^{\mathcal{N}}\sqrt{-\det g_1}\sum_{n=0}^{4}\beta_{k,n}e_n\Big(\sqrt{g_1^{-1}g_k}\Big). \tag{6.81}$$

Applying the spatial homogeneity ansatz and the ADM decomposition

$$ds_k^2=-N_k^2(t)dt^2+\gamma_{k,ij}(t)dx^idx^j \quad (k=1,2,...,\mathcal{N}), \tag{6.82}$$

each element of the interaction term can be written as

$$\sqrt{-\det g_1}\sum_{n=0}^{4}\beta_{k,n}e_n\Big(\sqrt{g_1^{-1}g_k}\Big)=N_1F_k\big(\gamma_1:\gamma_k\big)+N_kG_k\big(\gamma_1:\gamma_k\big), \tag{6.83}$$

where F_k and G_k are some functions. Summing them, the total interaction term is obtained

$$\sum_{k=2}^{\mathcal{N}} \sqrt{-\det g_1} \sum_{n=0}^{4} \beta_{k,n} e_n \Big(\sqrt{g_1^{-1} g_k} \Big) = N_1 \tilde{F}(\gamma_1 : \gamma_2, ..., \gamma_{\mathcal{N}}) + \sum_{k=2}^{\mathcal{N}} N_k G_k(\gamma_1 : \gamma_k), \tag{6.84}$$

$$\tilde{F}(\gamma_1 : \gamma_2, ..., \gamma_{\mathcal{N}}) := \sum_{k=2}^{\mathcal{N}} F_k(\gamma_1 : \gamma_k). \tag{6.85}$$

Here, we introduce canonical conjugate momentum π_k for γ_k, and obtain the Hamiltonian H

$$H = \sum_{k=1}^{\mathcal{N}} N_k C_k, \tag{6.86}$$

$$C_1 := C_1^0 \big(\gamma_1, \pi_1 \big) - \tilde{F}(\gamma_1 : \gamma_2, ..., \gamma_{\mathcal{N}}), \tag{6.87}$$

$$C_k := C_k^0 \big(\gamma_k, \pi_k \big) - G_k(\gamma_1 : \gamma_k) \quad (k = 2, 3, ..., \mathcal{N}). \tag{6.88}$$

In the above formulae, C_k^0 comes from the Einstein Hilbert term and contains γ_k and its canonical conjugate momentum π_k. The lapse functions N_k $(k = 1, 2, .., \mathcal{N})$ are Lagrange multipliers, and variation with respect to them leads to $\mathcal{N}$ primary constraints

$$C_k = 0 \ , \qquad (k = 1, 2, .., \mathcal{N}). \tag{6.89}$$

Since we notice $\{C_1, C_k\}_{PB} \not\approx 0$ for $k = 2, 3, ..., \mathcal{N}$ and $\{C_k, C_{k'}\}_{PB} = 0$ when $k, k' = 2, 3, ..., \mathcal{N}$, the consistency condition for the time evolution becomes

$$\begin{pmatrix} 0 & C_{1,2} & \cdots & C_{1,\mathcal{N}} \\ -C_{1,2} & & & \\ \vdots & & 0 & \\ -C_{1,\mathcal{N}} & & & \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_{\mathcal{N}} \end{pmatrix} \approx 0. \tag{6.90}$$

Thus, we have $\mathcal{N} - 1$ secondary constraints

$$C_{1,k} = \big\{ C_1, C_k \big\}_{PB} \approx 0 \quad (k = 2, 3, ..., \mathcal{N}). \tag{6.91}$$

These secondary constraints must be preserved along the time evolution. Hence, we impose the consistency condition

$$\dot{C}_{1,k} = \big\{ C_{1,k}, H \big\}_{PB} \approx \sum_{k'=1}^{\mathcal{N}} N_{k'} \big\{ C_{1,k}, C_{k'} \big\}_{PB} \approx 0 \quad (k = 2, 3, ..., \mathcal{N}), \tag{6.92}$$

which determines $\mathcal{N} - 1$ Lagrange multipliers. Eventually, we have $\mathcal{N}$ primary constraints, $\mathcal{N} - 1$ secondary constraints and one undetermined Lagrange multiplier. Therefore, the total number of degrees of freedom is

$$\frac{1}{2} \Big(2 \big(3 + 6(\mathcal{N} - 1) \big) - \mathcal{N} - (\mathcal{N} - 1) - 1 \Big) = 2 + 5(\mathcal{N} - 1). \tag{6.93}$$

We have no BD-ghost.

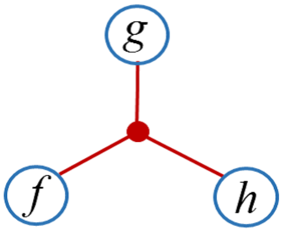

Figure 6.7: The diagram represents an example of the branching link type interaction containing three metrics.

6.4.2 Branching link type interaction

We close our investigation toward multimetric gravity with the analysis on the branching link type interaction. An example is sketched in Fig.6.7.

For matrices $X(I)^{\mu}{}_{\nu}$ $(I = 1, 2, 3, 4)$, we introduce a function $\mathcal{S}$ given by

$$\mathcal{S}\big(X(1), X(2), X(3), X(4)\big) := \epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\nu_2\nu_3\nu_4} X(1)^{\mu_1}{}_{\nu_1} X(2)^{\mu_2}{}_{\nu_2} X(3)^{\mu_3}{}_{\nu_3} X(4)^{\mu_4}{}_{\nu_4}, \quad (6.94)$$

where $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ and $\epsilon^{\nu_1\nu_2\nu_3\nu_4}$ are antisymmetrization symbols defined by $\epsilon_{0123} = 1$ and $\epsilon^{0123} = 1$ respectively. Here, we have four matrices since we are focusing on the four dimensional case. Interaction terms we have considered in previous sections have been constructed from the functions e_n $(n = 0, 1, 2, 3, 4)$, which can be expressed by S as

$$\begin{aligned}
e_0\big(\sqrt{g^{-1}f}\big) &\propto \mathcal{S}\big(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}\big), \\
e_1\big(\sqrt{g^{-1}f}\big) &\propto \mathcal{S}\big(\mathbf{1}, \mathbf{1}, \mathbf{1}, \sqrt{g^{-1}f}\big), \\
e_2\big(\sqrt{g^{-1}f}\big) &\propto \mathcal{S}\big(\mathbf{1}, \mathbf{1}, \sqrt{g^{-1}f}, \sqrt{g^{-1}f}\big), \qquad (6.95)\\
e_3\big(\sqrt{g^{-1}f}\big) &\propto \mathcal{S}\big(\mathbf{1}, \sqrt{g^{-1}f}, \sqrt{g^{-1}f}, \sqrt{g^{-1}f}\big), \\
e_4\big(\sqrt{g^{-1}f}\big) &\propto \mathcal{S}\big(\sqrt{g^{-1}f}, \sqrt{g^{-1}f}, \sqrt{g^{-1}f}, \sqrt{g^{-1}f}\big), \qquad (6.96)
\end{aligned}$$

where $\mathbf{1}$ means unit matrix.

On the spatial homogeneity ansatz, interaction terms in previous sections become linear in the lapse functions, which is necessary to exclude BD-ghosts. Otherwise, we have no constraints and cannot reduce any degrees of freedom. Thus, the interaction term of the branching link type should be constructed to be linear in the lapse functions.

In the case of trimetric gravity with metrics $g_{\mu\nu}$, $f_{\mu\nu}$ and $h_{\mu\nu}$, the following combinations are possible

$$\sqrt{-\det g} \times \begin{cases}
\mathcal{S}\big(\mathbf{1}, \mathbf{1}, \sqrt{g^{-1}f}, \sqrt{g^{-1}h}\big) \\
\mathcal{S}\big(\mathbf{1}, \sqrt{g^{-1}f}, \sqrt{g^{-1}f}, \sqrt{g^{-1}h}\big) \\
\mathcal{S}\big(\mathbf{1}, \sqrt{g^{-1}f}, \sqrt{g^{-1}h}, \sqrt{g^{-1}h}\big) \\
\mathcal{S}\big(\sqrt{g^{-1}f}, \sqrt{g^{-1}f}, \sqrt{g^{-1}h}, \sqrt{g^{-1}h}\big) \\
\mathcal{S}\big(\sqrt{g^{-1}f}, \sqrt{g^{-1}f}, \sqrt{g^{-1}f}, \sqrt{g^{-1}h}\big) \\
\mathcal{S}\big(\sqrt{g^{-1}f}, \sqrt{g^{-1}h}, \sqrt{g^{-1}h}, \sqrt{g^{-1}h}\big).
\end{cases} \qquad (6.97)$$

We can also include permutations of g, f and h. Since spatial homogeneity is assumed, three metrics are written as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 & 0 \\ 0 & \gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}, \tag{6.98}$$

$$f_{\mu\nu} = \begin{pmatrix} -L^2 & 0 \\ 0 & \omega_{ij} \end{pmatrix}, \quad f^{\mu\nu} = \begin{pmatrix} -1/L^2 & 0 \\ 0 & \omega^{ij} \end{pmatrix}, \tag{6.99}$$

$$h_{\mu\nu} = \begin{pmatrix} -Q^2 & 0 \\ 0 & \rho_{ij} \end{pmatrix}, \quad h^{\mu\nu} = \begin{pmatrix} -1/Q^2 & 0 \\ 0 & \rho^{ij} \end{pmatrix}, \tag{6.100}$$

from which basic elements of the interaction terms are expressed as

$$\left(g^{-1}f\right)^{\mu}_{\ \nu} = \begin{pmatrix} L^2/N^2 & 0 \\ 0 & \gamma^{il}\omega_{lj} \end{pmatrix} \implies \sqrt{g^{-1}f} = \begin{pmatrix} L/N & 0 \\ 0 & \sqrt{\gamma^{-1}\omega} \end{pmatrix}, \tag{6.101}$$

$$\left(g^{-1}h\right)^{\mu}_{\ \nu} = \begin{pmatrix} Q^2/N^2 & 0 \\ 0 & \gamma^{il}\rho_{lj} \end{pmatrix} \implies \sqrt{g^{-1}f} = \begin{pmatrix} Q/N & 0 \\ 0 & \sqrt{\gamma^{-1}\rho} \end{pmatrix}, \tag{6.102}$$

$$\sqrt{-\det g} = N\sqrt{\det\gamma}. \tag{6.103}$$

Then, we can immediately find that antisymmetrization in (6.94) actually makes interaction terms (6.97) linear in N, L and Q.

Now, the Hamiltonian can be read schematically as

$$H = NC_N + LC_L + QC_Q, \tag{6.104}$$

with

$$C_N := C_N^0(\gamma,\pi) + F_N(\gamma,\omega,\rho), \tag{6.105}$$

$$C_L := C_L^0(\omega,p) + F_L(\gamma,\omega,\rho), \tag{6.106}$$

$$C_Q := C_Q^0(\rho,\phi) + F_Q(\gamma,\omega,\rho), \tag{6.107}$$

where C_N^0, C_L^0 and C_Q^0 come from the Einstein-Hilbert terms while F_N, F_L and F_Q come from the interaction term. Notice that F_N, F_L and F_Q generally contain all of γ, ω and ρ. In fact, if we consider an interaction such as $\sqrt{-\det g}S\big(\mathbf{1}, \sqrt{g^{-1}f}, \sqrt{g^{-1}f}, \sqrt{g^{-1}h}\big)$, F_Q does not depend on ρ. Hence, F_N, F_L and F_Q may independent of γ, ω and ρ respectively, but this situation is not relevant to our analysis. The lapse functions N, L and Q are Lagrange multipliers, and variation with respect to them leads to three primary constraints

$$C_N = 0, \qquad C_L = 0, \qquad C_Q = 0. \tag{6.108}$$

Since they must be preserved along the time evolution, we impose the consistency condition

$$\dot{C}_N = \big\{C_N\,,\,H\big\}_{PB} = L\big\{C_N\,,\,C_L\big\}_{PB} + Q\big\{C_N\,,\,C_Q\big\}_{PB} \approx 0, \tag{6.109}$$

$$\dot{C}_L = \big\{C_L\,,\,H\big\}_{PB} = N\big\{C_L\,,\,C_N\big\}_{PB} + Q\big\{C_L\,,\,C_Q\big\}_{PB} \approx 0, \tag{6.110}$$

$$\dot{C}_Q = \big\{C_Q\,,\,H\big\}_{PB} = N\big\{C_Q\,,\,C_N\big\}_{PB} + L\big\{C_Q\,,\,C_L\big\}_{PB} \approx 0, \tag{6.111}$$

which is equivalent to

$$\begin{pmatrix} 0 & C_{NL} & C_{NQ} \\ -C_{NL} & 0 & C_{LQ} \\ -C_{NQ} & -C_{LQ} & 0 \end{pmatrix} \begin{pmatrix} N \\ L \\ Q \end{pmatrix} \approx 0. \tag{6.112}$$

Here, the Poisson brackets $C_{NL} := \{C_N, C_L\}_{PB}$, $C_{LQ} := \{C_L, C_Q\}_{PB}$ and $C_{NQ} := \{C_N, C_Q\}_{PB}$ does not vanish.

In the case of four metrics g_k $(k = 1, 2, 3, 4)$, the following combinations are possible

$$\sqrt{-\det g_1} \times \begin{cases} \mathcal{S}\big(\mathbf{1}, \sqrt{g_1{}^{-1}g_2}, \sqrt{g_1{}^{-1}g_3}, \sqrt{g_1{}^{-1}g_4}\big) \\ \mathcal{S}\big(\sqrt{g_1{}^{-1}g_2}, \sqrt{g_1{}^{-1}g_2}, \sqrt{g_1{}^{-1}g_3}, \sqrt{g_1{}^{-1}g_4}\big) \\ \mathcal{S}\big(\sqrt{g_1{}^{-1}g_3}, \sqrt{g_1{}^{-1}g_2}, \sqrt{g_1{}^{-1}g_3}, \sqrt{g_1{}^{-1}g_4}\big) \\ \mathcal{S}\big(\sqrt{g_1{}^{-1}g_4}, \sqrt{g_1{}^{-1}g_2}, \sqrt{g_1{}^{-1}g_3}, \sqrt{g_1{}^{-1}g_4}\big), \end{cases} \tag{6.113}$$

and permutations of g_k $(k = 1, 2, 3, 4)$. Applying the spatial homogeneity ansatz and the ADM decomposition $g_k = (N_k, \gamma_k)$, we obtain the Hamiltonian

$$H = \sum_{k=1}^{4} N_k C_k, \quad C_k := C_k^0(\gamma_k, \pi_k) + F_k(\gamma_1, \gamma_2, \gamma_3, \gamma_4), \tag{6.114}$$

where C_k^0 comes from the Einstein-Hilbert part while F_k comes from the interaction term. Notice that each F_k generally contains all kinds of spatial metrics. As mentioned before, F_k may not depend on γ_k in special situations, but our argument below does not change. Then, we have four primary constraints $C_k = 0$ $(k = 1, 2, 3, 4)$, and the consistency condition

$$\begin{pmatrix} 0 & C_{1,2} & C_{1,3} & C_{1,4} \\ -C_{1,2} & 0 & C_{2,3} & C_{2,4} \\ -C_{1,3} & -C_{2,3} & 0 & C_{3,4} \\ -C_{1,4} & -C_{2,4} & -C_{3,4} & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix} \approx 0 \tag{6.115}$$

is imposed. Here, $C_{k,l} := \{C_k, C_l\}_{PB}$ vanishes only for $k = l$.

In the case of five metrics g_k $(k = 1, 2, 3, 4, 5)$, possible combinations are

$$\sqrt{-\det g_1}\mathcal{S}\big(\sqrt{g_1{}^{-1}g_2}, \sqrt{g_1{}^{-1}g_3}, \sqrt{g_1{}^{-1}g_4}, \sqrt{g_1{}^{-1}g_5}\big) \tag{6.116}$$

and permutations of g_k $(k = 1, 2, 3, 4, 5)$. Then, the Hamiltonian is give by

$$H = \sum_{k=1}^{5} N_k C_k, \quad C_k := C_k^0(\gamma_k, \pi_k) + F_k(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5), \tag{6.117}$$

which leads to five primary constraints $C_k = 0$ $(k = 1, 2, 3, 4, 5)$, and we impose the consistency condition

$$\begin{pmatrix} 0 & C_{1,2} & C_{1,3} & C_{1,4} & C_{1,5} \\ -C_{1,2} & 0 & C_{2,3} & C_{2,4} & C_{2,5} \\ -C_{1,3} & -C_{2,3} & 0 & C_{3,4} & C_{3,5} \\ -C_{1,4} & -C_{2,4} & -C_{3,4} & 0 & C_{4,5} \\ -C_{1,5} & -C_{2,5} & -C_{3,5} & C_{4,5} & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{pmatrix} \approx 0. \tag{6.118}$$

Notice that $C_{k,l} := \{C_k, C_l\}_{PB}$ vanishes only for $k = l$.

In any case, consistency conditions (6.112), (6.115) and (6.118) can be read as a linear equation

$$\mathbb{C}N \approx 0, \tag{6.119}$$

where $\mathbb{C}$ is an antisymmetric matrix and N is a column vector. In general, an antisymmetric real matrix is a normal matrix because the Hermitian conjugate $\mathbb{C}^\dagger = -\mathbb{C}$ satisfies $\mathbb{C}\mathbb{C}^\dagger = -\mathbb{C}^2 = \mathbb{C}^\dagger\mathbb{C}$. Thus, the spectral decomposition is possible $\mathbb{C} = \sum_I a_I P_I$, where a_I is an eigenvalue and P_I is a projection operator. Taking the Hermitian conjugate of this spectral decomposition $\mathbb{C}^\dagger = \sum_I a_I^* P_I$, where a^* means complex conjugate of a, we find $a_I^* = -a_I$ to notice that eigenvalues a_I must be pure imaginary or zero. On the other hand, the trace of an antisymmetric matrix have to be zero. Hence, the number of non-zero eigenvalues must be even, and we come to the conclusion that the rank of matrix $\mathbb{C}$, which we denote by r, must be an even number.

In the following, we assume that we have $\mathcal{N}$ metrics, and $\mathbb{C}$ is a $\mathcal{N} \times \mathcal{N}$ matrix. In configuration space, the number of original dynamical variables is $3 + 6(\mathcal{N} - 1)$, but a no-ghost theory should have $2 + 5(\mathcal{N} - 1)$ ones. Thus, we have to remove extra $\mathcal{N}$ degrees of freedom. In phase space, we already have $\mathcal{N}$ primary constraints. Therefore, in order to exclude all of BD-ghosts, we need $\mathcal{N}$ additional conditions.

If the number of metrics $\mathcal{N}$ is odd, (6.119) determines r Lagrange multipliers while remaining $(\mathcal{N} - r)$ ones are left undetermined. Then, there is no secondary constraint, and we have only $\mathcal{N} - r$ gauge conditions, which cannot exclude all of BD-ghosts. Though we do not carry out explicit calculations, we expect $\mathcal{N} - r = 1$ because we have one overall invariance of the time coordinate change. Therefore, the total number of degrees of freedom should be

$$\frac{1}{2}\Big(2\big(3 + 6(\mathcal{N} - 1)\big) - \mathcal{N} - 1\Big) = 2 + 5(\mathcal{N} - 1) + \frac{\mathcal{N} - 1}{2}. \tag{6.120}$$

We have $(\mathcal{N} - 1)/2$ BD-ghosts.

When we have even number of metrics, we see two possibilities. One of them is a case where the rank r does not coincide with the number of metrics $r \neq \mathcal{N}$, and in the other case, we have $r = \mathcal{N}$. In the first case, we immediately find that we do not have enough conditions and some of BD-ghosts remain. In the second case, we must impose an additional condition $\det \mathbb{C} \approx 0$. Otherwise, only the solution $N = 0$ is allowed. Then, the rank of the matrix $\mathbb{C}$ reduces, which we denote by r' (r' is an even number.). Then, (6.119) determines r' Lagrange multipliers while remaining $\mathcal{N} - r'$ ones are left undetermined. Besides, we must impose the consistency condition on the secondary constraint $d \det \mathbb{C}/dt \approx 0$, which reduces the number of undetermined Lagrange multipliers from $\mathcal{N} - r'$ to $\mathcal{N} - r' - 1$. Therefore, we have one secondary constraint and $\mathcal{N} - r' - 1$ gauge conditions. In total, $\mathcal{N} - r'$ conditions are obtained, but they are not enough to exclude all of BD-ghosts. Here, we expect $\mathcal{N} - r' - 1 = 1$ which corresponds to the overall invariance of the time parameterization. Hence, the total number of degrees of freedom should be

$$\frac{1}{2}\Big(2\big(3 + 6(\mathcal{N} - 1)\big) - \mathcal{N} - 2\Big) = 5(\mathcal{N} - 1) + 2 + \frac{\mathcal{N} - 2}{2}. \tag{6.121}$$

We have $(\mathcal{N} - 2)/2$ BD-ghosts.

The conclusion is that the branching link type interaction always suffers from BD-ghosts.

Chapter 7

Applications

Applications of bimetric or multimetric gravity are diverse. For example, gravitational waves [39], inflationary universe [40], acceleration of the universe, dark matter,...etc. In this chapter, we consider an application to the AdS/CFT correspondence. This topic is based on our unpublished work [6].

7.1 The AdS/CFT correspondence and multimetric gravity

The AdS/CFT correspondence is a kind of holography where the world is understood as a boundary of a higher dimensional space-time. This is one of the most widely studied topics in modern theoretical physics [41, 42, 43, 44], covering general relativity, string theory, matter field theory and so on. In the AdS/CFT correspondence, we prepare a $(d+1)$-dimensional asymptotically AdS (anti de-Sitter) space-time and read information from the AdS-boundary. In the bulk space-time, we put a (classical) gravity theory. Then, a (quantum) matter field theory is reflected on the AdS-boundary. An important feature of the AdS/CFT correspondence is that when the bulk side is weakly coupled, the coupling of the boundary field gets strong. Thus, we can investigate the complicated matter field theory through analyzing rather simple equations in the bulk gravity theory. In standard settings, we use general relativity with other scalar, vector or spinor fields as a gravity theory. As is well known, we have a large number of applications of the AdS/CFT correspondence. For practical applications, a review [45] and references therein is useful.

Sometimes, massive fields play an important role in the AdS/CFT correspondence. For instance, a massive scalar field is used in holographic superconductor. Massive fields are also expected to describe dissipative systems, such as metals with impurities. Hence, it seems promising to investigate massive gravity in the context of the AdS/CFT correspondence. In a sense, a natural completion of massive gravity is bimetric gravity. It seems also interesting to study bimetric or more general multimetric gravity in this context. On the other hand, from a more theoretical view point, emergence of massive gravitons in the AdS/CFT correspondence has been reported in the past [46, 47, 48]. They say when we prepare several CFT boundaries and introduce interactions among them, some of gravitons on the bulk side become massive. This situation makes us recall multimetric gravity.

Motivated these facts, we attempt to apply bimetric gravity to the AdS/CFT correspondence. Multimetric cases are left as future works. In studying the AdS/CFT correspondence, one of the main difficulties is how to interpret a result. We put a gravity theory on the bulk space-time, and read information from the boundary. However, we cannot know in advance what kind of theory we obtain as a boundary field theory. We are often puzzled with what a result physically means. Therefore, we proceed our argument as close as possible to a well-known case where pure general relativity is used. Besides, we rely on the hydrodynamic limit, which makes analytic calculation possible. In these settings, the counterpart on the boundary side is interpreted as fluid of the supersymmetric Yang-Mills plasma [49, 50, 51]. Its transport coefficients such as sheer viscosity are calculated via the AdS/CFT prescription. Following this, we investigate the case of bimetric gravity and see that two-component fluid emerges. We also calculate values of the pressure and sheer viscosity.

7.2 dRGT massive gravity and the AdS/CFT correspondence

Prior to directly consider bimetric gravity, it is convenient to start with dRGT massive gravity. In this section, we firstly revisit the case of general relativity [49, 50], and then extend it to that of dRGT massive gravity. Our calculations follow a review [45].

In the AdS/CFT correspondence in general relativity, we encounter divergent terms. In order to cancel them, we add a counterterm and obtain a finite result. However, mass of a graviton gives rise to extra divergences which are absent in the case of general relativity. The main topic of this section is how to cancel these additional divergences.

7.2.1 The case of general relativity

We review the AdS/CFT correspondence in general relativity. Especially, we focus on the first order hydrodynamic limit. Hydrodynamic limit means a long wave length limit, where derivative expansion is effective. We take only the first order derivatives, and higher order ones are discarded. We can carry out analytic calculations owing to this approximation.

In these settings, the boundary field theory is interpreted as the supersymmetric Yang-Mills plasma, and the pressure and the sheer viscosity can be calculated via the AdS/CFT correspondence.

Now, we start with the five-dimensional Einstein-Hilbert action with a cosmological constant Λ:

$$S_{EH} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-\det g}\,(R - 2\Lambda). \tag{7.1}$$

Since we consider an asymptotically AdS space-time which has a boundary, we need to bring in the Gibbons-Hawking term. We denote the induced metric on the AdS-boundary as γ and introduce

$$S_{GH} := \frac{2}{16\pi G_5} \int_{AdS-bdy} d^4x \sqrt{-\det \gamma} K, \tag{7.2}$$

where K represents the extrinsic curvature and the subscript "$AdS - bdy$" means that the term is evaluated on the AdS-boundary. The Gibbons-Hawking term is necessary to obtain correctly the equation of motion. In the AdS/CFT correspondence, we have to add one more term

$$S_{ct} := \frac{1}{16\pi G_5}\int_{AdS-bdy} d^4x\sqrt{-\det\gamma}\Big(\frac{6}{L}+\frac{L}{2}\mathcal{R}+\cdots\Big), \tag{7.3}$$

namely a counterterm which plays a role to cancel divergences. In the above formula, $\mathcal{R}$ is the curvature constructed from γ, and L is the AdS-radius related to the cosmological constant as $\Lambda = -6/L^2$. Terms denoted by "$\cdots$" contain higher order derivatives such as $\mathcal{R}^2$. Including $\mathcal{R}$, they can be neglected because we leave only first order derivatives with respect to the coordinates on the AdS-boundary. This is the first order hydrodynamic limit. Then, we have set up the action

$$S = S_{EH} + S_{GH} + S_{ct}. \tag{7.4}$$

S_{EH} is a bulk term while S_{GH} and S_{ct} are boundary terms.

In general relativity, a lot of asymptotically AdS solutions are known. However, we focus only on five dimensional Schwarzschild AdS black hole (SAdS-BH) whose metric is given by

$$g_{\mu\nu}dx^\mu dx^\nu = \Big(\frac{r_0}{L}\Big)^2\frac{1}{u^2}\big(-hdt^2+dx^2+dy^2+dz^2\big)+\frac{L^2}{hu^2}du^2, \tag{7.5}$$

where r_0 is a constant and L is the AdS-radius. A function h is defined as $h := 1-u^4$ $(0 < u < 1)$. Note that the coordinates on the bulk space-time are denoted by $x^\mu = (t, x, y, x, u)$, and the AdS-boundary is located at $u = 0$ while the Black Hole horizon is on the region $u = 0$. If we set $h = 1$ and $r_0 = 1$, we restore the pure AdS space-time.

According to the ordinary prescription of the AdS/CFT correspondence, we consider a perturbation around a fixed background and expand the action up to the second order. Then, we solve the equation of motion, and substitute the solution back into the original action. To get this on-shell action is the first step. For this purpose, we take a perturbation $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ and set the background metric $\bar{g}$ to be SAdS-BH (7.5). We assume a simple ansatz for the fluctuation

$$\delta g^\mu{}_\nu = \bar{g}^{\mu\lambda}\delta g_{\lambda\nu} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&\phi&0&0\\ 0&\phi&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \qquad \phi=\phi(t,u). \tag{7.6}$$

Here, ϕ depends only on the coordinates t and u. Then, we expand the action $S = S_{EH}+S_{GH}+S_{ct}$ up to the second order in ϕ and perform the Fourier transform $\phi(t,u) = \int \frac{d\omega}{2\pi}e^{-i\omega t}\phi_\omega(u)$. The perturbed Einstein-Hilbert action is given by (B.44), but we should note that total derivatives cannot be discarded since we have boundaries. Including them, we write down the Einstein-Hilbert action

$$\begin{aligned} S_{EH} = &-\frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\int_0^1 du\frac{8}{u^5}\\ &+\frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int\frac{d\omega}{2\pi}\int_0^1 du\Big\{\frac{3}{2}\frac{h}{u^3}\phi'_{-\omega}\phi'_\omega+2\frac{h}{u^3}\phi_{-\omega}\phi''_\omega\\ &\qquad -\frac{8}{u^4}\phi_{-\omega}\phi'_\omega+\Big(\frac{1}{2u^3h}\Big(\frac{L^2}{r_0}\omega\Big)^2+\frac{4}{u^5}\Big)\phi_{-\omega}\phi_\omega\Big\}, \end{aligned} \tag{7.7}$$

where we have abbreviated $\int dxdydz = V_3$, $\partial_u\phi(u) = \phi'$ and $\int dtdxdydz = V_4$. Remaining parts S_{GH} and S_{ct} are easy to calculate. Using $\sqrt{-\gamma} \sim \left(\frac{r_0}{L}\right)^4 \frac{\sqrt{h}}{u^4}\left(1 - \frac{1}{2}\phi^2\right)$ and $K = -\frac{\partial_u\sqrt{\gamma}}{N\sqrt{\gamma}}$ with the lapse function $N^{-1} = u\sqrt{h}/L$, we expand S_{GH} as

$$
\begin{aligned}
S_{GH} =& -\frac{2}{16\pi G_5}\int d^4x \frac{1}{N}\partial_u\sqrt{-\gamma}\Big|_{u=0} \\
=& \frac{1}{16\pi G_5}\frac{r_0^4}{L^5}\int d^4x \left\{8\frac{h}{u^4} - \frac{h'}{u^3} + \left(\frac{h'}{2u^3} - 4\frac{h}{u^4}\right)\phi\phi + 2\frac{h}{u^3}\phi\phi'\right\}\Big|_{u=0} && (7.8)\\
=& \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\left(\frac{8}{u^4} - 4\right) + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int\frac{d\omega}{2\pi}\left\{\left(-\frac{4}{u^4} + 2\right)\phi_{-\omega}\phi_\omega + 2\left(\frac{1}{u^3} - u\right)\phi_{-\omega}\phi'_\omega\right\}\Bigg|_{u=0}. && (7.9)
\end{aligned}
$$

Noticing that the back ground value of $\mathcal{R}$ is zero, and taking into account the first order hydrodynamic limit , we also get

$$
\begin{aligned}
S_{ct} =& -\frac{1}{16\pi G_5}\int d^4x\sqrt{-\gamma}\frac{6}{L}\Big|_{u=0} \\
=& -6\frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\frac{\sqrt{h}}{u^4} + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int\frac{d\omega}{2\pi}3\frac{\sqrt{h}}{u^4}\phi_{-\omega}\phi_\omega\Big|_{u=0} && (7.10)\\
=& \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\left(3 - \frac{6}{u^4}\right) + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int\frac{d\omega}{2\pi}\left(\frac{3}{u^4} - \frac{3}{2} + O[u^4]\right)\phi_{-\omega}\phi_\omega\Big|_{u=0}, && (7.11)
\end{aligned}
$$

where $O[u^4]$ means higher than fourth order terms $u^{n\geq 4}$.

Boundary terms S_{GH} and S_{ct} do not contribute to the equation of motion. It is obtained from the bulk term S_{EH}. For convenience, we write the bulk term schematically as

$$
S_{bulk} = S_{EH} = \int_0^1 du\,\mathcal{L}(\phi\,,\,\phi'\,,\,\phi''), \tag{7.12}
$$

and take the variation $\phi \to \phi + \delta\phi$. We obtain the variation of the bulk action

$$
\begin{aligned}
\delta S_{bulk} =& \int_0^1 du\left\{\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)\delta\phi + \left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)\delta\phi' + \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)\delta\phi''\right\} \\
=& \left[\left\{\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right) - \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)'\right\}\delta\phi + \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)\delta\phi'\right]_0^1 + \int_0^1 du\left\{\left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)'' - \left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)' + \left(\frac{\partial\mathcal{L}}{\partial\phi}\right)\right\}\delta\phi.
\end{aligned} \tag{7.13}
$$

The $\delta\phi'$ term will be canceled by δS_{GH}, and we get the equation of motion

$$
\left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)'' - \left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)' + \left(\frac{\partial\mathcal{L}}{\partial\phi}\right) = 0. \tag{7.14}
$$

On the other hand, we can also express the bulk action as

$$
\begin{aligned}
S_{bulk} =& -\frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\int_0^1 du\frac{8}{u^5} + \frac{1}{2}\int_0^1 du\left\{\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)\phi + \left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)\phi' + \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)\phi''\right\} \\
=& \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\left[\frac{2}{u^4}\right]_0^1 + \frac{1}{2}\left[\left\{\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right) - \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)'\right\}\phi + \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)\phi'\right]_0^1 \\
&+ \frac{1}{2}\int_0^1 du\left\{\left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)'' - \left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)' + \left(\frac{\partial\mathcal{L}}{\partial\phi}\right)\right\}\phi,
\end{aligned} \tag{7.15}
$$

since we know that $\mathcal{L}(\phi, \phi', \phi'')$ is quadratic in ϕ, ϕ' and ϕ''. The remaining zeroth order term is explicitly written. According to the AdS/CFT prescription, we solve the equation of motion and substitute the solution into the original action, and get the on-shell action. Thus, the last term is discarded. Besides, we do not need the field value on the non AdS-boundary $\phi(u = 1)$, which we neglect [52, 53]. Then, the bulk action can be read as

$$S_{bulk} = \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\Big(2 - \frac{2}{u^4}\Big)\Big|_{u=0} - \frac{1}{2}\Big\{\Big(\Big(\frac{\partial\mathcal{L}}{\partial\phi'}\Big) - \Big(\frac{\partial\mathcal{L}}{\partial\phi''}\Big)'\Big)\phi + \Big(\frac{\partial\mathcal{L}}{\partial\phi''}\Big)\phi'\Big\}\Big|_{u=0}. \tag{7.16}$$

Therefore, recalling the explicit formula (7.7), we obtain the bulk action

$$S_{bulk} = \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\Big(2 - \frac{2}{u^4}\Big) + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int\frac{d\omega}{2\pi}\Big\{\frac{h}{u^4}\phi_{-\omega}\phi_\omega - \frac{3}{2}\frac{h}{u^3}\phi_{-\omega}\phi'_\omega\Big\}\Big|_{u=0} \tag{7.17}$$

$$= \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5}\Big(2 - \frac{2}{u^4}\Big) + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int\frac{d\omega}{2\pi}\Big\{\Big(\frac{1}{u^4} - 1\Big)\phi_{-\omega}\phi_\omega - \frac{3}{2}\Big(\frac{1}{u^3} - u\Big)\phi_{-\omega}\phi'_\omega\Big\}\Big|_{u=0}, \tag{7.18}$$

and the full action

$$\begin{aligned} S &= S_{bulk} + S_{GH} + S_{ct} \\ &= \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5} + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int\frac{d\omega}{2\pi}\Big\{\Big(-\frac{1}{2} + O[u^4]\Big)\phi_{-\omega}\phi_\omega + \frac{1}{2}\Big(\frac{1}{u^3} - u\Big)\phi_{-\omega}\phi'_\omega\Big\}\Big|_{u=0}. \end{aligned} \tag{7.19}$$

From (7.7) and (7.14), we also obtain explicitly the equation of motion

$$\Big(\frac{h}{u^3}\phi'_\omega\Big)' + \Big(\frac{L^2}{r_0}\omega\Big)^2\frac{1}{u^3 h}\phi_\omega = 0. \tag{7.20}$$

Now, we solve the equation of motion (7.20) and substitute the solution into the action (7.19) to get the on-shell action. Since our main interest is the first order hydrodynamic limit, we have only to solve (7.20) up to the first order expansion in ω. We expand $\phi_\omega(u)$ as

$$\phi_\omega(u) = \phi_0(u) + \omega\phi_1(u) + \omega^2\phi_2(u) + \cdots \tag{7.21}$$

and insert it into (7.20), which becomes

$$\Big(\frac{h}{u^3}\phi'_i\Big)' = 0 \qquad (i = 0, 1). \tag{7.22}$$

We can easily solve these equations

$$\phi_i = A_i + B_i \ln(1 - u^4) \qquad (i = 0, 1), \tag{7.23}$$

and obtain

$$\phi_\omega(u) = (A_0 + \omega A_1) + (B_0 + \omega B_1)\ln(1 - u^4) + O[\omega^2], \tag{7.24}$$

where A_i and B_i are constants. This solution is substituted into (7.19), but in (7.19) we need only the asymptotic formula near the AdS-boundary $u \sim 0$, which we approximate as

$$\phi_\omega(u) = A_\omega + B_\omega \ln(1-u^4) + O[\omega^2] = A_\omega + B_\omega(-u^4 + O[u^8]) + O[\omega^2], \tag{7.25}$$

where we have defined $A_\omega := A_0 + \omega A_1$ and $B_\omega := B_0 + \omega B_1$. Then, the on-shell action is obtained

$$S = \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5} + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int \frac{d\omega}{2\pi}\Big(-\frac{1}{2}A_{-\omega}A_\omega - 2A_{-\omega}B_\omega\Big). \tag{7.26}$$

The remaining constants $A_{0,1}$ and $B_{0,1}$ are fixed by a boundary condition on the Black Hole horizon ($u = 1$). Hence, we have to solve the equation of motion (7.20) near the $u \sim 1$ region. Approximating as $h = 1 - u^4 \sim 4(1-u)$, we get the near horizon equation from (7.20)

$$\phi_\omega'' - \frac{1}{1-u}\phi_\omega' + \Big(\frac{L^2}{4r_0}\omega\Big)^2 \frac{1}{(1-u)^2}\phi_\omega = 0. \tag{7.27}$$

Its solution is easy to find

$$\phi_\omega(u \sim 1) \propto (1-u)^{\pm i\frac{L^2}{4r_0}\omega}. \tag{7.28}$$

A boundary condition selects one of these branches. In order to see what they mean, we set $r_0 = L = 1$ and remember that background SAdS-BH has $ds^2 = -\frac{h}{u^2}dt^2 + \frac{du^2}{u^2 h}$. If we change the coordinate u to $u_* = \int_u^\infty \frac{du}{h} \sim \frac{1}{4}\ln(1-u)$, this metric reads as $ds^2 \propto (-dt^2 + du^2)$. Then, the solution is written as $\phi_\omega(u \sim 1) \propto (1-u)^{\pm\frac{i}{4}\omega} = e^{\pm i\omega u_*}$, and we notice $\phi(t, u \sim 1) \propto e^{-i\omega t}\phi_\omega(u \sim 1) = e^{-i\omega(t \mp u_*)}$. Because the Black Hole horizon is located at $u_* = -\infty$, the branch $e^{-i\omega(t+u_*)}$ represents an ingoing wave while the other $e^{+i\omega(t+u_*)}$ is outgoing. We select the ingoing wave condition according to the standard AdS/CFT prescription. Thus, we obtain the near horizon solution

$$\phi_\omega(u \sim 1) \propto (1-u)^{-i\frac{L^2}{4r_0}\omega} \sim (1-u^4)e^{-i\frac{L^2}{4r_0}\omega} = 1 - i\frac{L^2}{4r_0}\omega \ln(1-u^4) + O[\omega^2]. \tag{7.29}$$

The previously obtained solution (7.24) must match (7.29), which fixes the constants as $A_1 = 0$, $B_0 = 0$, $B_1 = -i\frac{L^2}{4r_0}A_0$. Renaming A_0 as $\phi^{(0)}$, we obtain the on-shell action from (7.26)

$$S = \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5} + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int \frac{d\omega}{2\pi}\Big\{ -\frac{1}{2}\phi_{-\omega}^{(0)}\phi_\omega^{(0)} + \frac{1}{2}\phi_{-\omega}^{(0)}\Big(i\frac{L^2}{r_0}\omega\Big)\phi_\omega^{(0)}\Big\}, \tag{7.30}$$

where we should notice that $i\int d\omega \phi_{-\omega}^{(0)}\omega\phi_\omega^{(0)}$ cannot be interpreted as zero [52].

The last step is to apply the GKP-Witten relation

$$\Big\langle \exp\Big(i\int \phi^{(0)}\mathcal{O}\Big)\Big\rangle = \exp\Big(iS\big[\phi|_{u=0} = \phi^{(0)}\big]\Big). \tag{7.31}$$

In the right hand side, we have an action $S[\phi]$. We solve the equation of motion for a bulk field ϕ and obtain its solution. The boundary value of the solution is important and denoted by $\phi^{(0)}$. We substitute the solution into the action and get the on-shell action $S\big[\phi|_{u=0} = \phi^{(0)}\big]$. The left hand side represents an expectation value of the boundary field theory, where $\phi^{(0)}$ becomes a source of an operator $\mathcal{O}$.

In our case, ϕ is a fluctuation of a gravitational field so that $\mathcal{O}$ is interpreted as a perturbed boundary energy-momentum tensor $\delta T_{\mu\nu}$. Thus, we obtain

$$< \delta T^{xy}_{\omega} >= \frac{\delta S}{\delta \phi^{(0)}_{-\omega}} = -\frac{1}{16\pi G_5}\frac{r_0^4}{L^5}\phi^{(0)}_{\omega} + i\frac{1}{16\pi G_5}\Big(\frac{r_0}{L}\Big)^3 \omega \phi^{(0)}_{\omega}, \tag{7.32}$$

where the functional derivative must be interpreted as $\frac{\delta}{\delta\phi_{-\omega}}\phi_{-\omega}\mathcal{F}_{\omega}\phi_{\omega} = 2\mathcal{F}_{\omega}\phi_{\omega}$ [52, 53]. In a long wave-length limit, any field theory can be effectively described by hydrodynamics. Hence, we assume that the energy-momentum tensor of the boundary field theory has the following form

$$T^{\mu\nu} = (\epsilon + P)u^{\mu}u^{\nu} + P\eta^{\mu\nu} + \tau^{\mu\nu}, \tag{7.33}$$

with energy density ϵ, pressure P and velocity field u^{μ}. Since the boundary field theory is supposed to be on the four dimensional uncurved space-time, we have $\mu = 0, 1, 2, 3$ and $\eta^{\mu\nu}$ is the Minkowski metric. The term $\tau_{\mu\nu}$ contains derivatives, but we need only first order ones. In the rest frame, $\tau_{\mu\nu}$ has no time component ($\mu = 0$) and spatial components are given by

$$\tau_{ij} = -\eta\Big(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u^k\Big) - \zeta\delta_{ij}\partial_k u^k \quad (i, j, k = 1, 2, 3). \tag{7.34}$$

η and ζ represent transport coefficients called sheer viscosity and bulk viscosity. Here, we assume that this fluid is firstly at rest $u^{\mu} = (1, 0, 0, 0)$, and then the background space-time is slightly distorted $\eta_{\mu\nu} \to g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$. Using a projection operator $\mathcal{P}^{\mu\nu} := g^{\mu\nu} + u^{\mu}u^{\nu}$, the energy-momentum tensor is written as

$$T^{\mu\nu} = (\epsilon + P)u^{\mu}u^{\nu} + Pg^{\mu\nu} - \mathcal{P}^{\mu\lambda}\mathcal{P}^{\nu\rho}\Big[\eta\Big(\nabla_{\lambda}u_{\rho} + \nabla_{\rho}u_{\lambda} - \frac{2}{3}g_{\lambda\rho}\partial_{\sigma}u^{\sigma}\Big) + \zeta g_{\lambda\rho}\nabla_{\sigma}u^{\sigma}\Big]. \tag{7.35}$$

We calculate the linear response of this tensor, but we are now concerned with a perturbation (7.6). Thus, we set

$$\delta g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \delta g_{xy}(t) & 0 \\ 0 & \delta g_{xy}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.36}$$

The velocity field $u^{\mu} = (1, 0, 0, 0)$ is not changed because of parity symmetry. We can easily calculate the linear level response. After the Fourier transformation, we find

$$\delta T^x_y = -P\delta g^x_y + i\omega\eta\delta g^x_y. \tag{7.37}$$

We compare (7.32) to (7.37) and identify $\phi^{(0)}$ with δg^x_y, from which we interpret that the boundary field theory has the pressure $P = \frac{1}{16\pi G_5}\frac{r_0^4}{L^5}$ and the shear viscosity $\eta = \frac{1}{16\pi G_5}\big(\frac{r_0}{L}\big)^3$.

On the Black Hole horizon, (x, y, z) components of the metric can be written as $ds^2 = \big(\frac{r_0}{L}\big)^2(dx^2 + dy^2 + dz^2)$. Employing the area law of Black Hole entropy, we can calculate the entropy density as $s = \frac{1}{4G_5}\left(\frac{r_0}{L}\right)^3$. This value is interpreted as the entropy density of the boundary field theory. Thus, we obtain the ratio $\eta/s = 1/4\pi$.

The pressure can be calculated in a different way. If we consider only background SAdS-BH with no perturbation, the Euclidean on-shell action is given by

$$S_E = -\frac{1}{16\pi G_5}\frac{r_0^4}{L^5}\int_0^\beta d\tau \int dxdydz = -\frac{\beta V_3}{16\pi G_5}\frac{r_0^4}{L^5}, \tag{7.38}$$

where β is the inverse temperature. Thus, we have the partition function $Z = e^{-S_E}$ and we can calculate the pressure as

$$P = \frac{1}{\beta}\partial_{V_3} \ln Z = \frac{1}{16\pi G_5}\frac{r_0^4}{L^5}. \tag{7.39}$$

This result is compatible with the value obtained from a perturbation.

This is the standard calculation of the first order hydrodynamics via the AdS/CFT correspondence. Other types of perturbations lead to other coefficients, but we will not treat them in this thesis.

7.2.2 The case of massive gravity

We apply the method explained in the case of general relativity to dRGT massive gravity. We see that mass of a graviton give rise to extra divergences, and how to cancel them is our concern. We show that not only a new counterterm have to be added but also a condition on graviton's mass must be imposed. For notational simplicity, we set $16\pi G_5 = 1$, $L = r_0 = 1$ and $V_3 = 1$ in this subsection.

In Section 3.3, we have obtained dRGT massive gravity, where we add the mass term

$$S_{mass} := 2m^2 \int d^5x \sqrt{-\det g} \sum_{n=0}^{5} \beta_n \, e_n\left(\sqrt{g^{-1}\bar{g}}\right) \tag{7.40}$$

to the usual Einstein-Hilbert action. In this mass term, $\bar{g}$ represents a background metric while g means a full metric $g = \bar{g} + \delta g$ (background +fluctuation). Parameters β_n are adjusted to restore the Fierz-Pauli mass term in the expansion up to the second order in δg

$$S_{mass} = -\frac{1}{4}m^2 \int d^5x \sqrt{-\det \bar{g}}\Big(\mathrm{Tr}(\delta g)^2 - \mathrm{Tr}^2(\delta g)\Big), \tag{7.41}$$

where we have abbreviated $\mathrm{Tr}^2 A = (\mathrm{Tr}A)^2$ and $\mathrm{Tr}A^2 = \mathrm{Tr}(A^2)$. Then, we start with the action given by

$$S = S_{EH} + S_{GH} + S_{ct} + S_{mass}, \tag{7.42}$$

and set the background metric $\bar{g}$ to be SAdS-BH. S_{EH}, S_{GH} and S_{ct} are the same as those in general relativity, and are constructed from the full metric g.

Now, we take a perturbation similar to (7.6)

$$\delta g^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \phi = \phi(t, u), \tag{7.43}$$

and expand the action up to the second order in ϕ. Calculations are almost the same as those in general relativity. The only difference is that S_{mass} is added to the bulk action (7.12)

$$S_{bulk} = S_{EH} + S_{mass} = \int du\, \mathcal{L}(\phi\,,\,\phi'\,,\,\phi''), \tag{7.44}$$

where we have

$$S_{mass} = -\int \frac{d\omega}{2\pi} \int_0^1 du \frac{m^2}{2}\frac{1}{u^5}\phi_{-\omega}\phi_\omega. \tag{7.45}$$

Since S_{mass} contains no derivative, it has no contribution to (7.16) and there is no change in (7.19)

$$S = S_{bulk} + S_{GH} + S_{ct} = V_4 + \int \frac{d\omega}{2\pi}\Big\{\Big(-\frac{1}{2} + O[u^4]\Big)\phi_{-\omega}\phi_\omega + \frac{1}{2}\Big(\frac{1}{u^3} - u\Big)\phi_{-\omega}\phi'_\omega\Big\}\Big|_{u=0}. \tag{7.46}$$

The difference occurs only in the equation of motion. From (7.14), we find

$$\Big(\frac{h}{u^3}\phi'_\omega\Big)' - m^2\frac{1}{u^5}\phi_\omega + \frac{\omega^2}{u^3 h}\phi_\omega = 0. \tag{7.47}$$

We solve (7.47) up to the first order expansion in ω. The result is

$$\phi_\omega(u) = A_\omega\Big\{u^{2-2\alpha} + \frac{1}{4}(1-\alpha)u^{6-2\alpha} + O[u^{10-2\alpha}]\Big\} + B_\omega\Big\{u^{2+2\alpha} + \frac{1}{4}(1+\alpha)u^{6+2\alpha} + O[u^{10+2\alpha}]\Big\}. \tag{7.48}$$

A_ω and B_ω are defined as $A_\omega := A_0 + \omega A_1$ and $B_\omega := B_0 + \omega B_1$ with constants $A_{0,1}$ and $B_{0,1}$. graviton's mass is contained in $\alpha := \sqrt{1 + m^2/4}$. We substitute this solution into the action (7.46) and obtain the on-shell action

$$\begin{aligned} S = V_4 + \int \frac{d\omega}{2\pi}\Big\{&(1+\alpha)A_{-\omega}B_\omega + (1-\alpha)B_{-\omega}A_\omega \\ &+ A_{-\omega}A_\omega\Big((1-\alpha)u^{-4\alpha} + \frac{1}{2}(\alpha^2-\alpha-1)u^{4-4\alpha} + O[u^{8-4\alpha}]\Big)\Big\}\Big|_{u=0}. \end{aligned} \tag{7.49}$$

However, we see that extra divergences arise from $u^{-4\alpha}|_{u=0}$, $u^{4-4\alpha}|_{u=0}$ and $O[u^{8-4\alpha}]|_{u=0}$.

Here, it should be noted that if we consider the pure AdS space-time, only one divergence occurs. The AdS space-time corresponds to setting $h = 1$ in the metric of SAdS-BH (7.5). In this case, the solution for (7.47) becomes $\phi_\omega(u) = A_\omega u^{2-2\alpha} + B_\omega u^{2+2\alpha}$. Higher order terms such as $O[u^{6-2\alpha}]$ or $O[u^{6+2\alpha}]$ in (7.48) come from the expansion of $h = 1 - u^4$ around $u \sim 0$. In addition, if we set $h = 1$ in (7.17), (7.8) and (7.10), the action (7.46) becomes $S = V_4 + \int \frac{d\omega}{2\pi}\frac{1}{2u^3}\phi_{-\omega}\phi'_\omega|_{u=0}$. Therefore, we have only one divergence $u^{-4\alpha}|_{u=0}$ in the pure AdS space-time.

In order to cancel divergences in (7.49), we attempt to add a new counterterm S_{mct}. This situation resembles that of a massive scalar field [54], in which extra divergences can be canceled by a mass term on the AdS-boundary. Therefore, we try to add a term

$$S_{mct} \propto \int_{Ads-bdy} d^4x\sqrt{-\det\gamma}\sum_{n=0}^{4}\beta'_n\, e_n\big(\gamma^{-1}\bar{\gamma}\big). \tag{7.50}$$

We chose parameters β'_n to return to the Fierz-Pauli mass term in the second order expansion

$$S_{mct} = -\frac{1}{2}(1-\alpha)\int_{Ads-bdy} d^4x\sqrt{-\det\bar{\gamma}}\Big(\mathrm{Tr}(\delta\gamma)^2 - \mathrm{Tr}^2(\delta\gamma)\Big). \tag{7.51}$$

The coefficient $(1-\alpha)$ is adjusted for our purpose. If we consider perturbations dependent on the spatial coordinates (x, y, z), we may need other counterterms [54]. However, we do not treat this topic in this thesis. In Section 7.2.3, we investigate the validity of this counterterm in a different perturbation.

Inserting the perturbation (7.43) and the solution (7.48), we find

$$\begin{aligned}
S_{mct} =& -\big(1-\alpha\big)\int_{Ads-bdy} d^4x\sqrt{-\det\gamma}\phi^2 & (7.52)\\
=& -\big(1-\alpha\big)\int\frac{d\omega}{2\pi}\frac{\sqrt{1-u^4}}{u^4}\phi_{-\omega}\phi_\omega\Big|_{u=0} & (7.53)\\
=& -\big(1-\alpha\big)\int\frac{d\omega}{2\pi}\Big(\frac{1}{u^4}-\frac{1}{2}+O[u^4]\Big)\phi_{-\omega}\phi_\omega\Big|_{u=0} & (7.54)\\
=& \int\frac{d\omega}{2\pi}\Big\{(\alpha-1)(A_{-\omega}B_\omega + B_{-\omega}A_\omega) & (7.55)\\
& + A_{-\omega}A_\omega\Big(-(1-\alpha)u^{-4\alpha} + \frac{1}{2}\alpha(1-\alpha)u^{4-4\alpha} + O[u^{8-2\alpha}]\Big)\Big\}\Big|_{u=0}. & (7.56)
\end{aligned}$$

Thus, we obtain

$$S + S_{mct} = V_4 + \int\frac{d\omega}{2\pi}\Big\{2\alpha A_{-\omega}B_\omega + A_{-\omega}A_\omega\Big(-\frac{1}{2}u^{4-4\alpha} + O[u^{8-2\alpha}]\Big)\Big\}\Big|_{u=0}. \tag{7.57}$$

The divergence from $u^{-4\alpha}$ has been canceled, but still other divergences remain. If we take the pure AdS space-time as a background, these remaining divergences do not appear and S_{mct} is enough. Cancellation of them requires a condition on graviton's mass. We have to set $-4 < m^2 < 0$, namely $0 < \alpha < 1$, which is a reminiscence of the BF-bound [55, 56, 57].

Then, the non-divergent on-shell action is given by

$$S + S_{mct} = V_4 + \int\frac{d\omega}{2\pi}(2\alpha A_{-\omega}B_\omega). \tag{7.58}$$

Finally, we attempt to fix constants $A_{0,1}$ and $B_{0,1}$. If we assume that the massive and massless solutions (7.48) and (7.29) coincide in the massless limit $\alpha = 1$, we should set $A_\omega = \phi^{(0)}_\omega$ and $B_\omega = \frac{i\omega}{4}\phi^{(0)}_\omega$, which leads to

$$\begin{aligned}
S + S_{mct} &= V_4 + \int\frac{d\omega}{2\pi}\Big(\frac{i\alpha\omega}{2}\Big)\phi^{(0)}_{-\omega}\phi^{(0)}_\omega, & (7.59)\\
< \delta T^{xy}_\omega > &= \frac{\delta S}{\delta\phi^{(0)}_{-\omega}} = i\omega\alpha\phi^{(0)}_\omega. & (7.60)
\end{aligned}$$

Compared to (7.37), we read $P = 0$ and $\eta = \alpha$. The pressure is zero which is not consistent with the value calculated from the background metric (7.39). We do not know how to interpret this result. We suspect that this peculiarity comes from the weird feature of dRGT massive gravity. It depends explicitly on the background metric, which seems artificial. We have a more natural theory for a massive graviton, that is bimetric gravity. Hence, we proceed to extend the method to bimetric gravity and expect that the peculiarity in dRGT massive gravity is modified.

7.2.3 Validity of the new counterterm

Before going to the case of bimetric gravity, we reconsider the counterterm introduced in (7.50) and (7.51). In Section 7.2.2, we have taken only a perturbation (7.43). Thus, it is worth considering whether the counterterm (7.50), (7.51) can cancel divergences in other types of perturbations. Recalling calculations in Section 7.2.2, we notice that the role of the counterterm (7.51) is to cancel the divergence coming from $u^{-4\alpha}$. This term is a leading order contribution in the expansion of $h = 1 - u^4$ around $u \sim 0$. Next order terms contribute as $u^{4-4\alpha}$, $u^{8-4\alpha}$,... and so on. If we consider the pure AdS space-time as a background, only $u^{-4\alpha}$ divergence is left and other divergences $O[u^{4-4\alpha}]$ do not occur. Hence, we need only to consider the AdS space-time for our purpose. In the following, we set $16\pi G_5 = 1$ and $L = r_0 = 1$ for notational simplicity.

We consider a perturbation dependent only on the coordinate u

$$\delta g^{\mu}{}_{\nu} = \begin{pmatrix} \chi_0(u) & -\theta_1(u) & -\theta_2(u) & -\theta_3(u) & -\Pi_0(u) \\ \theta_1(u) & \chi_1(u) & \phi_1(u) & \phi_2(u) & \Pi_1(u) \\ \theta_2(u) & \phi_1(u) & \chi_2(u) & \phi_3(u) & \Pi_2(u) \\ \theta_3(u) & \phi_2(u) & \phi_3(u) & \chi_3(u) & \Pi_3(u) \\ \Pi_0(u) & \Pi_1(u) & \Pi_2(u) & \Pi_3(u) & \chi_4(u) \end{pmatrix}, \tag{7.61}$$

and the background metric is set to be purely AdS. Minus signs are put to make $\delta g_{\mu\nu}$ symmetric.

Now, we expand the action as in Section 7.2.2. We can use (B.44) for the Einstein-Hilbert part. Then, we obtain the equation of motion. The solutions are substituted back into the action, and we see whether or not divergences remain. We skip the details, but we find that only diagonal components $\chi_{0,1,2,3,4}$ couple. Calculations for $\phi_{i=1,2,3}$ are the same as those in Section 7.2.2, so we omit this part. The simplest part is $\Pi_{i=0,1,2,3}$. The equation of motion turns out to be

$$m^2 \frac{1}{u^5}\Pi_0 = 0\ ,\ -m^2\frac{1}{u^5}\Pi_i = 0\ (i = 1,\, 2,\, 3), \tag{7.62}$$

and we have $\Pi_{i=0,1,2,3} = 0$. They have no contribution.

Each of $\theta_{i=1,2,3}$ obeys the equation of motion

$$-\frac{1}{u^3}\theta_i'' + \frac{3}{u^4}\theta_i' + \frac{m^2}{u^5}\theta_i = 0, \tag{7.63}$$

and has the action

$$S_{bulk} + S_{GH} + S_{ct} = -\frac{1}{2u^3}\theta_i\theta_i'\Big|_{u=0}, \tag{7.64}$$

$$S_{mct} = (1-\alpha)\frac{1}{u^4}\theta_i^2\Big|_{u=0}. \tag{7.65}$$

The solution for (7.63) is easy to obtain

$$\theta_i(u) = A_i u^{2-2\alpha} + B_i u^{2+2\alpha}, \tag{7.66}$$

where $\alpha := \sqrt{1 + m^2/4}$ is understood. Then, the divergent part of the on-shell action is given by

$$S_{bulk} + S_{GH} + S_{ct} = -A_i^2(1-\alpha)u^{-4\alpha}, \tag{7.67}$$
$$S_{mct} = (1-\alpha)A_i^2 u^{-4\alpha}, \tag{7.68}$$

and can be canceled.

Diagonal components $\chi_{i=0,1,2,3,4}$ are rather complicated. Their equations of motion are given by

$$(12+m^2)\chi_4 - 3u\chi_4' + m^2(\Gamma - \chi_i) + 3u(\Gamma' - \chi_i') - u^2(\Gamma'' - \chi_i'') = 0 \quad (i = 0,1,2,3), \tag{7.69}$$
$$12\chi_4 + m^2\Gamma + 3u\Gamma' = 0, \tag{7.70}$$

where we have defined $\Gamma := \chi_0 + \chi_1 + \chi_2 + \chi_3$. Summing up (7.69) for $i = 0,1,2,3$, we find

$$4(12+m^2)\chi_4 - 12u\chi_4' + 3m^2\Gamma + 9u\Gamma' - 3u^2\Gamma'' = 0. \tag{7.71}$$

From (7.70) and (7.71), we obtain

$$\Gamma = 0, \quad \chi_4 = 0. \tag{7.72}$$

Then, the equations of motion (7.69) are expressed as

$$-m^2\chi_i - 3u\chi_i' + u^2\chi_i'' = 0 \quad (i = 0,1,2,3), \tag{7.73}$$

and we have their solutions

$$\chi_i(u) = A_i u^{2-2\alpha} + B_i u^{2+2\alpha} \quad (i = 1,2,3), \tag{7.74}$$
$$\chi_0(u) = -(A_1 + A_2 + A_3)u^{2-2\alpha} - (B_1 + B_2 + B_3)u^{2+2\alpha}. \tag{7.75}$$

The action for χ part is, using $\Gamma = 0$,

$$S_{bulk} + S_{GH} + S_{ct} = \frac{1}{4u^4}(\chi_0\chi_0' + \chi_1\chi_1' + \chi_2\chi_2' + \chi_3\chi_3'), \tag{7.76}$$
$$S_{mct} = (1-\alpha)\frac{1}{u^4}(-\chi_0\chi_0 + \chi_1\chi_2 + \chi_2\chi_3 + \chi_3\chi_1). \tag{7.77}$$

We substitute the solutions (7.74) and (7.75), and obtain its divergent part

$$S_{bulk} + S_{GH} + S_{ct} = (1-\alpha)(A_1^2 + A_2^2 + A_3^2 + A_1A_2 + A_2A_3 + A_3A_1)u^{-4\alpha}\Big|_{u=0}, \tag{7.78}$$
$$S_{mct} = -(1-\alpha)(A_1^2 + A_2^2 + A_3^2 + A_1A_2 + A_2A_3 + A_3A_1)u^{-4\alpha}\Big|_{u=0}. \tag{7.79}$$

Therefore, all divergence can be canceled.

This fact supports the validity of the counter term introduced in (7.50) and (7.51).

7.3 Bimetric gravity and the AdS/CFT correspondence

In this section, we apply the AdS/CFT prescription studied in Section 7.2 to bimetric gravity. Our interest is what emerges as a boundary field theory. In Section 7.2, we have considered the case of dRGT massive gravity and obtained a weird result which we do not know how to interpret. Certainly, massive gravity itself is a peculiar theory. It explicitly contains a background metric, which we give outside the theory. In Section 7.2, we have set it to be SAdS-BH, but it has not been determined by an equation within massive gravity. Hence, it seems natural to make the reference metric dynamical and solve the equation of motion for both two metrics. This is nothing but bimetric gravity introduced in Section 3.5. Since bimetric gravity can be regarded as a completion of dRGT massive gravity, we expect that the strange result in massive gravity should be cured in bimetric gravity.

In the following, we denote two metrics in bimetric gravity as g and f, and their induced metrics on the AdS-boundary as γ and ρ respectively. Collecting the results in the previous section, we start with the action given by

$$\begin{aligned} S =& S_{EH}[g] + S_{GH}[\gamma] + S_{ct}[\gamma] \\ &+ S_{EH}[f] + S_{GH}[\rho] + S_{ct}[\rho] \\ &+ S_{int}[g,f] + S_{int,ct}[\gamma,\rho]. \end{aligned} \tag{7.80}$$

For one metric g, we have

$$\begin{aligned} & S_{EH}[g] + S_{GH}[\gamma] + S_{ct}[\gamma] \\ =& \frac{1}{16\pi G_g}\int d^5x\sqrt{-\det g}\big(R[g]-2\Lambda\big) \\ &+ \frac{2}{16\pi G_g}\int_{AdS-bdy} d^4x\sqrt{-\det\gamma}K[\gamma] + \frac{1}{16\pi G_g}\int_{AdS-bdy} d^4x\sqrt{-\det\gamma}\Big(\frac{6}{L}+\cdots\Big), \end{aligned} \tag{7.81}$$

and for the other metric f, we know

$$\begin{aligned} & S_{EH}[f] + S_{GH}[\rho] + S_{ct}[\rho] \\ =& \frac{1}{16\pi G_f}\int d^5x\sqrt{-\det f}\big(R[f]-2\Lambda\big) \\ &+ \frac{2}{16\pi G_f}\int_{AdS-bdy} d^4x\sqrt{-\det\rho}K[\rho] + \frac{1}{16\pi G_f}\int_{AdS-bdy} d^4x\sqrt{-\det\rho}\Big(\frac{6}{L}+\cdots\Big). \end{aligned} \tag{7.82}$$

In bimetric gravity, we can introduce different gravitational constants for two metrics, which we write as G_g and G_f. $R[g]$ is the scalar curvature for g and $R[f]$ is the scalar curvature for f. $K[\gamma]$ and $K[\rho]$ represent the extrinsic curvatures for each metric. In general, cosmological constants for g and f can be different, but we assume that they have the same value Λ and each metric has the same AdS-radius L. We impose this condition in order to take perturbations on the same background for both g and f. The interaction term $S_{int}[g,f]$ is given by

$$S_{int}[g,f] = \frac{2m^2}{16\pi G_g + 16\pi G_f}\int d^5x\sqrt{-\det g}\sum_{n=0}^{5}\beta_n\, e_n\big(\sqrt{g^{-1}f}\big), \tag{7.83}$$

and the counterterm $S_{int,ct}[\gamma,\rho]$ is an extension of the counterterm (7.50) we have added in massive gravity

$$S_{int,ct}[\gamma,\rho] \propto \frac{1}{16\pi G_g + 16\pi G_f}\frac{1}{L}\int_{AdS-bdy} d^4x\sqrt{-\det\gamma}\sum_{n=0}^{4}\beta'_n\, e_n\big(\sqrt{\gamma^{-1}\rho}\big). \tag{7.84}$$

As explained in Section 3.5, we suppose that $S_{int}[g,f]$ vanishes when we set $f=g$. Therefore, we have a solution $g=f=$(a solution in general relativity). In the following, we consider a perturbation $g=\bar{g}+\delta g$ and $f=\bar{f}+\delta f$ around the same background $\bar{g}=\bar{f}=$(SAdS-BH). Thus, the expansion of $S_{int}[g,f]$ and $S_{int,ct}[\gamma,\rho]$ up to the second order in δg and δf is given by

$$S_{int}[g,f] = -\frac{1}{16\pi G_g + 16\pi G_f}\Big(\frac{m^2}{4}\Big)\int d^5x\sqrt{-\det\bar{g}}\Big(\mathrm{Tr}(\delta g-\delta f)^2 - \mathrm{Tr}^2(\delta g-\delta f)\Big), \tag{7.85}$$

$$S_{int,ct}[\gamma,\rho] = -\Big(\frac{1-\alpha}{2L}\Big)\frac{1}{16\pi G_g + 16\pi G_f}\int_{Ads-bdy} d^4x\sqrt{-\det\bar{\gamma}}\Big(\mathrm{Tr}(\delta\gamma-\delta\rho)^2 - \mathrm{Tr}^2(\delta\gamma-\delta\rho)\Big), \tag{7.86}$$

where we have defined $\alpha := \sqrt{1+(mL)^2/4}$. The coefficient $(1-\alpha)$ is adjusted to cancel the leading order divergence.

Now, we take a perturbation such as (7.43)

$$\delta g^{\mu}{}_{\nu} = \bar{g}^{\mu\lambda}\delta g_{\lambda\nu} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&\phi&0&0\\ 0&\phi&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \qquad \phi=\phi(t,u), \tag{7.87}$$

$$\delta f^{\mu}{}_{\nu} = \bar{g}^{\mu\lambda}\delta f_{\lambda\nu} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&\psi&0&0\\ 0&\psi&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \qquad \psi=\psi(t,u). \tag{7.88}$$

The interaction term $S_{int}[g,f]$ is expanded to be

$$S_{int} = -\frac{1}{16\pi G_g + 16\pi G_f}\frac{(mL)^2}{2}\Big(\frac{r_0^4}{L^5}\Big)V_3\int\frac{d\omega}{2\pi}\int_0^1 du\frac{1}{u^5}(\phi_{-\omega}-\psi_{-\omega})(\phi_\omega-\psi_\omega), \tag{7.89}$$

and the bulk action $S_{bulk}=S_{EH}[g]+S_{EH}[f]+S_{int}[g,f]$ is obtained

$$\begin{aligned} &S_{bulk}\\ =&-\Big(\frac{1}{16\pi G_g}+\frac{1}{16\pi G_f}\Big)\Big(\frac{r_0^4}{L^5}\Big)V_4\int_0^1 du\frac{8}{u^5}\\ &+\frac{1}{16\pi G_g}\Big(\frac{r_0^4}{L^5}\Big)V_3\int\frac{d\omega}{2\pi}\int_0^1 du\Big\{\frac{3}{2}\frac{h}{u^3}\phi'_{-\omega}\phi'_\omega + 2\frac{h}{u^3}\phi_{-\omega}\phi''_\omega - \frac{8}{u^4}\phi_{-\omega}\phi'_\omega + \Big(\frac{1}{2u^3h}\Big(\frac{L^2}{r_0}\omega\Big)^2+\frac{4}{u^5}\Big)\phi_{-\omega}\phi_\omega\Big\}\\ &+\frac{1}{16\pi G_f}\Big(\frac{r_0^4}{L^5}\Big)V_3\int\frac{d\omega}{2\pi}\int_0^1 du\Big\{\frac{3}{2}\frac{h}{u^3}\psi'_{-\omega}\psi'_\omega + 2\frac{h}{u^3}\psi_{-\omega}\psi''_\omega - \frac{8}{u^4}\psi_{-\omega}\psi'_\omega + \Big(\frac{1}{2u^3h}\Big(\frac{L^2}{r_0}\omega\Big)^2+\frac{4}{u^5}\Big)\psi_{-\omega}\psi_\omega\Big\}\\ &-\frac{1}{16\pi G_g + 16\pi G_f}\frac{(mL)^2}{2}\Big(\frac{r_0^4}{L^5}\Big)V_3\int\frac{d\omega}{2\pi}\int_0^1 du\frac{1}{u^5}(\phi_{-\omega}-\psi_{-\omega})(\phi_\omega-\psi_\omega). \end{aligned} \tag{7.90}$$

We expand the counterterm

$$S_{int,ct} = -(1-\alpha)\frac{1}{16\pi G_g + 16\pi G_f}\Big(\frac{r_0^4}{L^5}\Big)V_3\int\frac{d\omega}{2\pi}\frac{\sqrt{h}}{u^4}(\phi_{-\omega}-\psi_{-\omega})(\phi_\omega-\psi_\omega)\Big|_{u=0}, \tag{7.91}$$

and including other boundary terms, we find

$$\begin{aligned} S =&\Big(\frac{1}{16\pi G_g}+\frac{1}{16\pi G_f}\Big)\Big(\frac{r_0^4}{L^5}\Big)V_4 \\ &+\frac{1}{16\pi G_g}\Big(\frac{r_0^4}{L^5}\Big)V_3\int\frac{d\omega}{2\pi}\Big\{\Big(-\frac{1}{2}+O[u^4]\Big)\phi_{-\omega}\phi_\omega+\frac{1}{2}\Big(\frac{1}{u^3}-u\Big)\phi_{-\omega}\phi'_\omega\Big\}\Big|_{u=0} \\ &+\frac{1}{16\pi G_f}\Big(\frac{r_0^4}{L^5}\Big)V_3\int\frac{d\omega}{2\pi}\Big\{\Big(-\frac{1}{2}+O[u^4]\Big)\psi_{-\omega}\psi_\omega+\frac{1}{2}\Big(\frac{1}{u^3}-u\Big)\psi_{-\omega}\psi'_\omega\Big\}\Big|_{u=0} \\ &-(1-\alpha)\frac{1}{16\pi G_g+16\pi G_f}\Big(\frac{r_0^4}{L^5}\Big)V_3\int\frac{d\omega}{2\pi}\Big(\frac{1}{u^4}-\frac{1}{2}+O[u^4]\Big)(\phi_{-\omega}-\psi_{-\omega})(\phi_\omega-\psi_\omega)\Big|_{u=0}. \end{aligned} \tag{7.92}$$

S_{int} has no contribution to (7.92) since S_{int} does not contain derivatives, but it has effects on the equation of motion obtained from (7.90). Here, we normalize $\tilde{\phi} := \phi/\sqrt{16\pi G_g}$ and $\tilde{\psi} := \psi/\sqrt{16\pi G_f}$, and introduce new variables

$$\Phi := \frac{\sqrt{16\pi G_f}\tilde{\phi}+\sqrt{16\pi G_g}\tilde{\psi}}{16\pi G_g+16\pi G_f}, \qquad \Psi := \frac{\sqrt{16\pi G_g}\tilde{\phi}-\sqrt{16\pi G_f}\tilde{\psi}}{16\pi G_g+16\pi G_f}. \tag{7.93}$$

Using relations such as $\tilde{\phi}^2+\tilde{\psi}^2 = \Phi^2+\Psi^2$, we notice

$$\begin{aligned} S =&\Big(\frac{1}{16\pi G_g}+\frac{1}{16\pi G_f}\Big)\Big(\frac{r_0^4}{L^5}\Big)V_4 \\ &+\frac{r_0^4}{L^5}V_3\int\frac{d\omega}{2\pi}\Big\{\Big(-\frac{1}{2}+O[u^4]\Big)\Phi_{-\omega}\Phi_\omega+\frac{1}{2}\Big(\frac{1}{u^3}-u\Big)\Phi_{-\omega}\Phi'_\omega\Big\}\Big|_{u=0} \\ &+\frac{r_0^4}{L^5}V_3\int\frac{d\omega}{2\pi}\Big\{\Big(-\frac{1}{2}+O[u^4]\Big)\Psi_{-\omega}\Psi_\omega+\frac{1}{2}\Big(\frac{1}{u^3}-u\Big)\Psi_{-\omega}\Psi'_\omega\Big\}\Big|_{u=0} \\ &-(1-\alpha)\frac{r_0^4}{L^5}V_3\int\frac{d\omega}{2\pi}\Big(\frac{1}{u^4}-\frac{1}{2}+O[u^4]\Big)\Psi_{-\omega}\Psi_\omega\Big|_{u=0}. \end{aligned} \tag{7.94}$$

On the other hand, we obtain the equation of motion for Φ and Ψ from the bulk action (7.90)

$$\Big(\frac{h}{u^3}\Phi'_\omega\Big)'+\Big(\frac{L^2}{r_0}\omega^2\Big)^2\frac{1}{u^3h}\Phi_\omega = 0, \tag{7.95}$$

$$\Big(\frac{h}{u^3}\Psi'_\omega\Big)'-(Lm)^2\frac{1}{u^5}\Psi_\omega+\Big(\frac{L^2}{r_0}\omega^2\Big)^2\frac{1}{u^3h}\Psi_\omega = 0. \tag{7.96}$$

We solve them up to the first order expansion in ω

$$\Phi_\omega(u) = A_\omega + B_\omega(u^4+O[u^8]), \tag{7.97}$$

$$\Psi_\omega(u) = C_\omega\Big\{u^{2-2\alpha}+\frac{1}{4}(1-\alpha)u^{6-2\alpha}+O[u^{10-2\alpha}]\Big\}+D_\omega\Big\{u^{2+2\alpha}+\frac{1}{4}(1+\alpha)u^{6+2\alpha}+O[u^{10+2\alpha}]\Big\}. \tag{7.98}$$

A_ω, B_ω, C_ω and D_ω are $A_\omega := A_0+\omega A_1$, $B_\omega := B_0+\omega B_1$, $C_\omega := C_0+\omega C_1$ and $D_\omega := D_0+\omega D_1$ with constants A_i, B_i, C_i and D_i $(i=0,1)$. Then, the on-shell action is found to be

$$\begin{aligned}S =&\Big(\frac{1}{16\pi G_g}+\frac{1}{16\pi G_f}\Big)\Big(\frac{r_0^4}{L^5}\Big)V_4\\&+\frac{r_0^4}{L^5}V_3\int\frac{d\omega}{2\pi}\Big(-\frac{1}{2}A_{-\omega}A_\omega+2A_{-\omega}B_\omega\Big)+\frac{r_0^4}{L^5}V_3\int\frac{d\omega}{2\pi}(2\alpha C_{-\omega}D_\omega),\end{aligned}\tag{7.99}$$

where we have imposed a condition $0<\alpha<1$ in order to make $O[u^{4-4\alpha}]|_{u=0}$ terms finite.

Now, we have obtained the on-shell action (7.99), but it is written by massless and massive modes Φ and Ψ. The original variables $g_{\mu\nu}$ and $f_{\mu\nu}$ are mixed. In the context of the gravity/fluid correspondence, how to interpret this mixture is not clear. Hence, we proceed with variables $g_{\mu\nu}$ and $f_{\mu\nu}$, namely ϕ and ψ. In addition, we have to consider a boundary condition on the Black Hole horizon to fix constants A, B, C and D. In the massless limit $m=0$, bimetric gravity decouples to a pair of independent general relativity, in which case we should select the ingoing wave condition. Therefore, it seems natural that the solutions for ϕ and ψ should match (7.29) in the massless limit.

When we set $\alpha=1$ $(m^2=0)$, we have

$$\begin{aligned}\phi_\omega =&\sqrt{16\pi G_g}\,\frac{\sqrt{16\pi G_f}\Phi_\omega+\sqrt{16\pi G_g}\Psi_\omega}{\sqrt{16\pi G_g+16\pi G_f}}\\=&\sqrt{16\pi G_g}\,\frac{\sqrt{16\pi G_f}A_\omega+\sqrt{16\pi G_g}C_\omega}{\sqrt{16\pi G_g+16\pi G_f}}+\sqrt{16\pi G_g}\,\frac{\sqrt{16\pi G_f}B_\omega+\sqrt{16\pi G_g}D_\omega}{\sqrt{16\pi G_g+16\pi G_f}}u^4+O[u^8],\end{aligned}\tag{7.100}$$

$$\begin{aligned}\psi_\omega =&\sqrt{16\pi G_f}\,\frac{\sqrt{16\pi G_g}\Phi_\omega-\sqrt{16\pi G_f}\Psi_\omega}{\sqrt{16\pi G_g+16\pi G_f}}\\=&\sqrt{16\pi G_f}\,\frac{\sqrt{16\pi G_g}A_\omega-\sqrt{16\pi G_f}C_\omega}{\sqrt{16\pi G_g+16\pi G_f}}+\sqrt{16\pi G_f}\,\frac{\sqrt{16\pi G_g}B_\omega-\sqrt{16\pi G_f}D_\omega}{\sqrt{16\pi G_g+16\pi G_f}}u^4+O[u^8].\end{aligned}\tag{7.101}$$

Thus, we put

$$\phi_\omega^{(0)}=\sqrt{16\pi G_g}\,\frac{\sqrt{16\pi G_f}A_\omega+\sqrt{16\pi G_g}C_\omega}{\sqrt{16\pi G_g+16\pi G_f}},\quad i\frac{L^2\omega}{4r_0}\phi_\omega^{(0)}=\sqrt{16\pi G_g}\,\frac{\sqrt{16\pi G_f}B_\omega+\sqrt{16\pi G_g}D_\omega}{\sqrt{16\pi G_g+16\pi G_f}},\tag{7.102}$$

$$\psi_\omega^{(0)}=\sqrt{16\pi G_f}\,\frac{\sqrt{16\pi G_g}A_\omega-\sqrt{16\pi G_f}C_\omega}{\sqrt{16\pi G_g+16\pi G_f}},\quad i\frac{L^2\omega}{4r_0}\psi_\omega^{(0)}=\sqrt{16\pi G_f}\,\frac{\sqrt{16\pi G_g}B_\omega-\sqrt{16\pi G_f}D_\omega}{\sqrt{16\pi G_g+16\pi G_f}},\tag{7.103}$$

and we obtain

$$A_\omega=\frac{16\pi G_f\phi_\omega^{(0)}+16\pi G_g\psi_\omega^{(0)}}{16\pi\sqrt{G_gG_f}\,\sqrt{16\pi G_g+16\pi G_f}},\quad B_\omega=\Big(i\frac{L^2\omega}{4r_0}\Big)\frac{16\pi G_f\phi_\omega^{(0)}+16\pi G_g\psi_\omega^{(0)}}{16\pi\sqrt{G_gG_f}\,\sqrt{16\pi G_g+16\pi G_f}},\tag{7.104}$$

$$C_\omega=\frac{(\phi_\omega^{(0)}-\psi_\omega^{(0)})}{\sqrt{16\pi G_g+16\pi G_f}},\quad D_\omega=\Big(i\frac{L^2\omega}{4r_0}\Big)\frac{(\phi_\omega^{(0)}-\psi_\omega^{(0)})}{\sqrt{16\pi G_g+16\pi G_f}}.\tag{7.105}$$

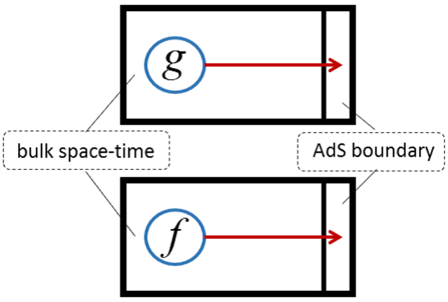

Figure 7.1: non-interacting case

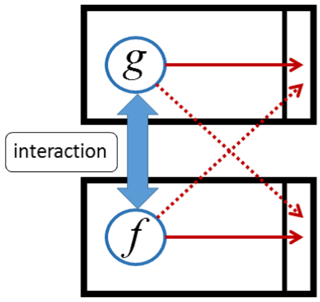

Figure 7.2: interacting case

Substituting these relations into (7.99), we obtain the on-shell action expressed by original variables ϕ and ψ

$$\begin{aligned}S =& \left(\frac{1}{16\pi G_g} + \frac{1}{16\pi G_f}\right)\left(\frac{r_0^4}{L^5}\right)V_4 \\ &+ \frac{1}{16\pi G_g + 16\pi G_f}\left(\frac{r_0^4}{L^5}\right)V_3 \int \frac{d\omega}{2\pi}\Bigg\{ -\frac{1}{2}\left(\frac{G_f}{G_g}\right)\phi^{(0)}_{-\omega}\phi^{(0)}_{\omega} + i\frac{L^2\omega}{2r_0}\left(\frac{G_f}{G_g} + \alpha\right)\phi^{(0)}_{-\omega}\phi^{(0)}_{\omega} \\ &\qquad -\frac{1}{2}\left(\frac{G_g}{G_f}\right)\psi^{(0)}_{-\omega}\psi^{(0)}_{\omega} + i\frac{L^2\omega}{2r_0}\left(\frac{G_g}{G_f} + \alpha\right)\psi^{(0)}_{-\omega}\psi^{(0)}_{\omega} \\ &\qquad -\frac{1}{2}\left(\phi^{(0)}_{-\omega}\psi^{(0)}_{\omega} + \psi^{(0)}_{-\omega}\phi^{(0)}_{\omega}\right) \\ &\qquad + i\frac{L^2\omega}{2r_0}(1-\alpha)\left(\phi^{(0)}_{-\omega}\psi^{(0)}_{\omega} + \psi^{(0)}_{-\omega}\phi^{(0)}_{\omega}\right)\Bigg\}. \end{aligned} \tag{7.106}$$

This on-shell action contains mixed terms such as $\phi\psi$, which suggests that two-component fluid emerges. If the metrics g and f do not interact, we have two independent AdS (bulk)/CFT (boundary) pairs. The fluctuation of g enters into one boundary and becomes a source to generate one field. The fluctuation of f goes into the other boundary and becomes a source of another field (FIG.7.1). For convenience, we call these boundaries as g-boundary and f-boundary. However, if the interaction between two metrics is switched on, perturbations begin to go into not only the original boundary but also the other. For example, the perturbation of metric g enters into f-boundary as well as g-boundary. As a result, two fields are generated on respective boundaries (FIG.7.2).

In this situation, the GKP-Witten relation can be written as

$$\left\langle \exp\left(i\int \phi^{(0)}\mathcal{O}_g + \phi^{(0)}\mathcal{O}_f + \psi^{(0)}\mathcal{Q}_g + \psi^{(0)}\mathcal{Q}_f\right)\right\rangle = \exp\left(iS\left[\phi,\psi|_{u=0} = \phi^{(0)},\psi^{(0)}\right]\right), \tag{7.107}$$

where $\mathcal{O}_g$ and $\mathcal{Q}_g$ are operators on g-boundary, and $\mathcal{O}_f$ and $\mathcal{Q}_f$ are on f-boundary. $\phi^{(0)}$ becomes a source of not only $\mathcal{O}_g$ on g-boundary but also $\mathcal{O}_f$ on f-boundary. $\psi^{(0)}$ becomes a source of $\mathcal{Q}_g$ as well as $\mathcal{Q}_f$. In our setting, these operators are interpreted as energy momentum tensors.

Here, we remember the discussion in the case of general relativity. We considered a perturbation around SAdS-BH and obtained an expectation value of the perturbed energy momentum tensor (7.32) via the AdS/CFT correspondence. On the other hand, we focused on the boundary field theory. We assumed that the boundary space-time was slightly distorted and calculated the linear response of the energy momentum tensor (7.37). We compared (7.32) with (7.37), and read the coefficients.

Now, we proceed in the same way. We have two boundaries, namely g-boundary and f-boundary. Each boundary field theory has some energy momentum tensor though we do not know their concrete formulae. Schematically, we write them as T[g] and T[f]. T[g] is an energy momentum tensor on g-boundary and T[f] is that on f-boundary. Then, we assume that the boundary space-times are slightly distorted from the flat background, and consider the linear response of these energy momentum tensors. We denote the distortion as $\eta_{\mu\nu} \to \eta_{\mu\nu} + \delta g_{\mu\nu}$ and $\eta_{\mu\nu} \to \eta_{\mu\nu} + \delta f_{\mu\nu}$ for g-boundary and f-boundary respectively. It seems natural to think that the response on g-boundary $\delta T[g]$ should consists of only $\delta g_{\mu\nu}$ and that on f-boundary $\delta T[f]$ should composed of only $\delta f_{\mu\nu}$. We seek concrete forms of these linear responses $\delta T[g] \propto \delta g$ and $\delta T[f] \propto \delta f$ from expectation values calculated via the AdS/CFT correspondence. Thus, expectation values on g-boundary $< \mathcal{O}_g >$ and $< \mathcal{Q}_g >$ should contain only the metric g (or ϕ) while $< \mathcal{O}_f >$ and $< \mathcal{Q}_f >$ should contain only the other metric f (or ψ).

Focusing on g-boundary, expectation values of the energy momentum tensors are calculated as

$$< \mathcal{O}_g >= \frac{\delta S}{\delta \phi^{(0)}_{-\omega}}\bigg|_{\psi=0} = -\left(\frac{r_0^4}{L^5}\right)\frac{G_f/G_g}{16\pi G_g + 16\pi G_f}\phi^{(0)}_\omega + i\omega\left(\frac{r_0^3}{L^3}\right)\frac{G_f/G_g+\alpha}{16\pi G_g + 16\pi G_f}\phi^{(0)}_\omega, \quad (7.108)$$

$$< \mathcal{Q}_g >= \frac{\delta S}{\delta \psi^{(0)}_{-\omega}}\bigg|_{\psi=0} = -\left(\frac{r_0^4}{L^5}\right)\frac{1}{16\pi G_g + 16\pi G_f}\phi^{(0)}_\omega + i\omega\left(\frac{r_0^3}{L^3}\right)\frac{1-\alpha}{16\pi G_g + 16\pi G_f}\phi^{(0)}_\omega. \quad (7.109)$$

These formulae and (7.37) have the same form. Thus, we compare them and conclude that we have two-component fluid. The pressure P and the sheer viscosity η of each component are given by

$$P[g]_\phi = \left(\frac{r_0^4}{L^5}\right)\frac{G_f/G_g}{16\pi G_g + 16\pi G_f}, \qquad P[g]_\psi = \left(\frac{r_0^4}{L^5}\right)\frac{1}{16\pi G_g + 16\pi G_f}, \quad (7.110)$$

$$\eta[g]_\phi = \left(\frac{r_0^3}{L^3}\right)\frac{G_f/G_g+\alpha}{16\pi G_g + 16\pi G_f}, \qquad \eta[g]_\psi = \left(\frac{r_0^3}{L^3}\right)\frac{1-\alpha}{16\pi G_g + 16\pi G_f}. \quad (7.111)$$

$P[g]_\phi$ represents the pressure on g-boundary generated by the fluctuation ϕ. We note that the total pressure is $P[g]_\phi + P[g]_\psi = \frac{1}{16\pi G_g}\frac{r_0^4}{L^5}$ which is compatible with the value calculated from the background metric (7.39). We recall that the entropy density on g-boundary is $s[g] = \frac{1}{4G_g}(\frac{r_0}{L})^3$ and calculate the ratios

$$\frac{\eta[g]_\phi}{s[g]} = \left(\frac{1}{4\pi}\right)\frac{G_f/G_g+\alpha}{G_f/G_g+1}, \quad \frac{\eta[g]_\psi}{s[g]} = \left(\frac{1}{4\pi}\right)\frac{1-\alpha}{G_f/G_g+1}. \quad (7.112)$$

If we set $G_g = G_f$, they depend only on graviton's mass

$$\frac{\eta[g]_\phi}{s[g]} = \left(\frac{1}{4\pi}\right)\frac{1+\alpha}{2}, \quad \frac{\eta[g]_\psi}{s[g]} = \left(\frac{1}{4\pi}\right)\frac{1-\alpha}{2}. \quad (7.113)$$

7.4 Summary and discussion

In this chapter, we applied the AdS/CFT correspondence to dRGT massive and bimetric gravity, especially in the first order hydrodynamic limit. Based on the well known case of general relativity, we first try dRGT massive gravity. Then, in contrast to the case of general relativity, we encountered additional divergences. In order to remove them, we added a new counterterm. Besides, we also need to imposed a condition on mass of a graviton. Certainly, we removed the divergences and obtained a finite result but how to interpret it was not clear. The AdS/CFT correspondence suggested that the pressure is zero, which contradicts the value calculated from the background metric. Thus, we further extended the AdS/CFT prescription to the case of bi-metric gravity, expecting this peculiarity is removed. As a result, we found that two-component fluid emerges, and the total pressure restores the background value. We also calculated their sheer viscosity, which is dependent on graviton's mass and ratio of gravitational constants.

However, what we studied in this chapter is only the simplest setting. Further detailed investigation is needed to clarify the features of the boundary field theory. It is worth studying more general perturbations. For example, diagonal perturbations which lead to other properties such as sound waves [51]. Applications to more general multimetric case can be also possible. Besides, the relation between bimetric or multimetric gravity and the deformation of boundary CFTs [46, 47] remains unanswered. We left these issues as future works.

Chapter 8

Conclusion

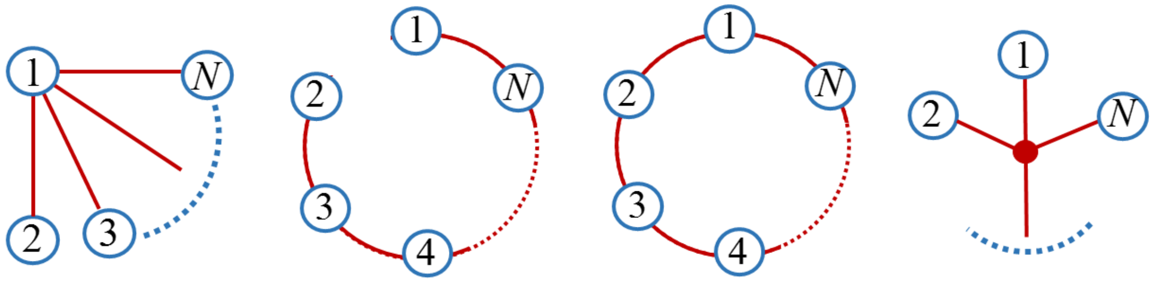

Figure 8.1: The diagrams represent branching node type, chain type, loop type and branching link type interactions.

The main purpose of this thesis is to determine when we can construct a healthy theory for interacting multiple gravitational fields. In general, interaction among gravitational fields generate extra ghost-like degrees of freedom, which we call BD-ghost. Thus, we need to construct a theory which excludes such ghosts and has the right number of degrees of freedom. In describing gravitational fields, we have two kinds of formulations. One of them is expressed by metrics, and the other is written by vielbeins. If there is only one field, these two formulations coincide. However, they do not necessarily overlap when we have multiple kinds of interacting gravitational fields. In any case, basic interaction patters are classified into four categories, namely branching node type, chain type, loop type and branching link type interactions (Fig.8.1). The former two patterns construct tree type interaction (Fig.8.2). Only under the tree type interaction, metric theories and vielbein theories coincide. In this thesis, we have addressed metric theories since vielbein theories are known to be ghost-free. Our strategy is to directly count the total number of degrees of freedom by using the ADM decomposition and the Hamiltonian analysis. However, the non-linear dependence of the lapse and shift obscures the constraint structure. Thus, we have employed the spatial homogeneous ansatz to make the lapse appear linearly. The result of our analysis is that, in a metric formulation, only the tree type interaction can exclude extra ghost degrees of freedom. Loop or branching link structures suffer from BD-ghosts and cannot be allowed.

Though we have determined when we can exclude extra ghost states, this is only for "extra" ones. Therefore, what we have obtained is a necessary condition for a healthy theory. In general settings, we may contain other types of ghost modes in ordinary degrees of freedom, which needs case by case treatment.

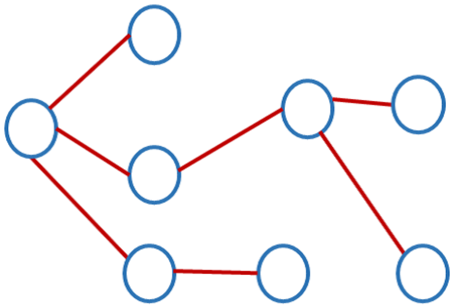

Figure 8.2: The diagrams represent an example of the tree type interaction.

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Appendix A

A note on the Poincaré group and degrees of freedom

In this Appendix, we consider the origin of degrees of freedom for massless and massive fields. We concentrate on the four-dimensional Minkowski space-time.

The number of degrees of freedom of a field is related to the Poincaré group, which is constructed from the translation and the Lorentz transformation. We denote its infinitesimal version as

$$x^\mu \to x'^\mu = x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu, \tag{A.1}$$

where parameters a^μ and $\omega^\mu{}_\nu$ correspond respectively to the translation and the Lorentz transformation. In the following, we raise or lower space-time indices by the Minkowski metric $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$, and a property $\omega^{\mu\nu} = -\omega^{\nu\mu}$ is understood. Under a transformation (A.1), a field $\Phi_A(x)$ transforms as $\Phi_A(x) \to \Phi'_A(x')$. In general, we can write it as $\Phi_A(x) = \Lambda_A{}^B \Phi'_B(x')$. A spin-1 vector field is a case of $A = \mu$ and $\Lambda_\mu{}^\nu = \frac{\partial x'^\nu}{\partial x^\mu}$ while a spin-2 tensor field has double index $A = (\mu\nu)$ and $\Lambda_{\mu\nu}{}^{\lambda\rho} = \frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu}$. In an infinitesimal transformation, we can approximate this matrix as $\Lambda_A{}^B \approx \delta_A^B - \frac{i}{2}\omega^{\mu\nu}(S_{\mu\nu})_A{}^B$. Explicit formulae are given by

$$(S_{\alpha\beta})_\mu{}^\nu = -i\big(\eta_{\alpha\mu}\delta^\nu_\beta - \eta_{\beta\mu}\delta^\nu_\alpha\big) \tag{A.2}$$

for a vector field, and

$$(S_{\alpha\beta})_{\mu\nu}{}^{\lambda\rho} = -i\big(\eta_{\alpha\nu}\delta^{(\lambda}_\mu\delta^{\rho)}_\beta + \eta_{\alpha\mu}\delta^{(\lambda}_\beta\delta^{\rho)}_\nu - \eta_{\beta\nu}\delta^{(\lambda}_\mu\delta^{\rho)}_\alpha - \eta_{\beta\mu}\delta^{(\lambda}_\alpha\delta^{\rho)}_\nu\big) \tag{A.3}$$

for a tensor field. The symbol $(\cdots)$ means symmetrization divided by the number of the elements. In any case, we obtain

$$\Phi'_A(x) - \Phi_A(x) = \big(-ia^\mu P_\mu \delta^B_A + \frac{i}{2}\omega^{\mu\nu}(M_{\mu\nu})_A{}^B\big)\Phi_B(x), \tag{A.4}$$

where we have defined

$$P_\mu := -i\partial_\mu, \quad (M_{\mu\nu})_A{}^B := (x_\mu P_\nu - x_\nu P_\mu)\delta^B_A + (S_{\mu\nu})_A{}^B. \tag{A.5}$$

They construct the generators of the Poincaré group. We regard contractions on indices A, B as matrix products, and find the following commutation relations

$$[P_\mu, P_\nu] = 0, \tag{A.6}$$

$$[M_{\mu\nu}, P_\alpha] = i\big(\eta_{\mu\alpha}P_\nu - \eta_{\alpha\nu}P_\mu\big), \tag{A.7}$$

$$[M_{\mu\nu}, M_{\alpha\beta}] = i\big(\eta_{\mu\alpha}M_{\nu\beta} + \eta_{\nu\beta}M_{\mu\alpha} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta}\big). \tag{A.8}$$

Note that $P^2 := P^\mu P_\mu$ commutes with all of the generators and is a casimir operator. The poincaré group has another casimir operator called the Pauli-Lubanski spin vector $W^\mu := \epsilon^{\mu\nu\lambda\rho} P_\nu M_{\lambda\rho}$. Representations of the Poincaré group are classified by these two casimir operators, which correspond to mass and spin respectively. $P^2 = 0$ means a massless case while $P^2 = -m^2$ stands for a massive case. $P^2 = m^2$ is a tachyon case and omitted as an unphysical situation. Since we know that momentum operators P^μ commute with each other, they can be diagonalized simultaneously. We write their eigenvalues as p^μ, but states with $p^0 < 0$ are discarded due to their negative energy.

Now, we assume that we have some large state space and need to find an irreducible representation. In general, irreducible states are interpreted as one particle states. Here, the representation of a transformation L is expressed as $U(L)$. A convenient way to classify representations for the Lorentz group is to use a notion of little group. Generators P_μ and $M_{\mu\nu}$ can be decomposed into time and spatial components as $H := P^0$, $J_i := \epsilon_{ijk} M_{jk}$ and $K_i := M_{0i}$, and they satisfy

$$[P_i, P_j] = 0, \quad [J_i, P_j] = i\epsilon_{ijk} P^k, \quad [K_i, P_j] = i\delta_{ij} H, \tag{A.9}$$

$$[P_i, H] = 0, \quad [J_i, H] = 0, \quad [K_i, H] = iP_i, \tag{A.10}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{A.11}$$

In a massive case $p^2 = -m^2$, we can chose a rest frame $p^\mu = p^{(0)\mu} := (m, 0, 0, 0)$. A general momentum p^μ can be restored by a Lorentz transformation $p = B(p)p^{(0)}$. Here, transformations which do not change the fixed momentum $p^{(0)}$ makes a group called little group. In this case, little group is composed of spatial rotations, whose generators are angular momenta J_1, J_2 and J_3. As is usual, we can simultaneously diagonalize $J_1^2 + J_2^2 + J_3^2$ and J_3, which we write $J_1^2 + J_2^2 + J_3^2 = s(s + 1)$ and $J_3 = n$ $(n = -s, -s + 1, ..., s - 1, s)$. Thus, states with $p^{(0)\mu}$ are labeled by spin s and its J_3 component $n = -s, ..., s$, which we denote by $|p^{(0)}, s, n\rangle$. These states span a subspace invariant under spatial rotations, and a rotation R is represented as

$$U(R)|p^{(0)}, s, n\rangle = \sum_{n'} D(R)_{n.n'} |p^{(0)}, s, n'\rangle, \tag{A.12}$$

where $D(R)_{n,n'}$ is the representation matrix for R. Then, we define states outside of this subspace as

$$|p, s, n\rangle := U(B(p))|p^{(0)}, s, n\rangle. \tag{A.13}$$

In fact, these states construct the irreducible representation of the Poincaré group. This is because an arbitrary Lorentz transformation L is represented as

$$\begin{aligned} U(L)|p, s, n\rangle = U(LB(p))|p^{(0)}, s, n\rangle =& U(B(Lp))U(B^{-1}(Lp)LB(p))|p^{(0)}, s, n\rangle \\ =& \sum_{n'} D(B^{-1}(Lp)LB(p))_{n.n'} U(B(Lp))|p^{(0)}, s, n'\rangle \\ =& \sum_{n'} D(B^{-1}(Lp)LB(p))_{n.n'} |Lp, s, n'\rangle, \end{aligned} \tag{A.14}$$

where we have used the fact that $B^{-1}(Lp)LB(p)$ does not change $p^{(0)}$ and thus it is interpreted as a spatial rotation. Therefore, states of a massive field are labeled by mass m, spin s and components $n = -s, ..., s$. A massive spin-1 (s=1) field has three ($n = -1, 0, 1$) degrees of freedom while a massive spin-2 (s=2) field has five $n = -2, -1, 0, 1, 2$ degrees of freedom. If we use $p^{(0)} = (m, 0, 0, 0)$, the Pauli-Lubanski spin vector becomes $W_0 = 0$ and $W_i = mJ_i$.

In a massless case, we cannot chose a rest frame, but we can chose such a frame $p^\mu = p'^{(0)\mu} := (1, 0, 0, 1)$. The original momentum p^μ is restored by a Lorentz transformation $p = B'(p)p'^{(0)}$. An infinitesimal Lorentz transformation which does not change $p'^{(0)}$ is specified by $\omega^\mu{}_\nu p'^{(0)\nu} = 0$. Thus, a condition $\omega^{\mu 0} = \omega^{\mu 3}$ corresponds to the little group, and we find

$$\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} = i\Big(\omega^{10}\big(M_{10} + M_{13}\big) + \omega^{20}\big(M_{20} + M_{23}\big) + \omega^{12}M_{12}\Big). \tag{A.15}$$

We notice that generators of the little group are $T_1 := -M_{10} - M_{13} = K_1 + J_2$, $T_2 := -M_{20} - M_{23} = K_2 - J_1$ and $J_3 = M_{12}$. Here, we define $T_+ := T_1 + iT_2$ and $T_- := T_1 - iT_2$. Then, commutation relations for these generators can be read as

$$[J_3, T_+] = T_+, \quad [J_3, T_-] = -T_-, \quad [T_+, T_-] = 0. \tag{A.16}$$

The above algebra has a finite dimensional representation

$$J_3|p'^{(0)}, \lambda\rangle = \lambda|p'^{(0)}, \lambda\rangle, \quad T_+|p'^{(0)}, \lambda\rangle = 0, \quad T_-|p'^{(0)}, \lambda\rangle = 0. \tag{A.17}$$

Other types of representations are continuously infinite dimensional and discarded as unphysical. This is because (A.16) means J_3 rotates T_+ and T_-, namely we have $e^{i\theta J_3}T_+e^{-i\theta J_3} = e^{i\theta}T_+$ and $e^{i\theta J_3}T_-e^{-i\theta J_3} = e^{-i\theta}T_-$. We can extend the states (A.17) to more general states

$$|p, \lambda\rangle := U(B'(p))|p'^{(0)}, \lambda\rangle \tag{A.18}$$

and construct the irreducible representation. This argument is the same as that in a massive case. Under the condition $p^\mu = p'^{(0)\mu}$, the Pauli-Lubanski spin vector satisfies $W^2 = W^\mu W_\mu \propto T_+T_-$ and can be interpreted as $W^2 = 0$. Besides, we immediately see $W^\mu P_\mu = 0$ from the definition of W^μ. Hence, W^μ must be proportional to p^μ. The proportional coefficient is nothing but λ, namely we have $W^\mu = 2\lambda p^\mu$, which says

$$\lambda = \frac{1}{2}\frac{W^0}{p^0} = \frac{\vec{p}\cdot\vec{J}}{\sqrt{\vec{p}\cdot\vec{p}}}. \tag{A.19}$$

An eigenvalue λ represents helicity of the field. In general, modes with opposite helicity λ and $-\lambda$ are related via spatial reversal. When the theory contains a symmetry about spatial reversal, opposite helicity modes are interpreted as two states of the same field. Theories for spin-1 electromagnetic field or spin-2 gravitational field have this symmetry. Therefore, we conclude that massless spin-1 or spin-2 field are composed of two degrees of freedom, which corresponds to helicity ± 1 and ± 2 modes respectively.

Appendix B

Linearization of the Einstein-Hilbert action

In this Appendix, we derive the action for linearized general relativity from the Einstein-Hilbert action

$$S_{EH}[g] = \frac{1}{16\pi G}\int d^D x\sqrt{-\det g}(R[g]-2\Lambda). \tag{B.1}$$

For a systematic calculation, we assume that we have two metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. We denote their Christoffel symbols as $\Gamma^{\mu}{}_{\nu\lambda}$ and $\bar{\Gamma}^{\mu}{}_{\nu\lambda}$ respectively. We also denote their covariant derivatives as ∇_μ and $\bar{\nabla}_\mu$. Since the difference of the Christoffel symbols plays an important role, we define

$$C^{\mu}{}_{\nu\lambda} := \Gamma^{\mu}{}_{\nu\lambda} - \bar{\Gamma}^{\mu}{}_{\nu\lambda}. \tag{B.2}$$

These two covariant derivatives operate on the metric $g_{\mu\nu}$ in the following way

$$\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^{\rho}{}_{\mu\nu}g_{\rho\lambda} - \Gamma^{\rho}{}_{\mu\lambda}g_{\nu\rho}, \tag{B.3}$$
$$\bar{\nabla}_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \bar{\Gamma}^{\rho}{}_{\mu\nu}g_{\rho\lambda} - \bar{\Gamma}^{\rho}{}_{\mu\lambda}g_{\nu\rho}, \tag{B.4}$$

from which we obtain

$$\nabla_\mu g_{\nu\lambda} - \bar{\nabla}_\mu g_{\nu\lambda} = -C^{\rho}{}_{\mu\nu}g_{\rho\lambda} - C^{\rho}{}_{\mu\lambda}g_{\nu\rho}. \tag{B.5}$$

In this formula, we set $\nabla_\mu g_{\nu\lambda} = 0$ and take permutations of indices (μ, ν, λ)

$$\begin{aligned}
\bar{\nabla}_\mu g_{\nu\lambda} &= C_{\lambda\mu\nu} + C_{\nu\mu\lambda}, \\
\bar{\nabla}_\nu g_{\lambda\mu} &= C_{\mu\nu\lambda} + C_{\lambda\nu\mu}, \\
\bar{\nabla}_\lambda g_{\mu\nu} &= C_{\nu\lambda\mu} + C_{\mu\lambda\nu},
\end{aligned} \tag{B.6}$$

where we have defined $C_{\lambda\mu\nu} := g_{\lambda\rho}C^{\rho}{}_{\mu\nu}$. We can solve the equations (B.6) for $C^{\lambda}{}_{\mu\nu}$ as

$$C^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\big(\bar{\nabla}_\mu g_{\rho\nu} + \bar{\nabla}_\nu g_{\mu\rho} - \bar{\nabla}_\rho g_{\mu\nu}\big). \tag{B.7}$$

Here, we remember the definition of the curvature tensors. The curvature tensors for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ can be defined by the following formulae, with an arbitrary vector u_μ,

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)u_\lambda =: R_{\mu\nu\lambda}{}^{\rho}u_\rho, \tag{B.8}$$
$$(\bar{\nabla}_\mu\bar{\nabla}_\nu - \bar{\nabla}_\nu\bar{\nabla}_\mu)u_\lambda =: \bar{R}_{\mu\nu\lambda}{}^{\rho}u_\rho. \tag{B.9}$$

The difference of these two curvatures is obtained when we express (B.8) by use of $\bar{\nabla}$. We can rewrite the covariant derivative ∇ by $\bar{\nabla}$ in such a way

$$\nabla_\mu u_\nu = \partial_\mu u_\nu - \Gamma^\lambda{}_{\mu\nu} u_\lambda = \bar{\nabla}_\mu - C^\lambda{}_{\mu\nu} u_\lambda. \tag{B.10}$$

Repeating such calculations, we can find

$$\begin{aligned}&(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)u_\lambda - (\bar{\nabla}_\mu\bar{\nabla}_\nu - \bar{\nabla}_\nu\bar{\nabla}_\mu)u_\lambda \\ =& -(\bar{\nabla}_\mu C^\rho{}_{\nu\lambda} - \bar{\nabla}_\nu C^\rho{}_{\mu\lambda})u_\rho + (C^\sigma{}_{\mu\lambda}C^\rho{}_{\nu\sigma} - C^\sigma{}_{\nu\lambda}C^\rho{}_{\mu\sigma})u_\rho.\end{aligned} \tag{B.11}$$

Hence, we obtain an important formula

$$R_{\mu\nu\lambda}{}^\rho = \bar{R}_{\mu\nu\lambda}{}^\rho - 2\bar{\nabla}_{[\mu}C^\rho{}_{\nu]\lambda} + 2C^\sigma{}_{\lambda[\mu}C^\rho{}_{\nu]\sigma}, \tag{B.12}$$

where $[\cdots]$ means antisymmetrization, for example $a_{[\mu}b_{\nu]} = \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu)$, divided by the factorial of the total number of the elements. We contract (B.12) with δ^ν_ρ and also obtain a formula for the difference of the Ricci tensors

$$R_{\mu\lambda} = \bar{R}_{\mu\lambda} - 2\bar{\nabla}_{[\mu}C^\rho{}_{\rho]\lambda} + 2C^\sigma{}_{\lambda[\mu}C^\rho{}_{\rho]\sigma}. \tag{B.13}$$

Thus far, we have not assumed any relation between two metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, but our present purpose is to obtain the linearized action of general relativity. Therefore, we assume that $\bar{g}_{\mu\nu}$ is a background metric and $g_{\mu\nu}$ is composed of the background plus fluctuation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}. \tag{B.14}$$

Then, we can obtain the perturbed curvature tensor for any order through the expansion of $C^\lambda{}_{\mu\nu}$ by $\delta g_{\mu\nu}$. For our purpose, it is enough to calculate up to the second. Using $\bar{\nabla}_\mu \bar{g}_{\nu\lambda} = 0$, the formula for $C^\lambda{}_{\mu\nu}$ (B.7) is now written as

$$C^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\big(\bar{\nabla}_\mu \delta g_{\rho\nu} + \bar{\nabla}_\nu \delta g_{\mu\rho} - \bar{\nabla}_\rho \delta g_{\mu\nu}\big), \tag{B.15}$$

from which we see that the higher order expansion comes from the inverse metric $g^{\mu\nu}$. In the following, we denote the inverse of the background metric $\bar{g}_{\mu\nu}$ as $\bar{g}^{\mu\nu}$, namely $\bar{g}^{\mu\lambda}\bar{g}_{\lambda\nu} = \delta^\mu{}_\nu$. The inverse of the metric $g_{\mu\nu}$ is defined by the formula $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu{}_\nu$, and we can calculate it order by order. We expand the inverse $g^{\mu\nu}$ as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + g^{\mu\nu}_{(1)} + g^{\mu\nu}_{(2)} + O[\delta^3], \tag{B.16}$$

and substitute it into the definition $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu{}_\nu$

$$\big(\bar{g}^{\mu\lambda} + g^{\mu\lambda}_{(1)} + g^{\mu\lambda}_{(2)} + O[\delta^3]\big)\big(\bar{g}_{\lambda\nu} + \delta g_{\lambda\nu}\big) = \delta^\mu{}_\nu. \tag{B.17}$$

We determine $g^{\mu\nu}_{(1)}$ and $g^{\mu\nu}_{(2)}$ perturbatively. The first order term is determined as $g^{\mu\nu}_{(1)} = -\bar{g}^{\mu\lambda}\bar{g}^{\nu\rho}\delta g_{\lambda\rho}$. If we perform index manipulations by the background metric $\bar{g}_{\mu\nu}$ and its inverse $\bar{g}^{\mu\nu}$, we can write $g^{\mu\nu}_{(1)}$ in a simple form $g^{\mu\nu}_{(1)} = -\delta g^{\mu\nu}$. We apply this index rule throughout this chapter. In the same way, the next second order term is determined as $g^{\mu\nu}_{(2)} = (\delta g^{\mu\rho})(\delta g_{\rho\lambda})\bar{g}^{\lambda\nu}$. Then, we can write the inverse metric $g^{\mu\nu}$ as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \delta g^{\mu\nu} + (\delta g^{\mu\rho})(\delta g_{\rho\lambda})\bar{g}^{\lambda\nu} + O[\delta^3]. \tag{B.18}$$

The action (B.1) contains the determinant of the metric $g_{\mu\nu}$. Thus, we also need to expand the determinant

$$\begin{aligned}\det(g..) =& \det(\bar{g}.. + \delta g..)\\ =&(\det \bar{g}..)\det(\mathbf{1} + \bar{g}^{-1}\delta g)\\ =&(\det \bar{g}..)\Big[1 + \delta g^{\mu}{}_{\mu} + \frac{1}{2}\big(\delta g^{\mu}{}_{\mu}\delta g^{\nu}{}_{\nu} - \delta g^{\mu}{}_{\nu}\delta g^{\nu}{}_{\mu}\big) + O[\delta^3]\Big],\end{aligned} \tag{B.19}$$

where we should notice $\delta g^{\mu}{}_{\nu} := \bar{g}^{\mu\lambda}\delta g_{\lambda\nu}$. Applying the Taylor expansion $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O[x^3]$, we find

$$\sqrt{-\det g..} = \sqrt{-\det \bar{g}..}\Big[1 + \frac{1}{2}\delta g^{\mu}{}_{\mu} + \frac{1}{8}\delta g^{\mu}{}_{\mu}\delta g^{\nu}{}_{\nu} - \frac{1}{4}\delta g^{\mu}{}_{\nu}\delta g^{\nu}{}_{\mu} + O[\delta^3]\Big]. \tag{B.20}$$

From these formulae, we can obtain the action for linearized general relativity. In the following, we denote $\delta g_{\mu\nu}$ as $h_{\mu\nu}$ for notational simplicity.

Substituting (B.18) into (B.15), we obtain the perturbed $C^{\lambda}{}_{\mu\nu}$

$$C^{\lambda(0)}_{\ \mu\nu} = 0, \tag{B.21}$$

$$C^{\lambda(1)}_{\ \mu\nu} = \frac{1}{2}\bar{g}^{\lambda\rho}\big(\bar{\nabla}_{\mu}h_{\rho\nu} + \bar{\nabla}_{\nu}h_{\mu\rho} - \bar{\nabla}_{\rho}h_{\mu\nu}\big), \tag{B.22}$$

$$C^{\lambda(2)}_{\ \mu\nu} = -\frac{1}{2}h^{\lambda\rho}\big(\bar{\nabla}_{\mu}h_{\rho\nu} + \bar{\nabla}_{\nu}h_{\mu\rho} - \bar{\nabla}_{\rho}h_{\mu\nu}\big). \tag{B.23}$$

We have attached the subscripts $(0), (1), ...$ to represent the order of the expansion. Then, the Ricci tensor (B.13) is calculated order by order

$$R^{(0)}_{\mu\lambda} = \bar{R}^{(0)}_{\mu\nu}, \tag{B.24}$$

$$R^{(1)}_{\mu\lambda} = \frac{1}{2}\bar{g}^{\alpha\beta}\big(\bar{\nabla}_{\alpha}\bar{\nabla}_{\mu}h_{\beta\lambda} - \bar{\nabla}_{\mu}\bar{\nabla}_{\lambda}h_{\alpha\beta} + \bar{\nabla}_{\alpha}\bar{\nabla}_{\lambda}h_{\mu\beta} - \bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}h_{\mu\lambda}\big), \tag{B.25}$$

$$\begin{aligned}R^{(2)}_{\mu\lambda} =& \frac{1}{2}\bar{\nabla}_{\mu}(h^{\alpha\beta}\bar{\nabla}_{\lambda}h_{\alpha\beta}) + \frac{1}{2}\bar{\nabla}_{\alpha}(h^{\alpha\beta}\bar{\nabla}_{\beta}h_{\mu\lambda}) - \frac{1}{2}\bar{\nabla}_{\alpha}(h^{\alpha\beta}\bar{\nabla}_{\mu}h_{\beta\lambda}) - \frac{1}{2}\bar{\nabla}_{\alpha}(h^{\alpha\beta}\bar{\nabla}_{\lambda}h_{\beta\mu})\\ &- \frac{1}{4}(\bar{\nabla}_{\mu}h^{\alpha\beta})(\bar{\nabla}_{\lambda}h_{\alpha\beta}) + \frac{1}{2}(\bar{\nabla}^{\beta}h_{\alpha\lambda})(\bar{\nabla}_{\beta}h_{\mu}{}^{\alpha}) - \frac{1}{2}(\bar{\nabla}^{\beta}h_{\alpha\lambda})(\bar{\nabla}^{\alpha}h_{\beta\mu})\\ &+ \frac{1}{4}(\bar{\nabla}_{\lambda}h_{\alpha\mu})(\bar{\nabla}^{\alpha}h^{\beta}{}_{\beta}) + \frac{1}{4}(\bar{\nabla}_{\mu}h_{\alpha\lambda})(\bar{\nabla}^{\alpha}h^{\beta}{}_{\beta}) - \frac{1}{4}(\bar{\nabla}_{\alpha}h_{\lambda\mu})(\bar{\nabla}^{\alpha}h^{\beta}{}_{\beta}).\end{aligned} \tag{B.26}$$

We proceed to calculate the scalar curvature $R = g^{\mu\lambda}R_{\mu\lambda}$. The zeroth and the first order terms are given by

$$R^{(0)} = \bar{g}^{\mu\lambda}R^{(0)}_{\mu\lambda} = \bar{R}, \tag{B.27}$$

$$R^{(1)} = \bar{g}^{\mu\lambda}R^{(1)}_{\mu\lambda} + g^{\mu\lambda}_{(1)}R^{(0)}_{\mu\lambda} = \bar{\nabla}^{\mu}\bar{\nabla}^{\lambda}h_{\mu\lambda} - \bar{\nabla}^{\mu}\bar{\nabla}_{\mu}h^{\lambda}{}_{\lambda} - h^{\mu\lambda}\bar{R}_{\mu\lambda}. \tag{B.28}$$

The second order term is composed of three elements

$$R^{(2)} = \bar{g}^{\mu\lambda}R^{(2)}_{\mu\lambda} + g^{\mu\lambda}_{(1)}R^{(1)}_{\mu\lambda} + g^{\mu\lambda}_{(2)}R^{(0)}_{\mu\lambda}, \tag{B.29}$$

and each of them is

$$
\begin{aligned}
\bar{g}^{\mu\lambda} R^{(2)}_{\mu\lambda} =& \frac{1}{4}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\alpha h^{\mu\nu}) - \frac{1}{2}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\nu h^{\alpha\mu}) + \frac{1}{2}(\bar{\nabla}_\mu h^\alpha{}_\alpha)(\bar{\nabla}_\nu h^{\mu\nu}) - \frac{1}{4}(\bar{\nabla}^\alpha h^\mu{}_\mu)(\bar{\nabla}^\alpha h^\nu{}_\nu) \\
&+ \frac{1}{2}\bar{\nabla}_\mu\big(h^{\alpha\beta}\bar{\nabla}^\mu h_{\alpha\beta} - 2h^{\mu\alpha}\bar{\nabla}^\beta h_{\alpha\beta} + h^{\mu\alpha}\bar{\nabla}_\alpha h^\beta{}_\beta\big), \qquad \text{(B.30)} \\
\bar{g}^{\mu\lambda}_{(1)} R^{(1)}_{\mu\lambda} =& (\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\nu h^{\alpha\mu}) - \frac{1}{2}(\bar{\nabla}_\mu h^\alpha{}_\alpha)(\bar{\nabla}_\nu h^{\mu\nu}) - \frac{1}{2}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\alpha h^{\mu\nu}) \\
&+ \frac{1}{2}\bar{\nabla}_\mu\big(-2h^{\alpha\beta}\bar{\nabla}_\alpha h^\mu{}_\beta + h^{\mu\alpha}\bar{\nabla}_\alpha h^\beta{}_\beta + h^{\alpha\beta}\bar{\nabla}^\mu h_{\alpha\beta}\big), \qquad \text{(B.31)} \\
\bar{g}^{\mu\lambda}_{(2)} R^{(0)}_{\mu\lambda} =& \bar{R}_{\mu\lambda} h^{\mu\alpha} h^\lambda{}_\alpha. \qquad \text{(B.32)}
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
R^{(2)} =& -\frac{1}{4}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\alpha h^{\mu\nu}) + \frac{1}{2}(\bar{\nabla}_\alpha h_{\mu\nu})(\bar{\nabla}^\nu h^{\alpha\mu}) - \frac{1}{4}(\bar{\nabla}^\alpha h^\mu{}_\mu)(\bar{\nabla}_\alpha h^\nu{}_\nu) + \bar{R}_{\mu\nu} h^{\mu\alpha} h^\nu{}_\alpha \\
&+ \bar{\nabla}_\mu\big(h^{\alpha\beta}\bar{\nabla}^\mu h_{\alpha\beta} - h^{\mu\alpha}\bar{\nabla}^\beta h_{\alpha\beta} + h^{\mu\alpha}\bar{\nabla}_\alpha h^\beta{}_\beta - h^{\alpha\beta}\bar{\nabla}_\alpha h^\mu{}_\beta\big). \qquad \text{(B.33)}
\end{aligned}
$$

Since our final purpose is to obtain the perturbed Lagrangian density $\sqrt{-\det g}(R - 2\Lambda)$, we must continue to calculate

$$
\begin{aligned}
\Big(\sqrt{-\det g}(R-2\Lambda)\Big)^{(0)} =& \sqrt{-\det g}^{(0)}(R^{(0)} - 2\Lambda) = \sqrt{-\det \bar{g}}(\bar{R} - 2\Lambda), \qquad \text{(B.34)} \\
\Big(\sqrt{-\det g}(R-2\Lambda)\Big)^{(1)} =& \sqrt{-\det g}^{(0)} R^{(1)} + \sqrt{-\det g}^{(1)} R^{(0)} - \sqrt{-\det g}^{(1)} \cdot 2\Lambda \\
=& \sqrt{-\det \bar{g}}\Big(\bar{\nabla}^\mu\big(\bar{\nabla}^\lambda h_{\mu\lambda} - \bar{\nabla}_\mu h^\lambda{}_\lambda\big) - h^{\mu\lambda}\big(\bar{R}_{\mu\lambda} - \frac{1}{2}\bar{g}_{\mu\lambda}\bar{R} + \Lambda\bar{g}_{\mu\lambda}\big)\Big), \qquad \text{(B.35)}
\end{aligned}
$$

and

$$
\Big(\sqrt{-\det g}(R-2\Lambda)\Big)^{(2)} = \sqrt{-\det g}^{(0)} R^{(2)} + \sqrt{-\det g}^{(1)} R^{(1)} - \sqrt{-\det g}^{(2)} (R - 2\Lambda)^{(0)}, \qquad \text{(B.36)}
$$

whose elements are

$$
\begin{aligned}
\sqrt{-\det g}^{(0)} R^{(2)} =& \sqrt{-\det \bar{g}}\, R^{(2)}, \qquad \text{(B.37)} \\
\sqrt{-\det g}^{(1)} R^{(1)} =& \sqrt{-\det \bar{g}}\Big[-\frac{1}{2}(\bar{\nabla}^\mu h^\alpha{}_\alpha)(\bar{\nabla}^\lambda h_{\mu\lambda}) + \frac{1}{2}(\bar{\nabla}^\mu h^\alpha{}_\alpha)(\bar{\nabla}_\mu h^\beta{}_\beta) - \frac{1}{2} h^\alpha{}_\alpha h^{\mu\lambda}\bar{R}_{\mu\lambda} \\
&+ \frac{1}{2}\bar{\nabla}^\mu\big(h^\alpha{}_\alpha \bar{\nabla}^\lambda h_{\lambda\mu} - h^\alpha{}_\alpha \bar{\nabla}_\mu h^\beta{}_\beta\big)\Big], \qquad \text{(B.38)} \\
\sqrt{-\det g}^{(2)} (R - 2\Lambda)^{(0)} =& \sqrt{-\det \bar{g}}\,\Big(\frac{1}{8} h^\alpha{}_\alpha h^\beta{}_\beta - \frac{1}{4} h^\alpha{}_\beta h^\beta{}_\alpha\Big)(\bar{R} - 2\Lambda). \qquad \text{(B.39)}
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\Big(\sqrt{-\det g}(R-2\Lambda)\Big)^{(2)}/\sqrt{-\det \bar{g}} \\
=&\frac{1}{4}\Big(h^{\alpha}{}_{\beta}h^{\beta}{}_{\alpha}-\frac{1}{2}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}\Big)(\bar{R}-2\Lambda)+\Big(\bar{R}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{R}+\Lambda\bar{g}_{\mu\nu}\Big)\big(h^{\mu\alpha}h_{\alpha}{}^{\nu}-\frac{1}{2}h^{\alpha}{}_{\alpha}h^{\mu\nu}\big) \\
&-\frac{1}{4}(\bar{\nabla}_{\alpha}h_{\mu\nu})(\bar{\nabla}^{\alpha}h^{\mu\nu})+\frac{1}{2}(\bar{\nabla}_{\alpha}h_{\mu\nu})(\bar{\nabla}^{\nu}h^{\alpha\mu})+\frac{1}{4}(\bar{\nabla}^{\mu}h^{\alpha}{}_{\alpha})(\bar{\nabla}_{\mu}h^{\beta}{}_{\beta})-\frac{1}{2}(\bar{\nabla}^{\mu}h^{\alpha}{}_{\alpha})(\bar{\nabla}^{\lambda}h_{\mu\lambda}) \\
&+\bar{\nabla}_{\mu}\Big(h^{\alpha\beta}\bar{\nabla}^{\mu}h_{\alpha\beta}-h^{\mu\alpha}\bar{\nabla}^{\beta}h_{\alpha\beta}+h^{\mu\alpha}\bar{\nabla}_{\alpha}h^{\beta}{}_{\beta}-h^{\alpha\beta}\bar{\nabla}_{\alpha}h^{\mu}{}_{\beta}+\frac{1}{2}h^{\beta}{}_{\beta}\bar{\nabla}^{\alpha}h^{\mu}{}_{\alpha}-\frac{1}{2}h^{\alpha}{}_{\alpha}\bar{\nabla}^{\mu}h^{\beta}{}_{\beta}\Big).
\end{aligned}
\tag{B.40}
$$

We can simplify the above formula by use of the Einstein equation for the back ground metric

$$
\bar{R}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{R}+\Lambda\bar{g}_{\mu\nu}=0, \tag{B.41}
$$

and its contracted form

$$
\Lambda=\frac{D-2}{2D}\bar{R}. \tag{B.42}
$$

Eventually, we obtain the action for linearized general relativity

$$
S_{EH}=\frac{1}{16\pi G}\int d^{D}x\mathcal{L}^{(2)}_{EH}, \tag{B.43}
$$

where the Lagrangian density $\mathcal{L}^{(2)}_{EH}$ is given by

$$
\begin{aligned}
\frac{\mathcal{L}^{(2)}_{EH}}{\sqrt{-\det\bar{g}}}=&\frac{2}{D}\bar{R}+\bar{\nabla}_{\mu}\big(\bar{\nabla}_{\lambda}h^{\mu\lambda}-\bar{\nabla}^{\mu}h^{\lambda}{}_{\lambda}\big) \\
&-\frac{1}{4}(\bar{\nabla}_{\alpha}h_{\mu\nu})(\bar{\nabla}^{\alpha}h^{\mu\nu})+\frac{1}{2}(\bar{\nabla}_{\alpha}h_{\mu\nu})(\bar{\nabla}^{\nu}h^{\alpha\mu})-\frac{1}{2}(\bar{\nabla}^{\mu}h^{\alpha}{}_{\alpha})(\bar{\nabla}^{\lambda}h_{\mu\lambda})+\frac{1}{4}(\bar{\nabla}^{\mu}h^{\alpha}{}_{\alpha})(\bar{\nabla}_{\mu}h^{\beta}{}_{\beta}) \\
&+\frac{\bar{R}}{2D}\Big(h^{\alpha}{}_{\beta}h^{\beta}{}_{\alpha}-\frac{1}{2}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}\Big) \\
&+\bar{\nabla}_{\mu}\Big(h^{\alpha\beta}\bar{\nabla}^{\mu}h_{\alpha\beta}-h^{\mu\alpha}\bar{\nabla}^{\beta}h_{\alpha\beta}+h^{\mu\alpha}\bar{\nabla}_{\alpha}h^{\beta}{}_{\beta}-h^{\alpha\beta}\bar{\nabla}_{\alpha}h^{\mu}{}_{\beta}+\frac{1}{2}h^{\beta}{}_{\beta}\bar{\nabla}^{\alpha}h^{\mu}{}_{\alpha}-\frac{1}{2}h^{\alpha}{}_{\alpha}\bar{\nabla}^{\mu}h^{\beta}{}_{\beta}\Big).
\end{aligned}
\tag{B.44}
$$

If we include a matter coupling

$$
S=S_{EH}+S_{coup}, \quad S_{coup}:=\int d^{D}x\sqrt{-\det g}\,\mathcal{L}_{coup}, \tag{B.45}
$$

its perturbation is interpreted as

$$
\delta S_{coup}=-\frac{1}{2}\int d^{D}x\,\sqrt{-\det g}\,T^{\mu\nu}\delta g_{\mu\nu}. \tag{B.46}
$$

Here, the energy momentum tensor $T^{\mu\nu}$ is defined by

$$
T^{\mu\nu}=-\frac{2}{\sqrt{-\det g}}\frac{\delta}{\delta g_{\mu\nu}}\big(\sqrt{-\det g}\mathcal{L}_{coup}\big). \tag{B.47}
$$

Appendix C

The Poisson bracket in general relativity

In Section 3.1, we have formulated the Einstein-Hilbert action in the ADM decomposition

$$S = \frac{1}{16\pi G}\int dt\, d^dx\, \Big[\dot{\gamma}_{ij}\pi^{ij} + N\mathcal{R}_0 + N^i\mathcal{R}_i\Big], \tag{C.1}$$

where each element has been defines as

$$\begin{aligned}\mathcal{R}_0 :=& \sqrt{\det\gamma}\,{}^{(d)}R - \frac{1}{\sqrt{\det\gamma}}\Big(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2\Big)\\ =& \sqrt{\det\gamma}\,{}^{(d)}R - \frac{1}{\sqrt{\det\gamma}}\Big(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\Big)\pi^{ij}\pi^{kl}, \end{aligned}\tag{C.2}$$

$$\mathcal{R}_i := 2\mathcal{D}_j\pi^j{}_i = 2\gamma_{ik}\mathcal{D}_j\pi^{jk}. \tag{C.3}$$

The Poisson bracket is determined by

$$\big\{F(x), G(y)\big\}_{PB} = \int d^dz\left[\frac{\delta F(x)}{\delta\gamma_{ij}(z)}\frac{\delta G(y)}{\delta\pi^{ij}(z)} - \frac{\delta F(x)}{\delta\pi^{ij}(z)}\frac{\delta G(y)}{\delta\gamma_{ij}(z)}\right], \tag{C.4}$$

and those between $\mathcal{R}_0$ and $\mathcal{R}_i$ are calculated as follows.

To begin with, we remember the calculation

$$\delta\det\gamma = (\det\gamma)\Big[\det(1+\gamma^{-1}\delta\gamma) - 1\Big] = (\det\gamma)\mathrm{Tr}(\gamma^{-1}\delta\gamma) = (\det\gamma)\gamma^{ij}\delta\gamma_{ij}, \tag{C.5}$$

$$\tag{C.6}$$

from which variation such as $\delta\sqrt{\det\gamma} = \frac{1}{2}\sqrt{\det\gamma}\gamma^{ij}\delta\gamma_{ij}$ can be easily obtained. We also remember the formula for the first order variation of the scalar curvature (B.28), and apply it to variation of the curvature ${}^{(d)}R$ with respect to γ_{ij}

$$\delta R = -R^{ij}\delta\gamma_{ij} + \mathcal{D}^i\mathcal{D}^j\delta\gamma_{ij} - \gamma^{ij}\mathcal{D}^k\mathcal{D}_k\delta\gamma_{ij}, \tag{C.7}$$

where we have omitted the subscript (d). Throughout this appendix, we neglect the subscript (d), and R means the scalar curvature for the spatial metric γ_{ij}.

Next, we calculate variation of $\mathcal{R}_0$ and $\mathcal{R}_i$. Variation of $\mathcal{R}_0$ with respect to γ_{ij} is given by

$$\begin{aligned}\delta_\gamma \mathcal{R}_0 =& \frac{1}{2}\sqrt{\det\gamma}\, R\gamma^{ij}\delta\gamma_{ij} + \sqrt{\det\gamma}\Big(- R^{ij}\delta\gamma_{ij} + \mathcal{D}^i\mathcal{D}^j\delta\gamma_{ij} - \gamma^{ij}\mathcal{D}^k\mathcal{D}_k\delta\gamma_{ij}\Big) \\ & - \frac{2}{\sqrt{\det\gamma}}\Big(\pi^{ik}\pi_k{}^j\delta\gamma_{ij} - \frac{1}{d-1}\pi\pi^{ij}\delta\gamma_{ij}\Big) + \frac{1}{2\sqrt{\det\gamma}}\Big(\pi^{kl}\pi_{kl} - \frac{1}{d-1}\pi^2\Big)\gamma^{ij}\delta\gamma_{ij}. \end{aligned} \tag{C.8}$$

In the above formula, we notice that derivatives are operating on the variation of the metric $\delta\gamma$. Hence, in the calculation of the Poisson bracket, we encounter derivatives on the delta function, $\mathcal{D}\frac{\delta\gamma(x)}{\delta\gamma(y)} = \mathcal{D}\delta(x-y)$. In such a situation, it is convenient to introduce integrated forms

$$\langle\langle f\mathcal{R}_0\rangle\rangle := \int d^dx f(x)\mathcal{R}_0(x), \qquad \langle\langle f^i\mathcal{R}_i\rangle\rangle := \int d^dx f^i(x)\mathcal{R}_i(x), \tag{C.9}$$

where $f(x)$ and $f^i(x)$ are arbitrary scalar and vector functions. Thus, neglecting total derivatives, we find

$$\begin{aligned}\delta_\gamma\langle\langle f\mathcal{R}_0\rangle\rangle = \int d^dx \Big[& \frac{1}{2}\sqrt{\det\gamma}\, fR\gamma^{ij} + \sqrt{\det\gamma}\Big(- fR^{ij} + \mathcal{D}^i\mathcal{D}^j f - \gamma^{ij}\mathcal{D}^k\mathcal{D}_k f\Big) \\ & - \frac{2}{\sqrt{\det\gamma}} f\Big(\pi^{ik}\pi_k{}^j - \frac{1}{d-1}\pi\pi^{ij}\Big) + \frac{1}{2\sqrt{\det\gamma}} f\Big(\pi^{kl}\pi_{kl} - \frac{1}{d-1}\pi^2\Big)\gamma^{ij}\Big]\delta\gamma_{ij}. \end{aligned} \tag{C.10}$$

Here, it should be noted that $[\cdots]^{ij}$ in (C.10) is symmetric for the indices ij because a relation $\mathcal{D}^i\mathcal{D}^j f = \mathcal{D}^j\mathcal{D}^i f$ holds. Variation of $\mathcal{R}_0$ with respect to π^{ij} is easy to obtain

$$\delta_\pi \mathcal{R}_0 = -\frac{2}{\sqrt{\det\gamma}}\Big(\pi_{ij} - \frac{1}{d-1}\pi\gamma_{ij}\Big)\delta\pi^{ij} \tag{C.11}$$

$$\delta_\pi\langle\langle f\mathcal{R}_0\rangle\rangle = \int d^dx\Big[- \frac{2}{\sqrt{\det\gamma}} f\Big(\pi_{ij} - \frac{1}{d-1}\pi\gamma_{ij}\Big)\Big]\delta\pi^{ij}. \tag{C.12}$$

We continue to variation of $\mathcal{R}_i$. Discarding total derivatives, the variation with respect to γ_{ij} is given by

$$\begin{aligned}\delta_\gamma\langle\langle f^i\mathcal{R}_i\rangle\rangle =& 2\delta_\gamma\langle\langle f^i\gamma_{ij}\mathcal{D}_k\pi^{kj}\rangle\rangle \\ =& -2\delta_\gamma\langle\langle\gamma_{ij}\pi^{jk}\mathcal{D}_k f^i\rangle\rangle \\ =& -2\langle\langle(\delta_\gamma\gamma_{ij})\pi^{jk}\mathcal{D}_k f^i\rangle\rangle - 2\langle\langle\gamma_{ij}\pi^{jk}\delta_\gamma(\mathcal{D}_k f^i)\rangle\rangle. \end{aligned} \tag{C.13}$$

The second term in (C.13) contains a factor

$$\delta_\gamma(\mathcal{D}_k f^i) = \delta_\gamma(\partial_k f^i + \Gamma^i{}_{kl} f^l) = (\delta_\gamma\Gamma^i{}_{kl}) f^i. \tag{C.14}$$

We apply the formula (B.22)

$$\delta_\gamma\Gamma^i{}_{kl} = \frac{1}{2}\gamma^{im}\big(\mathcal{D}_k\delta\gamma_{ml} + \mathcal{D}_l\delta\gamma_{km} - \mathcal{D}_m\delta\gamma_{kl}\big), \tag{C.15}$$

and find that the second term in (C.13) is written as

$$-2\langle\langle\gamma_{ij}\pi^{jk}\delta_\gamma(\mathcal{D}_k f^i)\rangle\rangle = \langle\langle-\pi^{ij}(\mathcal{D}_k\delta\gamma_{ij}) f^k\rangle\rangle = \langle\langle\mathcal{D}_k(\pi^{ij} f^k)\delta\gamma_{ij}\rangle\rangle. \tag{C.16}$$

Thus, we obtain

$$\delta_\gamma\langle\langle f^i R_i\rangle\rangle = \int d^dx\big[-2(\mathcal{D}_k f^{(i})\pi^{j)k} + \mathcal{D}_k(f^k\pi^{ij})\big]\delta\gamma_{ij}, \tag{C.17}$$

where the indices ij at $[\cdots]^{ij}$ are symmetrized. Variation of $\mathcal{R}_i$ with respect to π^{ij} leads to

$$\delta_\pi\langle\langle f^i R_i\rangle\rangle = \delta_\pi\langle\langle 2f^i\gamma_{ik}\mathcal{D}_j\pi^{jk}\rangle\rangle = \langle\langle -2\gamma_{k(i}(\mathcal{D}_{j)}f^k)\delta\pi^{ij}\rangle\rangle, \tag{C.18}$$

where the symmetrization is understood.

Now, we set about the calculation of the Poisson brackets. The first target is $\big\{\mathcal{R}_0,\mathcal{R}_0\big\}_{PB}$ which corresponds to

$$\big\{\langle\langle f\mathcal{R}_0\rangle\rangle,\langle\langle g\mathcal{R}_0\rangle\rangle\big\}_{PB} = \int d^dz\left[\frac{\delta\langle\langle f\mathcal{R}_0\rangle\rangle}{\delta\gamma_{ij}(z)}\frac{\delta\langle\langle g\mathcal{R}_0\rangle\rangle}{\delta\pi^{ij}(z)} - \frac{\delta\langle\langle f\mathcal{R}_0\rangle\rangle}{\delta\pi^{ij}(z)}\frac{\delta\langle\langle g\mathcal{R}_0\rangle\rangle}{\delta\gamma_{ij}(z)}\right]. \tag{C.19}$$

In the above Poisson bracket, contributions between non-derivative terms cancel out, and the remaining part is calculated to be

$$\begin{aligned}
&\int dz\frac{\delta\langle\langle f\mathcal{R}_0\rangle\rangle}{\delta\gamma_{ij}(z)}\frac{\delta\langle\langle g\mathcal{R}_0\rangle\rangle}{\delta\pi^{ij}(z)}\\
=&\int dx\int dy\int dz\left[\sqrt{\det\gamma}\,\Big(\mathcal{D}^i\mathcal{D}^j f - \gamma^{ij}\mathcal{D}^k\mathcal{D}_k f\Big)\right](x)\delta(x-z)\\
&\qquad\qquad\times\left[-\frac{2}{\sqrt{\det\gamma}}g\Big(\pi_{ij}-\frac{1}{d-1}\pi\gamma_{ij}\Big)\right](y)\delta(y-z)\\
=&-2\int dx\Big(\mathcal{D}^i\mathcal{D}^j f - \gamma^{ij}\mathcal{D}^k\mathcal{D}_k f\Big)g\Big(\pi_{ij}-\frac{1}{d-1}\pi\gamma_{ij}\Big)\\
=&-2\int dx\big[g\pi^{ij}(\mathcal{D}_i\mathcal{D}_j f)\big].
\end{aligned} \tag{C.20}$$

Combining the contribution from the exchange $f\leftrightarrow g$, we obtain

$$\begin{aligned}
\big\{\langle\langle f\mathcal{R}_0\rangle\rangle,\langle\langle g\mathcal{R}_0\rangle\rangle\big\}_{PB} =&-2\int dx\big[g\pi^{ij}(\mathcal{D}_i\mathcal{D}_j f)\big] + 2\int dx\big[f\pi^{ij}(\mathcal{D}_i\mathcal{D}_j g)\big]\\
=&2\int dx\big[g(\mathcal{D}^i\pi_{ij})(\mathcal{D}^j f) - f(\mathcal{D}^i\pi_{ij})(\mathcal{D}^j g)\big]\\
=&\int dx\big[(\mathcal{D}^i f)g - (\mathcal{D}^i g)f\big]\mathcal{R}_i.
\end{aligned} \tag{C.21}$$

Here, we write $f(x) = \int dy f(y)\delta(x-y)$ and $g(x) = \int dy f(y)\delta(x-y)$ to find

$$\begin{aligned}
&\big\{\langle\langle f\mathcal{R}_0\rangle\rangle,\langle\langle g\mathcal{R}_0\rangle\rangle\big\}_{PB}\\
=&\int dx\int dy[\mathcal{D}^i_{(x)}f(y)\delta(x-y)]g(x)\mathcal{R}_i(x) - \int dx\int dy[\mathcal{D}^i_{(x)}g(y)\delta(x-y)]f(x)\mathcal{R}_i(x)\\
=&\int dx\int dy f(x)g(y)\big[\mathcal{R}_i(y)\mathcal{D}^i_{(y)}\delta(x-y) - \mathcal{R}_i(x)\mathcal{D}^i_{(x)}\delta(x-y)\big],
\end{aligned} \tag{C.22}$$

where we have renamed integration variable as $x\leftrightarrow y$ at the first term. Hence, we conclude

$$\big\{\mathcal{R}_0(x),\mathcal{R}_0(y)\big\}_{PB} = \mathcal{R}_i(y)\mathcal{D}^i_{(y)}\delta^{(d)}(x-y) - \mathcal{R}_i(x)\mathcal{D}^i_{(x)}\delta^{(d)}(x-y). \tag{C.23}$$

We proceed to consider $\{\mathcal{R}_0, \mathcal{R}_i\}_{PB}$ which corresponds to

$$
\begin{aligned}
&\big\{\langle\langle f\mathcal{R}_0\rangle\rangle, \langle\langle g^i\mathcal{R}_i\rangle\rangle\big\}_{PB} \\
&= \int dx \int dy \int dz \Big[\frac{1}{2}\sqrt{\det\gamma}\, fR\gamma^{ij} + \sqrt{\det\gamma}\,\Big(- fR^{ij} + \mathcal{D}^i\mathcal{D}^j f - \gamma^{ij}\mathcal{D}^k\mathcal{D}_k f\Big) \\
&\quad - \frac{2}{\sqrt{\det\gamma}} f\Big(\pi^{ik}\pi_k{}^j - \frac{1}{d-1}\pi\pi^{ij}\Big) + \frac{1}{2\sqrt{\det\gamma}} f\Big(\pi^{kl}\pi_{kl} - \frac{1}{d-1}\pi^2\Big)\gamma^{ij}\Big](x)\delta(x-z) \\
&\quad \times \big[- 2\gamma_{mi}\mathcal{D}_j g^m\big](y)\delta(y-z) \\
&\quad - \int dx \int dy \int dz \Big[- \frac{2}{\sqrt{\det\gamma}} f\Big(\pi_{ij} - \frac{1}{d-1}\pi\gamma_{ij}\Big)\Big](x)\delta(x-z) \\
&\quad \times \big[- 2(\mathcal{D}_k g^i)\pi^{jk} + \mathcal{D}_k(g^k\pi^{ij})\big](y)\delta(y-z). \qquad\qquad (C.24)
\end{aligned}
$$

In the above formula, terms without π are

$$
\begin{aligned}
&\int dx \Big[\frac{1}{2}\sqrt{\det\gamma}\, fR\gamma^{ij} + \sqrt{\det\gamma}\,\Big(- fR^{ij} + \mathcal{D}^i\mathcal{D}^j f - \gamma^{ij}\mathcal{D}^k\mathcal{D}_k f\Big)\Big]\big[- 2\gamma_{mi}\mathcal{D}_j g^m\big] \\
&= \int dx \sqrt{\det\gamma}\,\Big[- fR(\mathcal{D}_i g^i) + 2fR^{ij}(\mathcal{D}_j g_i) - 2(\mathcal{D}_i\mathcal{D}^j f - \delta_i^j\mathcal{D}^k\mathcal{D}_k f)(\mathcal{D}_j g^i)\Big] \\
&= \int dx \sqrt{\det\gamma}\,\Big[- fR(\mathcal{D}_i g^i) + 2fR^{ij}(\mathcal{D}_j g_i) + 2(\mathcal{D}^j f)(\mathcal{D}_i\mathcal{D}_j - \mathcal{D}_j\mathcal{D}_i)g^i\Big] \\
&= \int dx \sqrt{\det\gamma}\,\Big[- fR(\mathcal{D}_i g^i) + 2fR^{ij}(\mathcal{D}_j g_i) + 2(\mathcal{D}^j f)R_{jk}g^k\Big] \\
&= \int dx \sqrt{\det\gamma}\,\Big[- fR(\mathcal{D}_i g^i) - 2f(\mathcal{D}^j R_{jk})g^k\Big] \\
&= \int dx \sqrt{\det\gamma}\,\Big[- fR(\mathcal{D}_i g^i) - f(\mathcal{D}_k R)g^k\Big] \\
&= \int dx \Big[- \sqrt{\det\gamma}\, f\mathcal{D}_i(g^i R)\Big], \qquad\qquad (C.25)
\end{aligned}
$$

where we have used the definition of the curvature and a property $\mathcal{D}^i R_{ij} = \frac{1}{2}\mathcal{D}_j R$. On the other hand, terms containing π are

$$
\begin{aligned}
&\int dx \Big[- \frac{2}{\sqrt{\det\gamma}} f\Big(\pi^{ik}\pi_k{}^j - \frac{1}{d-1}\pi\pi^{ij}\Big) + \frac{1}{2\sqrt{\det\gamma}} f\Big(\pi^{kl}\pi_{kl} - \frac{1}{d-1}\pi^2\Big)\gamma^{ij}\Big]\big[- 2\gamma_{mi}\mathcal{D}_j g^m\big] \\
&- \int dx \Big[- \frac{2}{\sqrt{\det\gamma}} f\Big(\pi_{ij} - \frac{1}{d-1}\pi\gamma_{ij}\Big)\Big]\big[- 2(\mathcal{D}_k g^i)\pi^{jk} + \mathcal{D}_k(g^k\pi^{ij})\big] \\
&= \int dx \frac{f}{\sqrt{\det\gamma}}\mathcal{D}_k\Big[\Big(\pi^{ij}\pi_{ij} - \frac{1}{d-1}\pi^2\Big)g^k\Big]. \qquad\qquad (C.26)
\end{aligned}
$$

Hence, we find

$$
\begin{aligned}
\big\{\langle\langle f\mathcal{R}_0\rangle\rangle, \langle\langle g^i\mathcal{R}_i\rangle\rangle\big\}_{PB} &= \int dx f\mathcal{D}_i\Big[- \sqrt{\det\gamma} R g^i + \frac{1}{\sqrt{\det\gamma}}\Big(\pi^{kl}\pi_{kl} - \frac{1}{d-1}\pi^2\Big)g^i\Big] \\
&= - \int dx f\mathcal{D}_i(\mathcal{R}_0 g^i). \qquad\qquad (C.27)
\end{aligned}
$$

If we write $\mathcal{R}_0(x)g^i(x) = \int dy \mathcal{R}_0(y)g^i(y)\delta(x-y)$, we see

$$\left\{\langle\langle f\mathcal{R}_0\rangle\rangle, \langle\langle g^i\mathcal{R}_i\rangle\rangle\right\}_{PB} = -\int dx \int dy f(x)g^i(y)\mathcal{R}_0(y)\mathcal{D}_i^{(x)}\delta(x-y), \tag{C.28}$$

from which we obtain

$$\left\{\mathcal{R}_0(x), \mathcal{R}_i(y)\right\}_{PB} = -\mathcal{R}_0(y)\mathcal{D}_i^{(x)}\delta(x-y). \tag{C.29}$$

The last calculation is $\left\{\mathcal{R}_i, \mathcal{R}_j\right\}_{PB}$. We write down the former half of this Poisson bracket

$$\begin{aligned}
&\int dz \frac{\delta\langle\langle f^k\mathcal{R}_k\rangle\rangle}{\delta\gamma_{ij}(z)}\frac{\delta\langle\langle f^l\mathcal{R}_l\rangle\rangle}{\delta\pi^{ij}(z)} \\
&= \int dx \int dy \int dz \big[-2(\mathcal{D}_k f^i)\pi^{jk} + \mathcal{D}_k(f^k\pi^{ij})\big](x)\delta(x-z)\big[-2\gamma_{l(i}\mathcal{D}_{j)}g^l\big](y)\delta(y-z) \\
&= \int dx \big[2\pi^{jk}(\mathcal{D}_k f^i)(\mathcal{D}_j g_i) + 2\pi^{jk}(\mathcal{D}_k f^i)(\mathcal{D}_i g_j) - 2(\mathcal{D}_j g_i)\mathcal{D}_k(f^k\pi^{ij})\big].
\end{aligned} \tag{C.30}$$

The second term in (C.30) is

$$\int 2\pi^{jk}(\mathcal{D}_k f^i)(\mathcal{D}_i g_j) = \int -2f^i(\mathcal{D}_k\pi^{jk})(\mathcal{D}_i g_j) - 2f^i\pi^{jk}\mathcal{D}_k\mathcal{D}_i g_j, \tag{C.31}$$

and the third term in (C.30) is

$$\begin{aligned}
\int -2(\mathcal{D}_j g_i)\mathcal{D}_k(f^k\pi^{ij}) &= \int 2f^k\pi^{ij}(\mathcal{D}_k\mathcal{D}_j g_i) \\
&= \int 2f^k\pi^{ij}\big[(\mathcal{D}_k\mathcal{D}_j - \mathcal{D}_j\mathcal{D}_k)g_i + \mathcal{D}_j\mathcal{D}_k g_i\big] \\
&= \int 2f^k\pi^{ij}R_{kjil}g^l + 2f^k\pi^{ij}\mathcal{D}_j\mathcal{D}_k g_i.
\end{aligned} \tag{C.32}$$

Thus, (C.30) can be read as

$$\int 2\pi^{jk}(\mathcal{D}_k f^i)(\mathcal{D}_j g_i) - 2f^i(\mathcal{D}_k\pi^{jk})(\mathcal{D}_i g_j) + 2f^k\pi^{ij}R_{kjil}g^l. \tag{C.33}$$

Combining the contribution from the exchange $f \leftrightarrow g$, we find

$$\begin{aligned}
\left\{\langle\langle f^i\mathcal{R}_i\rangle\rangle, \langle\langle g^j\mathcal{R}_j\rangle\rangle\right\}_{PB} &= \int 2f^k g^l\pi^{ij}(R_{kjil} - R_{ljik}) + g^i\mathcal{R}^j(\mathcal{D}_i f_j) - f^i\mathcal{R}^j(\mathcal{D}_i g_j) \\
&= \int dx \big[\mathcal{R}_j g^i(\mathcal{D}_i f^j) - \mathcal{R}_j f^i(\mathcal{D}_i g^j)\big],
\end{aligned} \tag{C.34}$$

where we have used a property $R_{kjil} = R_{ljik}$. If we write $f^j(x) = \int dy f^j(y)\delta(x-y)$ and $g^j(x) = \int dy g^j(y)\delta(x-y)$, we see

$$\begin{aligned}
\left\{\langle\langle f^i\mathcal{R}_i\rangle\rangle, \langle\langle g^j\mathcal{R}_j\rangle\rangle\right\}_{PB} &= \int dx \int dy \big[\mathcal{R}_j(x)g^i(x)\mathcal{D}_i^{(x)}f^j(y)\delta(x-y) - \mathcal{R}_j(x)f^i(x)\mathcal{D}_i^{(x)}g^j(y)\delta(x-y)\big] \\
&= \int dx \int dy f^i(x)g^j(y)\big[\mathcal{R}_i(y)\mathcal{D}_j^{(y)}\delta(x-y) - \mathcal{R}_j(x)\mathcal{D}_i^{(x)}\delta(x-y)\big].
\end{aligned} \tag{C.35}$$

Therefore, we conclude

$$\left\{\mathcal{R}_i(x), \mathcal{R}_j(y)\right\}_{PB} = \mathcal{R}_i(y)\mathcal{D}_j^{(y)}\delta^{(d)}(x-y) - \mathcal{R}_j(x)\mathcal{D}_i^{(x)}\delta^{(d)}(x-y). \tag{C.36}$$

Appendix D

Total derivatives

In this appendix, we think about total derivatives. In the main part of this thesis, we often rely on total derivative combinations constructed from polynomials of $\partial_\mu A^\nu$ or $\partial_\mu \partial_\nu \phi$. Here, the dimension is assumed to be D. We recall a relation

$$\begin{aligned}&\left(\partial_{\mu_1} A^{\nu_1}\right)\left(\partial_{\mu_2} A^{\nu_2}\right)\cdots\left(\partial_{\mu_n} A^{\nu_n}\right)\\ =&\partial_{\mu_1}\Big(A^{\nu_1}\left(\partial_{\mu_2} A^{\nu_2}\right)\cdots\left(\partial_{\mu_n} A^{\nu_n}\right)\Big)-\sum_{k=2}^{n}\Big(A^{\nu_1}\left(\partial_{\mu_2} A^{\nu_2}\right)\cdots\left(\partial_{\mu_1}\partial_{\mu_k} A^{\nu_k}\right)\cdots\left(\partial_{\mu_n} A^{\nu_n}\right)\Big),\end{aligned} \tag{D.1}$$

and find that antisymmetrization of indices on partial derivatives leads to a total derivative. Hence, we introduce the following polynomial for a matrix $\Pi^\mu{}_\nu$

$$\mathcal{L}_n^{TD}(\Pi) := \epsilon^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_n}\,\Pi^{\nu_1}{}_{\mu_1}\Pi^{\nu_2}{}_{\mu_2}\cdots\Pi^{\nu_n}{}_{\mu_n}, \tag{D.2}$$

where the symbol $\epsilon^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_n}$ is defined by

$$\epsilon^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_n} := n!\delta^{\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_n}_{\nu_n]}. \tag{D.3}$$

In the definition (D.3), $[\cdots]$ represents antisymmetrization divided by the number of the elements. When we set Π to be $\Pi^\mu{}_\nu = \partial_\nu A^\mu$, $\mathcal{L}_n^{TD}$ becomes a total derivative. We should notice that $\epsilon^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_n}$ satisfies $\epsilon^{\mu_1\mu_2\cdots\mu_n}_{\mu_1\mu_2\cdots\mu_n} = 1$ and returns zero if two sets $\{\mu_1,\mu_2,...,\mu_n\}$ and $\{\nu_1,\nu_2,...,\nu_n\}$ do not coincide. For example, we have $\epsilon^{123}_{124} = 0$ for $n=3$. It is also useful to introduce other antisymmetrization symbols $\epsilon^{\mu_1\mu_2\cdots\mu_D}$ and $\epsilon_{\nu_1\nu_2\cdots\nu_D}$. They are totally antisymmetric and determined by $\epsilon_{12\cdots D} = 1$ and $\epsilon^{12\cdots D} = 1$ respectively. We do not raise or lower their indices. We can easily show that relations

$$\epsilon^{\mu_1\mu_2\cdots\cdots\mu_D}\epsilon_{\nu_1\nu_2\cdots\cdots\nu_D} = D!\,\delta^{\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_D}_{\nu_D]}, \tag{D.4}$$

$$\epsilon^{\mu_1\cdots\mu_n\lambda_{n+1}\cdots\lambda_D}\epsilon_{\nu_1\cdots\nu_n\lambda_{n+1}\cdots\lambda_D} = (D-n)!\,\epsilon^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}, \tag{D.5}$$

hold, which are convenient for explicit calculations.

An important property for (D.2) is that the same index appears only up to two times. One is in the upper index and the other is in the lower index. Therefore, if we set $\Pi^\mu{}_\nu = \partial^\mu\partial_\nu\phi$, there is no higher order derivative with respect to time. Their concrete formulae are given by

$$\begin{aligned}
\mathcal{L}_1^{TD}(\Pi) :=& 1, && \text{(D.6)}\\
\mathcal{L}_1^{TD}(\Pi) =& \Pi^\mu{}_\mu = \mathrm{Tr}(\Pi), && \text{(D.7)}\\
\mathcal{L}_2^{TD}(\Pi) =& \Pi^\mu{}_\mu \Pi^\nu{}_\nu - \Pi^\mu{}_\nu \Pi^\nu{}_\mu = \mathrm{Tr}(\Pi)^2 - \mathrm{Tr}(\Pi^2), && \text{(D.8)}\\
\mathcal{L}_3^{TD}(\Pi) =& \mathrm{Tr}(\Pi)^3 - 3\mathrm{Tr}(\Pi)\mathrm{Tr}(\Pi^2) + 2\mathrm{Tr}(\Pi^3), && \text{(D.9)}\\
\mathcal{L}_4^{TD}(\Pi) =& \mathrm{Tr}(\Pi)^4 - 6\mathrm{Tr}(\Pi^2)\mathrm{Tr}(\Pi)^2 + 8\mathrm{Tr}(\Pi^3)\mathrm{Tr}(\Pi) + 3\mathrm{Tr}(\Pi^2)^2 - 6\mathrm{Tr}(\Pi^4), && \text{(D.10)}
\end{aligned}$$

and $\mathcal{L}_{n>D}^{TD}(\Pi) = 0$. The first one (D.6) is a definition.

In this thesis, instead of (D.2), we mainly use the following definition

$$e_n(\Pi) := \frac{1}{n!}\mathcal{L}_n^{TD}(\Pi). \tag{D.11}$$

We immediately find $e_D(\Pi) = \det \Pi$. When we have the inverse matrix Π^{-1}, namely $\Pi^\mu{}_\lambda (\Pi^{-1})^\lambda{}_\nu = (\Pi^{-1})^\mu{}_\lambda \Pi^\lambda{}_\nu = \delta^\mu_\nu$, we can express $e_n(\Pi)$ by $e_{D-n}(\Pi^{-1})$. Using (D.3) and (D.5), we calculate it as

$$\begin{aligned}
e_n(\Pi) =& \frac{1}{n!(D-n)!}\epsilon^{\mu_1\cdots\mu_n\lambda_{n+1}\cdots\lambda_D}\epsilon_{\nu_1\cdots\nu_n\lambda_{n+1}\cdots\lambda_D}\,\Pi^{\nu_1}{}_{\mu_1}\cdots\Pi^{\nu_n}{}_{\mu_n}\\
=& \frac{1}{n!(D-n)!}\epsilon^{\mu_1\cdots\mu_n\mu_{n+1}\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_n\nu_{n+1}\cdots\nu_D}\,\Pi^{\nu_1}{}_{\mu_1}\cdots\Pi^{\nu_n}{}_{\mu_n}\delta^{\nu_{n+1}}_{\mu_{n+1}}\cdots\delta^{\nu_D}_{\mu_D}\\
=& \frac{1}{n!(D-n)!}\epsilon^{\mu_1\cdots\mu_n\mu_{n+1}\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_n\nu_{n+1}\cdots\nu_D}\,\Pi^{\nu_1}{}_{\mu_1}\cdots\Pi^{\nu_n}{}_{\mu_n}\\
&\times\left(\Pi^{\nu_{n+1}}{}_{\alpha_{n+1}}(\Pi^{-1})^{\beta_{n+1}}{}_{\mu_{n+1}}\delta^{\alpha_{n+1}}_{\beta_{n+1}}\right)\cdots\left(\Pi^{\nu_D}{}_{\alpha_D}(\Pi^{-1})^{\beta_D}{}_{\mu_D}\delta^{\alpha_D}_{\beta_D}\right)\\
=& \frac{1}{n!(D-n)!(D-n)!}\epsilon^{\mu_1\cdots\mu_n\mu_{n+1}\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_n\nu_{n+1}\cdots\nu_D}\epsilon^{\alpha_{n+1}\cdots\alpha_D}_{\beta_{n+1}\cdots\beta_D}\\
&\times\left(\Pi^{\nu_1}{}_{\mu_1}\cdots\Pi^{\nu_n}{}_{\mu_n}\Pi^{\nu_{n+1}}{}_{\alpha_{n+1}}\cdots\Pi^{\nu_D}{}_{\alpha_D}\right)\left((\Pi^{-1})^{\beta_{n+1}}{}_{\mu_{n+1}}(\Pi^{-1})^{\beta_D}{}_{\mu_D}\right)\\
=& \frac{1}{n!(D-n)!(D-n)!\,{}_DC_n}\epsilon^{\mu_1\cdots\mu_n\mu_{n+1}\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_n\nu_{n+1}\cdots\nu_D}\epsilon^{\alpha_{n+1}\cdots\alpha_D}_{\beta_{n+1}\cdots\beta_D}\\
&\times\left(\Pi^{\nu_1}{}_{\mu_1}\cdots\Pi^{\nu_n}{}_{\mu_n}\Pi^{\nu_{n+1}}{}_{\mu_{n+1}}\cdots\Pi^{\nu_D}{}_{\mu_D}\right)\left((\Pi^{-1})^{\beta_{n+1}}{}_{\alpha_{n+1}}(\Pi^{-1})^{\beta_D}{}_{\alpha_D}\right)\\
=& \frac{1}{D!}\epsilon^{\mu_1\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_D}\left(\Pi^{\nu_1}{}_{\mu_1}\cdots\Pi^{\nu_D}{}_{\mu_D}\right)\times\frac{1}{(D-n)!}\epsilon^{\alpha_{n+1}\cdots\alpha_D}_{\beta_{n+1}\cdots\beta_D}\left((\Pi^{-1})^{\beta_{n+1}}{}_{\alpha_{n+1}}(\Pi^{-1})^{\beta_D}{}_{\alpha_D}\right)\\
=& e_D(\Pi)e_{D-n}(\Pi^{-1}).
\end{aligned}\tag{D.12}$$

In fact, we can include more general cases with matrices $\Pi(I)$ $(I = 1, 2, ..., n)$. A combination

$$\mathcal{S}_n^{TD}\big(\Pi(1), \pi(2), ..., \Pi(n)\big) := \epsilon^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_n}\,\Pi(1)^{\nu_1}{}_{\mu_1}\Pi(2)^{\nu_2}{}_{\mu_2}\cdots\Pi(n)^{\nu_n}{}_{\mu_n}, \tag{D.13}$$

becomes a total derivative when we set $\Pi(I)^\mu{}_\nu = \partial_\nu A(I)^\mu$ $(I = 1, 2, ..., n)$.

A derivative of (D.2) is also important

$$X(\Pi)^{(n)\nu}{}_\mu := \frac{1}{n+1}\frac{\delta}{\delta\Pi^\mu{}_\nu}\mathcal{L}_{n+1}^{TD}(\Pi) = \epsilon^{\mu\mu_1\mu_2\cdots\mu_n}_{\nu\nu_1\nu_2\cdots\nu_n}\,\Pi^{\nu_1}{}_{\mu_1}\Pi^{\nu_2}{}_{\mu_2}\cdots\Pi^{\nu_n}{}_{\mu_n}. \tag{D.14}$$

When we decompose the indices as $\mu=(0,i)$, we see that $X^{(n)i}{}_j$ contain up to two "0" indices, and $X^{(n)i}{}_j$ contains only one "0" index while $X^{(n)0}{}_0$ has no "0" index. The trace of (D.14) is easily calculated to be

$$X(\Pi)^{(n)\mu}{}_{\mu}=(D-n)\mathcal{L}^{TD}_n(\Pi). \tag{D.15}$$

Similarly, general $X^{\mu}{}_{\nu}$ can be expressed by $\mathcal{L}^{TD}$. Variation of $\mathcal{L}^{TD}$ is directly given by

$$\delta\mathcal{L}^{TD}_{n+1}(\Pi)=(n+1)\epsilon^{\mu_1\mu_2\cdots\mu_n+1}_{\nu_1\nu_2\cdots\nu_n+1}\left(\delta\Pi^{\nu_1}{}_{\mu_1}\right)\Pi^{\nu_2}{}_{\mu_2}\cdots\Pi^{\nu_{n+1}}{}_{\mu_{n+1}}, \tag{D.16}$$

but we know that the right hand side of (D.16) can be written by traces. Assuming that $\delta\Pi$ constructs a trace $\mathrm{Tr}\big((\delta\Pi)\Pi^m\big)$, the number of combinations to choose the elements of Π^m is ${}_nC_m=\frac{n!}{(n-m)!m!}$ and the number of their ordering is $m!$. In addition, we take notice that the signature of the permutation $(\lambda_2,\lambda_3,\cdots,\lambda_{m+1},\lambda_1)\to(\lambda_1,\lambda_2,\cdots,\lambda_{m+1})$ is $(-1)^m$. Thus, we obtain

$$\delta\mathcal{L}^{TD}_{n+1}(\Pi)=(n+1)\sum_{m=0}^{n}\frac{(-1)^m n!}{(n-m)!}\mathrm{Tr}\big((\delta\Pi)\Pi^m\big)\mathcal{L}^{TD}_{n-m}(\Pi), \tag{D.17}$$

from which we conclude

$$X(\Pi)^{(n)\mu}{}_{\nu}=\sum_{m=0}^{n}\frac{(-1)^m n!}{(n-m)!}(\Pi^m)^{\mu}{}_{\nu}\mathcal{L}^{TD}_{n-m}(\Pi). \tag{D.18}$$

In the same reasoning, we can also find

$$\mathcal{L}^{TD}_n(\Pi)=-\sum_{m=1}^{n}\frac{(-1)^m(n-1)!}{(n-m)!}\mathrm{Tr}(\Pi^m)\mathcal{L}^{TD}_{n-m}(\Pi). \tag{D.19}$$

These two relations (D.18) and (D.19) lead to a recursion formula

$$X(\Pi)^{(n)\mu}{}_{\nu}=-n\Pi^{\mu}{}_{\lambda}X(\Pi)^{(n-1)\lambda}{}_{\nu}+\Pi^{\lambda}{}_{\rho}X(\Pi)^{(n-1)\rho}{}_{\lambda}\delta^{\mu}{}_{\nu}. \tag{D.20}$$

Concrete formulae for $X^{(n)\mu}{}_{\nu}$ are given by

$$X(\Pi)^{(0)\mu}{}_{\nu}=\delta^{\mu}{}_{\nu}, \tag{D.21}$$
$$X(\Pi)^{(1)\mu}{}_{\nu}=\mathrm{Tr}(\Pi)\delta^{\mu}{}_{\nu}-\Pi^{\mu}{}_{\nu}, \tag{D.22}$$
$$X(\Pi)^{(2)\mu}{}_{\nu}=\big(\mathrm{Tr}(\Pi)^2-\mathrm{Tr}(\Pi^2)\big)\delta^{\mu}{}_{\nu}-2\mathrm{Tr}(\Pi)\Pi^{\mu}{}_{\nu}+2(\Pi^2)^{\mu}{}_{\nu} \tag{D.23}$$
$$\begin{aligned}X(\Pi)^{(3)\mu}{}_{\nu}=&\big(\mathrm{Tr}(\Pi)^3-3\mathrm{Tr}(\Pi)\mathrm{Tr}(\Pi^2)+2\mathrm{Tr}(\Pi^3)\big)\delta^{\mu}{}_{\nu}-3\big(\mathrm{Tr}(\Pi)^2-\mathrm{Tr}(\Pi^2)\big)\Pi^{\mu}{}_{\nu},\\&+6\mathrm{Tr}(\Pi)(\Pi^2)^{\mu}{}_{\nu}-6(\Pi^3)^{\mu}{}_{\nu},\end{aligned} \tag{D.24}$$

and $X(\Pi)^{(n\geq D-1)}=0$.

In Section 3.4, we encounter a case where we set $\Pi=\mathcal{K}$. The definition of $\mathcal{K}^{\mu}{}_{\nu}$ is found in (3.120). We need a derivative of $\sqrt{-\det g}\,\mathcal{L}^{TD}_n(\mathcal{K})$ with respect to $h_{\mu\nu}$. We recall the $h_{\mu\nu}$ dependence in $H^{\alpha}{}_{\beta}=g^{\alpha\lambda}H_{\lambda\beta}=(\eta^{\alpha\lambda}-h^{\alpha\lambda}+O[h^2])(h_{\lambda\beta}+\cdots)$ and find

$$\frac{\delta H^{\alpha}{}_{\beta}}{\delta h^{\mu\nu}}=\delta^{\alpha}_{(\mu}g_{\nu)\beta}-\delta^{\alpha}_{(\mu}H_{\nu)\beta}. \tag{D.25}$$

Then, using properties of the coefficient d_n such as $(n+1)d_{n+1} - nd_n = -\frac{1}{2}d_n$ and $d_1 = \frac{1}{2}$, we obtain

$$\frac{\delta}{\delta h^{\mu\nu}} \mathrm{Tr}\mathcal{K} = \frac{1}{2}(g_{\mu\nu} - \mathcal{K}_{\mu\nu}), \tag{D.26}$$

which leads to

$$\frac{\delta}{\delta h^{\mu\nu}} \mathrm{Tr}(\mathcal{K}^n) = \frac{n}{2}\big((\mathcal{K}^{n-1})_{\mu\nu} - (\mathcal{K}^n)_{\mu\nu}\big), \tag{D.27}$$

where the index is lowered by $g_{\mu\nu}$. From (D.17) and (D.27), we calculate as

$$\begin{aligned}
&\frac{\delta}{\delta h^{\mu\nu}}\big(\sqrt{-\det g}\,\mathcal{L}_n^{TD}(\mathcal{K})\big)\big|_{h=0,A=0} \\
=&\frac{1}{2}\eta^{\mu\nu}\mathcal{L}_n^{TD}(\partial\partial\phi) + \sum_{m=1}^{n}(-1)^{m-1}\frac{(n-1)!}{(n-m)!}\frac{n}{m}\Big[\frac{\delta}{\delta h^{\mu\nu}}\mathrm{Tr}(\mathcal{K}^m)\Big]\mathcal{L}_{n-m}^{TD}(\mathcal{K})\big|_{h=0,A=0} \\
=&\frac{1}{2}\eta^{\mu\nu}\mathcal{L}_n^{TD}(\partial\partial\phi) + \sum_{m=1}^{n}(-1)^{m-1}\frac{n!}{2(n-m)!}\Big[(\Pi^{m-1})_{\mu\nu} - (\Pi^m)_{\mu\nu}\Big]\mathcal{L}_{n-m}^{TD}(\Pi)\big|_{\Pi=\partial\partial\phi}.
\end{aligned} \tag{D.28}$$

Here, we introduce a notation $(\Pi^{-1})_{\mu\nu} := 0$ and use (D.18) to simplify the above formula. We come to the conclusion

$$\begin{aligned}
&\frac{\delta}{\delta h^{\mu\nu}}\big(\sqrt{-\det g}\,\mathcal{L}_n^{TD}(\mathcal{K})\big)\big|_{h=0,A=0} \\
=&\sum_{m=0}^{n}(-1)^{m}\frac{n!}{2(n-m)!}\Big[(\Pi^{m})_{\mu\nu} - (\Pi^{m-1})_{\mu\nu}\Big]\mathcal{L}_{n-m}^{TD}(\Pi)\big|_{\Pi=\partial\partial\phi} \\
=&\frac{1}{2}\big(X(\Pi)^{(n)}_{\mu\nu} + nX(\Pi)^{(n-1)}_{\mu\nu}\big)\big|_{\Pi=\partial\partial\phi}.
\end{aligned} \tag{D.29}$$

In the case of $\Pi^\mu{}_\nu = \partial^\mu\partial_\nu\phi$, we can also prove

$$\phi\mathcal{L}_n^{TD}(\partial\partial\phi) = -\frac{1}{2}(n+1)(\partial_\mu\phi)(\partial^\mu\phi)\mathcal{L}_{n-1}^{TD}(\partial\partial\phi) + (\text{total derivative}), \tag{D.30}$$

which is combined with (D.15) to give

$$\phi X(\partial\partial\phi)^{(n)\mu}_{\mu} = -\frac{1}{2}(D-n)(n+1)(\partial_\mu\phi)(\partial^\mu\phi)\mathcal{L}_{n-1}^{TD}(\partial\partial\phi) + (\text{total derivative}). \tag{D.31}$$

Similarly, we can show that the following relation

$$(\partial_\mu\phi)(\partial_\nu\phi)X(\partial\partial\phi)^{(n)\mu\nu} = \big(\frac{1}{2}n+1\big)(\partial_\mu\phi)(\partial^\mu\phi)\mathcal{L}_n^{TD}(\partial\partial\phi) + (\text{total derivative}) \tag{D.32}$$

holds. These relations are used in Section 3.4.

Appendix E

Linearized interaction in bimetric gravity

In D-dimensional space-times, the interaction term of bimetric gravity (3.150) is constructed from the following elements

$$\frac{\sqrt{-\det g}\, e_0\big(\sqrt{g^{-1}f}\big)}{\sqrt{-\det \bar{g}}} = 1 + \frac{1}{2}[h] - \frac{1}{4}[h^2] + \frac{1}{8}[h]^2, \tag{E.1}$$

$$\begin{aligned}\frac{\sqrt{-\det g}\, e_1\big(\sqrt{g^{-1}f}\big)}{\sqrt{-\det \bar{g}}} =& D + \Big(\frac{D}{2} - \frac{1}{2}\Big)[h] + \frac{1}{2}[l] + \Big(-\frac{D}{4} + \frac{3}{8}\Big)[h^2] - \frac{1}{4}[hl] - \frac{1}{8}[l^2]\\ &+ \Big(\frac{D}{8} - \frac{1}{4}\Big)[h]^2 + \frac{1}{4}[h][l],\end{aligned} \tag{E.2}$$

$$\begin{aligned}2 \cdot \frac{\sqrt{-\det g}\, e_2\big(\sqrt{g^{-1}f}\big)}{\sqrt{-\det \bar{g}}} =& D(D-1) + \Big(\frac{D}{2} - 1\Big)(D-1)[h] + (D-1)[l]\\ &+ \Big(-\frac{D^2}{4} + D - 1\Big)[h^2] - \Big(\frac{D}{2} - 1\Big)[hl] - \frac{D}{4}[l^2]\\ &+ \Big(\frac{D^2}{8} - \frac{5}{8}D + \frac{3}{4}\Big)[h]^2 + \Big(\frac{D}{2} - 1\Big)[h][l] + \frac{1}{4}[l]^2,\end{aligned} \tag{E.3}$$

$$\begin{aligned}&6 \cdot \frac{\sqrt{-\det g}\, e_3\big(\sqrt{g^{-1}f}\big)}{\sqrt{-\det \bar{g}}}\\ =& (D^3 - 3D^2 + 2D) + \Big(\frac{1}{2}D^3 - 3D^2 + \frac{11}{2}D - 3\Big)[h] + \Big(\frac{3}{2}D^2 - \frac{9}{2}D + 3\Big)[l]\\ &+ \Big(-\frac{D^3}{4} + \frac{15}{8}D^2 - \frac{37}{8}D + \frac{15}{4}\Big)[h^2] + \Big(-\frac{3}{4}D^2 + \frac{15}{4}D - \frac{9}{2}\Big)[hl] + \Big(-\frac{3}{8}D^2 + \frac{3}{8}D + \frac{3}{4}\Big)[l^2]\\ &+ \Big(\frac{D^3}{8} - \frac{9}{8}D^2 + \frac{13}{4}D - 3\Big)[h]^2 + \Big(\frac{3}{4}D^2 - \frac{15}{4}D + \frac{9}{2}\Big)[h][l] + \Big(\frac{3}{4}D - \frac{3}{2}\Big)[l]^2,\end{aligned} \tag{E.4}$$

$$
\begin{aligned}
&24\cdot\frac{\sqrt{-\det g}\,e_4\big(\sqrt{g^{-1}f}\big)}{\sqrt{-\det\bar{g}}}\\
=&(D^4-6D^3+11D^2-6D)+\Big(\frac{D^4}{2}-5D^3+\frac{35}{2}D^2-25D+12\Big)[h]+(2D^3-12D^2+22D-12)[l]\\
&+\Big(-\frac{D^4}{4}+3D^3-\frac{53}{4}D^2+\frac{51}{2}D-18\Big)[h^2]+(-D^3+9D^2-26D+24)[hl]\\
&+\Big(-\frac{D^3}{2}+\frac{3}{2}D^2+2D-6\Big)[l^2]+\Big(\frac{D^4}{8}-\frac{14}{8}D^3+\frac{71}{8}D^2-\frac{77}{4}D+15\Big)[h]^2\\
&+(D^3-9D^2+26D-24)[h][l]+\Big(\frac{3}{2}D^2-\frac{15}{2}D+9\Big)[l]^2, \qquad \text{(E.5)}
\end{aligned}
$$

where we have calculated up to e_4.

Appendix F

The Poisson bracket in dRGT massive/bimetric gravity

In this Appendix, we calculate the Poisson bracket which we need in Section 4.3. In calculating the Poisson bracket, it seems hard to take into account derivatives with respect to variables in $\hat{n}^i(\gamma, \pi, \omega)$. However, in fact, we do not need to trace variables hidden in $\hat{n}^i(\gamma, \pi, \omega)$. To see this fact, we recall variation of the Lagrangian density (4.102)

$$\frac{\partial \mathcal{L}}{\partial \hat{n}^k} = \mathcal{C}_i \frac{\partial}{\partial \hat{n}^k}(L\hat{n}^i + N\hat{D}^i{}_j \hat{n}^j), \tag{F.1}$$

where $\mathcal{C}_i$ represents the left hand side of (4.105), namely

$$\begin{aligned} \mathcal{C}_i := \mathcal{R}_i^{(g)} - 2m^2\sqrt{\det\gamma}\frac{1}{\sqrt{\hat{x}}}\hat{n}^T\omega\bigg\{&\beta_1\mathbf{1} + \beta_2\sqrt{\hat{x}}\big(\mathbf{1}\mathrm{Tr}\hat{D} - \hat{D}\big) \\ &+ \beta_3\hat{x}\Big[\hat{D}^2 - \hat{D}\mathrm{Tr}\hat{D} + \frac{1}{2}\mathbf{1}\big(\mathrm{Tr}^2\hat{D} - \mathrm{Tr}\hat{D}^2\big)\Big]\bigg\}_i. \end{aligned} \tag{F.2}$$

The variational principle leads to the equation of motion $\mathcal{C}_i = 0$, which we have solved as $\hat{n}^i = \hat{n}^i(\gamma, \pi, \omega)$. On the other hand, from the explicit formulae for the Lagrangian densities (4.112) and (4.115), we can express the left hand side of (F.1) as

$$\frac{\partial \mathcal{L}}{\partial \hat{n}^k} = -\frac{\partial \mathcal{H}_f}{\partial \hat{n}^k} + N\frac{\partial \mathcal{C}}{\partial \hat{n}^k}. \tag{F.3}$$

Thus, we obtain

$$\frac{\partial \mathcal{H}_f}{\partial \hat{n}^k} = -L\mathcal{C}_k, \qquad \frac{\partial \mathcal{C}}{\partial \hat{n}^k} = \mathcal{C}_i\frac{\partial}{\partial \hat{n}^k}(\hat{D}^i{}_j\hat{n}^j). \tag{F.4}$$

Since we have solved $\mathcal{C}_i = 0$ and substituted the solution, we conclude

$$\frac{\partial \mathcal{H}_f}{\partial \hat{n}^k} = 0, \qquad \frac{\partial \mathcal{C}}{\partial \hat{n}^k} = 0. \tag{F.5}$$

Therefore, when we concern $\mathcal{C}$ and $\mathcal{H}_f$, we need not to take care of derivatives with respect to variables contained in $\hat{n}^i(\gamma, \pi, \omega)$.

F.1 $\{\mathcal{C}(x), \mathcal{C}(y)\}_{PB}$

We perform the explicit calculation of the Poisson bracket $\{\mathcal{C}(x), \mathcal{C}(y)\}_{PB}$. We omit the subscript "PB" for convenience. In dRGT massive/bimetric gravity, the Poisson bracket is determined by

$$\{F(x), G(y)\} = \int d^3z \Big[\frac{\delta F(x)}{\delta \gamma_{mn}(z)} \frac{\delta G(y)}{\delta \pi^{mn}(z)} - \frac{\delta F(x)}{\delta \pi^{mn}(z)} \frac{\delta G(y)}{\delta \gamma_{mn}(z)} + (\omega \text{ and } p \text{ derivatives})\Big], \quad \text{(F.6)}$$

where we neglect the coefficients M_g and M_f. The above Poisson bracket contains functional derivatives with respect to ω_{ij} and p^{ij}. However, we have only to calculate derivatives with respect to γ_{ij} and π^{ij} because $\mathcal{C}$ does not contain p^{ij}. Therefore, there is no difference between the case of dRGT massive gravity and that of bimetric gravity. Results are translated each other via the replacement $m^2 \leftrightarrow M_{eff}^2 m^2/M_g^2$.

Now, we write down the Poisson bracket using the Leibniz rule

$$\begin{aligned}
\{\mathcal{C}(x), \mathcal{C}(y)\}_{PB} =& \{\mathcal{R}_0(x),\, \mathcal{R}_0(y)\} && \text{(F.7)}\\
&+ \{\mathcal{R}_0(x),\, \mathcal{R}_j \hat{D}^j{}_l \hat{n}^l(y)\} + \{\mathcal{R}_i \hat{D}^i{}_k \hat{n}^k(x),\, \mathcal{R}_0(y)\} && \text{(F.8)}\\
&+ \{\mathcal{R}_0(x),\, 2m^2\sqrt{\gamma}\, V(y)\} + \{2m^2\sqrt{\gamma}\, V(x),\, \mathcal{R}_0(y)\} && \text{(F.9)}\\
&+ \{\mathcal{R}_i \hat{D}^i{}_k \hat{n}^k(x),\, \mathcal{R}_j \hat{D}^j{}_l \hat{n}^l(y)\} && \text{(F.10)}\\
&+ \{\mathcal{R}_i \hat{D}^i{}_k \hat{n}^k(x),\, 2m^2\sqrt{\gamma}\, V(y)\} + \{2m^2\sqrt{\gamma}\, V(x),\, \mathcal{R}_j \hat{D}^j{}_l \hat{n}^l(y)\} && \text{(F.11)}\\
&+ \{2m^2\sqrt{\gamma}\, V(x),\, 2m^2\sqrt{\gamma}\, V(y)\}, && \text{(F.12)}
\end{aligned}$$

where γ is a shorthand notation for $\det\gamma$ and we have dropped the subscript (g) on $\mathcal{R}_{0,i}$. We can immediately find that (F.12) vanishes because both terms do not contain π outside $\hat{n}^i$. Similarly, (F.9) has no contribution, which can be shown as follows. Since $\sqrt{\gamma}\, V(y)$ does not contain π outside $\hat{n}$, we notice

$$\{\mathcal{R}_0(x),\, \sqrt{\gamma}\, V(y)\} \propto \frac{\delta \mathcal{R}_0}{\delta \pi} \frac{\delta(\sqrt{\gamma} V)}{\delta \gamma}. \quad \text{(F.13)}$$

Besides, there is no derivative on π in $\mathcal{R}_0$ and no derivative on γ in $\sqrt{\gamma}V$. Thus, we find $\{\mathcal{R}_0(x),\, \sqrt{\gamma}\, V(y)\} \propto \delta(x-y)$, and conclude

$$\{\mathcal{R}_0(x),\, \sqrt{\gamma}\, V(y)\} + \{\sqrt{\gamma}\, V(x),\, \mathcal{R}_0(y)\} \propto \delta(x-y) - \delta(y-x) = 0. \quad \text{(F.14)}$$

We can simplify (F.8). Applying the Leibniz rule, we have

$$\{\mathcal{R}_0(x),\, \mathcal{R}_j \hat{D}^j{}_l \hat{n}^l(y)\} = \{\mathcal{R}_0(x),\, \mathcal{R}_j(y)\} \hat{D}^j{}_l \hat{n}^l(y) + \{\mathcal{R}_0(x),\, \hat{D}^j{}_l \hat{n}^l(y)\} \mathcal{R}_j(y). \quad \text{(F.15)}$$

We notice that $\hat{D}^j{}_l \hat{n}^l$ does not contain π outside $\hat{n}$ and also does not have $\partial\gamma$ terms, while $\mathcal{R}_0$ contains no $\partial\pi$ terms. Then, we find $\{\mathcal{R}_0(x),\, \hat{D}^j{}_l \hat{n}^l(y)\} \mathcal{R}_j(y) \propto \delta(x-y)$, and (F.8) is simplified to be

$$\{\mathcal{R}_0(x),\, \mathcal{R}_j(y)\} \hat{D}^j{}_l \hat{n}^l(y) + \hat{D}^i{}_k \hat{n}^k(x) \{\mathcal{R}_i(x),\, \mathcal{R}_0(y)\}. \quad \text{(F.16)}$$

We proceed to take (F.10). Since $\hat{D}^j{}_l\hat{n}^l$ does not depend on π_{ij} outside $\hat{n}$, we find

$$\begin{aligned}\left\{\mathcal{R}_i\hat{D}^i{}_k\hat{n}^k(x),\ \mathcal{R}_j\hat{D}^j{}_l\hat{n}^l(y)\right\} =&\hat{D}^i{}_k\hat{n}^k(x)\left\{\mathcal{R}_i(x),\ \mathcal{R}_j(y)\right\}\hat{D}^j{}_l\hat{n}^l(y)\\ &+\hat{D}^i{}_k\hat{n}^k(x)\left\{\mathcal{R}_i(x),\ \hat{D}^j{}_l\hat{n}^l(y)\right\}\mathcal{R}_j(y)\\ &+\mathcal{R}_i(x)\left\{\hat{D}^i{}_k\hat{n}^k(x),\ \mathcal{R}_j(y)\right\}\hat{D}^j{}_l\hat{n}^l(y)\\ &+\mathcal{R}_i(x)\left\{\hat{D}^i{}_k\hat{n}^k(x),\ \hat{D}^j{}_l\hat{n}^l(y)\right\}\mathcal{R}_j(y)\end{aligned} \tag{F.17}$$

$$\begin{aligned}=&\hat{D}^i{}_k\hat{n}^k(x)\left\{\mathcal{R}_i(x),\ \mathcal{R}_j(y)\right\}\hat{D}^j{}_l\hat{n}^l(y)\\ &+\mathcal{R}_i(x)\frac{\delta\hat{D}^i{}_k\hat{n}^k(x)}{\delta\gamma_{mn}(x)}\frac{\delta\mathcal{R}_j(y)}{\delta\pi^{mn}(x)}\cdot\hat{D}^j{}_l\hat{n}^l(y)\\ &-\mathcal{R}_j(y)\frac{\delta\hat{D}^j{}_l\hat{n}^l(y)}{\delta\gamma_{mn}(y)}\frac{\delta\mathcal{R}_i(x)}{\delta\pi^{mn}(y)}\cdot\hat{D}^i{}_k\hat{n}^k(x).\end{aligned} \tag{F.18}$$

We can also write down (F.11) with functional derivatives

$$\left\{\mathcal{R}_i\hat{D}^i{}_k\hat{n}^k(x),\ 2m^2\sqrt{\gamma}\,V(y)\right\} = -2m^2\frac{\delta\sqrt{\gamma}V(y)}{\delta\gamma_{mn}(y)}\frac{\delta\mathcal{R}_i(x)}{\delta\pi^{mn}(y)}\cdot\hat{D}^i{}_k\hat{n}^k(x). \tag{F.19}$$

Collecting all of these formulae, we obtain

$$\left\{\mathcal{C}(x),\mathcal{C}(y)\right\} =\left\{\mathcal{R}_0(x),\mathcal{R}_0(y)\right\} \tag{F.20}$$

$$+\left\{\mathcal{R}_0(x),\ \mathcal{R}_j(y)\right\}\hat{D}^j{}_l\hat{n}^l(y)-\left\{\mathcal{R}_0(y),\ \mathcal{R}_j(x)\right\}\hat{D}^j{}_l\hat{n}^l(x) \tag{F.21}$$

$$+\hat{D}^i{}_k\hat{n}^k(x)\left\{\mathcal{R}_i(x),\ \mathcal{R}_j(y)\right\}\hat{D}^j{}_l\hat{n}^l(y) \tag{F.22}$$

$$+S^{mn}(x)\frac{\delta\mathcal{R}_j(y)}{\delta\pi^{mn}(x)}\cdot\hat{D}^j{}_l\hat{n}^l(y)-S^{mn}(y)\frac{\delta\mathcal{R}_i(x)}{\delta\pi^{mn}(y)}\cdot\hat{D}^i{}_k\hat{n}^k(x), \tag{F.23}$$

where S^{mn} is defined by

$$S^{mn}(x) := \mathcal{R}_i(x)\frac{\delta\hat{D}^i{}_k\hat{n}^k(x)}{\delta\gamma_{mn}(x)}+2m^2\frac{\delta\sqrt{\gamma}V(x)}{\delta\gamma_{mn}(x)}. \tag{F.24}$$

The Poisson brackets among $\mathcal{R}_0$ and $\mathcal{R}_i$ are obtained in Appendix C, which says

$$\left\{\mathcal{R}_0(x),\ \mathcal{R}_0(y)\right\} = -\left[\mathcal{R}^i(x)\frac{\partial}{\partial x^i}\delta^{(3)}(x-y)-\mathcal{R}^i(y)\frac{\partial}{\partial y^i}\delta^{(3)}(x-y)\right],\ \mathcal{R}^i:=\gamma^{ij}\mathcal{R}_j, \tag{F.25}$$

$$\left\{\mathcal{R}_0(x),\ \mathcal{R}_i(y)\right\} = -\mathcal{R}_0(y)\frac{\partial}{\partial x^i}\delta^{(3)}(x-y), \tag{F.26}$$

$$\left\{\mathcal{R}_i(x),\ \mathcal{R}_j(y)\right\} = -\left[\mathcal{R}_j(x)\frac{\partial}{\partial x^i}\delta^{(3)}(x-y)-\mathcal{R}_i(y)\frac{\partial}{\partial y^j}\delta^{(3)}(x-y)\right]. \tag{F.27}$$

We also need the following formula

$$\begin{aligned}\int d^3x\frac{\delta\mathcal{R}_j(x)}{\delta\pi^{mn}(y)}v^j(x) =&\int d^3x\frac{\delta}{\delta\pi^{mn}(y)}\left[2\gamma_{jl}\mathcal{D}_k\pi^{lk}\right]v^j(x)\\ =&-2\int d^3x\left(\mathcal{D}_kv_l\right)\frac{\delta\pi^{kl}}{\delta\pi^{mn}(y)}\\ =&-\mathcal{D}_mv_n(y)-\mathcal{D}_nv_m(y)\end{aligned} \tag{F.28}$$

for any vector $v^j = \gamma^{jk} v_k$. The symbol $\mathcal{D}_i$ represents the covariant derivative constructed from the spatial metric γ_{ij}. In the following calculation, it is convenient to consider the integrated Poisson bracket

$$\int d^3x \int d^3y f(x) g(y) \{\mathcal{C}(x), \mathcal{C}(y)\} \tag{F.29}$$

with some functions f and g. We multiply two functions $f(x)$ and $g(y)$ to the Poisson bracket $\{\mathcal{C}(x), \mathcal{C}(y)\}$ and integrate over x and y. If we define $\mathcal{C}(f) := \int d^3x f(x)\mathcal{C}(x)$ and $\mathcal{C}(g) := \int d^3y g(y)\mathcal{C}(y)$, this Poisson bracket is expressed as $\{\mathcal{C}(f), \mathcal{C}(g)\} = \int d^3x \int d^3y f(x)g(y)\{\mathcal{C}(x), \mathcal{C}(y)\}$. We integrate (F.20) with functions $f(x)$ and $g(y)$, and obtain

$$\begin{aligned}
&\int d^3x \int d^3y f(x) g(y) \big\{\mathcal{R}_0(x),\ \mathcal{R}_0(y)\big\} \\
&= \int d^3x \int d^3y f(x) g(y) \Big[-\mathcal{R}^i(x) \frac{\partial}{\partial x^i} \delta^{(3)}(x-y) + \mathcal{R}^i(y) \frac{\partial}{\partial y^i} \delta^{(3)}(x-y) \Big] \\
&= \int d^3x g \frac{\partial}{\partial x^i}(f\mathcal{R}^i) - \int d^3y f \frac{\partial}{\partial y^i}(g\mathcal{R}^i) \\
&= \int d^3x \big(g\partial_i f - f\partial_i g\big)\mathcal{R}^i
\end{aligned} \tag{F.30}$$

Similarly, we can see that integration of (F.21) leads to

$$\int d^3x \big(g\partial_i f - f\partial_i g\big) R^0 \hat{D}^i{}_k \hat{n}^k, \tag{F.31}$$

and (F.22) is integrated to give

$$\int d^3x \big(g\partial_i f - f\partial_i g\big) R_j \hat{D}^j{}_l \hat{n}^l \hat{D}^i{}_k \hat{n}^k. \tag{F.32}$$

We apply (F.28) to the integrated formula for (F.23), and find

$$-2\int d^3x \big(f\partial_m g - g\partial_m f\big) S^{mn} \gamma_{nj} \hat{D}^j{}_k \hat{n}^k. \tag{F.33}$$

Hence, we obtain

$$\{\mathcal{C}(f), \mathcal{C}(g)\} = -\int d^3x \big(f\partial_i g - g\partial_i f\big) P^i, \tag{F.34}$$

where P^i is determined by

$$P^i := R^i + R^0 \hat{D}^i{}_k \hat{n}^k + R_j \hat{D}^j{}_l \hat{n}^l \hat{D}^i{}_k \hat{n}^k + 2S^{il}\gamma_{lj} \hat{D}^j{}_k \hat{n}^k. \tag{F.35}$$

Here, we express $g(x)$ as $g(x) = \int d^3y g(y)\delta^{(3)}(x-y)$ in the first term on the right hand side of (F.34)

$$\int d^3x f(x) \Big[\frac{\partial}{\partial x^i} g(x)\Big] P^i(x) = \int d^3x \int d^3y f(x) g(y) \Big[\frac{\partial}{\partial x^i} \delta^{(3)}(x-y)\Big] P^i(x). \tag{F.36}$$

We also write $f(x)$ as $f(x)=\int d^3y f(y)\delta^{(3)}(x-y)$ in the second term on the right hand side of (F.34), and rename integration variables as $x\leftrightarrow y$

$$\int d^3x g(x)\Big[\frac{\partial}{\partial x^i}f(x)\Big]P^i(x)=\int d^3x\int d^3y f(x)g(y)\Big[\frac{\partial}{\partial y^i}\delta^{(3)}(x-y)\Big]P^i(y). \tag{F.37}$$

Therefore, we obtain

$$\{\mathcal{C}(f),\mathcal{C}(g)\}=-\int d^3x\int d^3y f(x)g(y)\Big[P^i(x)\frac{\partial}{\partial x^i}\delta^{(3)}(x-y)-P^i(y)\frac{\partial}{\partial y^i}\delta^{(3)}(x-y)\Big], \tag{F.38}$$

and read the Poisson bracket

$$\big\{\mathcal{C}(x),\mathcal{C}(y)\big\}=-\Big[P^i(x)\frac{\partial}{\partial x^i}\delta^{(3)}(x-y)-P^i(y)\frac{\partial}{\partial y^i}\delta^{(3)}(x-y)\Big]. \tag{F.39}$$

At first glance, P^i defined by (F.35) seems complicated, but we can show that it is proportional to $\mathcal{C}$. To see this fact, we write down S^{mn} explicitly

$$S^{mn}(x)=\mathcal{R}_i(x)\frac{\delta\hat{D}^i{}_k\hat{n}^k(x)}{\delta\gamma_{mn}(x)}+2m^2\frac{\delta\sqrt{\gamma}V(x)}{\delta\gamma_{mn}(x)}=\mathcal{R}_i\frac{\delta\hat{D}^i{}_k\hat{n}^k}{\delta\gamma_{mn}}+2m^2\Big(\frac{1}{2}\sqrt{\gamma}\,\gamma^{mn}V+\sqrt{\gamma}\,\frac{\delta V}{\delta\gamma_{mn}}\Big), \tag{F.40}$$

where we have used $\delta\sqrt{\det\gamma}=\frac{1}{2}\sqrt{\det\gamma}\,\mathrm{Tr}(\gamma^{-1}\delta\gamma)$. V is defined by (4.80), and in order to calculate the derivative on it, we prepare the following variation with respect to γ_{ij}

$$\begin{aligned}
\delta\sqrt{\hat{x}}\hat{D}=&\delta\mathrm{Tr}\sqrt{\big(\gamma^{-1}-\hat{D}\hat{n}\hat{n}^T\hat{D}^T\big)\omega}\\
=&\frac{1}{2}\mathrm{Tr}\Big[\sqrt{\big(\gamma^{-1}-\hat{D}\hat{n}\hat{n}^T\hat{D}^T\big)\omega}^{\,-1}\delta\big(\gamma^{-1}-\hat{D}\hat{n}\hat{n}^T\hat{D}^T\big)\omega\Big]\\
=&\frac{1}{2}\frac{1}{\sqrt{\hat{x}}}\big(\hat{D}^{-1}\big)^i{}_j\Big[\delta\gamma^{jk}\omega_{ki}-\delta(\hat{D}^j{}_k\hat{n}^k)\hat{n}^l\hat{D}^m{}_l\omega_{mi}-\hat{D}^j{}_k\hat{n}^k\delta(\hat{n}^l\hat{D}^m{}_l)\omega_{mi}\Big]\\
=&\frac{1}{2\sqrt{\hat{x}}}\Big[\omega_{ki}\big(\hat{D}^{-1}\big)^i{}_j\delta\gamma^{jk}-\big(\hat{D}^{-1}\big)^i{}_j\hat{D}^m{}_i\omega_{ml}\hat{n}^l\delta(\hat{D}^j{}_k\hat{n}^k)-\hat{n}^i\omega_{mi}\delta(\hat{n}^l\hat{D}^m{}_l)\Big]\\
=&\frac{1}{2\sqrt{\hat{x}}}\omega_{ij}\big(\hat{D}^{-1}\big)^j{}_k\delta\gamma^{ki}-\frac{1}{\sqrt{\hat{x}}}\hat{n}^i\omega_{ij}\delta(\hat{D}^j{}_k\hat{n}^k)\\
=&\frac{1}{2\sqrt{\hat{x}}}\mathrm{Tr}\big(\omega\hat{D}^{-1}\delta\gamma^{-1}\big)-\frac{1}{\sqrt{\hat{x}}}\hat{n}^T\omega\delta(\hat{D}\hat{n}),
\end{aligned} \tag{F.41}$$

and in the same way

$$\delta(\sqrt{\hat{x}}\hat{D})^2=\mathrm{Tr}(\omega\delta\gamma^{-1})-2\hat{n}^T\omega\hat{D}\delta(\hat{D}\hat{n}), \tag{F.42}$$

$$\delta(\sqrt{\hat{x}}\hat{D})^3=\frac{3}{2}\sqrt{\hat{x}}\,\mathrm{Tr}(\omega\hat{D}\delta\gamma^{-1})-3\sqrt{\hat{x}}\,\hat{n}^T\omega\hat{D}^2\delta(\hat{D}\hat{n}). \tag{F.43}$$

Here, it should be noted that the above calculation includes variation with respect to γ_{ij} within $\hat{n}^i(\gamma,\pi,\omega)$. We also need

$$\delta\gamma^{ij}=-\gamma^{ik}\,\delta\gamma_{kl}\,\gamma^{lj}, \tag{F.44}$$

which comes from $\gamma^{ik}\gamma_{kj} = \delta^i_j$. Then, we collect these variation, and after a rather lengthy calculation, we find

$$\delta V = -\frac{1}{2}\bar{V}^{mn}\delta\gamma_{mn} + \frac{\mathcal{C}_i - R_i}{2m^2\sqrt{\gamma}}\delta(\hat{D}\hat{n})^i, \tag{F.45}$$

where we have defined

$$\begin{aligned}\bar{V}^{mn} := \gamma^{mi}\Bigg[&\frac{\beta_1}{\sqrt{\hat{x}}}\omega\hat{D}^{-1} + \beta_2\big\{(\mathrm{Tr}\hat{D})\omega\hat{D}^{-1} - \omega\big\}\\ &+ \beta_3\sqrt{\hat{x}}\left\{\frac{1}{2}(\mathrm{Tr}^2\hat{D} - \mathrm{Tr}\hat{D}^2)\omega\hat{D}^{-1} + \omega\hat{D} - (\mathrm{Tr}\hat{D})\omega\right\}\Bigg]_{ij}\gamma^{jn}.\end{aligned} \tag{F.46}$$

Thus, we obtain

$$S^{mn} = \mathcal{C}_i\frac{\delta\hat{D}^i{}_k\hat{n}^k}{\delta\gamma_{mn}} + m^2\sqrt{\gamma}\big(\gamma^{mn}V - \bar{V}^{mn}\big). \tag{F.47}$$

We put $\mathcal{C}_i = 0$ and substitute (F.47) into (F.35), and we have

$$P^i = \mathcal{C}\hat{D}^i{}_k\hat{n}^k - 2m^2\sqrt{\det\gamma}\,\bar{V}^{il}\gamma_{lj}\hat{D}^j{}_k\hat{n}^k + \mathcal{R}^i. \tag{F.48}$$

We apply (F.46) to the second term on the right had side of (F.48)

$$\begin{aligned}\bar{V}^{il}\gamma_{lj}\hat{D}^j{}_k\hat{n}^k = \gamma^{im}\Bigg(&\frac{1}{\sqrt{\hat{x}}}\hat{n}^T\omega\Big[\beta_1\mathbf{1} + \beta_2\sqrt{\hat{x}}\left\{(\mathrm{Tr}\hat{D})\mathbf{1} - \hat{D}\right\}\\ &+ \beta_3\hat{x}\Big\{\frac{1}{2}\big(\mathrm{Tr}^2\hat{D} - \mathrm{Tr}\hat{D}^2\big)\mathbf{1} + \hat{D}^2 - (\mathrm{Tr}\hat{D})\hat{D}\Big\}\Big]\Bigg)_m,\end{aligned} \tag{F.49}$$

which leads to

$$\mathcal{R}^i - 2m^2\sqrt{\det\gamma}\,\bar{V}^{il}\gamma_{lj}\hat{D}^j{}_k\hat{n}^k = \gamma^{im}\mathcal{C}_m. \tag{F.50}$$

Therefore, we finally obtain

$$P^i = \mathcal{C}\hat{D}^i{}_jn^j + \gamma^{ij}\mathcal{C}_j = \mathcal{C}\hat{D}^i{}_jn^j. \tag{F.51}$$

F.2 $\big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\}_{PB}$

We perform the detailed calculation of $\big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\}_{PB}$. It is useful to rely on the integrated form

$$\int d^3y\big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\}_{PB}. \tag{F.52}$$

The calculation in Appendix F.1 has no difference between the case of dRGT massive gravity and that of bimetric gravity. Their results are translated each other under the replacement $m^2 \leftrightarrow M^2_{eff}m^2/M^2_g$. On the contrary, $\big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\}_{PB}$ picks up a difference. Since $\mathcal{H}_f$ contains momenta p^{ij}, we have such a result

$$\big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\}_{bimetric} = \big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\}_{massive} + \frac{\delta\mathcal{R}^{(f)}}{\delta p} \times (\cdots). \tag{F.53}$$

However, our main purpose is to show $\{\mathcal{C}(x), \mathcal{H}_f(y)\}_{PB} \not\approx 0$. Thus, we have only to see $\{\mathcal{C}(x), \mathcal{H}_f(y)\}_{massive} \not\approx 0$, and the following calculation is based on dRGT massive gravity. We assume that $\mathcal{C}_i = 0$ has been solved as $\hat{n}^i = \hat{n}^i(\gamma, \pi, \omega)$, and do not trace derivatives with respect to variables in $\hat{n}^i(\gamma, \pi, \omega)$

To begin with, we write down the Poisson bracket

$$\begin{aligned}\{\mathcal{C}(x), \mathcal{H}_f(y)\} =& - \{\mathcal{R}_0(x),\ (L\hat{n}^i + L^i)\mathcal{R}_i(y)\} \\ & - 2m^2\{\mathcal{R}_0(x),\ L\sqrt{\det\gamma}\,U(y)\} \\ & - \{\mathcal{R}_i\hat{D}^i{}_k\hat{n}^k(x),\ (L\hat{n}^j + L^j)\mathcal{R}_j(y)\} \\ & - 2m^2\{\mathcal{R}_i\hat{D}^i{}_k\hat{n}^k(x),\ L\sqrt{\det\gamma}\,U(y)\} \\ & - 2m^2\{\sqrt{\det\gamma}\,V(x),\ (L\hat{n}^j + L^j)\mathcal{R}_j(y)\} \\ & - 4m^2\{\sqrt{\det\gamma}\,V(x),\ L\sqrt{\det\gamma}\,U(y)\},\end{aligned} \tag{F.54}$$

which can be simplified via the Leibniz rule

$$\begin{aligned}\{\mathcal{C}(x), \mathcal{H}_f(y)\} =& - \{\mathcal{R}_0(x),\ \mathcal{R}_i(y)\}(L\hat{n}^i + L^i)(y) \\ & - 2m^2\{\mathcal{R}_0(x),\ L\sqrt{\det\gamma}\,U(y)\} \\ & - \hat{D}^i{}_k\hat{n}^k(x)\{\mathcal{R}_i(x),\ \mathcal{R}_j(y)\}(L\hat{n}^j + L^j)(y) \\ & - R_i(x)\{\hat{D}^i{}_k\hat{n}^k(x),\ \mathcal{R}_j(y)\}(L\hat{n}^j + L^j)(y) \\ & - 2m^2\{\mathcal{R}_i\hat{D}^i{}_k\hat{n}^k(x),\ L\sqrt{\det\gamma}\,U(y)\} \\ & - 2m^2\{\sqrt{\det\gamma}\,V(x),\ \mathcal{R}_j(y)\}(L\hat{n}^j + L^j)(y).\end{aligned} \tag{F.55}$$

We express the above formula with functional derivatives

$$\begin{aligned}\{\mathcal{C}(x), \mathcal{H}_f(y)\} =& - \{\mathcal{R}_0(x),\ \mathcal{R}_i(y)\}(L\hat{n}^i + L^i)(y) - \hat{D}^i{}_k\hat{n}^k(x)\{\mathcal{R}_i(x),\ \mathcal{R}_j(y)\}(L\hat{n}^j + L^j)(y) \\ & + 2m^2\frac{\delta\mathcal{R}_0(x)}{\delta\pi^{mn}(y)}\frac{\delta\sqrt{\gamma}\,U(y)}{\delta\gamma_{mn}(y)}\cdot L(y) \\ & - \mathcal{R}_i(x)\frac{\delta\hat{D}^i{}_k\hat{n}^k(x)}{\delta\gamma_{mn}(x)}\frac{\delta\mathcal{R}_j(y)}{\delta\pi^{mn}(x)}\cdot(L\hat{n}^j + L^j)(y) \\ & + 2m^2\hat{D}^i{}_k\hat{n}^k(x)\frac{\delta\mathcal{R}_i(x)}{\delta\pi^{mn}(y)}\frac{\delta\sqrt{\gamma}\,U(y)}{\delta\gamma_{mn}(y)}\cdot L(y) \\ & - 2m^2\frac{\delta\sqrt{\gamma}\,V(x)}{\delta\gamma_{mn}(x)}\frac{\delta\mathcal{R}_j(y)}{\delta\pi^{mn}(x)}\cdot(L\hat{n}^j + L^j)(y).\end{aligned} \tag{F.56}$$

Here, we define

$$U^{mn} := \frac{2}{\sqrt{\det\gamma}}\frac{\delta\sqrt{\det\gamma}\,U}{\delta\gamma_{mn}}, \tag{F.57}$$

and remember the definition of S^{mn} (F.24). Then, this Poisson bracket can be read as

$$
\begin{aligned}
&\big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\} \\
=& -\big\{\mathcal{R}_0(x),\, \mathcal{R}_i(y)\big\}(L\hat{n}^i + L^i)(y) - \hat{D}^i{}_k\hat{n}^k(x)\big\{\mathcal{R}_i(x),\, \mathcal{R}_j(y)\big\}(L\hat{n}^j + L^j)(y) && \text{(F.58)} \\
&+ m^2 L(y)\frac{\delta\mathcal{R}_0(x)}{\delta\pi^{mn}(y)}\sqrt{\gamma}\,U^{mn}(y) && \text{(F.59)} \\
&+ m^2 L(y)\hat{D}^i{}_k\hat{n}^k(x)\frac{\delta\mathcal{R}_i(x)}{\delta\pi^{mn}(y)}\sqrt{\gamma}\,U^{mn}(y) && \text{(F.60)} \\
&- S^{mn}(x)\frac{\delta\mathcal{R}_j(y)}{\delta\pi^{mn}(x)}(L\hat{n}^j + L^j)(y). && \text{(F.61)}
\end{aligned}
$$

We calculate functional derivatives

$$
\begin{aligned}
\frac{\delta\mathcal{R}_0(x)}{\delta\pi^{mn}(y)} &= \frac{1}{\sqrt{\det\gamma}}\Big(\gamma_{mn}(x)\pi^k{}_k(x) - 2\pi_{mn}(x)\Big)\delta^{(3)}(x-y) && \text{(F.62)} \\
\frac{\delta\mathcal{R}_i(x)}{\delta\pi^{mn}(y)} &= -\Big(\gamma_{im}(x)\mathcal{D}^{(y)}_n + \gamma_{in}(x)\mathcal{D}^{(y)}_m\Big)\delta^{(3)}(x-y), && \text{(F.63)}
\end{aligned}
$$

and substitute the Poisson brackets among $\mathcal{R}_0$ and $\mathcal{R}_i$. Integrating over y, (F.58) becomes

$$
\begin{aligned}
\int d^3y(F.58) =& \mathcal{D}_i\big(\mathcal{R}_0(L\hat{n}^i + L^i)\big) + \hat{D}^i{}_k\hat{n}^k\mathcal{R}_j\mathcal{D}_i(L\hat{n}^j + L^j) + \hat{D}^i{}_k\hat{n}^k\mathcal{D}_j\big(\mathcal{R}_i(L\hat{n}^j + L^j)\big) \\
=& \big(\mathcal{D}_i\mathcal{R}_0 + (\mathcal{D}_i\mathcal{R}_j)\hat{D}^j{}_k\hat{n}^k\big)(L\hat{n}^i + L^i) + \hat{D}^i{}_k\hat{n}^k\mathcal{R}_j\mathcal{D}_i(L\hat{n}^j + L^j) \\
&+ \hat{D}^i{}_k\hat{n}^k\mathcal{R}_i\mathcal{D}_j\big(L\hat{n}^j + L^j\big) + \mathcal{R}_0\mathcal{D}_i\big(L\hat{n}^i + L^i\big), \qquad \text{(F.64)}
\end{aligned}
$$

and (F.59) and (F.60) turn out to be

$$
\begin{aligned}
\int d^3y(F.59) =& m^2 L\big(\gamma_{mn}\pi^k{}_k - 2\pi_{mn}\big)U^{mn}, && \text{(F.65)} \\
\int d^3y(F.60) =& 2m^2\sqrt{\det\gamma}\,\gamma_{im}\mathcal{D}_n(LU^{mn})\hat{D}^i{}_k\hat{n}^k. && \text{(F.66)}
\end{aligned}
$$

Along with (F.47), we integrate (F.61) to find

$$
\begin{aligned}
\int d^3y(F.61) =& 2S^{mn}\gamma_{im}\nabla_n(L\hat{n}^i + L^i) \\
=& 2m^2\sqrt{\det\gamma}\,V\mathcal{D}_i(L\hat{n}^i + L^i) - 2m^2\sqrt{\det\gamma}\,\bar{V}^{mn}\gamma_{mi}\mathcal{D}_n(L\hat{n}^i + L^i). \qquad \text{(F.67)}
\end{aligned}
$$

Collecting these formulae, we obtain

$$
\begin{aligned}
&\int d^3y\big\{\mathcal{C}(x), \mathcal{H}_f(y)\big\} \\
=& m^2 L\big(\gamma_{mn}\pi^k{}_k - 2\pi_{mn}\big)U^{mn} + 2m^2\sqrt{\det\gamma}\,\gamma_{im}\mathcal{D}_n(LU^{mn})\hat{D}^i{}_k\hat{n}^k + \mathcal{C}\,\mathcal{D}_i(L\hat{n}^i + L^i) \\
&+ \big(\hat{D}^i{}_k\hat{n}^k\mathcal{R}_j - 2m^2\sqrt{\det\gamma}\,\bar{V}^{il}\gamma_{lj}\big)\mathcal{D}_i(L\hat{n}^j + L^j) + \big(\mathcal{D}_i\mathcal{R}_0 + \hat{D}^j{}_k\hat{n}^k\mathcal{D}_i\mathcal{R}_j\big)(L\hat{n}^i + L^i). \quad \text{(F.68)}
\end{aligned}
$$

Appendix G

The Poisson bracket in homogeneous bi/tri-metric gravity

In this appendix, we calculate the Poisson bracket among

$$
\begin{aligned}
C_N := &\frac{M_g^2}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^i{}_i\pi^j{}_j\Big) - M_g^2\sqrt{\det\gamma}\,{}^{(3)}\!R[\gamma] \\
&+ a_1\sqrt{\det\gamma}\big(\mathrm{tr}\sqrt{\gamma^{-1}\omega} - 3\big) + a_3\big(\sqrt{\det\rho} - \sqrt{\det\gamma}\big),
\end{aligned} \tag{G.1}
$$

$$
\begin{aligned}
C_L := &\frac{M_f^2}{\sqrt{\det\omega}}\Big(p^{ij}p_{ij} - \frac{1}{2}p^i{}_ip^j{}_j\Big) - M_f^2\sqrt{\det\omega}\,{}^{(3)}\!R[\omega] \\
&+ a_2\sqrt{\det\omega}\big(\mathrm{tr}\sqrt{\omega^{-1}\rho} - 3\big) + a_1\big(\sqrt{\det\gamma} - \sqrt{\det\omega}\big),
\end{aligned} \tag{G.2}
$$

and

$$
\begin{aligned}
C_Q := &\frac{M_h^2}{\sqrt{\det\rho}}\Big(\phi^{ij}\phi_{ij} - \frac{1}{2}\phi^i{}_i\phi^j{}_j\Big) - M_h^2\sqrt{\det\rho}\,{}^{(3)}\!R[\rho] \\
&+ a_3\sqrt{\det\rho}\big(\mathrm{tr}\sqrt{\rho^{-1}\gamma} - 3\big) + a_2\big(\sqrt{\det\omega} - \sqrt{\det\rho}\big),
\end{aligned} \tag{G.3}
$$

which we need in Chapter 6. In a homogeneous space, the Poisson bracket is defined as

$$
\begin{aligned}
\{F\,,\,G\}_{PB} = &\frac{1}{M_g^2}\Big(\frac{\partial F}{\partial\gamma_{mn}}\frac{\partial G}{\partial\pi^{mn}} - \frac{\partial F}{\partial\pi^{mn}}\frac{\partial G}{\partial\gamma_{mn}}\Big) + \frac{1}{M_f^2}\Big(\frac{\partial F}{\partial\omega_{mn}}\frac{\partial G}{\partial p^{mn}} - \frac{\partial F}{\partial p^{mn}}\frac{\partial G}{\partial\omega_{mn}}\Big) \\
&+ \frac{1}{M_h^2}\Big(\frac{\partial F}{\partial\rho_{mn}}\frac{\partial G}{\partial\phi^{mn}} - \frac{\partial F}{\partial\phi^{mn}}\frac{\partial G}{\partial\rho_{mn}}\Big).
\end{aligned} \tag{G.4}
$$

The calculation is done based on the trimetric case, but that of the bimetric case is obtained by setting parameters $a_1 \neq 0$, $a_2 = 0$ and $a_3 = 0$. In the following, we omit the subscript "PB".

To begin with, we calculate $C_{NL} := \{C_N, C_L\}$. Since C_N and C_L do not contain ϕ^{ij}, there is no contribution from the pair $(\rho_{ij},\, \phi^{ij})$.

$$\begin{aligned}\big\{C_N\,,\,C_L\big\} =&\Big\{\frac{M_g^2}{\sqrt{\det\gamma}}\Big(\frac{1}{2}\pi^i{}_i\pi^j{}_j-\pi^{ij}\pi_{ij}\Big)\,,\,-a_1\sqrt{\det\gamma}\Big\}\\&+\Big\{-a_1\sqrt{\det\gamma}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}\,,\,\frac{M_f^2}{\sqrt{\det\omega}}\Big(\frac{1}{2}p^i{}_ip^j{}_j-p^{ij}p_{ij}\Big)\Big\}\\=&a_1\Bigg(\frac{M_g^2}{\sqrt{\det\gamma}}\Big\{\sqrt{\det\gamma}\,,\,\frac{1}{2}\pi^i{}_i\pi^j{}_j-\pi^{ij}\pi_{ij}\Big\}-M_f^2\sqrt{\frac{\det\gamma}{\det\omega}}\Big\{\mathrm{tr}\sqrt{\gamma^{-1}\omega}\,,\,\frac{1}{2}p^i{}_ip^j{}_j-p^{ij}p_{ij}\Big\}\Bigg)\,.\end{aligned} \tag{G.5}$$

The point is that the result is proportional to a_1. Each term can be manipulated as

$$\begin{aligned}M_g^2\Big\{\sqrt{\det\gamma}\,,\,\frac{1}{2}\pi^i{}_i\pi^j{}_j-\pi^{ij}\pi_{ij}\Big\} &= \frac{\partial\sqrt{\det\gamma}}{\partial\gamma_{mn}}\frac{\partial}{\partial\pi^{mn}}\Big(\frac{1}{2}\pi^i{}_i\pi^j{}_j-\pi^{ij}\pi_{ij}\Big)\\&=\frac{1}{2}\sqrt{\det\gamma}\,\gamma^{mn}\left(\gamma_{mn}\pi^i{}_i-2\pi_{mn}\right)\\&=\frac{1}{2}\sqrt{\det\gamma}\,\pi^i{}_i\end{aligned} \tag{G.6}$$

and

$$\begin{aligned}M_f^2\Big\{\mathrm{tr}\sqrt{\gamma^{-1}\omega}\,,\,\frac{1}{2}p^i{}_ip^j{}_j-p^{ij}p_{ij}\Big\} &= \frac{\partial\mathrm{tr}\sqrt{\gamma^{-1}\omega}}{\partial\omega_{mn}}\frac{\partial}{\partial p^{mn}}\Big(\frac{1}{2}p^i{}_ip^j{}_j-p^{ij}p_{ij}\Big)\\&=\frac{1}{2}\Big(\sqrt{\gamma^{-1}\omega}^{-1}\gamma^{-1}\Big)^{mn}\left(\omega_{mn}p^i{}_i-2p_{mn}\right)\\&=\frac{1}{2}p^i{}_i\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}-\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big)\;,\end{aligned} \tag{G.7}$$

where p represents a matrix with components p^{mn}. Hence, we conclude

$$C_{NL}=\big\{C_N\,,\,C_L\big\}=a_1\Bigg[\frac{1}{2}\pi^i{}_i-\sqrt{\frac{\det\gamma}{\det\omega}}\left(\frac{1}{2}p^i{}_i\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}-\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big)\right)\Bigg]. \tag{G.8}$$

In this case, $C_{NL}\neq 0$ because there is an interaction between g and f, namely $a_1\neq 0$. Thus, if the Poisson bracket is non-trivial or not is determined by the interaction pattern. The others $C_{LQ}:=\{C_L,C_Q\}$ and $C_{QN}:=\{C_Q,C_N\}$ are obtained via permutations among $g=(N,\gamma)$, $f=(L,\omega)$ and $h=(Q,\rho)$

We proceed to the Poisson bracket $\{C_{NL},C_N\}$. Since C_{NL} does not contain ρ_{ij} and ϕ^{ij}, the pair $(\rho_{ij},\,\phi^{ij})$ has no contribution.

$$
\begin{aligned}
&\frac{1}{a_1}\{C_{NL},\,C_N\}\\
=&\frac{1}{2}M_g^2\Big\{\pi^k{}_k,\ \frac{1}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^i{}_i\pi^j{}_j\Big)-\sqrt{\det\gamma}\,{}^{(3)}R[\gamma]\Big\}\\
&+\frac{1}{2}a_1\Big\{\pi^k{}_k,\ \sqrt{\det\gamma}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big\}-\frac{1}{2}(3a_1+a_3)\Big\{\pi^k{}_k,\ \sqrt{\det\gamma}\Big\}\\
&-\frac{1}{2}M_g^2\frac{p^k{}_k}{\sqrt{\det\omega}}\Big\{\sqrt{\det\gamma}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega},\ \frac{1}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^i{}_i\pi^j{}_j\Big)\Big\}\\
&-\frac{1}{2}a_1\sqrt{\det\gamma}\sqrt{\frac{\det\gamma}{\det\omega}}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big\{p^k{}_k,\ \mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big\}\\
&+M_g^2\frac{1}{\sqrt{\det\omega}}\Big\{\sqrt{\det\gamma}\,\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big),\ \frac{1}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^i{}_i\pi^j{}_j\Big)\Big\}\\
&+a_1\sqrt{\det\gamma}\sqrt{\frac{\det\gamma}{\det\omega}}\Big\{\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big),\ \mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big\}. \qquad (G.9)
\end{aligned}
$$

Noticing that $\pi^k{}_k=\gamma_{kl}\pi^{kl}$ and $p^k{}_k=\omega_{kl}p^{kl}$, each term is calculated as

$$M_g^2\Big\{\pi^k{}_k,\ \frac{1}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^i{}_i\pi^j{}_j\Big)\Big\}=\frac{3}{2}\frac{1}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^i{}_i\pi^j{}_j\Big), \qquad (G.10)$$

$$M_g^2\Big\{\pi^k{}_k,\ \sqrt{\det\gamma}\,{}^{(3)}R[\gamma]\Big\}=-\frac{1}{2}\sqrt{\det\gamma}\,{}^{(3)}R[\gamma], \qquad (G.11)$$

$$M_g^2\Big\{\pi^k{}_k,\ \sqrt{\det\gamma}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big\}=-\sqrt{\det\gamma}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}, \qquad (G.12)$$

$$M_g^2\Big\{\pi^k{}_k,\ \sqrt{\det\gamma}\Big\}=-\frac{3}{2}\sqrt{\det\gamma}, \qquad (G.13)$$

$$
\begin{aligned}
&M_g^2\Big\{\sqrt{\det\gamma}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega},\ \frac{1}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^i{}_i\pi^j{}_j\Big)\Big\}\\
=&-\frac{1}{2}\pi^k{}_k-\frac{1}{\sqrt{\det\gamma}}\Big(\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,\pi\,\gamma\big)-\frac{1}{2}\pi^k{}_k\mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big), \qquad (G.14)
\end{aligned}
$$

$$M_f^2\Big\{p^k{}_k,\ \mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big\}=-\frac{1}{2}\mathrm{tr}\sqrt{\gamma^{-1}\omega}, \qquad (G.15)$$

$$
\begin{aligned}
&M_g^2\Big\{\sqrt{\det\gamma}\,\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big),\ \frac{1}{\sqrt{\det\gamma}}\Big(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^i{}_i\pi^j{}_j\Big)\Big\}\\
=&-\frac{1}{2}\pi^k{}_k\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big)-\big(\gamma_{mn}\pi^k{}_k-2\pi_{mn}\big)\frac{\partial}{\partial\gamma_{mn}}\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big), \qquad (G.16)
\end{aligned}
$$

$$M_f^2\Big\{\mathrm{tr}\big(\sqrt{\gamma^{-1}\omega}\,p\,\omega\big),\ \mathrm{tr}\sqrt{\gamma^{-1}\omega}\Big\}=-\frac{1}{2}\mathrm{tr}\big(\gamma^{-1}\omega\big), \qquad (G.17)$$

where π and p represent matrices with components π^{mn} and p^{mn}. Therefore, we conclude

$$\begin{aligned}
&\left\{C_{NL},\, C_N\right\}/a_1 \\
=&\frac{1}{4}\left[\frac{3}{\sqrt{\det\gamma}}\left(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^i{}_i\pi^j{}_j\right)+\sqrt{\det\gamma}\,{}^{(3)}R[\gamma]\right] \\
&-\frac{1}{2M_g^2}\sqrt{\det\gamma}\left[a_1\mathrm{tr}\sqrt{\gamma^{-1}\omega}-\frac{3}{2}(3a_1+a_3)\right] \\
&+\frac{1}{2}\frac{p^i{}_i}{\sqrt{\det\omega}}\left[\frac{1}{2}\pi^j{}_j+\frac{1}{\sqrt{\det\gamma}}\left(\mathrm{tr}\left(\sqrt{\gamma^{-1}\omega}\,\pi\,\gamma\right)-\frac{1}{2}\pi^k{}_k\mathrm{tr}\sqrt{\gamma^{-1}\omega}\right)\right] \\
&+\frac{1}{4M_f^2}a_1\frac{\det\gamma}{\sqrt{\det\omega}}\left[\mathrm{tr}^2\sqrt{\gamma^{-1}\omega}-2\mathrm{tr}\left(\gamma^{-1}\omega\right)\right] \\
&-\frac{1}{2}\frac{1}{\sqrt{\det\omega}}\left[\pi^k{}_k\mathrm{tr}\left(\sqrt{\gamma^{-1}\omega}\,p\,\omega\right)+2\left(\gamma_{mn}\pi^k{}_k-2\pi_{mn}\right)\frac{\partial}{\partial\gamma_{mn}}\mathrm{tr}\left(\sqrt{\gamma^{-1}\omega}\,p\,\omega\right)\right]. \qquad \text{(G.18)}
\end{aligned}$$

It is almost clear that $\left\{C_{NL},\, C_N\right\}\not\approx 0$.

The Poisson bracket $\left\{C_{NL},\, C_Q\right\}$ is easy to calculate

$$\begin{aligned}
\left\{C_{NL},\, C_Q\right\}/a_1 =&\frac{1}{2}a_3\sqrt{\det\rho}\left\{\pi^k{}_k,\ \mathrm{tr}\sqrt{\rho^{-1}\gamma}\right\} \\
&-a_2\sqrt{\frac{\det\gamma}{\det\omega}}\left(\frac{1}{2}\mathrm{tr}\sqrt{\gamma^{-1}\omega}\left\{p^k{}_k,\ \sqrt{\det\gamma}\right\}-\left\{\mathrm{tr}\left(\sqrt{\gamma^{-1}\omega}\,p\,\omega\right),\ \sqrt{\det\omega}\right\}\right).
\end{aligned} \qquad \text{(G.19)}$$

Each element is calculated to be

$$M_g^2\left\{\pi^k{}_k,\ \mathrm{tr}\sqrt{\rho^{-1}\gamma}\right\}=-\frac{1}{2}\mathrm{tr}\sqrt{\rho^{-1}\gamma}, \qquad \text{(G.20)}$$

$$M_f^2\left\{p^k{}_k,\ \sqrt{\det\omega}\right\}=-\frac{3}{2}\sqrt{\det\omega}, \qquad \text{(G.21)}$$

$$M_f^2\left\{\mathrm{tr}\left(\sqrt{\gamma^{-1}\omega}\,p\,\omega\right),\ \sqrt{\det\omega}\right\}=-\frac{1}{2}\sqrt{\det\omega}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}. \qquad \text{(G.22)}$$

Thus, we obtain

$$\left\{C_{NL},\, C_Q\right\}/a_1=-\frac{a_3}{4M_g^2}\sqrt{\det\rho}\,\mathrm{tr}\sqrt{\rho^{-1}\gamma}+\frac{a_2}{4M_f^2}\sqrt{\det\gamma}\,\mathrm{tr}\sqrt{\gamma^{-1}\omega}. \qquad \text{(G.23)}$$

Apparently, it is not automatically vanish $\left\{C_{NL},\, C_Q\right\}\not\approx 0$.

Other Poisson brackets such as $\left\{C_{NL},\, C_N\right\}$ or $\left\{C_{NQ},\, C_N\right\}$,...etc are obtained via permutations among $g=(N,\gamma)$, $f=(L,\omega)$ and $h=(Q,\rho)$.

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