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"Improving Efficiency Using Reserve Prices: An Equilibrium Analysis of Core-Selecting Auctions"

Ryuji Sano

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Improving Efficiency Using Reserve Prices: An Equilibrium Analysis of Core-Selecting Auctions*

Ryuji Sano†
Institute of Economic Research, Kyoto University

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Abstract

This study analyzes the equilibrium of core-selecting auctions under incomplete information. We consider the ascending proxy auction of Ausubel and Milgrom (2002) in a stylized environment with two goods and three bidders: two local and one global. Local bidders with a sufficiently high value bid almost truthfully. In contrast, local bidders with a sufficiently low value submit a zero bid because of free-riding incentive. We also provide equilibrium with reserve prices, and show that a reserve price for local bidders improve both allocative efficiency and revenue in equilibrium.

Keywords: core-selecting auctions, ascending proxy auction, reserve price
JEL code: D44, D47

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†Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan. E-mail address: sano@kier.kyoto-u.ac.jp

1 Introduction

When multiple interrelated goods are allocated efficiently, the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) is an important benchmark. It is known as a unique mechanism satisfying efficiency, incentive compatibility in dominant strategy, and individual rationality. However, the VCG mechanism has several drawbacks, such as low revenue and weakness against joint deviations when goods may be complements. In many real applications such as spectrum license auctions, complementarity naturally exists. The VCG mechanism may not be appropriate, and we require a careful design of an auction rule that considers these problems.[1]

Core-selecting auctions have been proposed by Day and Milgrom (2008) as an alternative to the VCG mechanism. A core-selecting auction selects the efficient allocation and payments such that the associated payoff profile lies in the core with respect to the reported bids.[2] The core-selecting property prevents some disadvantages of the VCG mechanism. Core-selecting auctions have been recently adopted and used for spectrum license auctions in many countries including the U.K., Australia, Canada and the Netherlands.[3]

A theoretical concern about core-selecting auctions is that they are not incentive compatible. A core-selecting auction is not guaranteed to implement a core outcome with respect to the true valuations in equilibrium. The incentive and equilibrium analysis of core-selecting auctions are still under development. Most preceding studies have considered complete information and analyzed full-information Nash equilibrium, because Bayesian analysis is too complex in a general setting. It is known that a core outcome is achievable in a full-information Nash equilibrium, which supports the use of core-selecting auctions in practice (Ausubel and Milgrom, 2002; Day and Milgrom, 2008; Beck and Ott,

[1] See Milgrom (2000) and Roth (2002) for an early design of spectrum license auctions. See Ausubel and Milgrom (2006a) for the theoretical drawbacks of the VCG mechanism.

[2] The core-selecting auction is not unique. See Erdil and Klemperer (2010), Day and Cramton (2012), and Lamy (2012) for designs of particular core-selecting auctions.

[3] See Cramton (2013) and Ausubel and Baranov (2014) for recent applications of core-selecting auctions to spectrum license allocations in the world.

2013). In the Bayesian setting, however, Goeree and Lien (2014) show that there is no incentive compatible core-selecting auction if the VCG mechanism is not core-selecting. In the presence of complementarity and information asymmetry, core-selecting auctions are inefficient in equilibrium. Equilibrium properties under incomplete information are of great importance for evaluating the performance of core-selecting auctions and considering future market design.

This study considers a simple stylized environment with two goods and three bidders, called the "LLG" model. There are two *local bidders* who each demands one of the goods and one *global bidder* who demands both goods. We examine a Bayesian Nash equilibrium of the ascending proxy auction introduced by Ausubel and Milgrom (2002; 2006b). The ascending proxy auction is a core-selecting auction whose outcome is determined using an ascending auction algorithm.[4] We show that although the global bidder has a weakly dominant strategy of truth-telling, the local bidders have an incentive to underbid. The incentive to underbid is strong when the value is low, and the local bidders submit a zero bid with positive probability in equilibrium. Consequently, the seller suffers zero revenue with positive probability. However, this negative result can be mitigated to some extent by setting reserve prices. A reserve price for local bidders increases the equilibrium bids, and improves both efficiency and revenue.

Reserve prices are commonly observed in real auctions of both single and multiple objects. In multiple-object auctions, it is possible to set a variety of reserve price rules (Ausubel and Cramton, 2004). Day and Cramton (2012) provide two methods for implementing reserve prices for core-selecting auctions. In the first method, a reserve price is treated as a dummy bid by the seller, thus that it directly influences the calculation of allocation and payments. In the second method, a reserve price is simply the lower bound of payments, and it is not used for calculation of allocation and payments. The former is called the "reserve bidder" rule, and the latter the "bounds only" rule. There are no robust preferences over these rules theoretically or practically.

We consider both reserve price rules and derive the equilibrium for each. We show

[4]The same ascending auction algorithm is proposed by Parkes and Ungar (2000).

that in the bounds only rule, a high reserve price for the global bidder reduces equilibrium bids of local bidders. Given a low or zero reserve price for the global bidder, the bounds only rule is more efficient than the reserve bidder rule. The socially optimal reserve price rule is the bounds only, in which a positive reserve price is imposed only on local bidders. We also show that among the ascending proxy auctions with reserve prices, the expected revenue is maximized by a combination of the reserve bidder and bounds only reserve prices.

Recently, several studies have independently examined Bayesian equilibrium analysis for the LLG model. Goeree and Lien (2014) derive a Bayesian equilibrium of another particular auction rule called the "nearest-VCG rule," in which core payments are determined by minimizing the Euclidean distance from those of the VCG mechanism. Ausubel and Baranov (2010) consider several core-selecting auction rules, including the ascending proxy and nearest-VCG. These papers focus only on the uniform distribution case and do not consider the effect of reserve prices.[5] Sano (2012) considers an ascending price core-selecting auction and a perfect Bayesian equilibrium. Hafalir and Yektas (2015) consider an incentive compatible mechanism that is not core-selecting but minimizes the distance from the core.

2 Model

We consider a stylized LLG model of two goods and three bidders. A seller wants to allocate two heterogeneous objects A and B. There are three buyers $\{1, 2, 3\}$. Bidder 1 wants only good A, whereas bidder 2 wants only good B. Bidders 1 and 2 are termed *local bidders*. Bidder 3, termed the *global bidder*, wants both A and B. All bidders are risk-neutral and have quasi-linear utilities. When a local bidder i obtains his desired good (A or B) and pays p_i, he earns a payoff $v_i - p_i$. Global bidder 3 earns a payoff $V_3 - p_3$ when he wins both A and B and pays p_3. The global bidder has a value of zero for each good individually: that is, goods are perfect complements for bidder 3. This situation is commonly known; however, the valuations for each bidder's desired goods are private

[5] Ausubel and Baranov allow a special type of correlation between local bidders' values.

information. Each value for a local bidder, v_i, is drawn from a distribution function F on the interval $[0, 1]$, whereas the global bidder's value V_3 is drawn from another distribution G on $[0, 2]$. These distributions have density functions f, $g > 0$.

A *core-selecting auction* is an auction mechanism where goods allocation and associated payments are determined such that the resulting payoff profile lies in the core with respect to the bids. Bidders simultaneously submit a sealed bid b_i for their desired goods. The seller allocates the goods efficiently with respect to the bids. That is, local bidders 1 and 2 win their desired goods if and only if $b_1 + b_2 \geq b_3$; otherwise bidder 3 wins both goods.[6] A payment rule of a core-selecting auction satisfies the following conditions: $p_1 + p_2 \geq b_3$ when bidders 1 and 2 win, and $p_3 \geq b_1 + b_2$ when bidder 3 wins, where p_i is bidder i's payment.

In the VCG mechanism payments are determined by each bidder's externality that he gives to the other bidders. The VCG payment rule here is given by $(p_1^V, p_2^V) = (\max\{0, b_3 - b_2\}, \max\{0, b_3 - b_1\})$ when bidders 1 and 2 win and by $p_3^V = b_1 + b_2$ when bidder 3 wins. The VCG mechanism is known to be incentive compatible in dominant strategy. However, the VCG mechanism is not core-selecting as $p_1^V + p_2^V < b_3$ when the local bidders win: the seller and bidder 3 form a blocking coalition and have an incentive to deviate from the current allocation. Moreover, the seller's revenue is $p_1^V + p_2^V = 0$ when the local bidders' bids are larger than b_3. This property is criticized in the literature and is one reason why the VCG mechanism has not been used for real spectrum license auctions (Ausubel and Milgrom, 2006a).

2.1 Ascending Proxy Auction

We consider a particular core-selecting auction, the ascending proxy auction, conceived by Ausubel and Milgrom (2002; 2006b). The ascending proxy auction is a sealed-bid auction, but the outcome is determined using an algorithm similar to an ascending price auction. There are proxy agents of bidders who participate in an ascending auction based on the bids.

[6] Ties are broken randomly when $b_1 + b_2 = b_3$.

We first describe the auction without a reserve price. Suppose that each bidder i submits a bid b_i for his desired good(s). Then, the ascending auction by proxy agents proceeds as follows.[7] At the initial round ($t = 1$), each (proxy) bidder places a bid of $b_i^1 = \epsilon$. Given b^1, in the revenue-maximizing allocation, bidders 1 and 2 obtain goods. In round 2, (proxy) bidder 3 raises the bid: $b_3^2 = 2\epsilon$. If the auctioneer selects bidder 3 as the tentative winner in round 2, bidders 1 and 2 raise the bid to 2ϵ in round 3, and so on. The proxy bidders raise the bids until the submitted bid.

Thus, as a result of the ascending auction algorithm, the auctioneer finally selects the efficient allocation with respect to bids. The payment rule $p(b) = (p_1(b), p_2(b), p_3(b))$ is specified as follows:

$$p(b) = \begin{cases} (\frac{1}{2}b_3, \frac{1}{2}b_3, 0) & \text{if } \min\{b_1, b_2\} \geq \frac{1}{2}b_3 \\ (b_1, b_3 - b_1, 0) & \text{if } 2b_1 < b_3 < b_1 + b_2 \\ (b_3 - b_2, b_2, 0) & \text{if } 2b_2 < b_3 < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if } b_3 > b_1 + b_2 \end{cases}. \tag{1}$$

Two kinds of reserve price rules, the bounds only (hereafter, BO) and the reserve bidder (hereafter, RB) rules, are introduced to the ascending proxy auction. Suppose (r_1, r_2, R_3) is the vector of reserve prices for bidders. For simplicity, we assume that $r_1 = r_2 = r$ and $R_3 \leq 2r$ in both rules. The followings are common to both rules. Each bidder needs to bid at least his reserve price when submitting a bid. The difference from the no reserve price case is that with reserve prices, the ascending auction algorithm starts from the reserve prices. The reserve prices do not affect the allocation and payments when all bids are sufficiently larger than the reserve prices. If the global bidder makes no bid and only local bidder(s) submit a bid, the local bidder(s) win with a payment of r. If two local bidders do not bid and the global bidder bids, the global bidder wins both goods with a payment of R_3.

Two reserve price rules generate different outcomes when only bidders 1 (or 2) and 3

[7] See Ausubel and Milgrom (2002; 2006b) for the precise and general definition of the ascending proxy auction.

submit bids. In the BO rule, the auction is reduced to a standard second-price auction with discriminatory reserve prices: i.e., bidder 1 wins good A if $b_1 \geq b_3$ with a payment of $p_1 = \max\{r, b_3\}$. Otherwise, bidder 3 wins both goods with a payment of $p_3 = \max\{R_3, b_1\}$. In the RB rule, reserve prices are treated as dummy bids. Hence, bidder 1 wins good A with a payment $p_1 = \max\{r, b_3 - r\}$ if $b_1 + r \geq b_3$. Otherwise, bidder 3 wins both goods with a payment of $p_3 = b_1 + r$. In the RB rule, the effective reserve price for the global bidder is $R_3 \geq 2r$ because even if $R_3 < 2r$, the "local reserve bidders" outbid any $b_3 < 2r$. Hence, from the assumption of $R_3 \leq 2r$, we set $R_3 = 2r$ in the RB rule.

Remark 1 Precisely speaking, a core-selecting auction with reserve prices is not core-selecting or efficient (with respect to bids) because reserve prices may clearly exclude low-value bidders. If we interpret the reserve prices as the seller's valuations for the goods, the auction with the RB rule is core-selecting in a precise sense. However, we assume that the seller's values for goods are all zero, and that the reserve prices generally generate the efficiency loss.

3 Analysis

The ascending proxy auction is identical to the VCG mechanism for bidder 3. Global bidder 3 wins if and only if $b_3 \geq b_1 + b_2$ and the payment is determined as the critical value $b_1 + b_2$. Thus, it is a weakly dominant strategy for bidder 3 to bid truthfully in the ascending proxy auction regardless of the reserve price.

Lemma 1 (Ausubel and Baranov, 2010; Sano, 2011; Goeree and Lien, 2014) *The global bidder, bidder 3, has a weakly dominant strategy of bidding truthfully.*

Hereafter, we regard the auction game as a two-player auction game by local bidders and focus on their incentives. We consider only pure strategies. A strategy β_i for (local) bidder i is a mapping from $[0, 1]$ to $\{\emptyset\} \cup [0, 1]$, where $\emptyset$ indicates no bid. We consider a symmetric Bayesian Nash equilibrium as the equilibrium concept.[8]

[8] We call an equilibrium symmetric if the equilibrium bidding functions are symmetric between local bidders.

3.1 No Reserve Price Case

We first consider the case of no reserve price.[9] The interim expected payoff function given truth-telling by bidder 3 and a strategy β_j of the other local bidder is denoted by $\pi_i(b_i, v_i)$, where v_i is i's value and b_i is bid. The winning probability of bidder 1 when bidding b_1 is denoted by

$$\Phi(b_1) \equiv \Pr\{b_1 + \beta_2(v_2) > V_3\}.$$

The interim expected payoff of bidder 1 is given by

$$\pi_1(b_1, v_1) = \Phi(b_1)v_i - E[p_1(b_1, \beta_2(v_2), V_3)].$$

Specifically,

$$\begin{aligned}\pi_1(b_1, v_1) =& \Phi(b_1)v_1 - b_1 \Pr\{2b_1 < V_3 < b_1 + \beta_2\} - E[\frac{1}{2}V_3 \mathbf{1}_{\{2b_1 \geq V_3,\ 2\beta_2 \geq V_3\}}] \\ & - E[(V_3 - \beta_2)\mathbf{1}_{\{2\beta_2 < V_3 < b_1 + \beta_2\}}].\end{aligned}$$

By differentiating with respect to b_1, we have the first-order condition for maximization,

$$(v_1 - b_1)\phi(b_1) - \Pr\{2b_1 < V_3 < b_1 + \beta_2(v_2)\} \leq 0, \tag{2}$$

where the equality holds for $b_1 > 0$. In addition,

$$\phi(b_1) = \Phi'(b_1) = \int_{v_2} f(v)g(b_1 + \beta_2(v))\mathrm{d}v \tag{3}$$

and

$$\Pr\{2b_1 < V_3 < b_1 + \beta_2(v_2)\} = \int_{b_1 < \beta_2} \int_{2b_1}^{b_1+\beta_2} f(v)g(s)\mathrm{d}s\mathrm{d}v. \tag{4}$$

Equations (2)–(4) determine an equilibrium strategy for bidders 1 and 2.

To obtain a closed-form equilibrium bidding function, we need to assume that the value of the global bidder is uniformly distributed.

Assumption 1 The value of the global bidder is uniformly distributed: $G(V) = \frac{V}{2}$.

The following theorem shows an equilibrium of the ascending proxy auction without a reserve price under the assumption. Ausubel and Baranov (2010) independently derive the result for the case where F is also uniform.

[9] The analysis in this subsection is based on an earlier version of the paper, Sano (2010).

Theorem 1 *Suppose Assumption 1. There exists a symmetric Bayesian Nash equilibrium of the ascending proxy auction with no reserve price,*

$$\beta(v) = \max\Big\{0, v - \int_v^1 \frac{1-F(s)}{F(s)}\mathrm{d}s\Big\}.$$

In particular, if F is also the uniform distribution,

$$\beta(v) = \max\{0, 1 + \log v\}. \tag{5}$$

Proof. See Appendix.

The equilibrium generates a serious low revenue problem. In the case of uniform distributions, bidders 1 and 2 submit zero if their values are less than $1/\mathrm{e} \doteqdot 0.37$. The probability that the seller earns no revenue is $(1/\mathrm{e})^2 \doteqdot 0.14$.

The payment rule (1) for local bidder i is likely to be the "first price" for the case $b_i < b_j$, whereas it is likely to be the VCG for the case $b_i > b_j$. Hence, when both the value and the bid are sufficiently high, his payment would hardly bind the bid; thus, he has little incentive to underbid and submits a bid close to the true value in equilibrium. Conversely, when a local bidder has a low value, he has a strong incentive to reduce his bid. Moreover, a local bidder has a chance to win a good even if he submits a zero bid. Hence, local bidders submit a zero bid when their values are sufficiently low and fully free-ride on the other local bidder.

This feature is most striking in a degenerate case where local values v_1 and v_2 are common knowledge among local bidders. Suppose that local bidders know the value for each other, that $v_1 > v_2$, and that V_3 is still private information of the global bidder and uniformly distributed. Then, there exists a unique Nash equilibrium in which bidder 1 submits his true value whereas bidder 2 submits zero and perfectly free-rides on bidder 1.

Proposition 1 *Suppose that the values of local bidders v_1 and v_2 are commonly known and that $v_1 > v_2$. When G is uniform, there exists a unique Nash equilibrium in which $\beta_1(v_1, v_2) = v_1$ and $\beta_2(v_1, v_2) = 0$.*[10]

[10] When $v_1 = v_2$, there exist many Nash equilibria such that $\beta_1 = v_1$ and $\beta_2 = \tilde{v}$, $\forall \tilde{v} \in [0, v_1]$. Thus,

Proof. See Appendix.

Even when G is not uniform, the equilibrium strategy possesses similar properties with the uniform case. First, local bidders submit almost truthfully when their values are sufficiently large. Suppose that there exists a symmetric equilibrium strategy β. Notice that $\Pr\{2\beta(v_1) < V_3 < \beta(v_1) + \beta(v_2)\}$ represents bidder 1's incentive to underbid. Since this probability fades away to 0 as v_1 goes to 1, we have $\beta(1) = 1$. Second, local bidders submit a zero bid with positive probability. As long as bidder 2 submits a non-zero bid $\beta(v_2) > 0$ with positive probability, we obtain $\Pr\{V_3 < \beta(v_2)\} > 0$. Hence, for a sufficiently small v_1, we have

$$\frac{\partial}{\partial b_1}\pi_1(b_1, v_1)|_{b_1=0} < 0, \tag{6}$$

that is, the first-order condition (2) does not hold with equality. Hence, $\beta(v_1) = 0$ in equilibrium for a sufficiently small v_1. These are summarized as follows.

Proposition 2 *When G may not be uniform, the symmetric Bayesian Nash equilibrium of the ascending proxy auction satisfies $\beta(1) = 1$ and $\exists \hat{v} > 0$, $\beta(v) = 0$ for all $v < \hat{v}$. Consequently, the seller suffers zero revenue with positive probability in equilibrium.*

3.2 The Reserve Bidder Rule

We now consider reserve prices. We first consider the RB rule. We can take the first-order approach in the same manner with no reserve price case.

When bidder 2 makes no bid, bidder 1 wins the good if and only if $b_1 + r \geq b_3$. The first-order condition for the RB rule is given by

$$(v_1 - b_1)\phi^{RB}(b_1; r) - \Pr\{2b_1 < V_3 < b_1 + \beta(v_2)\} \leq 0, \tag{7}$$

where

$$\phi^{RB}(b_1; r) = g(b_1 + r)F(r) + \int_r^1 g(b_1 + \beta(v_2))f(v_2)\mathrm{d}v_2 \tag{8}$$

truth-telling by all bidders is an equilibrium of a knife-edge case of $v_1 = v_2$. Ausubel and Baranov (2010) also report this truth-telling equilibrium.

and (4). Interestingly, the first-order condition is independent of r when G is uniform. The first-order condition is identical to the case of no reserve price. Thus, we immediately have the following theorem. Proof is similar to Theorem 1 and omitted.

Theorem 2 *Suppose Assumption 1 and the RB rule with local reserve bidders r. There exists a symmetric Bayesian Nash equilibrium of the ascending proxy auction,*

$$\beta^{RB}(v;r) = \max\Big\{r, v - \int_v^1 \frac{1-F(s)}{F(s)}\mathrm{d}s\Big\} \tag{9}$$

for $v \geq r$ and $\beta^{RB}(v;r) = \emptyset$ for $v < r$.

The only difference from the no reserve price case is that the minimum bid is raised by r. The equilibrium bidding function has almost the same form.

3.3 The Bounds Only Rule

We next consider the BO rule. A significant difference from the RB rules is that when bidder 2 makes no bid, bidder 1 wins the good if and only if $b_1 \geq b_3$ in the BO rule. The marginal payoff function $\frac{\partial}{\partial b_1}\pi_1(b_1, v_1)$ is given by

$$\frac{\partial}{\partial b_1}\pi_1(b_1, v_1; r, R_3) = (v_1 - b_1)\phi^{BO}(b_1; r, R_3) - \Pr\{2b_1 < V_3 < b_1 + \beta(v_2)\}, \tag{10}$$

where

$$\phi^{BO}(b_1; r, R_3) = \begin{cases} g(b_1)F(r) + \int_r^1 g(b_1 + \beta(v_2))f(v_2)\mathrm{d}v_2 & \text{if } b_1 \geq R_3 \\ \int_r^1 g(b_1 + \beta(v_2))f(v_2)\mathrm{d}v_2 & \text{if } b_1 < R_3 \end{cases} \tag{11}$$

and (4). Using uniform G, the marginal payoff function (10) yields

$$2 \cdot \frac{\partial}{\partial b_1}\pi_1(b_1, v_1; r, R_3) = \begin{cases} (v_1 - b_1) - \int_{b_1<\beta_2}(\beta(s) - b_1)f(s)\mathrm{d}s & \text{if } b_1 \geq R_3 \\ (v_1 - b_1)(1 - F(r)) - \int_{b_1<\beta_2}(\beta(s) - b_1)f(s)\mathrm{d}s & \text{if } b_1 < R_3 \end{cases}, \tag{12}$$

Again, we have the identical marginal payoff function to the case of no reserve price when G is uniform and $b_1 \geq R_3$. However, the marginal payoff function has a jump at $b_1 = R_3$, and we have a different marginal payoff for $b_1 < R_3$. We can follow the

first-order approach in two cases. First, when $R_3 \leq r$, we need not consider the case of $b_1 < R_3$ because $b_1 \geq r$. In this case, we have the same first-order condition with the no reserve price case, so that equilibrium bidding function coincides with the RB rule. Second, if $1 \leq R_3 (\leq 2r)$, any local bid is less than R_3. In this case, local bidders shade more in equilibrium than the case of low R_3.

Theorem 3 *Suppose Assumption 1 and the BO rule with reserve prices (r, R_3). If $R_3 \leq r$, there exists a symmetric Bayesian Nash equilibrium that is identical to the RB rule: $\beta^{BO}(v; r, R_3) = \beta^{RB}(v; r)$ for all v. If $1 \leq R_3 \leq 2r$, there exists a symmetric Bayesian Nash equilibrium*

$$\beta^{BO}(v; r, R_3) = \max\Big\{r, v - \int_v^1 \frac{1 - F(s)}{F(s) - F(r)} \mathrm{d}s\Big\} \tag{13}$$

for $v \geq r$ and $\beta^{BO}(v; r, R_3) = \emptyset$ for $v < r$.

Proof. See Appendix.

Suppose that $r < R_3 < 1$. We cannot straightforwardly take the first-order approach. To derive the equilibrium, assume that bidder 2 takes an "equilibrium strategy" β such that is increasing in the interior and jumps at most once (at $\hat{v}$). Let

$$\bar{h}(b_1, v_1) \equiv (v_1 - b_1) - \int_{b_1 < \beta_2} (\beta(s) - b_1) f(s) \mathrm{d}s \tag{14}$$

and

$$\underline{h}(b_1, v_1) \equiv (v_1 - b_1)(1 - F(r)) - \int_{b_1 < \beta_2} (\beta(s) - b_1) f(s) \mathrm{d}s.^{11} \tag{15}$$

It is clear that $\bar{h}(b_1, v_1) > \underline{h}(b_1, v_1)$ for all $b_1, v_1 \geq r$. Both $\bar{h}$ and $\underline{h}$ is decreasing in b_1. Hence, the optimal bid satisfies $\bar{h}(b_1, v_1) = 0$ as long as $\underline{h}(R_3, v_1) \geq 0$, whereas the optimal bid satisfies $\underline{h}(b_1, v_1) \leq 0$ as long as $\bar{h}(R_3, v_1) \leq 0$. As in the case of no reserve price and the RB rule, the local optimum condition $\bar{h}(\bar{\beta}(v), v) = 0$ yields

$$\bar{\beta}(v) = v - \int_v^1 \frac{1 - F(s)}{F(s)} \mathrm{d}s. \tag{16}$$

The optimal bid function jumps at $\hat{v}$ which satisfies $\underline{h}(R_3, \hat{v}) < 0 < \bar{h}(R_3, \hat{v})$. The optimal bid function has either form of the following two cases. In one case, the local optimum

[11] Both $\bar{h}$ and $\underline{h}$ depend on r and R_3 explicitly or implicitly through $\beta(v_2)$.

condition $\underline{h}(\beta(\hat{v}), \hat{v}) = 0$ holds. In the other case, the local optimum is a corner solution $\underline{h}(r, \hat{v}) \leq 0$. The following theorem states that the equilibrium bid function in each case. To state the theorem, we define

$$\hat{\beta}(v) \equiv \bar{\beta}(v) - \frac{F(r)}{F(v) - F(r)}\big(v - \bar{\beta}(v)\big) \tag{17}$$

where $\bar{\beta}(v)$ is defined by (16).

Theorem 4 *Suppose Assumption 1 and the BO rule with reserve prices (r, R_3) satisfying $r < R_3 \leq \min\{1, 2r\}$. If there exists $\hat{v}$ that solves*

$$F(\hat{v})(\bar{\beta}(\hat{v}) - R_3)^2 = (F(\hat{v}) - F(r))(R_3 - \hat{\beta}(\hat{v}))^2 \tag{18}$$

and satisfies

$$r \leq \hat{\beta}(\hat{v}) < R_3 < \bar{\beta}(\hat{v}), \tag{19}$$

then a symmetric equilibrium bidding function is given by

$$\beta^{BO}(v; r, R_3) = \begin{cases} \bar{\beta}(v) & \text{if } v \geq \hat{v} \\ \max\big\{r, v - \int_v^{\hat{v}} \frac{1-F(s)}{F(s)-F(r)} \mathrm{d}s - (\hat{v} - \hat{\beta}(\hat{v}))\big\} & \text{if } r \leq v < \hat{v} \end{cases} \tag{20}$$

and $\beta^{BO}(v; r, R_3) = \emptyset$ for $v < r$. Otherwise, if there is no $\hat{v}$ satisfying (18) and (19), then there exists $\tilde{v} > \bar{\beta}^{-1}(R_3)$ that satisfies

$$F(\tilde{v})(\bar{\beta}(\tilde{v}) - R_3)^2 = (R_3 - r)\big((F(\tilde{v}) - F(r))(R_3 + r) - 2F(\tilde{v})\bar{\beta}(\tilde{v}) + 2F(r)\tilde{v}\big) \tag{21}$$

and a symmetric equilibrium bidding function is given by

$$\beta^{BO}(v; r, R_3) = \begin{cases} \bar{\beta}(v) & \text{if } v \geq \tilde{v} \\ r & \text{if } r \leq v < \tilde{v} \end{cases} \tag{22}$$

and $\beta^{BO}(v; r, R_3) = \emptyset$ for $v < r$.

Proof. See Appendix.

The difference in equilibrium bidding between two reserve price rules appears when $r < R_3$. When bidder 1 submits $b_1 < R_3$ in the BO rule, he cannot solely outbid b_3 but

bidder 2's bid is necessary (except for the case where bidder 3 makes no bid). Hence, the marginal profit of increasing a bid is smaller than the case of $b_1 \geq R_3$, and bidders want to reduce their bids more:

$$(\beta^{BO})'(v) = \frac{1 - F(r)}{F(v) - F(r)} > \frac{1}{F(v)}$$

for $r < \beta^{BO}(v) < R_3$. Although reserve prices do not affect the incentive of the global bidder, a high R_3 increases the incentive to underbid for local bidders. In addition, a reserve price generates inefficiency of not allocating goods to bidders with a value under the reserve price. Therefore, R_3 should be low for achieving high efficiency in equilibrium.

Suppose $R_3 \leq r$. In this case, the BO and RB rules have the same equilibrium bidding function. This indicates that the BO rule is more efficient in equilibrium than the RB rule. In the RB rule with local reserve bidders r, the global bidder faces the reserve price of $2r$, which increases inefficiency. In addition, the allocation rule in the RB rule is distorted by a "handicap" r when one of local bidders does not make bid. Therefore, given a reserve price r for local bidders, the BO rule with $R_3 = 0$ is the most efficient in equilibrium. These observations are summarized as follows.

Corollary 1 *Suppose Assumption 1. In the BO rule (r, R_3), the equilibrium allocation is more efficient for $R_3 \leq r$ than for $r < R_3$. In addition, given r, the equilibrium allocation in the BO rule with $R_3 \leq r$ is more efficient than in the RB rule.*

3.4 Socially Optimal Reserve Price

Theorems 2 and 3 also imply that a positive reserve price for local bidders improves equilibrium efficiency. When local bidders have a low value, they have a strong incentive to underbid and submit the lowest possible bid in equilibrium. Hence, raising the reserve price for them directly increases the equilibrium bid, and both efficiency and revenue improve.

In what follows, we consider the socially optimal (symmetric) reserve price. Suppose that G is uniform distribution, and consider the BO rule with $R_3 = 0$. Consider the marginal welfare of increasing r. On the one hand, the positive effect emerges when

$V_3 \approx \beta(v_1; r) + \beta(v_2; r)$ and at least one of local bidders submit(s) r. Let $\hat{v}(r)$ be the maximum value with which a local bidder submits r in equilibrium. Then, the positive effect is given by

$$\begin{aligned}
&\int_r^{\hat{v}(r)} \int_r^{\hat{v}(r)} (v_1 + v_2 - 2r) g(2r) f(v_2) f(v_1) \mathrm{d}v_2 \mathrm{d}v_1 \\
&+2 \int_r^{\hat{v}(r)} \int_{\hat{v}(r)}^1 \big(v_1 + v_2 - (r + \beta(v_2))\big) g(r + \beta(v_2)) f(v_2) f(v_1) \mathrm{d}v_2 \mathrm{d}v_1 \\
&+2 \int_0^r \int_r^{\hat{v}(r)} (v_1 - r) g(r) f(v_1) f(v_2) \mathrm{d}v_2 \mathrm{d}v_1
\end{aligned} \tag{23}$$

On the other hand, the negative effect of increasing r is that a local bidder decides not to submit a bid when $v_i \approx r$. It is given by

$$\begin{aligned}
-2f(r) \int_r^1 \int_{\beta(v)}^{r+\beta(v)} (r + v - V_3) g(V_3) f(v) \mathrm{d}V_3 \mathrm{d}v - 2f(r) \int_r^1 \int_0^{\beta(v)} r g(V_3) f(v) \mathrm{d}V_3 \mathrm{d}v \\
-2f(r) \int_0^r \int_0^r (r - V_3) g(V_3) f(v) \mathrm{d}V_3 \mathrm{d}v.
\end{aligned} \tag{24}$$

Using uniform G and after some calculations, the marginal welfare $MW(r)$ of increasing r, the sum of (23) and (24), is given by

$$MW(r) = p(r)\Big(\mu(r) - r + \int_{\hat{v}(r)}^1 (v - \beta(v; r)) f(v) \mathrm{d}v\Big) - \frac{r f(r)}{2}\Big(r + 2\int_r^1 v f(v) \mathrm{d}v\Big), \tag{25}$$

where $p(r) = \Pr\{\beta(v; r) = r\}$ and $\mu(r) = E[v | \beta(v; r) = r]$.

Theorem 5 *Suppose Assumption 1 and the BO rule. The socially optimal reserve price for local bidders is strictly positive $r^* > 0$ and $R_3^* = 0$. In addition, when F is also uniform distribution, the socially optimal reserve price is numerically solved and $r^* \doteqdot 0.11$.*

Proof. The result immediately follows from $MW(0) > 0$. When F is uniform, we have $\hat{v}(r) = \mathrm{e}^{r-1}$, $p(r) = \mathrm{e}^{r-1} - r$, and $\mu(r) = \frac{\mathrm{e}^{r-1}+r}{2}$. Substituting these into (25), we solve $MW(r^*) = 0$. ■

3.5 Implication for Revenue Maximization

Reserve price is well known to be useful to increase expected revenue for the seller. Indeed, the seller's expected revenue under the socially optimal reserve price above is greater than that in the no reserve price case.[12] Applying the standard analysis of Myerson (1981), we obtain the revenue maximizing mechanism in this environment.[13] Let $J_F(v_i)$ and $J_G(V_3)$ be the virtual value function for local and global bidders, respectively. That is,

$$J_F(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)}$$

and

$$J_G(V_3) = V_3 - \frac{1 - G(V_3)}{g(V_3)} = 2V_3 - 2.$$

The optimal allocation rule for revenue maximization is efficient in terms of virtual values. The optimal reserve prices are given by $\hat{r} \equiv J_F^{-1}(0)$ for local bidders and $\hat{R}_3 \equiv J_G^{-1}(0) = 1$ for the global bidder. Because of asymmetric distributions of local and global bidders, the optimal allocation rule is not efficient even for sufficiently larger values in general. Hence, we focus on the case in which both G and F are uniform distributions so that the optimal allocation rule is efficient for sufficiently large values.

Lemma 2 *Suppose that both F and G are uniform distributions. Then, the optimal allocation rule that maximizes the seller's expected revenue is implemented by the VCG mechanism with the RB rule and $\hat{r} = \frac{1}{2}$.*

Proof. See Appendix.

An interesting question is the optimal reserve prices of the ascending proxy auction for revenue maximization. When the seller is restricted to using the ascending proxy auctions with reserve prices, she can maximize the expected revenue by adopting a combination of the RB and BO rules. Notice that the expected revenue is written by

$$E[J_F(v_1)\Phi_1^*(v_1)] + E[J_F(v_2)\Phi_2^*(v_2)] + E[J_G(V_3)\Phi_3^*(V_3)],$$

[12] This is simply because the reserve price increases the equilibrium bidding function, interpreting no bid for $v < r$ as a zero bid.

[13] See Ledyard (2007) for an analysis on revenue maximization in a multiple-object auction.

where Φ_i^* indicates i's winning probability in equilibrium. Thus, it is immediate that the optimal reserve prices satisfy $r \geq \frac{1}{2}$ and $R_3 \geq 1$, because allocating goods to bidders with negative virtual values reduces the expected revenue. By Lemma 2, the expected revenue is maximized by allocating goods efficiently among bidders with positive virtual values. By Theorem 5, positive reserve price for local bidders improves equilibrium efficiency, whereas the reserve price for the global bidder should be zero. Therefore, positive BO reserve price for local bidders in terms of virtual value increases the expected revenue, whereas the reserve price for the global bidder should be zero in terms of virtual value: $J_F(r^*) > 0$ and $J_G(R_3^*) = 0$. At the same time, to ensure the "virtually efficient" allocation in the case where one local bidder has $v_i < \frac{1}{2}$, local reserve bidders with $\hat{r} = \frac{1}{2}$ are also necessary. The local reserve bidders of $\hat{r}$ automatically impose the reserve price $R_3 = 1$ for the global bidder. This observation is summarized in the following proposition.

Proposition 3 *Suppose that both F and G are uniform distributions. Then, the ascending proxy auction that maximizes the seller's expected revenue adopt both the RB and BO rules. The combination of local reserve bidders $\hat{r} = \frac{1}{2}$ and a BO local reserve price $r^* > \frac{1}{2}$ is optimal.*

4 Concluding Remark

We examine the Bayesian Nash equilibrium of a core-selecting auction with and without reserve prices for a simple LLG model. Due to the free-riding problem between local bidders, they underbid in equilibrium and submit a zero bid with positive probability without a reserve price. A positive reserve price directly increases the local bidders' equilibrium bids and improves both efficiency and revenue.

There are many ways for future research. First, it is an open question how robust the effect of reserve prices on efficiency is in other core-selecting auction mechanisms. As Ausubel and Baranov (2010) report, local bidders submit a zero bid in equilibrium in several other core-selecting auctions. Therefore, reserve price would be beneficial for increasing bids by bidders with a low value. However, the effect of reserve prices on

overall bidding strategy is uncertain. The second question is how we can extend the LLG environment to a more general setting. In particular, a similar effect with a reserve price is obtained by introducing a new local bidder to the auction. When there are at least two local bidders for each individual good, the probability of zero-bidding must be zero in equilibrium. In such a case, a reserve price will just decrease the equilibrium efficiency because competition among local bidders will significantly mitigate the free-riding incentive.[14] The effect of reserve price would be considerable only when there is no competition among local bidders.

A Proofs

A.1 Proof of Theorem 1

Suppose $G(V) = \frac{V}{2}$. Suppose that there exists a symmetric equilibrium bidding function β which is continuous and increasing in the interior of its range. Then, the first order condition (2) for bidder 1 yields

$$v_1 - \beta(v_1)F(v_1) \leq \int_{v>v_1} \beta(v)f(v)\mathrm{d}v, \tag{26}$$

where equality holds if $\beta(v_1) > 0$. Evaluating (26) at $v_1 = 1$ yields $\beta(1) = 1$. Hence, (26) yields

$$\int_{v_1}^{1} \{\beta(v)f(v) + \beta'(v)F(v) - 1\}\mathrm{d}v \leq \int_{v_1}^{1} \beta(v)f(v)\mathrm{d}v,$$

and thus,

$$\int_{v_1}^{1} \{\beta'(v)F(v) - 1\}\mathrm{d}v \leq 0. \tag{27}$$

Since (27) holds for all v_1, $\beta'(v_1) = \frac{1}{F(v_1)}$. Using the initial condition $\beta(1) = 1$, we have

$$\beta(v_1) = \max\Big\{0, v_1 - \int_{v_1}^{1} \frac{1 - F(v)}{F(v)}\mathrm{d}v\Big\}. \tag{28}$$

The bidding function (28) is equilibrium since the marginal payoff function is decreasing in b_1 and satisfies the second-order condition. ■

[14] Sano (2012) shows that in an ascending core-selecting auction, increasing local bidders mitigate the free-riding incentive.

A.2 Proof of Proposition 1

When bidder i bids b_i and the other bidder bids b_j, bidder i's expected payoff is given by

$$\pi_i(b_i, b_j) = \begin{cases} v_i G(b_i + b_j) - \int_0^{2b_j} \frac{V_3}{2} g(V_3) \mathrm{d}V_3 - \int_{2b_j}^{b_i+b_j} (V_3 - b_j) g(V_3) \mathrm{d}V_3 & \text{if } b_i \geq b_j \\ v_i G(b_i + b_j) - \int_0^{2b_i} \frac{V_3}{2} g(V_3) \mathrm{d}V_3 - b_i(G(b_i + b_j) - G(2b_i)) & \text{if } b_i < b_j \end{cases}.$$

Substituting uniform distribution for G, the marginal payoff function is given by

$$\frac{\partial}{\partial b_i} \pi_i(b_i, b_j) = \begin{cases} (v_i - b_i) g(b_i + b_j) = \frac{1}{2}(v_i - b_i) & \text{if } b_i \geq b_j \\ (v_i - b_i) g(b_i + b_j) - (G(b_i + b_j) - G(2b_i)) = \frac{1}{2}(v_i - v_j) & \text{if } b_i < b_j \end{cases}.$$

Thus, the bidder submitting the higher bid should bid truthfully in equilibrium. It is not optimal for bidder 1 to submit any $b_1 < v_2$ when $b_2 = v_2$. The equilibrium bid of bidder 1 must be $b_1 = v_1$. Given that, the unique optimal bid for bidder 2 is $b_2 = 0$. ■

A.3 Proof of Theorem 3

Suppose $R_3 \leq r$. Because bidder 1's bid is $b_1 \geq r$, the first-order condition for bidder 1 is given by

$$\bar{h}(\beta(v_1), v_1) = (\leq) 0.$$

Given that G is uniform, this is the same as in the RB rule. Therefore, we have the equilibrium bidding function $\beta^{BO}(v; r, R_3) = \beta^{RB}(v, r)$.

Suppose $R_3 \geq 1$. Suppose that there exists a symmetric equilibrium bidding function β which is continuous and increasing in the interior of its range. Because bidder 1's bid is $b_1 \leq R_3$, the first-order condition for bidder 1 is given by

$$\underline{h}(\beta(v_1), v_1) = (\leq) 0. \tag{29}$$

By symmetry, (29) yields

$$(1 - F(r))v - (F(v) - F(r))\beta(v) \leq \int_v^1 \beta(s) f(s) \mathrm{d}s, \tag{30}$$

where equality holds if $\beta(v) > r$. Evaluating (30) at $v = 1$ yields $\beta(1) = 1$. Hence, (30) yields

$$\int_v^1 \{(F(s) - F(r))\beta'(s) - (1 - F(r))\}\mathrm{d}s \leq 0. \tag{31}$$

Since (31) holds for all v, $\beta'(v) = \frac{1-F(r)}{F(v)-F(r)}$. Using the initial condition $\beta(1) = 1$, we have

$$\beta(v) = \max\Big\{r, v - \int_v^1 \frac{1 - F(s)}{F(s) - F(r)}\mathrm{d}s\Big\}. \tag{32}$$

The bidding function (32) is equilibrium since $\underline{h}$ is decreasing in b_1 and satisfies the second-order condition. ■

A.4 Proof of Theorem 4

Suppose that bidder 2 takes a strategy β such that $\exists \hat{v}$,

$$\beta(v_2) = \begin{cases} \bar{\beta}(v_2) & \text{if } v > \hat{v} \\ \underline{\beta}(v_2) & \text{if } r \leq v < \hat{v} \end{cases}. \tag{33}$$

In the above strategy, $\bar{\beta}$ is given by (16) and $\underline{\beta}$ is some nondecreasing function and $\lim_{v \nearrow \hat{v}} \underline{\beta}(v) = \underline{\beta}(\hat{v}) < R_3 \leq \bar{\beta}(\hat{v})$. Given the strategy, define $\bar{h}$ and $\underline{h}$ is defined by (14) and (15). Then, bidder 1's marginal expected payoff of raising bid b_1 is given by

$$2 \cdot \frac{\partial}{\partial b_1}\pi_1(b_1, v_1) = \begin{cases} \bar{h}(b_1, v_1) & \text{if } b_1 \geq R_3 \\ \underline{h}(b_1, v_1) & \text{if } r \leq b_1 < R_3 \end{cases}.$$

By inspection, $\frac{\partial \bar{h}}{\partial b_1} \leq 0$, $\frac{\partial \underline{h}}{\partial b_1} \leq 0$, and $\bar{h}(b_1, v_1) > \underline{h}(b_1, v_1)$ for all $v_1 > r$ and all $b_1 \in [r, v_1)$. Therefore, given v_1, the optimal bid b^* satisfies $b^* > R_3$ and $\bar{h}(b^*, v_1) = 0$ if $\underline{h}(R_3, v_1) \geq 0$. In addition, the optimal bid satisfies $b^* < R_3$ and $\underline{h}(b^*, v_1) \leq 0$ (equality holds for $b^* > r$) if $\bar{h}(R_3, v_1) \leq 0$.

Suppose that the optimal bid satisfies the upper local first-order condition $\bar{h} = 0$; i.e.,

$$v_1 - b - \int_{b<\beta_2} (\beta(s) - b)f(s)\mathrm{d}s = 0. \tag{34}$$

By Theorem 1, the symmetric bid $b_1 = \bar{\beta}(v_1)$ satisfies the local first-order condition.

Consider the case $\underline{h}(R_3, v) < 0 < \bar{h}(R_3, v)$. For such v, there are two locally optimal bids: one is $\bar{\beta}(v)$ that satisfies $\bar{h} = 0$, and the other satisfies $\underline{h}(b, v) \leq 0$. We assume for now that there exists an interior solution b satisfying $\underline{h}(b, v) = 0$. To show that bidder 2's strategy β forms an equilibrium, the two locally optimal bids yield the same expected payoff at the jump point $\hat{v}$. The upper locally optimal bid is given by $\bar{\beta}(\hat{v})$. For now, we consider that the lower locally optimal bid satisfies $\underline{h}(b, \hat{v}) = 0$: i.e.,

$$(1 - F(r))(\hat{v} - b) - \int_{b<\beta_2} (\beta(s) - b) f(s) \mathrm{d}s = 0. \tag{35}$$

Notice that $b < \beta(v_2)$ implies $v_2 \geq \hat{v}$. Hence, (35) yields

$$(1 - F(r))\hat{v} - (F(\hat{v}) - F(r))b - \int_{\hat{v}}^{1} \beta(s) f(s) \mathrm{d}s = 0. \tag{36}$$

Since $\beta(s) = \bar{\beta}(s)$ for $s > \hat{v}$ and using

$$\hat{v} - \bar{\beta}(\hat{v}) F(\hat{v}) = \int_{\hat{v}}^{1} \bar{\beta}(s) f(s) \mathrm{d}s, \tag{37}$$

we have

$$(1 - F(r))\hat{v} - (F(\hat{v}) - F(r))b - (\hat{v} - \bar{\beta}(\hat{v}) F(\hat{v})) = 0, \tag{38}$$

which yields

$$b = \hat{\beta}(\hat{v}) = \bar{\beta}(\hat{v}) - \frac{F(r)}{F(\hat{v}) - F(r)} (\hat{v} - \bar{\beta}(\hat{v})). \tag{39}$$

Suppose that $\hat{v}$ satisfies $\hat{\beta}(\hat{v}) < R_3 < \bar{\beta}(\hat{v})$. Notice that $\forall b \in (\hat{\beta}(\hat{v}), \bar{\beta}(\hat{v}))$, $b < \beta(v_2)$ indicates $v_2 \geq \hat{v}$. Hence, using (37), the marginal payoff of bidder 1 for $b \in (\hat{\beta}(\hat{v}), \bar{\beta}(\hat{v}))$ is given by

$$2 \cdot \frac{\partial}{\partial b_1} \pi_1(b, \hat{v}) = \begin{cases} \bar{h}(b_1, \hat{v}) = F(\hat{v})(\bar{\beta}(\hat{v}) - b) & \text{if } b_1 > R_3 \\ \underline{h}(b, \hat{v}) = F(\hat{v})\bar{\beta}(\hat{v}) - F(r)\hat{v} - (F(\hat{v}) - F(r))b & \text{if } r \leq b_1 < R_3 \end{cases}. \tag{40}$$

Since $\hat{\beta}(\hat{v})$ and $\bar{\beta}(\hat{v})$ give the same expected payoff, $\hat{v}$ is specified by

$$\int_{\hat{\beta}(\hat{v})}^{R_3} \underline{h}(b, \hat{v}) \mathrm{d}b + \int_{R_3}^{\bar{\beta}(\hat{v})} \bar{h}(b, \hat{v}) \mathrm{d}b = 0. \tag{41}$$

Equation (41) is equivalent to (18). Since the light hand side of (41) is increasing in $\hat{v}$, (41) has at most one solution.

Case 1. There exists the solution of (41), $\hat{v}$, and $r \leq \hat{\beta}(\hat{v}) < R_3 < \bar{\beta}(\hat{v})$.

Define $\hat{v}$ as the solution, and consider that bidder 2's strategy is such that $\beta(v_2) = \bar{\beta}(v_2)$ for $v_2 > \hat{v}$ and

$$\lim_{v_2 \nearrow \hat{v}} \underline{\beta}(v_2) = \hat{\beta}(\hat{v}).$$

Suppose $v_1 < \hat{v}$. Consider that local bidders take a symmetric strategy derived from the local optimum condition $\underline{h}(\underline{\beta}(v), v) \leq 0$. By symmetry, we have

$$(1 - F(r)) - (F(v) - F(r))\underline{\beta}(v) - \int_v^1 \beta(s) f(s) \mathrm{d}s \leq 0. \tag{42}$$

Using (38), (42) yields

$$\int_v^{\hat{v}} \left\{ (F(s) - F(r))\underline{\beta}'(s) - (1 - F(r)) \right\} \mathrm{d}s \leq 0. \tag{43}$$

Thus, the locally optimal bidding function is given by $\underline{\beta}'(v) = \frac{1-F(r)}{F(v)-F(r)}$ with the initial condition $\underline{\beta}(\hat{v}) = \hat{\beta}(\hat{v})$. Therefore, the symmetric bidding function β is specified by (20).

Finally, we verify that the derived locally optimal bid is globally optimal. Let $\bar{\Pi}(v) \equiv \pi_1(\bar{\beta}(v), v)$ be the expected payoff when bidding $\bar{\beta}$ satisfying $\bar{h}(\bar{\beta}, v) = 0$. Similarly, $\underline{\Pi}(v) \equiv \pi_1(\underline{\beta}(v), v)$ is the expected payoff when bidding $\underline{\beta}$ satisfying $\underline{h}(\underline{\beta}, v) \leq 0$. It holds that $\bar{\Pi}(\hat{v}) = \underline{\Pi}(\hat{v})$. By the envelope theorem, $\bar{\Pi}'(v) = \Phi(\bar{\beta}(v))$ and $\underline{\Pi}'(v) = \Phi(\underline{\beta}(v))$ where Φ is winning probability of bidder 1. Since $\bar{\beta}(v) > \underline{\beta}(v)$, we have $\bar{\Pi}(v) > \underline{\Pi}(v)$ for $v > \hat{v}$, and $\bar{\Pi}(v) < \underline{\Pi}(v)$ for $v < \hat{v}$. Therefore, strategy (20) is an equilibrium strategy.

Case 2. There exists no $\hat{v}$ that satisfies (41) and $r \leq \hat{\beta}(\hat{v}) < R_3 < \bar{\beta}(\hat{v})$.

Now define $\tilde{v}$ such that

$$\int_r^{R_3} \underline{h}(b, \tilde{v}) \mathrm{d}b + \int_{R_3}^{\bar{\beta}(\tilde{v})} \bar{h}(b, \tilde{v}) \mathrm{d}b = 0 \tag{44}$$

Equation (44) is equivalent to (21). The light hand side of (44) is increasing in $\tilde{v}$, and (44) has a unique solution $\tilde{v} > \bar{\beta}^{-1}(R_3)$.

Specify bidder 2's strategy as

$$\beta_2(v) = \begin{cases} \bar{\beta}(v) & \text{if } v > \tilde{v} \\ r & \text{if } r \leq v \leq \tilde{v} \end{cases}. \tag{45}$$

Given β_2, it is locally optimal for bidder 1 to bid $\bar{\beta}(v_1)$ for $v > \tilde{v}$. By definition of $\tilde{v}$, both $\bar{\beta}(\tilde{v})$ and r are optimal and indifferent for $v_1 = \tilde{v}$. For $v_1 < \tilde{v}$, bidding r is locally optimal because $\underline{h}(r, v) < 0$.

Finally, we verify that the derived strategy is globally optimal. Let $\bar{\Pi}(v) \equiv \pi_1(\bar{\beta}(v), v)$ be the expected payoff when bidding $\bar{\beta}$ satisfying $\bar{h}(\bar{\beta}, v) = 0$. Let $\underline{\Pi}(v) \equiv \pi_1(r, v)$. It holds that $\bar{\Pi}(\tilde{v}) = \underline{\Pi}(\tilde{v})$. The envelope theorem implies $\bar{\Pi}'(v) = \Phi(\bar{\beta}(v))$ and $\underline{\Pi}'(v) = \Phi(r)$. Hence, $\bar{\Pi}(v) > \underline{\Pi}(v)$ for $v > \tilde{v}$, and $\bar{\Pi}(v) < \underline{\Pi}(v)$ for $v < \tilde{v}$. Therefore, strategy (22) is optimal. ■

A.5 Proof of Lemma 2

Suppose that both F and G are uniform distributions. Hence, $J_F(v_i) = 2v_i - 1$ and $J_G(V_3) = 2V_3 - 2$. By the standard argument of Myerson (1981) and Ledyard (2007), the optimal allocation rule is the efficient allocation rule in terms of virtual values. That is, goods are never allocated if $v_i \leq \frac{1}{2}$ for local bidders, or if $V_3 \leq 1$ for the global bidder. When all the bidders have positive virtual values, then local bidders obtain goods if and only if

$$J_F(v_1) + J_F(v_2) \geq J_G(V_3) \Leftrightarrow v_1 + v_2 \geq V_3,$$

i.e., the optimal allocation is efficient. When only bidders 1 and 3 have positive virtual values, then bidder 1 obtains good A if and only if

$$J_F(v_1) \geq J_G(V_3) \Leftrightarrow v_1 + \frac{1}{2} \geq V_3.$$

Hence, the optimal allocation rule is the efficient allocation rule with local reserve bidders $r = \frac{1}{2}$. ■

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