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RESEARCH ARTICLE

Conversion of Linear Time-Invariant Time-Delay Feedback Systems into Delay-Differential Equations with Commensurate Delays

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A new stability analysis method of time-delay systems (TDSs) called the monodromy operator approach has been studied under the assumption that a TDS is represented as a time-delay feedback system consisting of a finite-dimensional linear time-invariant (LTI) system and a pure delay. For applying this approach to TDSs described by delay-differential equations (DDEs), the problem of converting DDEs into representation as time-delay feedback systems has been studied. With regard to such a problem, it was shown that, under discontinuous initial functions, it is natural to define the solutions of DDEs in two different ways, and the above conversion problem was solved for each of these two definitions. More precisely, the solution of a DDE was represented as either the state of the finite-dimensional part of a time-delay feedback system or a part of the output of another time-delay feedback system, depending on which definition of the DDE solution one is talking about. Motivated by the importance in establishing a thorough relationship between time-delay feedback systems and DDEs, this paper discusses the opposite problem of converting time-delay feedback systems into representation as DDEs, including the discussions about the conversion of the initial conditions. We show that the state of (the finite-dimensional part of) a time-delay feedback system can be represented as the solution of a DDE in the sense of one of the two definitions, while its "essential" output can be represented as that of another DDE in the sense of the other type of definition. Rigorously speaking, however, it is also shown that the latter representation is possible regardless of the initial conditions, while some initial condition could prevent the conversion into the former representation. This study hence establishes that the representation of TDSs as time-delay feedback systems possesses higher ability than that with DDEs, as description methods for LTI TDSs with commensurate delays.

Keywords: time-delay systems; delay-differential equations; concatenated solutions; system representation

1. Introduction

This paper is concerned with mathematical descriptions of time-delay systems (TDSs); delay-differential equations (DDEs) and time-delay feedback systems. The analysis of TDSs represented as linear time-invariant (LTI) DDEs with commensurate delays described by

$$\dot{x}(t) = Jx(t) + \sum_{i=1}^{\eta} K_i \dot{x}(t-ih) + \sum_{i=1}^{\eta} L_i x(t-ih) \tag{1}$$

has been studied for many years (Bellen & Zennaro 2003, Bellman & Cooke 1963, Hale 1977, Hale & Lunel 1993, Kolmanovskii & Nosov 1986, Kolmanovskii & Myshkis 1999). These conventional studies dealt with the stability analysis of DDEs (1) based on the characteristics equation or the Lyapunov-type stability theory.

On the other hand, a new approach to TDSs has been proposed recently (Hirata & Kokame 2003). In this approach, continuous-time TDSs are viewed as a sort of discrete-time systems through the lifting technique (Bamieh & Pearson 1992, Yamamoto 1994) developed for sampled

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data systems. This study has been further extended, and a new bounded-operator-theoretic approach called the monodromy operator approach has been developed (Hagiwara 2008, Hagiwara & Fujinami 2010, Hagiwara & Inui 2010, Hagiwara & Hirata 2011). This approach is based on the assumption that a TDS is represented as a time-delay feedback system Σ as in Figure 1. Here, F is a finite dimensional (FD) LTI system described by

$$F: \begin{cases} \dot{q}(t) = Aq(t) + Bu(t) & \text{(2a)} \\ y(t) = Cq(t) + Du(t) & \text{(2b)} \end{cases}$$

and H is a pure delay with delay length h;

$$H: \quad u(t) = y(t-h), \quad t \geq h \tag{3}$$

To exploit the results on the monodromy operator approach in the analysis of TDSs described by DDEs, one needs to convert DDEs into representation as Σ. Such conversion has been studied in Hagiwara & Kobayashi (2011), in which it has been shown that the conversion problem can be studied adequately enough only after introducing appropriate definitions of the solutions of DDEs under discontinuous initial functions. As such, this conversion is not a trivial problem.

In fact, the pseudo concatenated solution and continuous concatenated solution (and the regular solution as a special case of these solutions) of the DDE (1) are defined under discontinuous initial functions in Hagiwara & Kobayashi (2011) (the definitions of these solutions are reviewed in Section 2). The discussions there have established that each type of solution can be represented as a signal of an associated time-delay feedback system, and have given explicit conversion methods. To be more precise, the pseudo concatenated solution of a given DDE was described as a part of the output y of an appropriate Σ, while the continuous concatenated solution as the state q of F in another Σ. Furthermore, such a conversion problem has been extended in Yamazaki & Hagiwara (2011) to encompass the case with external input and output.

This paper studies the opposite problem of converting a time-delay feedback system Σ into representation as a DDE, including the conversion of the initial condition. More precisely, two such methods are given, each of which corresponds to whether the pseudo concatenated solution or continuous concatenated solution of the resulting DDE plays the role of representing a target signal in Σ; roughly speaking, the former solution corresponds to the case when the target signal is the output y of F in Σ (Subsection 3.1), while the latter solution to the case when the state q of F is the target (Subsection 3.2). In these methods, the coefficient matrices and (possibly discontinuous) initial functions of the resulting DDEs (which, in general, result in commensurate delays) are given explicitly. Some further extension of such arguments is also provided in Section 4. The studies on the forward conversion problem in the preceding study (Hagiwara & Kobayashi 2011), together with the backward conversion problem in the present study, clarify an entire picture on the mutual relationship between the representation of TDSs with DDEs and that with time-delay feedback systems Σ. In particular, the arguments in the present paper suggest the existence of a time-delay feedback system Σ such that, under some initial condition, a target signal in Σ cannot be converted into representation as a DDE. As such, the opposite conversion problem tackled in this paper is also nontrivial and significant, and to the best knowledge of the authors, this paper is the first to have discussed such a mutual relationship deeply and explicitly enough, and is believed to extend our understanding on TDSs. For example, for the reason stated just above, the arguments in this paper (together with the

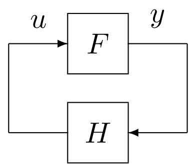

Figure 1. Time-delay feedback system Σ.

preceding study on the forward conversion) show that a time-delay feedback system Σ is more versatile than a DDE as a model for representing a general TDS with a general initial condition (see, e.g., Example 3). Clarifying this fact is one of the important contributions of this paper. Nevertheless, it is definitely true that there exist vast wisdom and powerful tools for DDEs collected and established over the history on the study of time-delay systems. Hence, one would very naturally face with a desire of converting a model with a time-delay feedback system Σ into a DDE representation, to draw benefit from the knowledge on DDEs and their treatment collected over the history. This paper establishes general methods for such (structural) conversion for any time-delay feedback system Σ (even though the resulting DDE may only preserve a structural aspect of Σ but might fail to retain the information on the initial condition, because preserving both the structure and initial condition is generally impossible, as mentioned above and as it turns out from the arguments in this paper). Such conversion methods would be very useful on their own, and can be very useful tools in practical studies of time-delay feedback systems.

2. Definitions of the solutions of DDE

In the discussions about the conversion of Σ into a DDE, discontinuous initial functions arise frequently, for which defining the solutions of DDEs contains subtle theoretical issues. Such issues are quite relevant and important in the arguments of this paper. Hence, this section is devoted to reviewing the definitions of the solutions of DDEs under possibly discontinuous initial functions; see Hagiwara & Kobayashi (2011) for details.

2.1. *Neutral DDE*

Let us first consider the neutral DDE with commensurate delays given by (1) under the possibly discontinuous initial condition given by

$$x(t) = \phi(t), \quad -\eta h \le t < 0 \tag{4a}$$

$$x(0) = \xi \tag{4b}$$

where $\phi(t)$ is defined on the closed interval $[-\eta h, 0]$; note that $\phi(t)$ is regarded to be defined also at $t = 0$ to facilitate subsequent arguments. In conventional studies, it is customary to assume that $\phi(t)$ is continuously differentiable but in this paper, we only assume that $\phi(t)$ is bounded and continuously differentiable on each of the intervals $[-ih, -(i-1)h)$ $(i = 1, \cdots, \eta)$. Furthermore, we do not assume that $\phi(0) = \xi$. Therefore, the initial function $x(t)$, $-\eta h \le t \le 0$ is not necessarily left-continuous at $t = -ih$ $(i = 0, \cdots, \eta - 1)$, and $\phi(t)$, $-h \le t \le 0$ is not necessarily identical with the initial function (with possible discrepancy at $t = 0$). The initial value problem under such an initial function is deeply related to the arguments of this paper.

We first recall a solution of (1) in the strongest sense, i.e., a differentiable solution, assuming continuous differentiability of the initial function

Definition 1: $x(t)$, $t \ge -\eta h$ is said to be a regular solution of (1) if the following three conditions are satisfied: (i) it satisfies the initial condition (4) for ξ $(= \phi(0))$ and $\phi(t)$ that is continuously differentiable on the interval $[-\eta h, 0]$; (ii) it is differentiable for $t \ge -\eta h$; (iii) it satisfies (1) for $t \ge 0$.

A regular solution is continuous at $t = 0$ by definition. Hence, it exists only if $\xi = \phi(0)$, but this is not sufficient for its existence. Therefore, to allow a solution of (1) only under the continuous differentiability assumption of $\phi(t)$, a weaker definition is necessary for its solution. There are two directions for introducing such a weaker definition, both of which will admit possible discontinuities of the initial function $\phi(t)$ at the same time.

The first direction is to consider the following modified DDE.

$$\dot{v}(t) = Jx(t) + \sum_{i=1}^{\eta} L_i x(t - ih), \quad v(t) = x(t) - \sum_{i=1}^{\eta} K_i x(t - ih) \tag{5}$$

If each term on the right hand side of the second equation of (5) were differentiable, then we would immediately be led equivalently to (1), but we proceed to the following definition (Hagiwara & Kobayashi 2011) without explicitly assuming such differentiability.

Definition 2: $x(t),\ t \geq -\eta h$ is said to be a pseudo concatenated solution of the modified DDE (5) if the following three conditions are satisfied: (i) it satisfies the initial condition (4) for ξ and $\phi(t)$ that is bounded and continuous on each of the intervals $[-ih,\ -(i-1)h)$ $(i = 1, \cdots, \eta)$; (ii) $v(t)$ is continuous for $t \geq 0$; (iii) $v(t)$ is differentiable and satisfies the first equation of (5) for $t \geq 0$ except possibly at $t = kh$ $(k \in \mathbb{N})$. In particular, such $x(t)$ is said to be a pseudo concatenated solution of the original DDE (1) if (i)' $\phi(t)$ is bounded and continuously differentiable on each of the intervals $[-ih,\ -(i-1)h)$ $(i = 1, \cdots, \eta)$.

Under (i)', a pseudo concatenated solution of the modified DDE (5) is differentiable except possibly at $t = kh$ $(k \in \mathbb{N})$, which is the rationale for the latter part of the above definition. It is obvious that $v(t),\ t \geq 0$ is differentiable except possibly at $t = kh$ if and only if $x(t)$ is. The continuity requirement of $v(t)$, however, does not necessarily mean that of $x(t)$, and hence the pseudo concatenated solution $x(t)$ of (5) (or (1)) is not necessarily continuous; see the example below. In contrast, the second direction (Hagiwara & Kobayashi 2011) in the following introduces a weaker definition of solutions that are continuous, without referring to (5).

Definition 3: $x(t),\ t \geq -\eta h$ is said to be a continuous concatenated solution of (1) if the following three conditions are satisfied: (i) it satisfies the initial condition (4) for ξ and $\phi(t)$ that is bounded and continuously differentiable on each of the intervals $[-ih,\ -(i-1)h)$ $(i = 1, \cdots, \eta)$ and has limits $\lim_{t \to -(i-1)h-0} \phi(t)$ $(i = 1, \cdots, \eta)$; (ii) it is continuous for $t \geq 0$; (iii) it is differentiable and satisfies (1) for $t \geq 0$ except possibly at $t = kh$ $(k \in \mathbb{N})$.

We remark that the existence assumption of the limits in (i) is introduced to ensure the existence of the limits $\lim_{t \to kh-0} x(t)$, because the definition does not make sense (no continuous $x(t)$ exists) if such limits do not exist.

Under an initial function (4) consistent with each of the above definitions, the existence of a unique associated solution has been established (Hagiwara & Kobayashi 2011). If a regular solution of (1) exists under some initial function, then it is at the same time a pseudo concatenated solution and a continuous concatenated solution for the same initial condition. However, even if a continuous concatenated solution exists, it is not necessarily a pseudo concatenated solution at the same time. Hence, Definitions 2 and 3 are generalizations of Definition 1 that are independent of each other, in general.

Example 1: Let us consider the DDE $\dot{x}(t) = \dot{x}(t-h)$ (which corresponds to $\eta = 1,\ J = 0,\ K_1 = 1,\ L_1 = 0$) under the initial function given by $x(t) = 0$ $(-h \leq t < 0)$, $x(0) = 1$. We can readily verify that this DDE has the pseudo concatenated solution $x(t) = k+1$ $(kh \leq t < (k+1)h, k = -1, 0, 1, \cdots)$ and continuous concatenated solution $x(t) = \begin{cases} 0 & (-h \leq t < 0) \\ 1 & (0 \leq t) \end{cases}$.

The above definitions consider discontinuities of the initial function only possibly at $t = -ih$ $(i = 0, \cdots, \eta-1)$. This is because considering some sort of discontinuity particularly at those time instants seems quite important in studying the conversion of Σ with general initial conditions into a DDE, as we shall see in Section 3.

2.2. *Retarded DDE*

We next consider the particular case of $K_i = 0$ $(i = 1, \cdots, \eta)$ in (1), i.e. the retarded DDE given by

$$\dot{x}(t) = Jx(t) + \sum_{i=1}^{\eta} L_i x(t - ih) \tag{6}$$

In this case, a regular solution can be defined as follows, without assuming differentiability of the initial function.

Definition 4: $x(t),\ t \geq -\eta h$ is said to be a regular solution of (6) if the following three conditions are satisfied: (i) it satisfies the initial conditions (4) for ξ $(= \phi(0))$ and $\phi(t)$ that is continuous on the interval $[-\eta h, 0]$; (ii) it is differentiable for $t \geq 0$; (iii) it is satisfies (6) for $t \geq 0$.

Since $K_i = 0$ $(i = 1, \cdots, \eta)$, it follows from the second equation of (5) that $v(t) = x(t)$. Hence, unlike the case of neutral DDEs, the definitions of pseudo and continuous concatenated solutions degenerate to an identical one as given below (Hagiwara & Kobayashi 2011).

Definition 5: $x(t),\ t \geq -\eta h$ is said to be a (continuous) concatenated solution of (6) if the following three conditions are satisfied: (i) it satisfies the initial conditions (4) for ξ and $\phi(t)$ that is bounded and continuous on each of the intervals $[-ih, -(i-1)h)$ $(i = 1, \cdots, \eta)$; (ii) $x(t)$ is continuous for $t \geq 0$; (iii) $x(t)$ is differentiable and satisfies (6) for $t \geq 0$ except possibly at $t = kh$ $(k \in \mathbb{N})$.

Remark 1: As in the preceding subsection, existence and uniqueness of a regular/concatenated solution can be ensured in the retarded case, too, under such initial conditions as in the above definitions (Hagiwara & Kobayashi 2011).

3. Conversion of a time-delay feedback system into a DDE

Let us consider the time-delay feedback system Σ in Figure 1. We assume that F in Σ is an FDLTI system given by (2), where $q(t) \in \mathbb{R}^n$, $u(t)$, $y(t) \in \mathbb{R}^\mu$, and H is the pure delay with retardation h given by (3). We also denote the initial conditions of Σ by

$$q(0) = \zeta \tag{7a}$$

$$u(t) = \psi(t), \quad 0 \leq t < h \tag{7b}$$

where the initial input $\psi(t)$ to F is defined on the interval $[0, h)$, continuous on this interval, and $\Psi(t) := \int_0^t \psi(\tau) d\tau$ has limit $\lim_{t \to h-0} \Psi(t)$, so that $q(h)$ can be well-defined. The problem we study in this paper is to represent the signals of Σ as the solution of some appropriate DDEs. We remark that we might be led to $\eta \geq 2$ in the resulting DDE (1), even though H is a (multi-channel) delay with single retardation h.

Rigorously speaking, however, we see that we must make the above problem formulation more precise, because a DDE generally has two different types of solutions (i.e., pseudo and continuous concatenated solutions; a regular solution can be regarded as a special case of these solutions). Hence, we must be specific as to whether a target signal in Σ is to be represented as a pseudo concatenated solution of an associated DDE or as a continuous concatenated solution of another associated DDE, when we tackle the conversion problem of Σ into DDEs. We will tackle both cases separately in Subsections 3.1 and 3.2.

Specifically, Subsection 3.1 studies representing an "essential" output of F as the pseudo concatenated solution of a DDE, while Subsection 3.2 studies representing the state of F as the continuous concatenated solution of another DDE. Considering these two combinations between the output/state and the pseudo/continuous concatenated solutions (and not the other two remaining) is believed to be essential since it is consistent with the preceding study on the conversion of DDEs into time-delay feedback systems (Hagiwara & Kobayashi 2011); it has been shown that a pseudo concatenated solution of a DDE is represented as a part of the output of F in an appropriate time-delay feedback system Σ, while a continuous concatenated solution of a DDE is represented as the state of F in another Σ.

We remark that the conversion methods provided in these two subsections can be confirmed to lead to a reasonable consequence: the time-delay feedback system Σ obtained from a given DDE (via the conversion method provided in Hagiwara & Kobayashi (2011) with respect to its pseudo/continuous concatenated solution) can always be converted back into the original DDE with respect to the same type of solutions, together with the same initial condition as the original one.

3.1. *Conversion into pseudo concatenated solution of DDE*

In this subsection, we confine ourselves, without loss of generality, to the case when F in the time-delay feedback system Σ is given by the form

$$F: \begin{cases} \dot{q}(t) = Aq(t) + \begin{bmatrix} B_1 \; B_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} & \text{(8a)} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} q(t) + \begin{bmatrix} D_{11} \; D_{12} \\ D_{21} \; D_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} & \text{(8b)} \end{cases}$$

where $q(t) \in \mathbb{R}^n$, and satisfies the following assumption.

Assumption 1: The pair

$$\left(\begin{bmatrix} D_{11} \; D_{12} \\ D_{21} \; D_{22} \end{bmatrix}, \begin{bmatrix} I \\ 0 \end{bmatrix} \right) \tag{9}$$

is controllable, its controllability index is η, and

$$D_{22}^{\eta-1} = 0 \tag{10}$$

When $\eta = 1$, the matrix D_{22} is empty, and we regard that (10) is vacuously satisfied.

In fact, we can transform a general F into an "equivalent system" satisfying the above assumption, as will be discussed shortly. Since the state $q(t)$ of F is directly reflected on $y_1(t)$ but is not reflect on $y_2(t)$ at all, we say that $y_1(t)$ is an "essential part" of $y(t)$ (or an essential output of F). This subsection tackles the problem of representing the essential output of F in the time-delay feedback system Σ as a pseudo concatenated solution of a DDE, given the initial condition of Σ in (7).

3.1.1. Rationale for introducing Assumption 1

We begin by discussing why we assume the special form of the matrix "C" in (8b) and why the above assumption does not cause loss of generality, provided that (D, C) (of F before equivalent transformation into (8)) is assumed controllable. For the case when (D, C) is uncontrollable, see the further arguments in Subsection 4.1.

To begin with, we note that the pure delay H commutes with multiplication by constant matrices. This implies that any Σ with F given by (2) whose matrix C is of full column rank can be equivalently transformed, through similarity transformation and input-output scaling of F, into another Σ whose C is in the above special form. Even if the original C has rank deficiency, we can introduce fictitious elements in $y(t)$ and $u(t)$ in such a way that the fictitious elements in the augmented $y(t)$ are, essentially, not fed back to the corresponding elements in $u(t)$. Hence, C can always be made to have full column rank in this way without essentially changing the original Σ. For example, if $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $D = d$, and $C = [1\ 1]$, which is not of full column rank, then replace C with $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, B with $B = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \end{bmatrix}$, and D with $D = \begin{bmatrix} d & 0 \\ * & * \end{bmatrix}$. These observations imply that assuming the above special form of C does not lead to any loss of generality.

In connection with the above process of augmenting $y(t)$ (and thus $u(t)$, too), suppose that the number of fictitious elements introduced into $y(t)$ is taken as small as possible (i.e., equal to the rank deficiency of the original C). Then, we can show that the pair (9) constructed from the matrices in (8b) is controllable if and only if the original (D, C) is. Furthermore, we can show the following result relevant to the assumption (10).

Lemma 1: If the pair (9) is controllable and its controllability index is η, then there exists a similarity transformation matrix T that converts the pair (9) into

$$\left(\begin{bmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} \\ \widetilde{D}_{21} & \widetilde{D}_{22} \end{bmatrix}, \begin{bmatrix} I \\ 0 \end{bmatrix} \right), \quad \widetilde{D}_{22} = (\widetilde{D}_{22,ij})_{i,j=1}^{\eta-1} \tag{11}$$

with square $\widetilde{D}_{22,ii}$ and $\widetilde{D}_{22,ij} = 0$ $(1 \le i \le j \le \eta - 1)$.

The proof is given in Appendix A, which in fact shows that $\widetilde{D}_{21}$ and $\widetilde{D}_{22}$ can be made to have sparse forms corresponding to a sort of controllable canonical form.

Similarity transformation of the pair (D, C) (and thus T in the above lemma) can be regarded as scaling the input and output of F, which does not essentially change the feedback system Σ. Hence, we may assume without loss of generality that the matrices in (8b) are already in such forms as in (11). Then, (10) is satisfied due to the strict block lower triangular form of D_{22}. These observations imply that Assumption 1 does not lead to loss of generality.

3.1.2. *Determining the matrices in DDE*

On the basis of the above preparation, the conversion of Σ into DDE proceeds as follows by paying attention to the response of the essential output $y_1(t)$.

It follows from (8b) and (3) that

$$y_1(t) = q(t) + D_{11} y_1(t-h) + D_{12} u_2(t) \tag{12a}$$

$$y_2(t) = D_{21} y_1(t-h) + D_{22} u_2(t) \tag{12b}$$

We are immediately led from (12b) and (3) to

$$u_2(t) = D_{21} y_1(t-2h) + D_{22} u_2(t-h) \tag{13}$$

Applying (13) recursively and noting (10) lead to

$$u_2(t) = \sum_{i=2}^{\eta} D_{22}^{i-2} D_{21} y_1(t-ih) \tag{14}$$

Substituting the above into (12a) yields

$$q(t) = y_1(t) - D_{11} y_1(t-h) - D_{12} \sum_{i=2}^{\eta} D_{22}^{i-2} D_{21} y_1(t-ih) \tag{15}$$

Furthermore, substituting (3), (14) and (15) into (8a) leads to

$$\dot{q}(t) = Ay_1(t) + (B_1 - AD_{11})\, y_1(t-h) + (B_2 - AD_{12}) \sum_{i=2}^{\eta} D_{22}^{i-2} D_{21} y_1(t-ih) \tag{16}$$

By comparing (15) and (16) with (5), we see that if

$$J = A,\ K_1 = D_{11},\ K_i = D_{12}D_{22}^{i-2}D_{21},\ L_1 = B_1 - AD_{11},\ L_i = (B_2 - AD_{12})\, D_{22}^{i-2}D_{21} \\ (i = 2, \cdots, \eta) \tag{17}$$

in (5), then the essential output $y_1(t)$ and the state $q(t)$ of F in Σ is expected to coincide, over $t \geq 0$, with $x(t)$ and $v(t)$ of the DDE (5), respectively. However, the above discussions leave issues on the initial conditions. From a viewpoint of dealing with the response of Σ under the prescribed initial condition (7), we can only see that (12) hold for $t \geq h$, and (14) for $t \geq \eta h$, and thus (15) and (16) in fact hold only for $t \geq \eta h$. Hence, we need to rigorously establish that $y_1(t)$ of Σ could really coincide with the (pseudo concatenated) solution $x(t)$ of the modified DDE (5) (or that of the original DDE (1)) with the coefficient matrices (17) under a suitably determined initial function.

3.1.3. Determining the initial function of DDE

The unresolved issue raised above can be given the answer described by the following theorems, whose proofs are given in Appendix B.

Theorem 1: *Suppose in the feedback system Σ that (i) F is described by (8), and (ii) the initial input $\psi(t)$, $0 \leq t < h$ is bounded and continuous. Then, the essential output $y_1(t)$ of F in Σ coincides, over $t \geq 0$, with the pseudo concatenated solution $x(t)$ of the modified DDE (5) with the coefficient matrices (17) under the initial function given by*

$$x(t) = \phi(t), \quad -\eta h \leq t < 0 \tag{18a}$$

$$x(0) = \zeta + \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} \psi(0) \tag{18b}$$

where $\phi(t)$ is such a function satisfying the following conditions: $\phi(t)$ is bounded and continuous on each of the intervals $[-ih, -(i-1)h)$ $(i = 1, \cdots, \eta)$ and satisfies

$$\begin{bmatrix} \phi(t-h) \\ \displaystyle\sum_{i=2}^{\eta} D_{22}^{i-2} D_{21} \phi(t-ih) \end{bmatrix} = \psi(t), \quad 0 \leq t < h \tag{19}$$

Theorem 2: *Suppose in the feedback system Σ that the condition (i) in Theorem 1 is satisfied and (ii)' the initial input $\psi(t)$, $0 \leq t < h$ is bounded and continuously differentiable. Then, the essential output $y_1(t)$ of F in Σ coincides, over $t \geq 0$, with the regular solution $x(t)$ resulting from the neutral DDE (1) with (17) under the initial function given by (18), provided that there exists $\phi(t)$ satisfying the following conditions: $\phi(t)$ is continuously differentiable on the interval $[-\eta h, 0]$ and satisfies (19) and*

$$\phi(0) = \zeta + \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} \psi(0) \tag{20a}$$

$$\dot{\phi}(-0) = A\zeta + \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} \dot{\psi}(+0) + B\psi(0) \tag{20b}$$

The additional hypotheses in Theorem 2 on $\phi(t)$ correspond to the splicing condition (Bellen & Zennaro 2003) (or the sewing condition (Kolmanovskii & Nosov 1986, Kolmanovskii & Myshkis

1999)) which is known to ensure the existence of a regular solution (as a special case of a pseudo concatenated solution).

Remark 2: Since (D_{22}, D_{21}) is controllable with controllability index $\eta - 1$ by Assumption 1, existence of $\phi(t)$ satisfying the conditions in Theorem 1 is always ensured. Indeed, $[\phi(t-h)^T, \cdots, \phi(t-\eta h)^T]^T = \mathrm{diag}[I, U_c^\dagger]\psi(t)$ $(0 \le t < h)$ is a solution, where $U_c^\dagger = U_c^T(U_cU_c^T)^{-1}$ denotes the pseudo inverse of $U_c := [D_{21}, D_{22}D_{21}, \cdots, D_{22}^{\eta-2}D_{21}]$. Hence, the conversion problem considered in this subsection always has an answer (by allowing pseudo concatenated solutions, rather than considering only regular solutions).

Remark 3: In Theorem 1, if the assumption (ii) is replaced by the stronger assumption (ii)' in Theorem 2, then $y(t),\ t \ge 0$ coincides with the pseudo concatenated solution $x(t)$ of (1), provided that the continuity assumption of $\phi(t)$ in the theorem is also strengthened to the continuous differentiability assumption.

Remark 4: Suppose that we are first given a DDE (1) and its initial condition, and suppose that we convert its pseudo concatenated solution into representation as a time-delay feedback system via the method provided in Hagiwara & Kobayashi (2011). If we further consider converting the essential output of this time-delay feedback system Σ back into a DDE representation, we can confirm that the above theorems lead to exactly the same DDE and its initial condition as the original ones. Even though this is not surprising at all, it implies that these theorems as well as the results in Hagiwara & Kobayashi (2011) give a complete solution to the mutual conversion and comparison problems between DDEs and time-delay feedback systems in the sense of pseudo concatenated solutions.

3.1.4. The retarded case

Under Assumption 1, (D_{22}, D_{21}) is controllable with controllability index $\eta - 1$. Hence, K_i given by (17) reduces to $K_i = 0$ $(i = 1, \cdots, \eta)$ (and thus the resulting DDE degenerates to a retarded DDE (6)) if and only if $D_{11} = 0$ and $D_{12} = 0$. This fact leads to the following two theorems (corresponding to Theorems 1 and 2, respectively).

Theorem 3: *Suppose in the feedback system Σ that (i) F is described by (8) with $D_{11} = 0$ and $D_{12} = 0$, and (ii) the initial input $\psi(t),\ 0 \le t < h$ is bounded and continuous. Then, the essential output $y_1(t)$ of F in Σ coincides, over $t \ge 0$, with the (continuous) concatenated solution $x(t)$ resulting from the retarded DDE (6) with (17) under the initial function given by (18), where $\phi(t)$ is such a function satisfying the following conditions: $\phi(t)$ is bounded and continuous on the each of the intervals $[-ih, -(i-1)h)$ $(i = 1, \cdots, \eta)$ and satisfies (19).*

Theorem 4: *Suppose in the feedback system Σ that the conditions (i) and (ii) in Theorem 3 are satisfied. Then, the essential output $y_1(t)$ of F in Σ coincides, over $t \ge 0$, with the regular solution $x(t)$ resulting from the retarded DDE (6) with (17) under the initial function given by (18), provided that there exists $\phi(t)$ satisfying the following conditions: $\phi(t)$ is continuous on the interval $[-\eta h, 0]$ and satisfies (19) and*

$$\phi(0) = \zeta \tag{21}$$

3.2. *Conversion into continuous concatenated solution of DDE*

In contrast to Subsection 3.1, this subsection tackles the problem of representing the state of F in the time-delay feedback system Σ as a continuous concatenated solution of a DDE. To this end, we introduce the following assumption.

Assumption 2: The pair (B, D) is observable and its observability index is η.

If we recall the fact that Assumption 1 is essentially equivalent to the mere controllability assumption of the pair (D, C), the above is dual to Assumption 1. For the case when (B, D) is unobservable, see the further arguments in Subsection 4.2.

3.2.1. Determining the matrices in DDE

The conversion of Σ into DDE proceeds as follows by paying attention to the response of the state $q(t)$.

It follows from (2b) and (3) that

$$u(t) = Cq(t-h) + Du(t-h) \tag{22}$$

A recursive use of (22) leads to

$$u(t) = \sum_{i=1}^{k} D^{i-1}Cq(t-ih) + D^k u(t-kh) \tag{23}$$

Substituting the above equation into (2a) leads to

$$\dot{q}(t) = Aq(t) + \sum_{i=1}^{k} BD^{i-1}Cq(t-ih) + BD^k u(t-kh) \tag{24}$$

Replacing t by $t-(\eta-k)h$ in (24), we have

$$\dot{q}(t-(\eta-k)h) = Aq(t-(\eta-k)h) + BD^k u(t-\eta h) + \sum_{i=1}^{k} BD^{i-1}Cq(t-(\eta-k+i)h) \tag{25}$$

Taking (25) with $k = 0, \cdots, \eta-1$ leads to

$$\begin{bmatrix} \dot{q}(t-h) \\ \dot{q}(t-2h) \\ \vdots \\ \dot{q}(t-\eta h) \end{bmatrix} = \widetilde{A}_\eta \begin{bmatrix} q(t-h) \\ q(t-2h) \\ \vdots \\ q(t-\eta h) \end{bmatrix} + U_o u(t-\eta h) \tag{26}$$

where

$$\widetilde{A}_\eta = \begin{bmatrix} A & BC & \cdots & BD^{\eta-2}C \\ & A & \ddots & \vdots \\ & & \ddots & BC \\ & & & A \end{bmatrix}, \quad U_o = \begin{bmatrix} BD^{\eta-1} \\ BD^{\eta-2} \\ \vdots \\ B \end{bmatrix} \tag{27}$$

Here, the pseudo inverse $U_o^\dagger = (U_o^T U_o)^{-1} U_o^T$ of U_o exists by Assumption 2. Hence, we have

$$u(t-\eta h) = U_o^\dagger \begin{bmatrix} \dot{q}(t-h) \\ \vdots \\ \dot{q}(t-\eta h) \end{bmatrix} - U_o^\dagger \widetilde{A}_\eta \begin{bmatrix} \dot{q}(t-h) \\ \vdots \\ \dot{q}(t-\eta h) \end{bmatrix} \tag{28}$$

from (26). By introducing the partitioning $U_o^\dagger =: [V_1 \cdots V_\eta]$ and $U_o^\dagger \widetilde{A}_\eta =: [W_1 \cdots W_\eta]$ ($V_i,\ W_i \in \mathbb{R}^{\mu \times n},\ i = 1, \cdots, \eta$), we can rewrite the above equation as

$$u(t-\eta h) = \sum_{i=1}^{\eta} V_i \dot{q}(t-ih) - \sum_{i=1}^{\eta} W_i q(t-ih) \tag{29}$$

Substituting (29) into (25) with $k = \eta$ leads to the DDE

$$\dot{q}(t) = Aq(t) + \sum_{i=1}^{\eta} BD^{\eta}V_i\dot{q}(t-ih) + \sum_{i=1}^{\eta} B\left(D^{i-1}C - D^{\eta}W_i\right)q(t-ih) \tag{30}$$

This implies that if

$$J = A, \quad K_i = BD^{\eta}V_i, \quad L_i = B\left(D^{i-1}C - D^{\eta}W_i\right) \quad (i = 1, \cdots, \eta) \tag{31}$$

in (1), then the state $q(t)$ of the FDLTI system F in the time-delay feedback system Σ is expected to coincide, over $t \geq 0$, with the (continuous concatenated) solution $x(t)$ of the DDE (1).

3.2.2. Determining the initial function of DDE

As in the preceding subsection, however, the above observation needs to be established rigorously. The following two theorems indeed give complete answers to the conversion problem of the state of F into the continuous concatenated solution of the DDE (1), whose proofs are given in Appendix B. For the ease in the statement of the theorems, we note the fact that $u(t)$ and $q(t)$ $(0 \leq t < \eta h)$ of Σ are determined by the initial conditions ζ and ψ given by (7), and introduce the following notation $f^k(t, \psi, \zeta)$ to denote functions relevant to the response of Σ:

$$f^k(t,\psi,\zeta) := Bu(t) - \sum_{i=1}^{k} K_i\dot{q}(t-ih) - \sum_{i=1}^{k} L_i q(t-ih), \quad kh \leq t < (k+1)h \quad (k = 0, \cdots, \eta-1) \tag{32}$$

Theorem 5: *Suppose in the feedback system Σ that $\psi(t)$ is continuous for $0 \leq t < h$, and $\Psi(t) := \int_0^t \psi(\tau)d\tau$ has limit $\lim_{t\to h-0}\Psi(t)$. Then, the state $q(t)$ of F in Σ coincides, over $t \geq 0$, with the continuous concatenated solution $x(t)$ resulting from the neutral DDE (1) with the coefficient matrices (31) under the initial function given by*

$$x(t) = \phi(t), \quad -\eta h \leq t < 0 \tag{33a}$$

$$x(0) = \zeta \tag{33b}$$

provided that there exists $\phi(t)$ satisfying the following conditions: $\phi(t)$ is bounded and continuously differentiable on each of the intervals $[-ih, -(i-1)h)$ $(i = 1, \cdots, \eta)$, has limits $\lim_{t\to-(i-1)h-0}\phi(t)$ $(i = 1, \cdots, \eta)$, and satisfies

$$\sum_{i=k+1}^{\eta} K_i\dot{\phi}(t-ih) + \sum_{i=k+1}^{\eta} L_i\phi(t-ih) = f^k(t,\psi,\zeta), \quad kh \leq t < (k+1)h \quad (k = 0, \cdots, \eta-1) \tag{34}$$

where $f^k(t, \psi, \zeta)$ is given by (32).

Theorem 6: *Suppose in the feedback system Σ that the conditions in Theorem 5 are satisfied. Then, the state $q(t)$ of F in Σ coincides, over $t \geq 0$, with the regular solution $x(t)$ resulting from the neutral DDE (1) with (31) under the initial function given by (33), provided that there exists $\phi(t)$ satisfying the following conditions: $\phi(t)$ is continuously differentiable on the interval $[-\eta h, 0]$ and satisfies (34) and*

$$\phi(0) = \zeta \tag{35a}$$

$$\dot{\phi}(-0) = A\zeta + B\psi(0) \tag{35b}$$

Remark 5: In contrast with Theorem 1 in the preceding subsection, existence of $\phi(t)$ satisfying the conditions in Theorem 5 is not necessarily ensured (see Example 3). However, if K_η determined from (31) is invertible, there indeed exists such $\phi(t)$; to see this, it would suffice to consider solving (34) for $\dot{\phi}$ by rearranging it into a form involving a block triangular matrix. For K_η to be invertible, the matrix D must be invertible as shown in the following lemma The proof is given in Appendix A.

Lemma 2: If B is of full row rank, the pair (B, D) is observable with observability index η, and K_η determined from (31) is invertible, then D is invertible.

Even if D is invertible, however, K_η is not necessarily invertible. For example, this is the case if $B = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \end{bmatrix}$, $D = \begin{bmatrix} 0\,0\,1 \\ 1\,0\,0 \\ 0\,1\,0 \end{bmatrix}$, for which B is of full column rank, D is invertible and the observability index of $(B,\ D)$ is $\eta = 2$, while $K_\eta = \begin{bmatrix} 0\,1 \\ 0\,0 \end{bmatrix}$ is not invertible.

Remark 6: Suppose that we first convert a continuous concatenated solution of a given DDE (1) under a given initial function into representation as a time-delay feedback system via the method provided in Hagiwara & Kobayashi (2011). If we further consider converting the state of this time-delay feedback system Σ back into a DDE representation, the above theorems lead to the original DDE and the original initial function.

3.2.3. The retarded case

The coefficient matrices K_i in (31) satisfies $K_i = 0$ $(i = 1, \cdots, \eta)$ if and only if $BD^\eta = 0$. Thus we obtain the following two theorems for the retarded case corresponding to Theorems 5 and 6, respectively.

Theorem 7 : *Suppose in the feedback system Σ that (i) $BD^\eta = 0$ and (ii) $\psi(t)$ is continuous for $0 \le t < h$, and $\Psi(t) := \int_0^t \psi(\tau)d\tau$ has limit $\lim_{t\to h-0} \Psi(t)$. Then, the state $q(t)$ of F in Σ coincides, over $t \ge 0$, with the continuous concatenated solution $x(t)$ resulting from the DDE (6) with (31) under the initial function given by (33), provided that there exists $\phi(t)$ satisfying the following conditions: $\phi(t)$ is bounded and continuous on each of the intervals $[-ih,\ -(i-1)h)$ $(i = 1, \cdots, \eta)$ and satisfies (34).*

Theorem 8 : *Suppose in the feedback system Σ that the conditions (i) and (ii) in Theorem 7 are satisfied. Then, the state $q(t)$ of F in Σ coincides, over $t \ge 0$, with the regular solution $x(t)$ resulting from the DDE (6) with (31) under the initial function given by (33), provided that there exists $\phi(t)$ satisfying the following conditions: $\phi(t)$ is continuous on the interval $[-\eta h,\ 0]$ and satisfies (34) and (21).*

Remark 7: For the retarded case, if $L_\eta = BD^{\eta-1}C$ (determined from (31)) is invertible, there exists $\phi(t)$ satisfying the conditions in Theorem 7.

3.3. *Illustrative Examples*

This subsection is devoted to illustrating the conversion methods given in this section.

We first give the following example, illustrating the situation that even if the same time-delay feedback system and same initial conditions are considered, the coefficient matrices and initial functions of the corresponding DDEs can differ according to the two methods (or, depending on what type of solution we are dealing with about the DDEs).

Example 2: Consider the time-delay feedback system Σ with

$$F: \begin{cases} \dot{q}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} q(t) + u(t) & \text{(36a)} \\ y(t) = q(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u(t) & \text{(36b)} \end{cases}$$

We first give explicit representations of the responses of Σ under the initial condition given by

$$\psi(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (0 \le t < h), \quad \zeta = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{37}$$

It follows from (36) and (37) that $q(t) = [0,\ -1]^T$ and $y(t) = [1,\ 0]^T$ for $0 \le t < h$. As far as $q(t)$ is assumed continuous, we have $q(h) = [0,\ -1]^T$. Since $u(t) = [1,\ 0]^T (= \psi(t-h))$ for $h \le t < 2h$ and $q(h) = \zeta$, the above discussions are repeatable by regarding $t = h$ as the initial time instant. Thus we have

$$q(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad y(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (t \ge 0) \tag{38}$$

Note that the essential output $y_1(t)$ coincides with $y(t)$ itself in this example.

Next, we consider converting the above Σ through the methods in Subsections 3.1 and 3.2, respectively.

(i) Conversion of the essential output into a pseudo concatenated solution of DDE

The DDE and initial function given by Theorem 1 are

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \dot{x}(t-h) + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} x(t-h) \tag{39}$$

$$x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad -h \le t \le 0 \tag{40}$$

We can readily verify that $\phi(t)$ corresponding to the above initial function satisfies the additional conditions in Theorem 2, too, and the DDE (39) under this initial function has the regular solution $x(t) = [1,\ 0]^T$, $t \ge -h$ (as a special case of pseudo concatenated solutions). This regular solution coincides, over $t \ge 0$, with the above $y(t)$, and this implies that the conversion is indeed successful.

(ii) Conversion of the state into a continuous concatenated solution of DDE

According to Theorem 5, it asserts that the above Σ can be converted into the different neutral DDE

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \dot{x}(t-h) + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} x(t-h) \tag{41}$$

where the initial function $x(t),\ -h \le t \le 0$ must satisfy

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad -h \le t < 0 \tag{42a}$$

$$x(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{42b}$$

The DDE (41) together with (42) leads to $x(t) = [0,\ -1]^T$, $0 \le t < h$. If we note $x(h) = [0,\ -1]^T$

by continuity by the definition of continuous concatenated solutions, we can repeat the arguments and readily have the continuous concatenated solution of (41) given by $x(t) = [0,\ -1]^T$ for $t \geq 0$, where $x(t),\ -h \leq t < 0$ is an arbitrary function satisfying (42a), such as[1] $x(t) = [x_1,\ x_1 - 1]^T,\ -h \leq t < 0$ with an arbitrary x_1. Since this continuous concatenated solution $x(t)$ is identical, over $t \geq 0$, with $q(t)$ in (38), we see that the conversion is indeed successful.

In addition to what has been described just before the above example, the verification with this example has two meanings. The first meaning is obviously that we have demonstrated the validity of our theorems, which would be useful if one is interested in analyzing a time-delay feedback system through the vast wisdom and various tools available for DDEs. The second meaning is relevant to Remarks 2 and 5. This example corresponds to $\eta = 1$ both in Assumptions 1 and 2 (since D_{22} is empty), and Remark 2 always ensures the existence of $\phi(t-h),\ 0 \leq t < h$ for the first part of the example, which in fact is unique by (19). For the second part of the example dealing with continuous concatenated solutions, on the other hand, Remark 5 has suggested a possibility that no adequate ϕ could be found, leading to failure in converting Σ into a DDE. This might lead to a suspicion that the condition in Theorem 5 might in fact hold only in a very restrictive situations. We have given some observation for such a suspicion by showing a situation in which there actually exist infinitely many ϕ satisfying the required condition. This is in sharp contrast with the first part of the example in which ϕ was unique, and might suggest an essential difference in the two independent conversion problems that is worth investigating in the future.

As stated earlier, there is a slight difference in the feasibility of the above two conversion methods; if no $\phi(t)$ satisfies the conditions in Theorem 5 (or Theorem 6), these theorems cannot give an answer to the conversion problem considered in Subsection 3.2, unlike that in Subsection 3.1. This fact essentially implies that the state $q(t)$ of a time-delay feedback system Σ is not always convertible into representation as a continuous concatenated solution of a DDE. We give an example illustrating such a situation.

Example 3: Consider Σ with

$$F : \begin{cases} \dot{q}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} q(t) + u(t) & \text{(43a)} \\ y(t) = q(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} u(t) & \text{(43b)} \end{cases}$$

Here, it is obvious that (A, B) and (D, C) are controllable, and (C, A) and (B, D) are observable. The corresponding DDE with respect to its regular/continuous-concatenated solution determined by (31) is given by

$$\dot{x}(t) = Ax(t) + D\dot{x}(t-h) + (C - DA)\,x(t-h) \tag{44}$$

Here, $f^0(t, \psi, \zeta)$ is defined as $f^0(t, \psi, \zeta) := \psi(t),\ 0 \leq t < h$ by (32). Hence, by (34), the initial function $x(t) = \phi(t),\ -h \leq t < 0$ should satisfy

$$\begin{aligned} \psi(t) &= D\dot{\phi}(t-h) + (C - DA)\,\phi(t-h) \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{\phi}_1(t-h) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \phi_1(t-h), \quad 0 \leq t < h \end{aligned} \tag{45}$$

[1]We could take $x(t) = [0,\ -1]^T,\ -h \leq t < 0$ by letting $x_1 = 0$, in which case the above continuous concatenated solution of (41) in fact becomes a regular solution.

If $\psi(t) = [1,\ 1]^T$, $0 \le t < h$, however, no $\phi(t)$, $-h \le t < 0$ satisfies the above. Hence, Σ under such an initial input cannot be converted into a DDE in the sense of regular/continuous-concatenated solutions (through Theorems 5 and 6), regardless of $\zeta(= q(0))$.

4. Further extension by the introduction of auxiliary signals

This section extends the discussions in the preceding section by introducing auxiliary signals. Such arguments indeed give an answer to the conversion problems discussed in the preceding section without assuming the controllability of (D, C) and the observability of (B, D).

4.1. *Conversion into pseudo concatenated solution of DDE without controllability assumption*

This subsection deals with the case when F is described by (8), and relate the behavior of Σ with the pseudo concatenated solution of the modified DDE (5). Unlike in Subsection 3.1, we do not assume that the pair (D, C) is controllable.

First, we define $q_1(t) := q(t)$ and $q_2(t) := 0$ for $t \ge 0$, where $q_2(t) \in \mathbb{R}^{\mu - n}$ (note that we may assume $\mu \ge n$ by following the arguments in 3.1.1). With the introduction of this auxiliary signal $q_2(t)$, F given by (8) is described equivalently as

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} A & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{46a}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{\mu - n} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{46b}$$

where A_2 and A_3 are arbitrary matrices with compatible sizes. The above system in turn is equivalent to

$$\dot{\widetilde{q}}(t) = \widetilde{A}\widetilde{q}(t) + \widetilde{B}u(t) \tag{47a}$$

$$y(t) = \widetilde{q}(t) + \widetilde{D}u(t) \tag{47b}$$

if we define $\widetilde{q}(t) = [q_1(t)^T,\ q_2(t)^T]^T$ and

$$\widetilde{A} = \begin{bmatrix} A & A_2 \\ 0 & A_3 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{D} = D \tag{48}$$

The equivalent system of F described by (47) corresponds to (8) with D_{22} being an empty matrix, and thus it satisfies Assumption 1 with $\eta = 1$. Hence by Theorem 1, the signal $y(t)$ of F coincides, over $t \ge 0$, with the pseudo concatenated solution $\widetilde{x}(t)$ resulting from

$$\dot{\widetilde{v}}(t) = \widetilde{A}\widetilde{x}(t) + (\widetilde{B} - \widetilde{A}\widetilde{D})\widetilde{x}(t-h), \quad \widetilde{v}(t) = \widetilde{x}(t) - \widetilde{D}\dot{\widetilde{x}}(t-h) \tag{49}$$

under the initial function given by

$$\widetilde{x}(t) := \psi(t+h), \quad -h \le t < 0 \tag{50a}$$

$$\widetilde{x}(0) := \widetilde{q}(0) + \widetilde{D}\psi(0) = \begin{bmatrix} \zeta \\ 0 \end{bmatrix} + \widetilde{D}\psi(0) \tag{50b}$$

provided that $\psi(t)$ is bounded and continuous. In particular, if $\psi(t)$ is bounded and continuously differentiable, if $\lim_{t\to h-0}\psi(t)$ exists and coincides with (50b), and if $\lim_{t\to h-0}\dot{\psi}(t)$ exists and

$$\dot{\psi}(h-0)=\widetilde{A}\begin{bmatrix}\zeta\\0\end{bmatrix}+\widetilde{D}\dot{\psi}(+0)+\widetilde{B}\psi(0) \tag{51}$$

is satisfied, then by Theorem 2, $y(t)$ coincides with the regular solution of (49) under (50).

Note that the above conversion procedure is free from the controllability assumption of (the original) (D,C), and follows only from the unrestrictive (and non-essential) assumption that $C=[I,\,0]^T$. From the viewpoint of describing the behavior of the essential part $y_1(t)\in\mathbb{R}^n$ of the output $y(t)$ of F, however, this is achieved at the price of introducing an augmented vector $\widetilde{x}(t)\in\mathbb{R}^\mu$ $(\mu\geq n)$ to give an associated (higher-order or augmented) DDE, unlike in the arguments in Subsection 3.1.

4.2. *Conversion into continuous concatenated solution of DDE without observability assumption*

This subsection gives a conversion method related to the continuous concatenated solution of (1), without assumption on the observability of (B,D). We first note by arguments similar to the second paragraph of 3.1.1 that F in (2) may be assumed, without loss of generality, to be given by the form

$$\dot{q}(t)=Aq(t)+\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix} \tag{52a}$$

$$\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}=\begin{bmatrix}C_1\\C_2\end{bmatrix}q(t)+\begin{bmatrix}D_{11} & D_{12}\\D_{21} & D_{22}\end{bmatrix}\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix} \tag{52b}$$

We define $q_1(t):=q(t)$ and introduce $q_2(t)\in\mathbb{R}^{\mu-n}$ to describe the above system equivalently as

$$\begin{bmatrix}\dot{q}_1(t)\\\dot{q}_2(t)\end{bmatrix}=\begin{bmatrix}A & 0\\A_2 & A_3\end{bmatrix}\begin{bmatrix}q_1(t)\\q_2(t)\end{bmatrix}+\begin{bmatrix}I_n & 0\\0 & I_{\mu-n}\end{bmatrix}\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix} \tag{53a}$$

$$\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}=\begin{bmatrix}C_1 & 0\\C_2 & 0\end{bmatrix}\begin{bmatrix}q_1(t)\\q_2(t)\end{bmatrix}+\begin{bmatrix}D_{11} & D_{12}\\D_{21} & D_{22}\end{bmatrix}\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix} \tag{53b}$$

where A_2 and A_3 are arbitrary matrices with compatible sizes (note that $q_2(t)$ is not identically zero, in general). The above system is further equivalent to

$$\dot{\widetilde{q}}(t)=\widetilde{A}\widetilde{q}(t)+u(t) \tag{54a}$$

$$y(t)=\widetilde{C}\widetilde{q}(t)+\widetilde{D}u(t) \tag{54b}$$

where we define $\widetilde{q}(t)=[q_1(t)^T,\,q_2(t)^T]^T$ and

$$\widetilde{A}=\begin{bmatrix}A & 0\\A_2 & A_3\end{bmatrix},\quad \widetilde{C}=\begin{bmatrix}C_1 & 0\\C_2 & 0\end{bmatrix},\quad \widetilde{D}=D \tag{55}$$

The system F in the above equivalent form satisfies Assumption 2 with $\eta=1$. Hence by Theorem 5, the above signal $\widetilde{q}(t)$ coincides, over $t\geq 0$, with the continuous concatenated solution $\widetilde{x}(t)$ resulting from the DDE

$$\dot{\widetilde{x}}(t)=\widetilde{A}\widetilde{x}(t)+\widetilde{D}\dot{\widetilde{x}}(t-h)+(\widetilde{C}-\widetilde{D}\widetilde{A})\widetilde{x}(t-h) \tag{56}$$

under the initial function given by

$$\widetilde{x}(t) = \widetilde{\phi}(t), \quad -h \le t < 0 \tag{57a}$$

$$\widetilde{x}(0) = \begin{bmatrix} \zeta \\ q_2(0) \end{bmatrix} \tag{57b}$$

provided that there exists bounded and continuously differentiable $\widetilde{\phi}(t)$ that has the limit $\widetilde{\phi}(-0)$ and satisfies

$$(\widetilde{C} - \widetilde{D}\widetilde{A})\widetilde{\phi}(t) + \widetilde{D}\dot{\widetilde{\phi}}(t) = \psi(t+h), \quad -h \le t < 0 \tag{58}$$

If $\psi(t)$ satisfies the condition in Theorem 5, there exists such $\widetilde{\phi}(t)$ if $\widetilde{D}$ is invertible. However, if it is not invertible, then it could occur that no such $\widetilde{\phi}(t)$ exists whatever matrices A_2 and A_3 we may choose. For example, this is indeed the case if $A = C_1 = C_2 = D_{11} = 1$, $D_{12} = D_{21} = D_{22} = 0$ and $\psi(t) = [1,\ 0]^T$ $(0 \le t < h)$, for which (B, D) is unobservable in (52), while it is observable in the augmented representation by (54).

This situation is in sharp contrast with the parallel arguments in the case of the pseudo concatenated solutions of (5), in which introducing the auxiliary signal $q_2(t) \in \mathbb{R}^{\mu-n}$ always made it possible to convert Σ into an (augmented) DDE.

The discussions in Section 3 are based on the implicit assumption that $x(t)$ in the DDE converted from Σ must have the same number of elements as the state $q(t)$ of F. On the other hand, this section has revealed the fact that relaxing this assumption could enable us to convert Σ with more general F into a DDE by introducing the auxiliary signals.

5. Conclusion

This paper studied the problem of converting linear time-invariant (LTI) time-delay systems (TDS) described by time-delay feedback systems Σ as in Figure 1 into representation as delay-differential equations (DDEs) described by (1), including the discussions of initial conditions. Two such methods were given by assuming the controllability of the pair (D, C) or observability of (B, D). In the first method, an essential part of the output of Σ was made to coincide, over $t \ge 0$, with the pseudo concatenated solution of the neutral DDE (1) with commensurate delays (or the modified DDE (5), or the concatenated solution of the retarded DDE (6), depending on the parameters and initial condition of Σ), while the second method showed a method to make the state of Σ coincide, over $t \ge 0$, with the continuous concatenated solution of the neutral DDE (1) (or that of the retarded DDE (6)). We also gave an example illustrating the fact that even if the same feedback system Σ and same initial conditions are considered, the coefficient matrices (and initial functions) of the corresponding DDEs can differ according to the two methods (or equivalently, which signal in Σ is to be converted into a DDE representation).

Next, this paper extended such conversion methods by introducing auxiliary signals (without changing Σ essentially). The augmented systems obtained by this treatment can always be converted into (augmented) DDEs, without the above-mentioned controllability/observability assumption. In short, we showed that if we relax the implicit assumption that $x(t)$ in the DDEs converted from Σ must have the same number of elements as the state $q(t)$ of F, we can enlarge the class of Σ that can be converted into (higher-order) DDEs.

Nonetheless, we also gave an example illustrating that some time-delay feedback systems Σ cannot be converted into DDEs. This implies that some TDS can be represented as Σ but not as a DDE. This, together with the fact that LTI DDEs with commensurate delays can always be converted into Σ regardless of initial functions (Hagiwara & Kobayashi 2011), establishes that representation as LTI time-delay feedback systems Σ possesses higher ability than that as DDEs, as description methods for TDSs with commensurate delays.

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Appendix A. Proofs of lemmas

We introduce the following lemma for the preparation of the proof of Lemma 1.

Lemma 3: If (A_0, B_0) is controllable and its controllability index is $\nu \geq 2$, then there exists T_0 such that

$$(T_0^{-1}A_0T_0, T_0^{-1}B_0) = \left(\begin{bmatrix} \widetilde{A}_{0,11} & \widetilde{A}_{0,12} \\ \widetilde{A}_{0,21} & \widetilde{A}_{0,22} \end{bmatrix}, \begin{bmatrix} \widetilde{B}_{0,1} \\ 0 \end{bmatrix} \right) \tag{A1}$$

where $\widetilde{B}_{0,1}$ is of full row rank and $(\widetilde{A}_{0,22}, \widetilde{A}_{0,21})$ is controllable with controllability index $\nu - 1$.

Proof: Let the singular value decomposition of B_0 be $B_0 = U_0\Sigma_0V_0$. Then, $T_0 = U_0$ leads to the assertion. This is especially easy to see when $\nu = 2$; it is obvious from the definition of singular value decomposition that $T_0^{-1}B_0$ has the desired form as in (A1) with full-row-rank $\widetilde{B}_{0,1}$ (for any ν). Furthermore, by the full-row-rank property of the partial controllability matrix $\mathcal{C}_k(A_0, B_0) := [B_0, A_0B_0, \cdots, A_0^{k-1}B_0]$ with $k = \nu = 2$, or equivalently $\mathcal{C}_2(T_0^{-1}A_0T_0, T_0^{-1}B_0) =$

$\begin{bmatrix} \widetilde{B}_{0,1} & \widetilde{A}_{0,11}\widetilde{B}_{0,1} \\ 0 & \widetilde{A}_{0,21}\widetilde{B}_{0,1} \end{bmatrix}$, it follows that $\widetilde{A}_{0,21}$ is of full row rank. This implies that $(\widetilde{A}_{0,22}, \widetilde{A}_{0,21})$ is controllable with controllability index 1 $(= \nu - 1)$.

To confirm the assertion for $\nu = 3$, we have to verify that $\mathcal{C}_k(\widetilde{A}_{0,21}, \widetilde{A}_{0,22})$ is of full row rank for $k = \nu - 1 = 2$ but not for $k < 2$. Taking account of the structure of the matrix

$$\mathcal{C}_3(T_0^{-1}A_0T_0, T_0^{-1}B_0) = \begin{bmatrix} \widetilde{B}_{0,1} & \widetilde{A}_{0,11}\widetilde{B}_{0,1} & \widetilde{A}_{0,11}^2\widetilde{B}_{0,1} + \widetilde{A}_{0,12}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \\ 0 & \widetilde{A}_{0,21}\widetilde{B}_{0,1} & \widetilde{A}_{0,21}\widetilde{A}_{0,11}\widetilde{B}_{0,1} + \widetilde{A}_{0,22}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \end{bmatrix} \tag{A2}$$

(which has full row rank by the assumption that $\nu = 3$), it follows from the full-row-rank property of $\widetilde{B}_{0,1}$ that the second block-column can be used to cancel the first terms on the third block-column (through column manipulations using a matrix X_1 such that $\widetilde{B}_{0,1}X_1 = -\widetilde{A}_{0,11}\widetilde{B}_{0,1}$). This implies that the following matrix has full row rank.

$$\begin{bmatrix} \widetilde{B}_{0,1} & \widetilde{A}_{0,11}\widetilde{B}_{0,1} & \widetilde{A}_{0,12}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \\ 0 & \widetilde{A}_{0,21}\widetilde{B}_{0,1} & \widetilde{A}_{0,22}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \end{bmatrix} \tag{A3}$$

It then follows from the structure of the above matrix that $[\widetilde{A}_{0,21}\widetilde{B}_{0,1} \ \ \widetilde{A}_{0,22}\widetilde{A}_{0,21}\widetilde{B}_{0,1}] = \mathcal{C}_2(\widetilde{A}_{0,22}, \widetilde{A}_{0,21}\widetilde{B}_{0,1})$ is of full row rank. Since $\widetilde{B}_{0,1}$ has full row rank, this obviously implies that $\mathcal{C}_2(\widetilde{A}_{0,22}, \widetilde{A}_{0,21})$ also has full row rank. However, $\mathcal{C}_1(\widetilde{A}_{0,22}, \widetilde{A}_{0,21}) = \widetilde{A}_{0,21}$ does not have full row rank, because otherwise it follows from (A2) that $\mathcal{C}_2(T_0^{-1}A_0T_0, T_0^{-1}B_0)$ also has full row rank; this obviously contradicts the assumption that $\nu = 3$. This completes the arguments for confirming the assertion for the case $\nu = 3$.

For $\nu = 4$, we can apply similar arguments by further introducing matrices X_{21} and X_{22} such that $\widetilde{B}_{0,1}X_{21} = -\widetilde{A}_{0,11}^2\widetilde{B}_{0,1}$ and $\widetilde{B}_{0,1}X_{22} = -\widetilde{A}_{0,12}\widetilde{A}_{0,21}\widetilde{B}_{0,1}$ to confirm that $\mathcal{C}_3(\widetilde{A}_{0,22}, \widetilde{A}_{0,21})$ has full row rank but $\mathcal{C}_k(\widetilde{A}_{0,22}, \widetilde{A}_{0,21})$ does not for $k < 3$. Alternatively, by taking the third block-column of (A3) and noting that

$$\begin{bmatrix} \widetilde{A}_{0,11} & \widetilde{A}_{0,12} \\ \widetilde{A}_{0,21} & \widetilde{A}_{0,22} \end{bmatrix} \begin{bmatrix} \widetilde{A}_{0,12}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \\ \widetilde{A}_{0,22}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{0,11} \\ \widetilde{A}_{0,21} \end{bmatrix} \widetilde{A}_{0,12}\widetilde{A}_{0,21}\widetilde{B}_{0,1} + \begin{bmatrix} \widetilde{A}_{0,12} \\ \widetilde{A}_{0,22} \end{bmatrix} \widetilde{A}_{0,22}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \tag{A4}$$

$$= -\begin{bmatrix} \widetilde{A}_{0,11} \\ \widetilde{A}_{0,21} \end{bmatrix} \widetilde{B}_{0,1}X_{22} + \begin{bmatrix} \widetilde{A}_{0,12} \\ \widetilde{A}_{0,22} \end{bmatrix} \widetilde{A}_{0,22}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \tag{A5}$$

where the first term on the right hand side is a result of column manipulations on the second block-column of (A3), we can see that $\mathcal{C}_4(T_0^{-1}A_0T_0, T_0^{-1}B_0)$ has the same rank as

$$\begin{bmatrix} \widetilde{B}_{0,1} & \widetilde{A}_{0,11}\widetilde{B}_{0,1} & \widetilde{A}_{0,12}\widetilde{A}_{0,21}\widetilde{B}_{0,1} & \widetilde{A}_{0,12}\widetilde{A}_{0,22}\widetilde{A}_{0,21}\widetilde{B}_{0,1} \\ 0 & \widetilde{A}_{0,21}\widetilde{B}_{0,1} & \widetilde{A}_{0,22}\widetilde{A}_{0,21}\widetilde{B}_{0,1} & \widetilde{A}_{0,22}^2\widetilde{A}_{0,21}\widetilde{B}_{0,1} \end{bmatrix} \tag{A6}$$

We can then easily see that $(\widetilde{A}_{0,22}, \widetilde{A}_{0,21}\widetilde{B}_{0,1})$ has controllability index 3 $(= \nu - 1)$, and so does $(\widetilde{A}_{0,22}, \widetilde{A}_{0,21})$.

The assertion for general ν can be confirmed along the same line. That is, we can confirm the assertion through

$$\operatorname{rank} \mathcal{C}_k(A_0, B_0) = \operatorname{rank} \begin{bmatrix} \widetilde{B}_{0,1} & * \\ 0 & \mathcal{C}_{k-1}(\widetilde{A}_{0,22}, \widetilde{A}_{0,21}\widetilde{B}_{0,1}) \end{bmatrix} \tag{A7}$$

□

Proof of Lemma 1: If $B_0 = [I,\, 0]^T$ in Lemma 3 (note that the pair (9) under consideration corresponds to such a case), it follows from the above proof that we can choose $T_0 = I$. This implies that (D_{22}, D_{21}) is controllable with controllability index $\eta - 1$. We can then apply Lemma 3 again to $(A_0, B_0) = (D_{22}, D_{21})$ (unless D_{21} is of full column rank so that the controllability index of (D_{22}, D_{21}) is 1); let us denote by T_{01} the resulting similarity transformation. Let $\widehat{T}_{01} := \mathrm{diag}[I, T_{01}]$ and apply the similarity transformation with $\widetilde{T}_{01}$ to the pair (9). Then, we are led to

$$\left(\left[\begin{array}{c:cc} * & * & * \\ \hdashline \widehat{D}_1 & * & * \\ 0 & \widehat{A}_{21} & \widehat{A}_{22} \end{array} \right], \left[\begin{array}{c} I \\ \hdashline 0 \\ 0 \end{array} \right] \right) \tag{A8}$$

for some $\widehat{D}_1$, $\widehat{A}_{21}$ and $\widehat{A}_{22}$, where $\widehat{D}_1$ is of full column rank and $(\widehat{A}_{22}, \widehat{A}_{21})$ is controllable with controllability index $\eta - 2$. If $\widehat{A}_{21}$ is not of full column rank (or equivalently, if the controllability index of $(\widehat{A}_{22}, \widehat{A}_{21})$ is greater than 1), then we can further apply Lemma 3 to $(A_0, B_0) = (\widehat{A}_{22}, \widehat{A}_{21})$. Hence, applying Lemma 3 to the pair (9) recursively yields its similarity transform given by

$$\left(\left[\begin{array}{cccc:c} * & * & * & * & * \\ \widehat{D}_1 & * & * & * & * \\ 0 & \ddots & * & * & * \\ \vdots & \ddots & \widehat{D}_{\eta-2} & * & * \\ \hdashline 0 & \cdots & 0 & \widehat{D}_{\eta-1} & * \end{array} \right], \left[\begin{array}{c} I \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \end{array} \right] \right) \tag{A9}$$

where $\widehat{D}_i$ $(i = 1, \cdots, \eta - 1)$ are of full row rank.

Since $\widehat{D}_{\eta-1}$ is of full row rank, we can determine X such that the similarity transformation

$$\mathrm{diag}[I, T_1], \quad T_1 = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \tag{A10}$$

applied to the above pair leads to

$$\left(\left[\begin{array}{cccc:c} * & * & * & *' & *' \\ \widehat{D}_1 & * & * & *' & *' \\ 0 & \ddots & * & *' & *' \\ \vdots & \ddots & \widehat{D}_{\eta-2} & *' & *' \\ \hdashline 0 & \cdots & 0 & \widehat{D}_{\eta-1} & 0 \end{array} \right], \left[\begin{array}{c} I \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \end{array} \right] \right) \tag{A11}$$

Next, since $\widehat{D}_{\eta-2}$ is of full row rank, we can have

$$\left(\left[\begin{array}{cccc:c} * & * & *' & *'' & *'' \\ \widehat{D}_1 & * & *' & *'' & *'' \\ 0 & \ddots & *' & *'' & *'' \\ \vdots & \ddots & \widehat{D}_{\eta-2} & 0 & 0 \\ \hdashline 0 & \cdots & 0 & \widehat{D}_{\eta-1} & 0 \end{array} \right], \left[\begin{array}{c} I \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \end{array} \right] \right) \tag{A12}$$

through the similarity transformation

$$\mathrm{diag}[I, T_2], \quad T_2 = \left[\begin{array}{cc:c} I & Y_1 & Y_2 \\ 0 & I & 0 \\ \hdashline 0 & 0 & I \end{array}\right] \tag{A13}$$

with Y_1 and Y_2 determined appropriately. Repeating similar arguments leads to

$$\left(\left[\begin{array}{c:cccc} \star & \star & \star & \star & \star \\ \hdashline \widehat{D}_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \widehat{D}_{\eta-2} & 0 & 0 \\ 0 & \cdots & 0 & \widehat{D}_{\eta-1} & 0 \end{array}\right], \left[\begin{array}{c} I \\ \hdashline 0 \\ \vdots \\ 0 \\ 0 \end{array}\right] \right) \tag{A14}$$

This completes the proof. □

Proof of Lemma 2 : We prove the assertion by contradiction. Without loss of generality, we may assume $B = [I\ 0]$. If D is not invertible, there exists nonzero $\xi = [\xi_1^T, \xi_2^T]^T$ such that $D\xi = 0$. By the above form of B, it follows from the observability of (B, D) that $\xi_1 \neq 0$. Multiplying $U_o\xi$ from the right of $[K_1\ \cdots\ K_\eta] = BD^\eta U_o^\dagger$ leads to

$$\left[K_1 \cdots K_\eta\right] U_o\xi = BD^\eta\xi = 0 \tag{A15}$$

Here, recalling that $B = [I\ 0]$, the left-hand side is

$$\left[K_1 \cdots K_\eta\right] \begin{bmatrix} BD^{\eta-1} \\ \vdots \\ BD \\ B \end{bmatrix} \xi = \left[K_1 \cdots K_\eta\right] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \xi_1 \end{bmatrix} = K_\eta\xi_1 \tag{A16}$$

This implies that K_η is not invertible. □

Appendix B. Proofs of theorems

Proof of Theorem 1 : We have seen in 3.1.2 that a pseudo concatenated solution of DDE (5) (or that of DDE (1)) is expected to coincide with the essential output $y_1(t)$ of F in Σ. However, there are some issues to be resolved; discussions there do not clarify how the suitable initial condition should be given. Roughly speaking, this proof shows that under the initial function given in Theorem 1, the DDE admits a pseudo concatenated solution and it indeed coincides with the essential output $y_1(t)$.

For $0 \leq t < h$, substituting (17) and (19) into (8b) leads to

$$y_1(t) = q(t) + \sum_{i=1}^{\eta} K_i\phi(t - ih) \tag{B1}$$

Then substituting the above into (8a) yields

$$\dot{q}(t) = Jy_1(t) + \sum_{i=1}^{\eta} L_i\phi(t - ih) \tag{B2}$$

if we note (17) and (19). Here, it follows from (8a) and (18a) that $q(t)$ is (continuous and) differentiable for $0 \leq t < h$. Comparing (B1) and (B2) with the modified DDE (5) leads to $v(t) = q(t)$ and $x(t) = y_1(t)$ for $0 \leq t < h$, if we note that $v(0) = q(0)$ and $x(0) = y_1(0)$ by (7), (17), (18b) and (19). Hence, for $h \leq t < 2h$, it follows from (8b) and (3) that

$$u_1(t) = y_1(t - h) = x(t - h), \quad h \leq t < 2h \tag{B3a}$$

$$u_2(t) = [D_{21}\ D_{22}]\psi(t - h), \quad h \leq t < 2h \tag{B3b}$$

Substituting (19) into (B3b) leads to

$$\begin{aligned} u_2(t) &= D_{21}\phi(t - 2h) + D_{22}\sum_{i=2}^{\eta} D_{22}^{i-2}D_{21}\phi(t - (i+1)h) \\ &= \sum_{i=2}^{\eta+1} D_{22}^{i-2}D_{21}\phi(t - ih) \\ &= \sum_{i=2}^{\eta} D_{22}^{i-2}D_{21}\phi(t - ih), \quad h \leq t < 2h \end{aligned} \tag{B4}$$

if we note (10). It follows from (B3a) and (B4) that (a) the "initial input" (under the time advanced by h) of F for $h \leq t < 2h$ is given by the relation corresponding to (19) with the time advances by h (in other words, "the same ϕ" satisfies (19) under the time advance by h).

Furthermore, it follows from boundedness of $\psi(t)$ that $\lim_{t\to h-0} q(t)$ exists, and as far as we consider only continuous $q(t)$ and $v(t)$ according to the underlying definition, we have $v(h) = q(h) := \lim_{t\to h-0} q(t)$. It then follows from the second equation of (5), (17), (B3a) and (B4) that the pseudo concatenated solution of (5) satisfies $x(h) = v(h) + K_1x(0) + \sum_{i=2}^{\eta} K_i\phi(-(i-1)h)$ $= q(h) + D_{11}x(0) + \sum_{i=2}^{\eta} D_{12}D_{22}^{i-2}D_{21}\phi(-(i-1)h) = q(h) + [D_{11}\ D_{12}]u(h)$. This implies that (b) the "initial value" of q that we view by advancing the time by h is given by the relation corresponding to (18b) with the time advanced by h. The above two facts (a) and (b) about the time advance by h enable us to repeat the arguments, which immediately completes the proof of the theorem. □

Proof of Theorem 2 : The existence of the (unique) regular solution of neutral DDE (1) is ensured by the following conditions (Hagiwara & Kobayashi 2011): $\phi(t)$ is continuously differentiable on the interval $[-\eta h, 0]$ and satisfies

$$\phi(0) = \xi \tag{B5a}$$

$$\dot{\phi}(-0) = J\phi(0) + \sum_{i=1}^{\eta} K_i\dot{\phi}(-ih + 0) + \sum_{i=1}^{\eta} L_i\phi(-ih) \tag{B5b}$$

We can readily verify that the additional assumptions in Theorem 2 correspond to such requirements. Hence the pseudo concatenated solution introduced in the above proof actually becomes the regular solution (because a regular solution is a pseudo concatenated solution at the same time and a pseudo concatenated solution is unique). □

Proof of Theorem 5 : As discussed in 3.2.2, a continuous concatenated solution of DDE (1) is expected to coincide with the state $q(t)$ of F in Σ. Roughly speaking, this proof verifies that under the initial function given in Theorem 5, the DDE admits a continuous concatenated solution and it indeed coincides with the state $q(t)$.

For $0 \le t < h$, it follows from (2a) and (31) together with (32) and (34) with $k = 0$ that

$$\dot{q}(t) = Jq(t) + \sum_{i=1}^{\eta} K_i\dot{\phi}(t - ih) + \sum_{i=1}^{\eta} L_i\phi(t - ih) \tag{B6}$$

By (7a) and (33b), we readily have $x(0) = q(0)$, and then (B6) immediately leads to $x(t) = q(t)$ on the interval $[0,\, h)$. If we note the assumptions on $\phi(t)$, it is obvious that $q(t)$ is (continuous and) continuously differentiable on this interval. Furthermore, the assumptions on $\psi(t)$ ensure the existence of $\lim_{t\to h-0} q(t)$, and as far as we confine ourselves to continuous $q(t)$ and $x(t)$ for $t \ge 0$, we are led to defining $x(h) = q(h)$ by the above limit. Since (34) holds on each of the intervals $[kh,\, (k+1)h)$ $(k = 1, \cdots, \eta - 1)$, we can repeat similar arguments to establish that $x(t) = q(t)$ is (continuous and) continuously differentiable on each of these intervals. In particular, we are led to $x(\eta h) = q(\eta h)$ by continuity.

On the other hand, for $t \ge \eta h$, it follows from (23) with $k = \eta$, (29) and (31) that

$$Bu(t) = \sum_{i=1}^{\eta} K_i\dot{q}(t - ih) + \sum_{i=1}^{\eta} L_i q(t - ih),\ t \ge \eta h \tag{B7}$$

This together with $x(\eta h) = q(\eta h)$ immediately leads to $x(t) = q(t)$ also for $t \ge \eta h$, where such $x(t)$ is (continuous and) continuously differentiable on each of the intervals $[kh,\, (k+1)h)$ $(k = 0, 1, \cdots)$. Hence, it is obviously a continuous concatenated solution of the DDE (1). This leads to the assertion of the theorem. □

Proof of Theorem 6 : As in the proof of Theorem 2, we can readily verify that the additional hypotheses in Theorem 6 ensure the existence of the regular solution. Hence the continuous concatenated solution introduced in the above proof actually becomes the regular solution. □

Theorems for the retarded case (Theorems 3 and 4, and Theorems 7 and 8) can be proved similarly to the neutral case, and the details are omitted.