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<th>Title</th>
<th>Indivisibility of central values of L-functions for modular forms</th>
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INDIVISIBILITY OF CENTRAL VALUES OF $L$-FUNCTIONS FOR MODULAR FORMS

MASATAKA CHIDA

Abstract. In this paper, we generalize works of Kohnen-Ono [7] and James-Ono [5] on indivisibility of (algebraic part of) central critical values of $L$-functions to higher weight modular forms.

1. INTRODUCTION

In this article, we show an indivisibility result on central critical values of $L$-functions associated to quadratic twists of modular forms using a method of Kohnen-Ono [7] and James-Ono [5].

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. For a fundamental discriminant $D$ with $(D,N)=1$, we define the $D$-th quadratic twist of $f$ by

$$f \otimes \chi_D = \sum_{n=1}^{\infty} a(n)\chi_D(n)q^n,$$

where $\chi_D$ is the quadratic character corresponding to the quadratic extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Then $f \otimes \chi_D$ is a newform of weight $2k$ for $\Gamma_0(D^2N)$. Similarly, the $D$-th quadratic twist of the $L$-function $L(f,s)$ is given by

$$L(f \otimes \chi_D,s) = \sum_{n=1}^{\infty} \frac{a(n)\chi_D(n)}{n^s}.$$

Let $E$ be the number field generated by all Fourier coefficient of $f$ and $\mathbb{Q}$. Then it is known that there exists a period $\Omega_f \in \mathbb{C}^*$ satisfying that $\frac{L(f \otimes \chi_D,k)D_0k^{-1/2}}{\Omega_f}$ are integers in $E$ for all fundamental discriminant $D$ with $\delta(f) \cdot D > 0$, where $\delta(f) \in \{\pm1\}$ is the sign defined in Ono-Skinner [10, p. 655] and $D_0$ is given by

$$D_0 = \begin{cases} |D| & \text{if } D \text{ is odd,} \\ |D|/4 & \text{if } D \text{ is even.} \end{cases}$$

We fix such a period $\Omega_f$.

For convenience, we denote

$$S(X) = \{D \in \mathbb{Z} \mid |D| < X, D : \text{fundamental discriminant}\},$$

and if functions $f$, $g$ on $\mathbb{R}$ satisfy that there is a positive constant $c$ such that $f(X) \geq c \cdot g(X)$ for sufficiently large $X > 0$, then we write $f(X) \gg g(X)$.

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Theorem 1.1. Let \( f(z) = \sum_{n=1}^{\infty} a(n)q^n \) be a normalized newform of weight \( 2k \) for \( \Gamma_0(N) \) with trivial character. Then, for all but finitely many primes \( \lambda \) of \( E \), we have
\[
\# \left\{ D \in S(X) \mid \delta(f) \cdot D > 0, \lambda \nmid D \text{ and } \frac{L(f \otimes \chi_D, k)D_0^{k-1}}{\Omega_f} \neq 0 \mod \lambda \right\} \gg f, \lambda \frac{\sqrt{X}}{\log X}.
\]

This result is a refinement of results of Bruinier [2] and Ono-Skinner [10]. The proof is based on a generalization of a method of Kohnen-Ono [7] and James-Ono [5]. In the above theorem, we do not assume that the Fourier coefficients of \( f \) belong to \( \mathbb{Z} \), therefore it does not hold the surjectivity of the residual Galois representation associated to \( f \) for almost all places in general. This makes some technical difficulty on the proof. To solve this problem, we may use a result of Ribet [12] on the image of Galois representations associated to modular forms. This is an ingredient in our proof. In the last section, we also consider an indivisibility result on non-central critical values of \( L \)-functions for higher weight modular forms using congruences of modular form with different weights.

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2. Modular forms of half-integral weight

We denote the space of modular forms of weight \( k + 1/2 \), level \( N \) with character \( \chi \) by \( M_{k+1/2}(N, \chi) \), and the space of cusp forms of weight \( k + 1/2 \), level \( N \) with character \( \chi \) by \( S_{k+1/2}(N, \chi) \). Then \( M_{k+1/2}(N, \chi) \) and \( S_{k+1/2}(N, \chi) \) are complex vector spaces.

For a modular form of half-integral weight \( g(z) = \sum_{n=0}^{\infty} b(n)q^n \in M_{k+1/2}(N, \chi) \), we define the action of Hecke operator \( T_{p^2} \) by
\[
T_{p^2}(g)(z) = \sum_{n=0}^{\infty} b'(n)q^n,
\]
where \( b'(n) \) are given by
\[
b'(n) = b(p^2n) + \chi(p) \left( \frac{-1}{p} \right)^k \left( \frac{n}{p} \right) p^{k-1}b(n) + \chi(p^2)p^{2k-1}b(n/p^2)
\]
and \( b(n/p^2) \) are zero if \( p^2 \nmid n \).

Now we give a short review of the theory of the Shimura correspondence. Let \( N \) be a positive integer which is divisible by four and \( \chi \) a Dirichlet character mod \( N \). Then we define a vector space \( S_{3/2}^0(N, \chi) \) to be the subspace of \( S_{3/2}(N, \chi) \) generated by
\[
\left\{ f(z) = \sum_{n=1}^{\infty} \psi(n)nq^{\text{cond}(\psi)tn} \mid N = 4 \text{ cond}(\psi)^2t|N, \chi = \psi\chi^{-t} \text{ and } \psi(-1) = -1 \right\}
\]
and denote the orthogonal complement by \( S_{3/2}'(N, \chi) \). Then we assume
\[
g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)
\]
if $k \geq 2$, and
\[
g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{g/2}(N, \chi)
\]
if $k = 1$. Let $t$ be a square-free positive integer. Define a number $A_t(n)$ to be
\[
\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = \left( \sum_{n=1}^{\infty} \frac{\chi(n)(-1)^k \left( \frac{t}{n} \right)}{n^{s-k+1}} \right) \left( \sum_{n=1}^{\infty} \frac{b(tn^2)}{n^s} \right).
\]
Then Shimura [14] proved that there is a positive integer $M$ such that $SH_t(g(z)) = f_t(z) = \sum_{n=1}^{\infty} A_t(n) q^n \in S_{2k}(M, \chi^2)$. (In fact, one can prove that $M = N/2$). Furthermore for any $t, t'$, the difference between $SH_t(g)$ and $SH_{t'}$ is only constant multiple, so essentially this correspondence is independent of choice of $t$. This correspondence between modular forms is called the Shimura correspondence. Moreover if $g$ is an eigenform for all Hecke operators $T_p$ with $(p, 2N) = 1$, then the image of $g$ under the Shimura correspondence is also an eigenform for all Hecke operators $T_p$ with $(p, 2N) = 1$ and the Hecke eigenvalue of $T_p g$ for $g$ coincides with the Hecke eigenvalue for $T_p g$ for $SH_t(g)$.

We recall the following result which is a useful version of Waldspurger’s formula ([17, Théorém 1]) by Ono-Skinner. This formula gives a relation between the Fourier coefficients of modular forms of half-integral weight and the central values of twisted $L$-functions for modular forms.

**Theorem 2.1** (Ono-Skinner [9], (2a),(2b)). Let $f(z) = \sum_{n=1}^{\infty} a(n) q^n$ be a normalized newform of weight $2k$, level $M$ with trivial character. Then there is $\delta(f) \in \{ \pm 1 \}$, a positive integer $N$ with $4M \mid N$, a Dirichlet character $\chi$ modulo $N$, a period $\Omega_f \in \mathbb{C}^\times$ and a non-zero eigenform
\[
g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{k+1/2}(N, \chi)
\]
with the property that $g(z)$ maps to a twist of $f$ under the Shimura correspondence and for all fundamental discriminant $D$ with $\delta(f)D > 0$ we have
\[
b(D_0)^2 = \begin{cases} 
\alpha_D \frac{L(f \otimes \chi_D, k) D_0^{k-1/2}}{\Omega_f} & \text{if } (D, N) = 1, \\
0 & \text{otherwise},
\end{cases}
\]
where $\alpha_D$ and $b(n)$ are algebraic integers in some finite extension of $\mathbb{Q}$. Moreover, there exists a finite set of primes $S$ such that if $D$ is a fundamental discriminant for which
\begin{enumerate}
\item $\delta(f)D > 0,$
\item $(D, N) = 1,$
\end{enumerate}
then we have $|L(f \otimes \chi_D, k) D_0^{k-1/2}/\Omega_f|^\lambda = |b(D_0)^2|^\lambda$ for $\lambda \not\in S$.

3. Some properties of Fourier coefficients of modular forms and Galois representations

In this section we generalize some results of Serre [13] and Swinnerton-Dyer [16] using a result of Ribet [12]. These results should be well-known for specialists. However we give a short review for them, since it does not seem to be available in the literature. Let $f = \sum_{n=1}^{\infty} a(n) q^n$ be a normalized newform of weight $2k$ for $G_0(N)$ with trivial character. Let $E$ be the subfield of $\mathbb{C}$ generated by the Fourier coefficients $a(n)$ of $f$. Then $E$ is a finite extension of $\mathbb{Q}$. Let $\mathcal{O}_E$ be the ring of integers of $E$. For each prime $\ell$, we let $\mathcal{O}_{E, \ell} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. 

Theorem 3.1 (Deligne [3]). For each prime $\ell$, there exists a continuous representation 
\[ \rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O}_{E,\ell}) \subset \text{GL}_2(E_{\ell}) \]
unramified at all primes $p \nmid N\ell$ such that $\text{trace}_{f,\ell}(\text{Frob}_p) = a(p)$ and $\text{det}_{f,\ell}(\text{Frob}_p) = p^{2k-1}$ for all primes $p \nmid N\ell$, where $\text{Frob}_p$ is the arithmetic Frobenius at $p$.

For each prime $\ell$, denote 
\[ A_{\ell} = \left\{ g \in \text{GL}_2(\mathcal{O}_{E,\ell}) \mid \det(g) \in \mathbb{Z}_\ell^{\times(2k-1)} \right\}, \]
where $\mathbb{Z}_\ell^{\times(2k-1)}$ is the group of $(2k-1)$-th powers of elements in $\mathbb{Z}_\ell^{\times}$. Replacing $\rho_{f,\ell}$ by an isomorphic representation, we may assume that for almost all $\rho_{f,\ell}$ sends $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $A_{\ell}$.

Then Ribet proved the following theorem.

Theorem 3.2 (Ribet [12]). Assume that $f$ has no complex multiplication. Then for almost all $\ell$, we have $\rho_{f,\ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = A_{\ell}$.

We call the set of primes $\ell$ with the property $\rho_{f,\ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \neq A_{\ell}$ by the exceptional primes for $f$. Let $S$ be the set of exceptional places for $f$. Let $\varepsilon_{\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^{\times}$ be the $\ell$-adic cyclotomic character. Then by a similar argument with Swinnerton-Dyer [16], one can see that the image of 
\[ (\rho_{f,\ell}, \varepsilon_{\ell}) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^{\times} \]
is $\{ (g, \alpha) \in \text{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^{\times} \mid \det(g) = \alpha^{2k-1} \}$ if $\ell$ is not exceptional. Since $A_{\ell}$ contains an element with the form 
\[ \begin{pmatrix} \text{trace}_{f,\ell}(\sigma) & -1 \\ \text{det}_{f,\ell}(\sigma) & 0 \end{pmatrix}, \]
the map $(\text{trace}_{f,\ell}, \varepsilon_{\ell}) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^{\times}$ is surjective. Moreover by a ramification argument, one can see that the map 
\[ \prod_{\ell \in S} (\text{trace}_{f,\ell}, \varepsilon_{\ell}) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \prod_{\ell \in S} (\mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^{\times}) \]
is also surjective. Therefore we have the following result which is a generalization of a result of Serre [13, THÉORÈM 11] using Chebotarev density theorem.

Theorem 3.3. Assume that $f$ has no complex multiplication. Let $t$ be a positive integer and $\alpha$ a non-zero integer in $E$. Fix $\beta \in \mathcal{O}_E/\alpha \mathcal{O}_E$ and $r \in (\mathbb{Z}/t \mathbb{Z})^{\times}$. Suppose that $\alpha$ does not contain a prime divisor which divides an exceptional prime for $f$. Then the set of prime $p$ with the properties $a(p) \equiv \beta \mod \alpha$ and $p \equiv r \mod t$ has positive density.

4. Indivisibility of Fourier coefficients of modular forms of half-integral weight

In this section, we give a result on modulo $\ell$ indivisibility of Fourier coefficients of half-integral weight modular forms using a method of Kohnen-Ono [7] and James-Ono [5]. Our result is a refinement of a result of Bruinier [2] and Ono-Skinner [10].

To consider the indivisibility of Fourier coefficients of half-integral weight modular forms, we will use the following results.

Theorem 4.1 (Sturm [15]). Let 
\[ g(z) = \sum_{n=1}^{\infty} b(n)q^n \in M_k(N, \chi) \]
be a half-integral or integral weight modular form for which the coefficients \(b(m)\) are algebraic integers contained in a number field \(E\). Let \(v\) be a finite place of \(E\) and let
\[
\text{ord}_v(g) = \begin{cases} 
+\infty & \text{if } b(n) \equiv 0 \mod v \text{ for all } n, \\
\min\{ n \mid b(n) \neq 0 \mod v\} & \text{otherwise}
\end{cases}
\]
Moreover put
\[
\mu = \frac{k}{12} [\Gamma_0(1) : \Gamma_0(N)] = \frac{kN}{12} \prod_{p \mid N} p + 1 \frac{1}{p}.
\]
Assume that
\[
\text{ord}_v(g) > \mu,
\]
then \(\text{ord}_v(g) = +\infty\).

**Remark 4.2** (cf. [5] Proposition 5). In [15], Sturm proved this theorem for integral weight modular forms with trivial character, but the general case follows by taking an appropriate power of \(g\).

**Lemma 4.3** (Shimura, [14] Section 1). Suppose
\[
g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)
\]
is a half integral weight cusp form and \(p\) is a prime. We define \((U_p g)(z), (V_p g)(z)\) by
\[
(U_p g)(z) = \sum_{n=1}^{\infty} u_p(n)q^n = \sum_{n=1}^{\infty} b(pn)q^n,
\]
\[
(V_p g)(z) = \sum_{n=1}^{\infty} v_p(n)q^n = \sum_{n=1}^{\infty} b(n)q^{pn}.
\]
Then
\[
(U_p g)(z), (V_p g)(z) \in S_{k+1/2}\left(Np, \chi\left(\frac{4p}{.}\right)\right).
\]
Let
\[
f(z) = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N, \chi)
\]
be an integral weight modular form for which the coefficients \(a(m)\) are algebraic integers in \(E\). For a prime \(\lambda\) of \(E\) and positive integers \(r, t\) with \((r, t) = 1\), define \(T(r, t)\) and \(T(\lambda, r, t)\) by
\[
T(r, t) = \{p : \text{prime} \mid a(p) = 0, p \equiv r \mod t\}
\]
and
\[
T(\lambda, r, t) = \{p : \text{prime} \mid a(p) \equiv 0 \mod \lambda, p \equiv r \mod t\}.
\]
For a positive real number \(X\), we also denote \(T(r, t, X) = \{p \in T(r, t) \mid p \leq X\}\) and \(T(\lambda, r, t, X) = \{p \in T(\lambda, r, t) \mid p \leq X\}\).

For \(g = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi) \cap \mathcal{O}_{E,\chi}[[q]]\), denote \(s_\lambda(g) = \min\{\text{ord}_\lambda(b(n)) \mid n \in \mathbb{Z}_{>0}\}\). The following two lemmas give an estimate for indivisibility of Fourier coefficients of modular forms of half integral weight.
Lemma 4.4. Let \( \ell \) be a prime greater than 3. Let \( f(z) = \sum_{n=1}^{\infty} a(n)q^n \) be a normalized Hecke eigen newform of weight \( 2k \), level \( M \) with trivial character and let

\[
g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)
\]

be the eigenform given in Theorem 2.1. Assume that \( f \) has complex multiplication in the sense of Ribet [11] and \( \lambda \) be a prime in \( E \) above \( \ell \). If there exists an integer \( D' \) such that \( \delta(f)D' > 0 \), \( (D', N) = 1 \), \( \varepsilon = \left( \frac{D'}{\ell} \right) \neq 0 \) and \( \ord \chi_D(b(|D'|)) = s_\chi(g) \), then

\[
\# \left\{ D \in S(X) \left| \left( \frac{D}{\ell} \right) = \varepsilon, \ord \chi_D(b(D)) = s_\chi(g) \right. \right\} \gg f, \log X.
\]

Proof. By dividing \( g \) by \( \chi_\lambda \), we may assume \( s_\chi(g) = 0 \). If we put

\[
b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left( \frac{n}{\ell} \right) = \varepsilon, \\ 0 & \text{otherwise}, \end{cases}
\]

then

\[
g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')
\]

for a suitable character \( \chi' \). Since \( f \) has complex multiplication, so there exists a imaginary quadratic field \( K \) such that for every prime \( p \) satisfying \( p \equiv 3 \mod 4 \), \( (p, N) = 1 \) and \( \left( \frac{\Delta_K}{p} \right) = -1 \) we have \( a(p) = 0 \), where \( \Delta_K \) is the discriminant of \( K \). Therefore, for such \( p \), using the formulae for the action of Hecke operator \( T_p \), we find that

\[
b(p^2n) + \chi'(p)p^{k-1} \left( \frac{(-1)^{k_1}n}{p} \right) b(n) + \chi'(p^2)p^{2k-1}b(n/p^2) = 0.
\]

Hence if \( (r, t) = 1, 4 \mid t, r \equiv 3 \mod 4 \), then

\[
\#T(r, t, X) = \# \{ p \in T(r, t) \left| \right. p \leq X \} \gg f \frac{X}{\log X}
\]

and for any \( p \in T(r, t) \) we have

\[
b(p^2n) = -\chi'(p)p^{k-1} \left( \frac{(-1)^{k_1}n}{p} \right) b(n) - \chi'(p^2)p^{2k-1}b(n/p^2).
\]

Put \( \kappa = (k + \frac{1}{2}) \frac{\Gamma_0(11)\Gamma_0(N\ell^2)}{12} + 1 \). Now, we choose \( (r_0, t_0) \) satisfying the following properties:

1. \( N\ell^2 | t_0, (r_0, t_0) = 1, \chi'(r_0) = 1 \) and \( p \equiv 3 \mod 4 \).
2. If \( p \) is a prime with \( p \equiv r_0 \mod t_0 \), then \( \left( \frac{(-1)^{k_1}n}{p} \right) = -1 \) for any \( 1 \leq n \leq \kappa \) with \( (n, N\ell^2) = 1 \).
3. For each prime \( p \equiv r_0 \mod t_0 \) we have \( \left( \frac{\Delta_K}{p} \right) = -1 \).
4. Each prime \( p \equiv r_0 \mod t_0 \) satisfies \( \left| \chi'(p^2)p - \chi'(p) \left( \frac{(-1)^{k_1}|D'|}{p}\right) \right| = 1 \).
If $p \in T(r_0, t_0)$ is a sufficiently large prime, for all $1 \leq n \leq \kappa$

$$u_p(pm) = b_0(p^2n) = -\chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right)b_0(n) - p^{2k-1}\chi^2(p)b_0(n/p^2)$$

Since $b_0(n/p^2) = 0$, we have $u_p(pm) = \chi'(p)p^{k-1}b_0(n) = p^{k-1}b_0(p) = p^{k-1}v_p(pm)$. By the relation (4.1),

$$v_p(p^3|D'|) = b_0(p^2|D'|) = -\chi'(p)p^{k-1}\left(\frac{(-1)^k|D'|}{p}\right)b_0(|D'|)$$

and

$$u_p(p^3|D'|) = b_0(p^4|D'|) = -p^{2k-1}\chi'(p^2)b_0(|D'|).$$

Therefore by the assumption and the choice of $(r_0, t_0)$,

$$|u_p(p^3|D'|) - p^{k-1}v_p(p^3|D'|)|_{\lambda} = \left|\chi'(p^2)p^{2k-1} - \chi'(p)p^{2k-2}\left(\frac{(-1)^k|D'|}{p}\right)\right|_{\lambda} = 1.$$ 

Hence

$$\text{ord}_{\lambda}(U_pg_0 - p^{-k-1}V_pg_0) < +\infty.$$ 

By Theorem 4.1 and Lemma 4.3, there exists an integer $n_p$ such that

$$1 \leq n_p \leq \left(\frac{k + \frac{1}{2}}{2}\right)\frac{[\Gamma_0(1) : \Gamma_0(NI^2p)]}{12} = \kappa(p + 1), \ (n_p, p) = 1$$

and

$$b_0(n_pp_p) = u_p(n_p) \neq p^{-k-1}v_p(n_p) = 0 \mod \lambda.$$ 

Consequently, let $D_{sf}$ be the square-free part of $D = n_pp$, then

$$|b_0(D_{sf})|_{\lambda} = 1.$$ 

For convenience, let $p_i$ be the primes in $T(r_0, t_0)$ in increasing order, and let $D_i$ be the square-free part of $p_ip_i$. If $r < s < t$ and $D_r = D_s = D_t$, then $p_ip_sp_i|D_r$. However this can only occur for finitely many $r, s$ and $t$ since $|D_i| < \kappa p_i(p_i + 1)$. Therefore, the number of distinct $|D_i| < X$ is at least half the number of $p \in T(r_0, t_0)$ with $p \leq \sqrt{X/\kappa}$. Therefore the lemma follows from $\#T(r_0, t_0, X) \gg_{f, \lambda} X/\log X$. 

**Lemma 4.5.** Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigen newform of weight $2k$, level $M$ with trivial character. Denote $E = \mathbb{Q}(\{a(n)|n \geq 1\})$ and let

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1. We fix a prime number $\ell$ greater than 3 and let $\lambda$ be a prime in $E$ above $\ell$. Assume that $f$ does not have complex multiplication and the image of the Galois representation associated to $f$

$$\rho_{f, \ell} : \text{Gal}({\overline{\mathbb{Q}}}/\mathbb{Q}) \rightarrow \text{GL}_2(O_{E, \ell})$$

coincides with $A_\ell$. If there exists an integer $D'$ such that $\delta(f)D' > 0$, $(D', N) = 1$, $\varepsilon = \left(D'\atop \ell\right) \neq 0$ and $\text{ord}_{\lambda}(b(|D'|)) = s_\lambda(g)$, then

$$\# \left\{D \in S(X) \mid \left(D'\atop \ell\right) = \varepsilon, \text{ord}_{\lambda}(b(D)) = s_\lambda(g) \right\} \gg_{f, \lambda} \sqrt{X/\log X}.$$
Proof. First, we may assume \( \text{ord}_\lambda(g) = 0 \). If we put

\[
b_0(n) = \begin{cases} 
  b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left( \frac{n}{\ell} \right) = \varepsilon, \\
  0 & \text{otherwise,}
\end{cases}
\]

then

\[
g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')
\]

for a suitable character \( \chi' \). If \( a(p) \equiv 0 \mod \lambda \), by the formula for the action of Hecke operator \( T_p \) we find that

\[
b(p^2n) + \chi'(p)p^{k-1} \left( \frac{(-1)^{k_n}b}{p} \right) b(n) + \chi'^2(p)p^{2k-1}b(n/p^2) \equiv 0 \mod \lambda.
\]

By the assumption, \( \ell \) is not exceptional. Hence Theorem 3.3 implies

\[
\#T(\lambda, r, t, X) = \# \{ p \in T(\lambda, r, t) \mid p \leq X \} \gg_{f, \lambda} \frac{X}{\log X}
\]

and for each \( p \in T(\lambda, r, t) \)

\[
b(p^2n) \equiv -\chi'(p)p^{k-1} \left( \frac{(-1)^{k_n}b}{p} \right) b(n) - \chi'^2(p)p^{2k-1}b(n/p^2) \mod \lambda.
\]

(4.2)

Let \( \kappa \) be the number as in the proof of Lemma 4.4. Now, we choose \((r_0, t_0)\) satisfying the following properties:

1. \( N\ell^2|t_0, (r_0, t_0) = 1, \chi'(r_0) = 1 \).
2. If \( p \) is a prime with \( p \equiv r_0 \mod t_0 \), then \( \left( \frac{(-1)^{k_n}}{p} \right) = -1 \) for any \( 1 \leq n \leq \kappa \) with \( (n, N\ell^2) = 1 \).
3. For each prime \( p \equiv r_0 \mod t_0 \) we have \( \left( \frac{(-1)^{k|D'|}}{p} \right) = -1 \).
4. Each prime \( p \equiv r_0 \mod t_0 \) has the property that \( 1 + p \equiv 0 \mod \lambda \).

If \( p \in T(\lambda, r_0, t_0) \) is a sufficiently large prime, for all \( 1 \leq n \leq \kappa \) with \( (n, N\ell^2) = 1 \), one has

\[
u_p(pn) = b_0(p^2n) \equiv -p^{k-1} \left( \frac{(-1)^{k_n}}{p} \right) b_0(n) - p^{2k-1}b_0(n/p^2) = p^{k-1}b_0(n) = p^{k-1}v_p(pn) \mod \lambda.
\]

By the relation (4.2), we have

\[
u_p(p^3|D'|) = b_0(p^2|D'|) \equiv p^{k-1}b_0(|D'|) \mod \lambda,
\]

also

\[
u_p(p^3|D'|) = b_0(p^4|D'|) \equiv -p^{2k-1}b_0(|D'|) \mod \lambda.
\]

Therefore by assumption and the choice of \((r_0, t_0)\),

\[
p^{k-1}v_p(p^3|D'|) - u_p(p^3|D'|) \equiv p^{2k-2}(1 + p)b_0(|D'|) \not\equiv 0 \mod \lambda.
\]

Hence

\[
\text{ord}_\lambda(U_pg_0 - p^{k-1}V_pg_0) < +\infty.
\]

By Theorem 4.1 and Lemma 4.3, there exists an integer \( n_p \) such that

\[
1 \leq n_p \leq (k + 1/2)[\Gamma_0(1) : \Gamma_0(N\ell^2p)]/12 = \kappa(p + 1), \quad (n_p, p) = 1
\]
and
\[ b_0(n_p) = u_p(n_p) \neq p^{k-1} v_p(n_p) = 0 \mod \lambda. \]
In particular, let \( D_{sf} \) be the square-free part of \( D = n_p \), then
\[ |b_0(D_{sf})|_{\lambda} = 1. \]

Now the lemma follows from the same argument with the proof of the previous lemma using Theorem 3.3. \( \square \)

**Proof of Theorem 1.1.**

Now we give the proof of Theorem 1.1. Let
\[ g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{k+1/2}(N, \chi) \]
be the eigenform given in Theorem 2.1 for \( f \).

By replacing \( f \) by a suitable quadratic twist of \( f \) if necessary, we may assume that \( \varepsilon = \delta(f) \), where \( \varepsilon \) is the sign of the functional equation of \( L(f, s) \). By the result of Friedberg and Hoffstein [4], we can take an integer \( D' \) such that \( \varepsilon = 1 \), \( (D', 2N) = 1 \) and \( b(D') \neq 0 \). In particular, for almost all finite places \( \lambda \) of \( E \) we have
\[ |b(D')|_{\lambda} = 1. \]

Thus by Lemmas 4.4, 4.5, Theorem 2.1 and Theorem 3.3, for all but finitely many primes \( \lambda \) we have
\[ \# \{ D \in \mathcal{S}(X) \mid \delta(f) \cdot D > 0, (\ell, D) = 1 \text{ and } \left| \frac{L(f \otimes \chi_{D'}, k)}{\Omega_f} \right|_{\lambda} = 1 \} \gg_{f, \lambda} \frac{\sqrt{X}}{\log X}. \]

This completes the proof.

5. **Indivisibility for the non-central critical values**

In this section, we consider a special case for non-central values of \( L \)-functions for modular forms. We fix a prime \( \ell \) greater than 7 and let \( f = \sum_{n=1}^{\infty} a(n) q^n \) be a normalized Hecke eigenform of weight \( \ell + 1 \) for \( \text{SL}_2(\mathbb{Z}) \). Let \( \lambda \) be a prime in a number field \( E \). We assume that the integer ring of \( E \) contains all Fourier coefficients of \( f \) and choose a period \( \Omega_f \) as in Ash-Stevens [1, Theorem 4.5]. Then for any Dirichlet character \( \chi \), the quotient
\[ \tau(\chi^{-1}) \frac{L(f \otimes \chi, 1)}{(2\pi i)\Omega_f^+} \]
is an integer in \( E_\lambda(\chi) \) where \( \tau \) is the Gauss sum and \( \pm = \chi(-1) \).

**Theorem 5.1.** Let \( \lambda \) be a prime in \( E \) above \( \ell \). We assume the following conditions.

1. There exists a unique eigenform \( F \) of weight 2 for \( \Gamma_0(\ell) \) such that \( F \equiv f \mod \lambda \).
2. \( \ell \) is not exceptional.
3. There exists an square-free negative integer \( d_0 \) such that \( (d_0, 2\ell) = 1 \), \( \chi_{d_0}(\ell) = -\varepsilon(F) \), where \( \varepsilon(F) \) is the sign of functional equation of \( L(F, s) \) and
\[ \frac{L(f \otimes \chi_{d_0}, 1)\sqrt{d_0}}{(2\pi i)\Omega_f^+} \neq 0 \mod \lambda. \]

Then we have
\[ \# \{ D \in \mathcal{S}(X) \mid \frac{L(f \otimes \chi_{D'}, 1)\sqrt{D}}{(2\pi i)\Omega_f^+} \neq 0 \mod \lambda \} \gg_{f, \lambda} \frac{\sqrt{X}}{\log X}. \]
Theorem 5.1 implies the assumptions of Theorem 5.2. Since $f$ corresponds to some eigenform $q$ existence of a prime $D < \ell$ and $(\Omega_q)$ then there exists a complex number $\Omega_F^{\pm}$ such that for any Dirichlet character $\chi$ satisfying $(\text{cond } \chi, p) = 1$, we have
\[
\frac{\tau(\chi^{-1})L(f \otimes \chi, 1)}{(2\pi i)\Omega_F^{\pm}} \equiv \frac{\tau(\chi^{-1})L(F \otimes \chi, 1)}{(2\pi i)\Omega_F^{\pm}} \mod \lambda.
\]

Now we prove Theorem 5.1. By the Kohnen-Zagier formula [6], there exists an eigenform $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{3/2}(\Gamma_0(4\ell))$ such that for any negative square-free integer $D$ satisfying $\left(\frac{D}{\ell}\right) = -\varepsilon(F)$,
\[
|b(|D|)|^2 = 2 \cdot \frac{\sqrt{D}}{\pi} \cdot \langle g, g \rangle L(F \otimes \chi_D, 1),
\]
where $\langle \cdot, \cdot \rangle$ is the Petersson inner product. We can normalize $g$ by the relation $\langle f, g \rangle = \Omega_F^{\pm}$. Taking a linear combination of twists of $g$, one may assume $b(|D|) = 0$ if $\left(\frac{D}{\ell}\right) \neq -\varepsilon(F)$ and $D < \ell$. From the assumptions of the theorem, $\ell$ is not exceptional. This implies the existence of a prime $q$ satisfying $a(q) \neq q^{k-1} + 1 \mod \lambda$, therefore the assumptions of Theorem 5.1 implies the assumptions of Theorem 5.2. Since $\tau(\chi_D)^{-1} = \pm 1/\sqrt{D}$, one can see that
\[
L(f \otimes \chi, 1)\sqrt{D} \equiv L(F \otimes \chi, 1)\sqrt{D} \mod \lambda
\]
with a $\lambda$-adic unit $c$. By the assumption (3), we have
\[
\text{ord}_\lambda \left(\frac{L(f \otimes \chi_{d\ell}, 1)\sqrt{D}}{(2\pi i)\Omega_F^{\pm}}\right) = 0,
\]
therefore $\text{ord}_\lambda(b(d_0)) = \min\{\text{ord}_\lambda(b(n)) \mid n : \text{square-free, } \chi_{d_\ell}(\ell) = -\varepsilon(f)\}$. Hence Lemma 4.5 implies
\[
\#\left\{D \in S(X) \mid \chi_{d\ell}(\ell) = -\varepsilon(f), \text{ord}_\lambda(b(D)) = s\right\} \gg_f \lambda \frac{\sqrt{X}}{\log X},
\]
thus we have
\[
\#\left\{D \in S(X) \left| \frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_F^{\pm}} \neq 0 \mod \lambda \right.\right\} \gg_f \lambda \frac{\sqrt{X}}{\log X}.
\]
This completes the proof.

Remark 5.3. Lemma 4.5 states only for $g$ given in Theorem 2.1, but one can show the similar result for any eigenform $g \in S_{k+1/2}(N, \chi)$ if $k \geq 2$ ($S_{3/2}(N, \chi)$ if $k = 1$) corresponding to some eigenform $f \in S_{2k}(\Gamma_0(M))$ under the Shimura correspondence.
Example 5.4. Let

\[ f = \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\Gamma_0(1)) \]

and

\[ F = q \prod_{n=1}^{\infty} (1 - q^n)^2(1 - q^{11n})^2 \in S_2(\Gamma_0(11)). \]

Then it is well-known that \( f \equiv F \mod 11 \), \( \dim S_2(\Gamma_0(11)) = 1 \) and the mod 11 Galois representation associated to \( f \) is surjective. Moreover one can check that

\[ \frac{L(\Delta \otimes \chi_{-3}, 1)}{\Omega^+_{\Delta \otimes \chi_{-3}}} = 36741600 \not\equiv 0 \mod 11 \]

by using MAGMA. So the assumptions of Theorem 5.1 are satisfied for \( f = \Delta \). Hence we have

\[ \# \left\{ D \in S(X) \left| \frac{L(\Delta \otimes \chi_D, 1)\sqrt{D}}{(2\pi)^2\Omega^+_{\Delta}} \not\equiv 0 \mod 11 \right. \right\} \gg \frac{\sqrt{X}}{\log X}. \]

References


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