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Kyoto University
INDIVISIBILITY OF CENTRAL VALUES OF $L$-FUNCTIONS FOR MODULAR FORMS

MASATAKA CHIDA

Abstract. In this paper, we generalize works of Kohnen-Ono [7] and James-Ono [5] on indivisibility of (algebraic part of) central critical values of $L$-functions to higher weight modular forms.

1. Introduction

In this article, we show an indivisibility result on central critical values of $L$-functions associated to quadratic twists of modular forms using a method of Kohnen-Ono [7] and James-Ono [5].

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. For a fundamental discriminant $D$ with $(D, N) = 1$, we define the $D$-th quadratic twist of $f$ by

$$f \otimes \chi_D = \sum_{n=1}^{\infty} a(n)\chi_D(n)q^n,$$

where $\chi_D$ is the quadratic character corresponding to the quadratic extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Then $f \otimes \chi_D$ is a newform of weight $2k$ for $\Gamma_0(D^2N)$. Similarly, the $D$-th quadratic twist of the $L$-function $L(f, s)$ is given by

$$L(f \otimes \chi_D, s) = \sum_{n=1}^{\infty} \frac{a(n)\chi_D(n)}{n^s}.$$

Let $E$ be the number field generated by all Fourier coefficient of $f$ and $\mathbb{Q}$. Then it is known that there exists a period $\Omega_f \in \mathbb{C}^*$ satisfying that $\frac{L(f \otimes \chi_D, k)D_0^{-k+1/2}}{\Omega_f}$ are integers in $E$ for all fundamental discriminant $D$ with $\delta(f) \cdot D > 0$, where $\delta(f) \in \{\pm 1\}$ is the sign defined in Ono-Skinner [10, p. 655] and $D_0$ is given by

$$D_0 = \begin{cases} |D| & \text{if } D \text{ is odd,} \\ |D|/4 & \text{if } D \text{ is even.} \end{cases}$$

We fix such a period $\Omega_f$.

For convenience, we denote

$$S(X) = \{D \in \mathbb{Z} \mid |D| < X, D : \text{fundamental discriminant}\},$$

and if functions $f$, $g$ on $\mathbb{R}$ satisfy that there is a positive constant $c$ such that $f(X) \geq c \cdot g(X)$ for sufficiently large $X > 0$, then we write $f(X) \gg g(X)$. 

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Theorem 1.1. Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. Then, for all but finitely many primes $\lambda$ of $E$, we have

$$\# \left\{ D \in S(X) \mid \delta(f) \cdot D > 0, \lambda \nmid D \text{ and } \frac{L(f \otimes \chi_D, k)D_0^{k-\frac{1}{2}}}{\Omega_f} \neq 0 \mod \lambda \right\} \gg f, \lambda \frac{\sqrt{X}}{\log X}.$$  

This result is a refinement of results of Bruinier [2] and Ono-Skinner [10]. The proof is based on a generalization of a method of Kohnen-Ono [7] and James-Ono [5]. In the above theorem, we do not assume that the Fourier coefficients of $f$ belong to $\mathbb{Z}$, therefore it does not hold the surjectivity of the residual Galois representation associated to $f$ for almost all places in general. This makes some technical difficulty on the proof. To solve this problem, we may use a result of Ribet [12] on the image of Galois representations associated to modular forms. This is an ingredient in our proof. In the last section, we also consider an indivisibility result on non-central critical values of $L$-functions for higher weight modular forms using congruences of modular form with different weights.

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2. Modular forms of half-integral weight

We denote the space of modular forms of weight $k + 1/2$, level $N$ with character $\chi$ by $M_{k+1/2}(N, \chi)$, and the space of cusp forms of weight $k + 1/2$, level $N$ with character $\chi$ by $S_{k+1/2}(N, \chi)$. Then $M_{k+1/2}(N, \chi)$ and $S_{k+1/2}(N, \chi)$ are complex vector spaces.

For a modular form of half-integral weight $g(z) = \sum_{n=0}^{\infty} b(n)q^n \in M_{k+1/2}(N, \chi)$, we define the action of Hecke operator $T_{p^2}$ by

$$T_{p^2}(g)(z) = \sum_{n=0}^{\infty} b'(n)q^n,$$

where $b'(n)$ are given by

$$b'(n) = b(p^2n) + \chi(p)\left(\frac{-1}{p}\right)^k \left(\frac{n}{p}\right) p^{k-1}b(n) + \chi(p^2)p^{2k-1}b(n/p^2)$$

and $b(n/p^2)$ are zero if $p^2 \nmid n$.

Now we give a short review of the theory of the Shimura correspondence. Let $N$ be a positive integer which is divisible by four and $\chi$ a Dirichlet character mod $N$. Then we define a vector space $S^0_{3/2}(N, \chi)$ to be the subspace of $S_{3/2}(N, \chi)$ generated by

$$\left\{ f(z) = \sum_{n=1}^{\infty} \psi(n)nq^{tm^2} \mid N = 4 \text{ cond}(\psi)^2t|N, \chi = \psi\chi_{-t} \text{ and } \psi(-1) = -1 \right\}$$

and denote the orthogonal complement by $S'_{3/2}(N, \chi)$. Then we assume

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$
if \( k \geq 2 \), and
\[
g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{3/2}(N, \chi)
\]
if \( k = 1 \). Let \( t \) be a square-free positive integer. Define a number \( A_t(n) \) to be
\[
\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = \left( \sum_{n=1}^{\infty} \frac{\chi(n)(-1)^k \left( \frac{n}{t} \right)}{n^{s-k+1}} \right) \left( \sum_{n=1}^{\infty} \frac{b(n^2)}{n^s} \right).
\]
Then Shimura [14] proved that there is a positive integer \( M \) such that \( \text{SH}_t(g(z)) = f_t(z) = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}(M, \chi^2) \). (In fact, one can prove that \( M = N/2 \)). Furthermore for any \( t, t' \), the difference between \( \text{SH}_t(g) \) and \( \text{SH}_{t'} \) is only constant multiple, so essentially this correspondence is independent of choice of \( t \). This correspondence between modular forms is called the Shimura correspondence. Moreover if \( g \) is an eigenform for all Hecke operators \( T_p \) with \( (p, 2N) = 1 \), then the image of \( g \) under the Shimura correspondence is also an eigenform for all Hecke operators \( T_p \) with \( (p, 2N) = 1 \) and the Hecke eigenvalue of \( T_p g \) coincides with the Hecke eigenvalue for \( T_p \) for \( \text{SH}_t(g) \).

We recall the following result which is a useful version of Waldspurger’s formula ([17, Théorème 1]) by Ono-Skinner. This formula gives a relation between the Fourier coefficients of modular forms of half-integral weight and the central values of twisted \( L \)-functions for modular forms.

**Theorem 2.1** (Ono-Skinner [9], (2a),(2b)). Let \( f(z) = \sum_{n=1}^{\infty} a(n)q^n \) be a normalized newform of weight \( 2k \), level \( M \) with trivial character. Then there is \( \delta(f) \in \{ \pm 1 \} \), a positive integer \( N \) with \( 4M \mid N \), a Dirichlet character \( \chi \) modulo \( N \), a period \( \Omega_f \in \mathbb{C}^* \) and a non-zero eigenform
\[
g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)
\]
with the property that \( g(z) \) maps to a twist of \( f \) under the Shimura correspondence and for all fundamental discriminant \( D \) with \( \delta(f)D > 0 \) we have
\[
b(D_0)^2 = \begin{cases} 
\alpha_D \frac{L(f \otimes \chi_D, k)D_0^{k-1/2}}{\Omega_f} & \text{if } (D, N) = 1, \\
0 & \text{otherwise},
\end{cases}
\]
where \( \alpha_D \) and \( b(n) \) are algebraic integers in some finite extension of \( \mathbb{Q} \). Moreover, there exists a finite set of primes \( S \) such that if \( D \) is a fundamental discriminant for which
\begin{enumerate}
\item \( \delta(f)D > 0 \),
\item \( (D, N) = 1 \),
\end{enumerate}
then we have \( |L(f \otimes \chi_D, k)D_0^{k-1/2}/\Omega_f|_\lambda = |b(D_0)^2|_\lambda \) for \( \lambda \notin S \).

**3. Some properties of Fourier coefficients of modular forms and Galois representations**

In this section we generalize some results of Serre [13] and Swinnerton-Dyer [16] using a result of Ribet [12]. These results should be well-known for specialists. However we give a short review for them, since it does not seem to be available in the literature. Let \( f = \sum_{n=1}^{\infty} a(n)q^n \) be a normalized newform of weight \( 2k \) for \( \Gamma_0(N) \) with trivial character. Let \( E \) be the subfield of \( \mathbb{C} \) generated by the Fourier coefficients \( a(n) \) of \( f \). Then \( E \) is a finite extension of \( \mathbb{Q} \). Let \( \mathcal{O}_E \) be the ring of integers of \( E \). For each prime \( \ell \), we let \( \mathcal{O}_{E, \ell} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \) and \( E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \).
Theorem 3.1 (Deligne [3]). For each prime $\ell$, there exists a continuous representation 

$$\rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O}_{E,\ell}) \subset \text{GL}_2(E_{\ell})$$

unramified at all primes $p \nmid N\ell$ such that $

\text{trace}_{f,\ell}(\text{Frob}_p) = a(p) \quad \text{and} \quad \det_{f,\ell}(\text{Frob}_p) = p^{2k-1}$

for all primes $p \nmid N\ell$, where $\text{Frob}_p$ is the arithmetic Frobenius at $p$.

For each prime $\ell$, denote 

$$A_{\ell} = \left\{ g \in \text{GL}_2(\mathcal{O}_{E,\ell}) \mid \det(g) \in \mathbb{Z}_\ell^{\times(2k-1)} \right\},$$

where $\mathbb{Z}_\ell^{\times(2k-1)}$ is the group of $(2k-1)$-th powers of elements in $\mathbb{Z}_\ell^{\times}$. Replacing $\rho_{f,\ell}$ by an isomorphic representation, we may assume that for almost all $\rho_{f,\ell}$ sends $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $A_{\ell}$.

Then Ribet proved the following theorem.

Theorem 3.2 (Ribet [12]). Assume that $f$ has no complex multiplication. Then for almost all $\ell$, we have 

$$\rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to A_{\ell}.$$ 

We call the set of primes $\ell$ with the property $\rho_{f,\ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = A_{\ell}$ by the exceptional primes for $f$. Let $S$ be the set of exceptional places for $f$. Let $\varepsilon_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^{\times}$ be the $\ell$-adic cyclotomic character. Then by a similar argument with Swinnerton-Dyer [16], one can see that the image of 

$$(\rho_{f,\ell}, \varepsilon_\ell) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^{\times}$$

is $$\left\{ (g, \alpha) \in \text{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^{\times} \mid \det(g) = \alpha^{2k-1} \right\}$$

if $\ell$ is not exceptional. Since $A_{\ell}$ contains an element with the form 

$$\left( \begin{array}{cc} \text{trace}_{f,\ell}(\sigma) & -1 \\ \det_{f,\ell}(\sigma) & 0 \end{array} \right),$$

the map $(\text{trace}_{f,\ell}, \varepsilon_\ell) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^{\times}$ is surjective. Moreover by a ramification argument, one can see that the map 

$$\prod_{\ell \in S}(\text{trace}_{f,\ell}, \varepsilon_\ell) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \prod_{\ell \in S}(\mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^{\times})$$

is also surjective. Therefore we have the following result which is a generalization of a result of Serre [13, THÉORÊM 11] using Chebotarev density theorem.

Theorem 3.3. Assume that $f$ has no complex multiplication. Let $t$ be a positive integer and $\alpha$ a non-zero integer in $E$. Fix $\beta \in \mathcal{O}_E/\alpha \mathcal{O}_E$ and $r \in (\mathbb{Z}/t\mathbb{Z})^{\times}$. Suppose that $\alpha$ does not contain a prime divisor which divides an exceptional prime for $f$. Then the set of prime $p$ with the properties $a(p) \equiv \beta \mod \alpha$ and $p \equiv r \mod t$ has positive density.

4. INDIVISIBILITY OF FOURIER COEFFICIENTS OF MODULAR FORMS OF HALF-INTEGRAL WEIGHT

In this section, we give a result on modulo $\ell$ indivisibility of Fourier coefficients of half-integral weight modular forms using a method of Kohnen-Ono [7] and James-Ono [5]. Our result is a refinement of a result of Bruinier [2] and Ono-Skinner [10].

To consider the indivisibility of Fourier coefficients of half-integral weight modular forms, we will use the following results.

Theorem 4.1 (Sturm [15]). Let 

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in M_k(N, \chi)$$
be a half-integral or integral weight modular form for which the coefficients $b(m)$ are algebraic integers contained in a number field $E$. Let $v$ be a finite place of $E$ and let

$$\text{ord}_v(g) = \begin{cases} +\infty & \text{if } b(n) \equiv 0 \mod v \text{ for all } n, \\ \min \{ n \mid b(n) \neq 0 \mod v \} & \text{otherwise} \end{cases}$$

Moreover put

$$\mu = \frac{k}{12}[\Gamma_0(1) : \Gamma_0(N)] = \frac{kN}{12} \prod_{p \mid N} p + 1 \frac{1}{p}.$$

Assume that

$$\text{ord}_v(g) > \mu,$$

then $\text{ord}_v(g) = +\infty$.

**Remark 4.2** (cf. [5] Proposition 5). In [15], Sturm proved this theorem for integral weight modular forms with trivial character, but the general case follows by taking an appropriate power of $g$.

**Lemma 4.3** (Shimura, [14] Section 1). Suppose

$$g(z) = \sum_{n=1}^{\infty} b(n) q^n \in \mathcal{S}_{k+1/2}(N, \chi)$$

is a half integral weight cusp form and $p$ is a prime. We define $(U_p g)(z), (V_p g)(z)$ by

$$(U_p g)(z) = \sum_{n=1}^{\infty} u_p(n) q^n = \sum_{n=1}^{\infty} b(pn) q^n,$$

$$(V_p g)(z) = \sum_{n=1}^{\infty} v_p(n) q^n = \sum_{n=1}^{\infty} b(n) q^{pn}.$$

Then

$$(U_p g)(z), (V_p g)(z) \in \mathcal{S}_{k+1/2} \left( Np, \chi \left( \frac{4p}{.} \right) \right).$$

Let

$$f(z) = \sum_{n=1}^{\infty} a(n) q^n \in \mathcal{M}_k(N, \chi)$$

be an integral weight modular form for which the coefficients $a(m)$ are algebraic integers in $E$. For a prime $\lambda$ of $E$ and positive integers $r, t$ with $(r, t) = 1$, define $T(r, t)$ and $T(\lambda, r, t)$ by

$$T(r, t) = \{ p : \text{prime} \mid a(p) = 0, p \equiv r \mod t \}$$

and

$$T(\lambda, r, t) = \{ p : \text{prime} \mid a(p) \equiv 0 \mod \lambda, p \equiv r \mod t \}.$$

For a positive real number $X$, we also denote $T(r, t, X) = \{ p \in T(r, t) \mid p \leq X \}$ and $T(\lambda, r, t, X) = \{ p \in T(\lambda, r, t) \mid p \leq X \}$.

For $g = \sum_{n=1}^{\infty} b(n) q^n \in \mathcal{S}_{k+1/2}(N, \chi) \cap \mathcal{O}_{E, \chi}[q]$, denote $s_\lambda(g) = \min \{ \text{ord}_\lambda(b(n)) \mid n \in \mathbb{Z}_{>0} \}$. The following two lemmas give an estimate for indivisibility of Fourier coefficients of modular forms of half integral weight.
Lemma 4.4. Let ℓ be a prime greater than 3. Let \( f(z) = \sum_{n=1}^{\infty} a(n)q^n \) be a normalized Hecke eigen newform of weight 2k, level \( M \) with trivial character and let

\[
g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)
\]

be the eigenform given in Theorem 2.1. Assume that \( f \) has complex multiplication in the sense of Ribet [11] and \( \lambda \) be a prime in \( E \) above \( \ell \). If there exists an integer \( D' \) such that \( \delta(f)D' > 0 \), \( (D', N) = 1 \), \( \varepsilon = \left( \frac{D'}{\ell} \right) \neq 0 \) and \( \text{ord}_\lambda(b(|D'|)) = s_\lambda(g) \), then

\[
\# \left\{ D \in S(X) \mid \left( \frac{D}{\ell} \right) = \varepsilon, \text{ord}_\lambda(b(D)) = s_\lambda(g) \right\} \gg f, \frac{X}{\log X}.
\]

Proof. By dividing \( g \) by \( \lambda^{s_\lambda(g)} \), we may assume \( s_\lambda(g) = 0 \). If we put

\[
b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left( \frac{n}{\ell} \right) = \varepsilon, \\ 0 & \text{otherwise}, \end{cases}
\]

then

\[
g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')
\]

for a suitable character \( \chi' \). Since \( f \) has complex multiplication, so there exists an imaginary quadratic field \( K \) such that for every prime \( p \) satisfying \( p \equiv 3 \) mod 4, \( (p, N) = 1 \) and \( \left( \frac{\Delta_K}{p} \right) = -1 \) we have \( a(p) = 0 \), where \( \Delta_K \) is the discriminant of \( K \). Therefore, for such \( p \), using the formulae for the action of Hecke operator \( T_q \), we find that

\[
b(p^2n) + \chi'(p)p^{k-1} \left( \frac{(-1)^kn}{p} \right) b(n) + \chi'(p^2)p^{2k-1}b(n/p^2) = 0.
\]

Hence if \( (r, t) = 1, 4 \mid t, r \equiv 3 \mod 4 \), then

\[
\#T(r, t, X) = \#\{ p \in T(r, t) \mid p \leq X \} \gg f, \frac{X}{\log X}
\]

and for any \( p \in T(r, t) \) we have

\[
b(p^2n) = -\chi'(p)p^{k-1} \left( \frac{(-1)^kn}{p} \right) b(n) - \chi'^2(p)p^{2k-1}b(n/p^2).
\]

Put \( \kappa = (k + \frac{1}{2}) \frac{\Gamma(1)\Gamma(N/2)}{12} + 1 \). Now, we choose \( (r_0, t_0) \) satisfying the following properties:

1. \( N\ell^2|t_0, (r_0, t_0) = 1, \chi'(r_0) = 1 \text{ and } p \equiv 3 \mod 4 \).
2. If \( p \) is a prime with \( p \equiv r_0 \mod t_0 \), then \( \left( \frac{(-1)^kn}{p} \right) = -1 \) for any \( 1 \leq n \leq \kappa \) with \( (n, N\ell^2) = 1 \).
3. For each prime \( p \equiv r_0 \mod t_0 \) we have \( \left( \frac{\Delta_K}{p} \right) = -1 \).
4. Each prime \( p \equiv r_0 \mod t_0 \) satisfies \( \left| \chi'(p^2)p - \chi'(p) \left( \frac{(-1)^k|D'|}{p} \right) \right| = 1 \).
If \( p \in T(r_0, t_0) \) is a sufficiently large prime, for all \( 1 \leq n \leq \kappa \)
\[
u_p(p^2n) = b_0(p^2n) = -\chi'(p)p^{k-1} \left( \frac{(-1)^k n}{p} \right) b_0(n) - p^{2k-1} \chi^2(p)b_0(n/p^2)
\]
Since \( b_0(n/p^2) = 0 \), we have \( u_p(pm) = \chi'(p)p^{k-1}b_0(n) = p^{k-1}b_0(p) = p^{k-1}v_p(pm) \). By the relation (4.1),
\[
u_p(p^2|D'|) = b_0(p^2|D'|) = -\chi'(p)p^{k-1} \left( \frac{(-1)^k |D'|}{p} \right) b_0(|D'|),
\]
and
\[
u_p(p^3|D'|) = b_0(p^3|D'|) = -p^{2k-1} \chi'(p^2)b_0(|D'|).
\]
Therefore by the assumption and the choice of \((r_0, t_0)\),
\[
\left| u_p(p^3|D'|) - p^{k-1}v_p(p^3|D'|) \right| \lambda = \left| (\chi'(p^2)p^{2k-1} - \chi'(p)p^{2k-2} \left( \frac{(-1)^k |D'|}{p} \right))b_0(|D'|) \right| = 1.
\]
Hence
\[
\text{ord}_\lambda(U_pg_0 - p^{k-1}V_pg_0) < +\infty.
\]
By Theorem 4.1 and Lemma 4.3, there exists an integer \( n_p \) such that
\[
1 \leq n_p \leq \left( k + \frac{1}{2} \right) \frac{[\Gamma_0(1) : \Gamma_0(N\ell^2p)]}{12} = \kappa(p+1), (n_p, p) = 1
\]
and
\[
b_0(n_pp) = u_p(n_p) \neq p^{k-1}v_p(n_p) = 0 \mod \lambda.
\]
Consequently, let \( D_{sf} \) be the square-free part of \( D = n_pp \), then
\[
\left| b_0(D_{sf}) \right| \lambda = 1.
\]
For convenience, let \( p_i \) be the primes in \( T(r_0, t_0) \) in increasing order, and let \( D_i \) be the square-free part of \( p_in_{p_i} \). If \( r < s < t \) and \( D_r = D_s = D_t \), then \( p_{i}p_{j}p_{k}|D_r \). However this can only occur for finitely many \( r, s \) and \( t \) since \( |D_i| < \kappa p_i(p_i + 1) \). Therefore, the number of distinct \( |D_i| < X \) is at least half the number of \( p \in T(r_0, t_0) \) with \( p \leq \sqrt{X/\kappa} \). Therefore the lemma follows from \( \#T(r_0, t_0, X) \gg f,\lambda X/\log X \).  
\[\square\]

**Lemma 4.5.** Let \( f(z) = \sum_{n=1}^{\infty} a(n)q^n \) be a normalized Hecke eigen newform of weight \( 2k \), level \( M \) with trivial character. Denote \( E = \mathbb{Q} \{ \{a(n)|n \geq 1\} \} \) and let
\[
g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)
\]
be the eigenform given in Theorem 2.1. We fix a prime number \( \ell \) greater than 3 and let \( \lambda \) be a prime in \( E \) above \( \ell \). Assume that \( f \) does not have complex multiplication and the image of the Galois representation associated to \( f \)
\[
\rho_{f,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O}_{E,\ell})
\]
coincides with \( A_{\ell} \). If there exists an integer \( D' \) such that \( \delta(f)D' > 0 \), \( (D', N) = 1 \), \( \varepsilon = \left( \frac{D'}{\ell} \right) \neq 0 \) and \( \text{ord}_\lambda(b(|D'|)) = s_\lambda(g) \), then
\[
\# \left\{ D \in S(X) \left| \left( \frac{D}{\ell} \right) = \varepsilon, \text{ord}_\lambda(b(D)) = s_\lambda(g) \right. \right\} \gg f,\lambda \frac{\sqrt{X}}{\log X}.
\]
Proof. First, we may assume $\text{ord}_\lambda(g) = 0$. If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise}, \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character $\chi'$. If $a(p) \equiv 0 \mod p$, by the formula for the action of Hecke operator $T_{p^2}$ we find that

$$b(p^2n) + \chi'(p)b(p^{k-1}) \left(\frac{(-1)^{k_n}}{p}\right) b(n) + \chi'^2(p)p^{2k-1}b(n/p^2) \equiv 0 \mod \lambda.$$ 

By the assumption, $\ell$ is not exceptional. Hence Theorem 3.3 implies

$$\#T(\lambda, r, t, X) = \# \{ p \in T(\lambda, r, t) \mid p \leq X \} \gg \frac{X}{\log X}$$

and for each $p \in T(\lambda, r, t)$

$$(4.2) \quad b(p^2n) \equiv -\chi'(p)p^{k-1} \left(\frac{(-1)^{k_n}}{p}\right) b(n) - \chi'^2(p)p^{2k-1}b(n/p^2) \mod \lambda.$$ 

Let $\kappa$ be the number as in the proof of Lemma 4.4. Now, we choose $(r_0, t_0)$ satisfying the following properties:

1. $N\ell^2|r_0$, $(r_0, t_0) = 1$, $\chi'(r_0) = 1$.
2. If $p$ is a prime with $p \equiv r_0 \mod t_0$, then $\left(\frac{(-1)^{k_n}}{p}\right) = -1$ for any $1 \leq n \leq \kappa$ with $(n, N\ell^2) = 1$.
3. For each prime $p \equiv r_0 \mod t_0$ we have $\left(\frac{(-1)^{k_n}}{p}\right) = -1$.
4. Each prime $p \equiv r_0 \mod t_0$ has the property that $1 + p \not\equiv 0 \mod \lambda$.

If $p \in T(\lambda, r_0, t_0)$ is a sufficiently large prime, for all $1 \leq n \leq \kappa$ with $(n, N\ell^2) = 1$, one has

$$u_p(pn) = b_0(p^2n) \equiv -p^{k-1} \left(\frac{(-1)^{k_n}}{p}\right) b_0(n) - p^{2k-1}b_0(n/p^2) = p^{k-1}b_0(n) = p^{k-1}v_p(pn) \mod \lambda.$$ 

By the relation (4.2), we have

$$v_p(p^3|D'|) = b_0(p^2|D'|) \equiv p^{k-1}b_0(|D'|) \mod \lambda,$$

also

$$u_p(p^3|D'|) = b_0(p^4|D'|) \equiv -p^{2k-1}b_0(|D'|) \mod \lambda.$$ 

Therefore by assumption and the choice of $(r_0, t_0)$,

$$p^{k-1}v_p(p^3|D'|) - u_p(p^3|D'|) \equiv p^{2k-2}(1 + p)b_0(|D'|) \neq 0 \mod \lambda.$$ 

Hence

$$\text{ord}_\lambda(U_{p^2}g_0 - p^{k-1}V_{p^2}g_0) < +\infty.$$ 

By Theorem 4.1 and Lemma 4.3, there exists a integer $n_p$ such that

$$1 \leq n_p \leq (k + 1/2)[\Gamma_0(1) : \Gamma_0(N\ell^2p)]/12 = \kappa(p + 1), (n_p, p) = 1$$
and 

\[ b_0(n_p) = u_p(n_p) \neq p^{k-1}v_p(n_p) = 0 \mod \lambda. \]

In particular, let \( D_{sq} \) be the square-free part of \( D = n_p \), then

\[ |b_0(D_{sq})|_\lambda = 1. \]

Now the lemma follows from the same argument with the proof of the previous lemma using Theorem 3.3. □

**Proof of Theorem 1.1.**

Now we give the proof of Theorem 1.1. Let

\[ g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi) \]

be the eigenform given in Theorem 2.1 for \( f \).

By replacing \( f \) by a suitable quadratic twist of \( f \) if necessary, we may assume that \( \epsilon = \delta(f) \), where \( \epsilon \) is the sign of the functional equation of \( L(f, s) \). By the result of Friedberg and Hoffstein [4], we can take an integer \( D' \) such that \( (f', 2N) = 1 \) and \( b(D') \neq 0 \). In particular, for almost all finite places \( \lambda \) of \( E \) we have

\[ |b(D')|_\lambda = 1. \]

Thus by Lemmas 4.4, 4.5, Theorem 2.1 and Theorem 3.3, for all but finitely many primes \( \lambda \) we have

\[
\# \left\{ D \in S(X) \mid \delta(f) \cdot D > 0, (\ell, D) = 1 \text{ and } \left| \frac{L(f \otimes \chi, 1)D^{k-1/2}}{\Omega_f} \right|_\lambda = 1 \right\} \gg f, \sqrt{X} \log X.
\]

This completes the proof.

5. **Indivisibility for the non-central critical values**

In this section, we consider a special case for non-central values of \( L \)-functions for modular forms. We fix a prime \( \ell \) greater than 7 and let \( f = \sum_{n=1}^{\infty} a(n)q^n \) be a normalized Hecke eigenform of weight \( \ell + 1 \) for \( SL_2(\mathbb{Z}) \). Let \( \lambda \) be a prime in a number field \( E \). We assume that the integer ring of \( E \) contains all Fourier coefficients of \( f \) and choose a period \( \Omega_f \) as in Ash-Stevens [1, Theorem 4.5]. Then for any Dirichlet character \( \chi \), the quotient

\[
\tau(\chi^{-1}) \frac{L(f \otimes \chi, 1)}{(2\pi i)^{\frac{1}{2}}\Omega_f^{\pm}}
\]

is an integer in \( E_\lambda(\chi) \) where \( \tau \) is the Gauss sum and \( \pm = \chi(-1) \).

**Theorem 5.1.** Let \( \lambda \) be a prime in \( E \) above \( \ell \). We assume the following conditions.

1. There exists a unique eigenform \( F \) of weight 2 for \( \Gamma_0(\ell) \) such that
   \[ F \equiv f \mod \lambda. \]

2. \( \ell \) is not exceptional.

3. There exists an square-free negative integer \( d_0 \) such that \( (d_0, 2\ell) = 1 \), \( \chi_{d_0}(\ell) = -\epsilon(F) \), where \( \epsilon(F) \) is the sign of functional equation of \( L(F, s) \) and
   \[
   \frac{L(f \otimes \chi_{d_0}, 1)\sqrt{d_0}}{(2\pi i)^{1/2}\Omega_f^{\pm}} \neq 0 \mod \lambda.
   \]

Then we have

\[
\# \left\{ D \in S(X) \mid \frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)^{1/2}\Omega_f^{\pm}} \neq 0 \mod \lambda \right\} \gg f, \sqrt{X} \log X.
\]
Theorem 5.1 implies the assumptions of Theorem 5.2. Since $f$ corresponding to some eigenform existence of a prime $q$ $D < q$ and $(2)$ $\exists$ such that for any negative square-free integer $D$ satisfying $\sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(1))$ an eigenform satisfying the assumptions of Theorem 5.1. We fix a prime $\lambda$ above $\ell$ in a number field $E$ which contains all Fourier coefficients of $f$. Assume that

1. There exists a prime $q$ satisfying $a(q) \neq q^{k-1} + 1 \mod \lambda$.
2. There exists an unique eigenform $F \in S_2(\Gamma_1(\ell))$ such that $f \equiv F \mod \lambda$.

Then there exists a complex number $\Omega_F^\pm$ such that for any Dirichlet character $\chi$ satisfying $(\cond \chi, p) = 1$, we have

$$\frac{\tau(\chi^{-1})L(f \otimes \chi, 1)}{(2\pi i)\Omega_F^\pm} \equiv \frac{\tau(\chi^{-1})L(F \otimes \chi, 1)}{(2\pi i)\Omega_F^\pm} \mod \lambda.$$ 

Now we prove Theorem 5.1. By the Kohnen-Zagier formula [6], there exists an eigenform $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{3/2}(\Gamma_0(4\ell))$ such that for any negative square-free integer $D$ satisfying $(D \ell) = -\varepsilon(F)$,

$$|b(|D|)|^2 = 2 \cdot \sqrt{D} \cdot \frac{(g, g)}{(F, F)} L(F \otimes \chi_D, 1),$$

where $(\cdot, \cdot)$ is the Petersson inner product. We can normalize $g$ by the relation $(F, F) = \Omega_F^\pm$. Taking a linear combination of twists of $g$, one may assume $b(|D|) = 0$ if $(D \ell) \neq -\varepsilon(F)$ and $D < 0$. From the assumptions of the theorem, $\ell$ is not exceptional. This implies the existence of a prime $q$ satisfying $a(q) \neq q^{k-1} + 1 \mod \lambda$, therefore the assumptions of Theorem 5.1 implies the assumptions of Theorem 5.2. Since $\tau(\chi_D)^{-1} = \pm 1/\sqrt{D}$, one can see that

$$\frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_F^\pm} \equiv \frac{L(F \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_F^\pm} = |b(|D|)|^2 \cdot c \mod \lambda$$

with a $\lambda$-adic unit $c$. By the assumption (3), we have

$$\ord_\lambda \left( \frac{L(f \otimes \chi_{d_0}, 1)\sqrt{D_0}}{(2\pi i)\Omega_F^\pm} \right) = 0,$$

therefore $\ord_\lambda(b(d_0)) = \min\{\ord_\lambda(b(n)) \mid n \text{ square-free, } \chi_{d_0}(\ell) = -\varepsilon(f)\}$. Hence Lemma 4.5 implies

$$\# \left\{ D \in S(X) \mid \chi_D(\ell) = -\varepsilon(f), \ord_\lambda(b(D)) = s \right\} \gg f \lambda \frac{\sqrt{X}}{\log X},$$

thus we have

$$\# \left\{ D \in S(X) \mid \frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_F^\pm} \neq 0 \mod \lambda \right\} \gg f \lambda \frac{\sqrt{X}}{\log X}.$$ 

This completes the proof.

Remark 5.3. Lemma 4.5 states only for $g$ given in Theorem 2.1, but one can show the similar result for any eigenform $g \in S_{k+1/2}(N, \chi)$ if $k \geq 2$ ($S_{k/2}(N, \chi)$ if $k = 1$) corresponding to some eigenform $f \in S_{2k}(\Gamma_0(M))$ under the Shimura correspondence.
Example 5.4. Let

$$f = \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\Gamma_0(1))$$

and

$$F = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 \in S_2(\Gamma_0(11)).$$

Then it is well-known that $f \equiv F \mod 11$, $\dim S_2(\Gamma_0(11)) = 1$ and the mod 11 Galois representation associated to $f$ is surjective. Moreover one can check that

$$\frac{L(\Delta \otimes \chi_3, 1)}{\Omega^+_{\Delta \otimes \chi_3}} = 36741600 \not\equiv 0 \mod 11$$

by using MAGMA. So the assumptions of Theorem 5.1 are satisfied for $f = \Delta$. Hence we have

$$\# \left\{ D \in S(X) \left| \frac{L(\Delta \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)^6 \Omega^+_{\Delta}} \not\equiv 0 \mod 11 \right. \right\} \gg \frac{\sqrt{X}}{\log X}.$$

References


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