INDIVISIBILITY OF CENTRAL VALUES OF L-FUNCTIONS FOR MODULAR FORMS

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Abstract. In this paper, we generalize works of Kohnen-Ono [7] and James-Ono [5] on indivisibility of (algebraic part of) central critical values of L-functions to higher weight modular forms.

1. Introduction

In this article, we show an indivisibility result on central critical values of L-functions associated to quadratic twists of modular forms using a method of Kohnen-Ono [7] and James-Ono [5].

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. For a fundamental discriminant D with (D, N)=1, we define the D-th quadratic twist of f by

$$f \otimes \chi_D = \sum_{n=1}^{\infty} a(n)\chi_D(n)q^n,$$

where χ_D is the quadratic character corresponding to the quadratic extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Then $f \otimes \chi_D$ is a newform of weight $2k$ for $\Gamma_0(D^2N)$. Similarly, the D-th quadratic twist of the L-function $L(f, s)$ is given by

$$L(f \otimes \chi_D, s) = \sum_{n=1}^{\infty} \frac{a(n)\chi_D(n)}{n^s}.$$

Let E be the number field generated by all Fourier coefficient of f and $\mathbb{Q}$. Then it is known that there exists a period $\Omega_f \in \mathbb{C}^{\times}$ satisfying that $\frac{L(f\otimes\chi_D,k){D_0}^{k-1/2}}{\Omega_f}$ are integers in E for all fundamental discriminant D with $\delta(f) \cdot D > 0$, where $\delta(f) \in \{\pm 1\}$ is the sign defined in Ono-Skinner [10, p. 655] and D_0 is given by

$$D_0 = \begin{cases} |D| & \text{if } D \text{ is odd,} \\ |D|/4 & \text{if } D \text{ is even.} \end{cases}$$

We fix such a period Ω_f.

For convenience, we denote

$$S(X) = \{D \in \mathbb{Z} \,\big|\, |D| < X, D : \text{fundamental discriminant}\},$$

and if functions f, g on $\mathbb{R}$ satisfy that there is a positive constant c such that $f(X) \geq c \cdot g(X)$ for sufficiently large $X > 0$, then we write $f(X) \gg g(X)$.

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Theorem 1.1. *Let $f(z)=\sum_{n=1}^{\infty}a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. Then, for all but finitely many primes λ of E, we have*

$$\#\left\{D\in S(X)\,\middle|\,\delta(f)\cdot D>0,\lambda\nmid D\text{ and }\frac{L(f\otimes\chi_D,k)D_0^{k-\frac{1}{2}}}{\Omega_f}\not\equiv 0\mod\lambda\right\}\gg_{f,\lambda}\frac{\sqrt{X}}{\log X}.$$

This result is a refinement of results of Bruinier [2] and Ono-Skinner [10]. The proof is based on a generalization of a method of Kohnen-Ono [7] and James-Ono [5]. In the above theorem, we do not assume that the Fourier coefficients of f belong to $\mathbb{Z}$, therefore it does not hold the surjectivity of the residual Galois representation associated to f for almost all places in general. This makes some technical difficulty on the proof. To solve this problem, we may use a result of Ribet [12] on the image of Galois representations associated to modular forms. This is an ingredient in our proof. In the last section, we also consider an indivisibility result on non-central critical values of L-functions for higher weight modular forms using congruences of modular form with different weights.

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2. Modular forms of half-integral weight

We denote the space of modular forms of weight $k+1/2$, level N with character χ by $M_{k+1/2}(N,\chi)$, and the space of cusp forms of weight $k+1/2$, level N with character χ by $S_{k+1/2}(N,\chi)$. Then $M_{k+1/2}(N,\chi)$ and $S_{k+1/2}(N,\chi)$ are complex vector spaces.

For a modular form of half-integral weight

$$g(z)=\sum_{n=0}^{\infty}b(n)q^n\in M_{k+1/2}(N,\chi),$$

we define the action of Hecke operator T_{p^2} by

$$T_{p^2}(g)(z)=\sum_{n=0}^{\infty}b'(n)q^n,$$

where $b'(n)$ are given by

$$b'(n)=b(p^2n)+\chi(p)\left(\frac{-1}{p}\right)^k\left(\frac{n}{p}\right)p^{k-1}b(n)+\chi(p^2)p^{2k-1}b(n/p^2)$$

and $b(n/p^2)$ are zero if $p^2\nmid n$.

Now we give a short review of the theory of the Shimura correspondence. Let N be a positive integer which is divisible by four and χ a Dirichlet character mod N. Then we define a vector space $S^0_{3/2}(N,\chi)$ to be the subspace of $S_{3/2}(N,\chi)$ generated by

$$\left\{f(z)=\sum_{n=1}^{\infty}\psi(n)nq^{tm^2}\,\middle|\,N=4\,\mathrm{cond}(\psi)^2t|N,\ \chi=\psi\chi_{-t}\text{ and }\psi(-1)=-1\right\}$$

and denote the orthogonal complement by $S'_{3/2}(N,\chi)$. Then we assume

$$g(z)=\sum_{n=1}^{\infty}b(n)q^n\in S_{k+1/2}(N,\chi)$$

if $k \geq 2$, and

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S'_{3/2}(N, \chi)$$

if $k = 1$. Let t be a square-free positive integer. Define a number $A_t(n)$ to be

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\chi(n)(\frac{-1}{n})^k \left(\frac{t}{n}\right)}{n^{s-k+1}} \right) \left(\sum_{n=1}^{\infty} \frac{b(tn^2)}{n^s} \right).$$

Then Shimura [14] proved that there is a positive integer M such that $\mathrm{SH}_t(g(z)) = f_t(z) = \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2k}(M, \chi^2)$. (In fact, one can prove that $M = N/2$). Furthermore for any t, t', the difference between $\mathrm{SH}_t(g)$ and $\mathrm{SH}_{t'}$ is only constant multiple, so essentially this correspondence is independent of choice of t. This correspondence between modular forms is called the Shimura correspondence. Moreover if g is an eigenform for all Hecke operators T_{p^2} with $(p, 2N) = 1$, then the image of g under the Shimura correspondence is also an eigenform for all Hecke operators T_p with $(p, 2N) = 1$ and the Hecke eigenvalue of T_{p^2} for g coincides with the Hecke eigenvalue for T_p for $\mathrm{SH}_t(g)$.

We recall the following result which is a useful version of Waldspurger's formula ([17, Thèorém 1]) by Ono-Skinner. This formula gives a relation between the Fourier coefficients of modular forms of half-integral weight and the central values of twisted L-functions for modular forms.

Theorem 2.1 (Ono-Skinner [9], (2a),(2b))**.** *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$, level M with trivial character. Then there is $\delta(f) \in \{\pm 1\}$, a positive integer N with $4M \mid N$, a Dirichlet character χ modulo N, a period $\Omega_f \in \mathbb{C}^\times$ and a non-zero eigenform*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

with the property that $g(z)$ maps to a twist of f under the Shimura correspondence and for all fundamental discriminant D with $\delta(f)D > 0$ we have

$$b(D_0)^2 = \begin{cases} \alpha_D \dfrac{L(f \otimes \chi_D, k){D_0}^{k-1/2}}{\Omega_f} & \text{if } (D, N) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where α_D and $b(n)$ are algebraic integers in some finite extension of $\mathbb{Q}$. Moreover, there exists a finite set of primes S such that if D is a fundamental discriminant for which

(1) $\delta(f)D > 0$,
(2) $(D, N) = 1$,

then we have $|L(f \otimes \chi_D, k){D_0}^{k-1/2}/\Omega_f|_\lambda = |b(D_0)^2|_\lambda$ for $\lambda \notin S$.

3. Some properties of Fourier coefficients of modular forms and Galois representations

In this section we generalize some results of Serre [13] and Swinnerton-Dyer [16] using a result of Ribet [12]. These results should be well-known for specialists. However we give a short review for them, since it does not seem to be available in the literature. Let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. Let E be the subfield of $\mathbb{C}$ generated by the Fourier coefficients $a(n)$ of f. Then E is a finite extension of $\mathbb{Q}$. Let $\mathcal{O}_E$ be the ring of integers of E. For each prime ℓ, we let $\mathcal{O}_{E,\ell} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Theorem 3.1 (Deligne [3]). *For each prime ℓ, there exists a continuous representation*

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E,\ell}) \subset \mathrm{GL}_2(E_\ell)$$

unramified at all primes $p \nmid N\ell$ such that $\mathrm{trace}\rho_{f,\ell}(\mathrm{Frob}_p) = a(p)$ *and* $\det\rho_{f,\ell}(\mathrm{Frob}_p) = p^{2k-1}$ *for all primes $p \nmid N\ell$, where* Frob_p *is the arithmetic Frobenius at p.*

For each prime ℓ, denote

$$A_\ell = \left\{ g \in \mathrm{GL}_2(\mathcal{O}_{E,\ell}) \,\middle|\, \det(g) \in \mathbb{Z}_\ell^{\times(2k-1)} \right\},$$

where $\mathbb{Z}_\ell^{\times(2k-1)}$ is the group of $(2k-1)$-th powers of elements in $\mathbb{Z}_\ell^\times$. Replacing $\rho_{f,\ell}$ by an isomorphic representation, we may assume that for almost all $\rho_{f,\ell}$ sends $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to A_ℓ. Then Ribet proved the following theorem.

Theorem 3.2 (Ribet [12]). *Assume that f has no complex multiplication. Then for almost all ℓ, we have $\rho_{f,\ell}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = A_\ell$.*

We call the set of primes ℓ with the property $\rho_{f,\ell}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \neq A_\ell$ by the exceptional primes for f. Let S be the set of exceptional places for f. Let $\varepsilon_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$ be the ℓ-adic cyclotomic character. Then by a similar argument with Swinnerton-Dyer [16], one can see that the image of

$$(\rho_{f,\ell}, \varepsilon_\ell) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^\times$$

is $\left\{ (g, \alpha) \in \mathrm{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^\times \,\middle|\, \det(g) = \alpha^{2k-1} \right\}$ if ℓ is not exceptional. Since A_ℓ contains an element with the form

$$\begin{pmatrix} \mathrm{trace}\rho_{f,\ell}(\sigma) & -1 \\ \det\rho_{f,\ell}(\sigma) & 0 \end{pmatrix},$$

the map $(\mathrm{trace}\rho_{f,\ell}, \varepsilon_\ell) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^\times$ is surjective. Moreover by a ramification argument, one can see that the map

$$\prod_{\ell \notin S} (\mathrm{trace}\rho_{f,\ell}, \varepsilon_\ell) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \prod_{\ell \notin S} (\mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^\times)$$

is also surjective. Therefore we have the following result which is a generalization of a result of Serre [13, THÈORÉM 11] using Chebotarev density theorem.

Theorem 3.3. *Assume that f has no complex multiplication. Let t be a positive integer and α a non-zero integer in E. Fix $\beta \in \mathcal{O}_E/\alpha\mathcal{O}_E$ and $r \in (\mathbb{Z}/t\mathbb{Z})^\times$. Suppose that α does not contain a prime divisor which divides an exceptional prime for f. Then the set of prime p with the properties $a(p) \equiv \beta \mod \alpha$ and $p \equiv r \mod t$ has positive density.*

4. Indivisibility of Fourier coefficients of modular forms of half-integral weight

In this section, we give a result on modulo ℓ indivisibility of Fourier coefficients of half-integral weight modular forms using a method of Kohnen-Ono [7] and James-Ono [5]. Our result is a refinement of a result of Bruinier [2] and Ono-Skinner [10].

To consider the indivisibility of Fourier coefficients of half-integral weight modular forms, we will use the following results.

Theorem 4.1 (Sturm [15]). *Let*

$$g(z) = \sum_{n=1}^{\infty} b(n) q^n \in M_k(N, \chi)$$

be a half-integral or integral weight modular form for which the coefficients $b(m)$ are algebraic integers contained in a number field E. Let v be a finite place of E and let

$$\mathrm{ord}_v(g) = \begin{cases} +\infty & \text{if } b(n) \equiv 0 \mod v \text{ for all } n, \\ \min\{ n \mid b(n) \not\equiv 0 \mod v\} & \text{otherwise} \end{cases}$$

Moreover put

$$\mu = \frac{k}{12}[\Gamma_0(1) : \Gamma_0(N)] = \frac{kN}{12}\prod_{p|N}\frac{p+1}{p}.$$

Assume that

$$\mathrm{ord}_v(g) > \mu,$$

then $\mathrm{ord}_v(g) = +\infty$.

Remark 4.2 (cf. [5] Proposition 5)**.** *In* [15]*, Sturm proved this theorem for integral weight modular forms with trivial character, but the general case follows by taking an appropriate power of g.*

Lemma 4.3 (Shimura, [14] Section 1)**.** *Suppose*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

is a half integral weight cusp form and p is a prime. We define $(U_p g)(z)$, $(V_p g)(z)$ by

$$\begin{aligned} (U_p g)(z) &= \sum_{n=1}^{\infty} u_p(n)q^n = \sum_{n=1}^{\infty} b(pn)q^n, \\ (V_p g)(z) &= \sum_{n=1}^{\infty} v_p(n)q^n = \sum_{n=1}^{\infty} b(n)q^{pn}. \end{aligned}$$

Then

$$(U_p g)(z),\ (V_p g)(z) \in S_{k+1/2}\left(Np, \chi\left(\frac{4p}{\cdot}\right)\right).$$

Let

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N, \chi)$$

be an integral weight modular form for which the coefficients $a(m)$ are algebraic integers in E. For a prime λ of E and positive integers r, t with $(r, t) = 1$, define $T(r, t)$ and $T(\lambda, r, t)$ by

$$T(r, t) = \{p : \text{prime} \mid a(p) = 0,\ p \equiv r \bmod t\}$$

and

$$T(\lambda, r, t) = \{p : \text{prime} \mid a(p) \equiv 0 \bmod \lambda,\ p \equiv r \bmod t\}.$$

For a positive real number X, we also denote $T(r, t, X) = \{p \in T(r, t) \mid p \leq X\}$ and $T(\lambda, r, t, X) = \{p \in T(\lambda, r, t) \mid p \leq X\}$.

For $g = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi) \cap \mathcal{O}_{E,\lambda}[[q]]$, denote $s_\lambda(g) = \min\{\mathrm{ord}_\lambda(b(n)) \mid n \in \mathbb{Z}_{>0}\}$. The following two lemmas give an estimate for indivisibility of Fourier coefficients of modular forms of half integral weight.

Lemma 4.4. *Let ℓ be a prime greater than 3. Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigen newform of weight $2k$, level M with trivial character and let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1. Assume that f has complex multiplication in the sense of Ribet [11] *and λ be a prime in E above ℓ. If there exists an integer D' such that $\delta(f)D' > 0$, $(D', N) = 1$, $\varepsilon = \left(\frac{D'}{\ell}\right) \neq 0$ and $\mathrm{ord}_\lambda(b(|D'|)) = s_\lambda(g)$, then*

$$\#\left\{D \in S(X) \,\middle|\, \left(\frac{D}{\ell}\right) = \varepsilon,\ \mathrm{ord}_\lambda(b(D)) = s_\lambda(g)\right\} \gg_{f,\ell} \frac{\sqrt{X}}{\log X}.$$

Proof. By dividing g by $\lambda^{s_\lambda(g)}$, we may assume $s_\lambda(g) = 0$. If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character χ'. Since f has complex multiplication, so there exists a imaginary quadratic field K such that for every prime p satisfying $p \equiv 3 \mod 4$, $(p, N) = 1$ and $\left(\frac{\Delta_K}{p}\right) = -1$ we have $a(p) = 0$, where Δ_K is the discriminant of K. Therefore, for such p, using the formulae for the action of Hecke operator T_{p^2}, we find that

$$b(p^2 n) + \chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right) b(n) + \chi'(p^2)p^{2k-1} b(n/p^2) = 0.$$

Hence if $(r, t) = 1$, $4 \,|\, t$, $r \equiv 3 \mod 4$, then

$$\#T(r, t, X) = \#\{p \in T(r, t) \,|\, p \leq X\} \gg_f \frac{X}{\log X}$$

and for any $p \in T(r, t)$ we have

$$b(p^2 n) = -\chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right) b(n) - \chi'^2(p)p^{2k-1} b(n/p^2). \tag{4.1}$$

Put $\kappa = (k+\frac{1}{2})\frac{[\Gamma_0(1):\Gamma_0(N\ell^2)]}{12}+1$. Now, we choose (r_0, t_0) satisfying the following properties:

(1) $N\ell^2 | t_0$, $(r_0, t_0) = 1$, $\chi'(r_0) = 1$ and $p \equiv 3 \mod 4$.
(2) If p is a prime with $p \equiv r_0 \mod t_0$, then $\left(\frac{(-1)^k n}{p}\right) = -1$ for any $1 \leq n \leq \kappa$ with $(n, N\ell^2) = 1$.
(3) For each prime $p \equiv r_0 \mod t_0$ we have $\left(\frac{\Delta_K}{p}\right) = -1$.
(4) Each prime $p \equiv r_0 \mod t_0$ satisfies $\left|\chi'(p^2)p - \chi'(p)\left(\frac{(-1)^k |D'|}{p}\right)\right|_\lambda = 1$.

If $p \in T(r_0, t_0)$ is a sufficiently large prime, for all $1 \leq n \leq \kappa$

$$u_p(pn) = b_0(p^2 n) = -\chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right) b_0(n) - p^{2k-1}\chi'^2(p)b_0(n/p^2)$$

Since $b_0(n/p^2) = 0$, we have $u_p(pn) = \chi'(p)p^{k-1}b_0(n) = p^{k-1}b_0(p) = p^{k-1}v_p(pn)$. By the relation (4.1),

$$v_p(p^3|D'|) = b_0(p^2|D'|) = -\chi'(p)p^{k-1}\left(\frac{(-1)^k|D'|}{p}\right) b_0(|D'|),$$

and

$$u_p(p^3|D'|) = b_0(p^4|D'|) = -p^{2k-1}\chi'(p^2)b_0(|D'|).$$

Therefore by the assumption and the choice of (r_0, t_0),

$$\left|u_p(p^3|D'|) - p^{k-1}v_p(p^3|D'|)\right|_\lambda = \left|\left(\chi'(p^2)p^{2k-1} - \chi'(p)p^{2k-2}\left(\frac{(-1)^k|D'|}{p}\right)\right) b_0(|D'|)\right|_\lambda = 1.$$

Hence

$$\mathrm{ord}_\lambda(U_p g_0 - p^{k-1}V_p g_0) < +\infty.$$

By Theorem 4.1 and Lemma 4.3, there exists a integer n_p such that

$$1 \leq n_p \leq \left(k + \frac{1}{2}\right)\frac{[\Gamma_0(1) : \Gamma_0(N\ell^2 p)]}{12} = \kappa(p+1),\ (n_p, p) = 1$$

and

$$b_0(n_p p) = u_p(n_p) \not\equiv p^{k-1}v_p(n_p) = 0 \mod \lambda.$$

Consequently, let D_{sf} be the square-free part of $D = n_p p$, then

$$|b_0(D_{\mathrm{sf}})|_\lambda = 1.$$

For convenience, let p_i be the primes in $T(r_0, t_0)$ in increasing order, and let D_i be the square-free part of $p_i n_{p_i}$. If $r < s < t$ and $D_r = D_s = D_t$, then $p_r p_s p_t | D_r$. However this can only occur for finitely many r, s and t since $|D_i| < \kappa p_i(p_i + 1)$. Therefore, the number of distinct $|D_i| < X$ is at least half the number of $p \in T(r_0, t_0)$ with $p \leq \sqrt{X/\kappa}$. Therefore the lemma follows from $\#T(r_0, t_0, X) \gg_{f,\lambda} X/\log X$. $\square$

Lemma 4.5. *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigen newform of weight $2k$, level M with trivial character. Denote $E = \mathbb{Q}(\{a(n)|n \geq 1\})$ and let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1. We fix a prime number ℓ greater than 3 and let λ be a prime in E above ℓ. Assume that f does not have complex multiplication and the image of the Galois representation associated to f

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E,\ell})$$

coincides with A_ℓ. If there exists an integer D' such that $\delta(f)D' > 0$, $(D', N) = 1$, $\varepsilon = \left(\frac{D'}{\ell}\right) \neq 0$ and $\mathrm{ord}_\lambda(b(|D'|)| = s_\lambda(g)$, then

$$\#\left\{D \in S(X) \,\middle|\, \left(\frac{D}{\ell}\right) = \varepsilon,\ \mathrm{ord}_\lambda(b(D)) = s_\lambda(g)\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

Proof. First, we may assume $\mathrm{ord}_\lambda(g) = 0$. If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n) q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character χ'. If $a(p) \equiv 0 \mod \lambda$, by the formula for the action of Hecke operator T_{p^2} we find that

$$b(p^2 n) + \chi'(p) p^{k-1} \left(\frac{(-1)^k n}{p}\right) b(n) + \chi'^2(p) p^{2k-1} b(n/p^2) \equiv 0 \mod \lambda.$$

By the assumption, ℓ is not exceptional. Hence Theorem 3.3 implies

$$\# T(\lambda, r, t, X) = \#\{p \in T(\lambda, r, t) \mid p \le X\} \gg_{f,\lambda} \frac{X}{\log X}$$

and for each $p \in T(\lambda, r, t)$

$$b(p^2 n) \equiv -\chi'(p) p^{k-1} \left(\frac{(-1)^k n}{p}\right) b(n) - \chi'^2(p) p^{2k-1} b(n/p^2) \mod \lambda. \tag{4.2}$$

Let κ be the number as in the proof of Lemma 4.4. Now, we choose (r_0, t_0) satisfying the following properties:

(1) $N\ell^2 | t_0$, $(r_0, t_0) = 1$, $\chi'(r_0) = 1$.
(2) If p is a prime with $p \equiv r_0 \mod t_0$, then $\left(\frac{(-1)^k n}{p}\right) = -1$ for any $1 \le n \le \kappa$ with $(n, N\ell^2) = 1$.
(3) For each prime $p \equiv r_0 \mod t_0$ we have $\left(\frac{(-1)^k |D'|}{p}\right) = -1$.
(4) Each prime $p \equiv r_0 \mod t_0$ has the property that $1 + p \not\equiv 0 \mod \lambda$.

If $p \in T(\lambda, r_0, t_0)$ is a sufficiently large prime, for all $1 \le n \le \kappa$ with $(n, N\ell^2) = 1$, one has

$$u_p(pn) = b_0(p^2 n) \equiv -p^{k-1} \left(\frac{(-1)^k n}{p}\right) b_0(n) - p^{2k-1} b_0(n/p^2) = p^{k-1} b_0(n) = p^{k-1} v_p(pn) \mod \lambda.$$

By the relation (4.2), we have

$$v_p(p^3 |D'|) = b_0(p^2 |D'|) \equiv p^{k-1} b_0(|D'|) \mod \lambda,$$

also

$$u_p(p^3 |D'|) = b_0(p^4 |D'|) \equiv -p^{2k-1} b_0(|D'|) \mod \lambda.$$

Therefore by assumption and the choice of (r_0, t_0),

$$p^{k-1} v_p(p^3 |D'|) - u_p(p^3 |D'|) \equiv p^{2k-2}(1+p) b_0(|D'|) \not\equiv 0 \mod \lambda.$$

Hence

$$\mathrm{ord}_\lambda(U_p g_0 - p^{k-1} V_p g_0) < +\infty.$$

By Theorem 4.1 and Lemma 4.3, there exists a integer n_p such that

$$1 \le n_p \le (k + 1/2)[\Gamma_0(1) : \Gamma_0(N\ell^2 p)]/12 = \kappa(p+1), \ (n_p, p) = 1$$

and

$$b_0(n_p p) = u_p(n_p) \not\equiv p^{k-1} v_p(n_p) = 0 \mod \lambda.$$

In particular, let D_{sf} be the square-free part of $D = n_p p$, then

$$|b_0(D_{\mathrm{sq}})|_\lambda = 1.$$

Now the lemma follows from the same argument with the proof of the previous lemma using Theorem 3.3. □

Proof of Theorem 1.1.

Now we give the proof of Theorem 1.1. Let

$$g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1 for f.

By replacing f by a suitable quadratic twist of f if necessary, we may assume that $\varepsilon = \delta(f)$, where ε is the sign of the functional equation of $L(f, s)$. By the result of Friedberg and Hoffstein [4], we can take an integer D' such that $\delta(f)D' > 0$, $(D', 2N) = 1$ and $b(D') \neq 0$. In particular, for almost all finite places λ of E we have

$$|b(D')|_\lambda = 1.$$

Thus by Lemmas 4.4, 4.5, Theorem 2.1 and Theorem 3.3, for all but finitely many primes λ we have

$$\#\left\{ D \in S(X) \,\middle|\, \delta(f) \cdot D > 0,\ (\ell, D) = 1 \text{ and } \left| \frac{L(f \otimes \chi_D, k) D_0^{k-1/2}}{\Omega_f} \right|_\lambda = 1 \right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

This completes the proof.

5. Indivisibility for the non-central critical values

In this section, we consider a special case for non-central values of L-functions for modular forms. We fix a prime ℓ greater than 7 and let $f = \sum_{n=1}^{\infty} a(n) q^n$ be a normalized Hecke eigenform of weight $\ell + 1$ for $\mathrm{SL}_2(\mathbb{Z})$. Let λ be a prime in a number filed E. We assume that the integer ring of E contains all Fourier coefficients of f and choose a period $\Omega_f^{\pm}$ as in Ash-Stevens [1, Theorem 4.5]. Then for any Dirichlet character χ, the quotient $\tau(\chi^{-1}) \dfrac{L(f \otimes \chi, 1)}{(2\pi i)\Omega_f^{\pm}}$ is an integer in $E_\lambda(\chi)$ where τ is the Gauss sum and $\pm = \chi(-1)$.

Theorem 5.1. *Let λ be a prime in E above ℓ. We assume the following conditions.*

(1) *There exists a unique eigenform F of weight 2 for $\Gamma_0(\ell)$ such that*

$$F \equiv f \mod \lambda.$$

(2) *ℓ is not exceptional.*

(3) *There exists an square-free negative integer d_0 such that $(d_0, 2\ell) = 1$, $\chi_{d_0}(\ell) = -\varepsilon(F)$, where $\varepsilon(F)$ is the sign of functional equation of $L(F, s)$ and*

$$\frac{L(f \otimes \chi_{d_0}, 1)\sqrt{d_0}}{(2\pi i)\Omega_f^{\pm}} \not\equiv 0 \mod \lambda.$$

Then we have

$$\#\left\{ D \in S(X) \,\middle|\, \frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}} \not\equiv 0 \mod \lambda \right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

For the proof, we recall a result of Ash and Stevens.

Theorem 5.2 (Ash-Stevens, [1]). *Let k be a positive integer less than $\ell+2$ and $f=\sum_{n=1}^{\infty}a(n)q^n \in S_k(\Gamma_0(1))$ an eigenform satisfying the assumptions of Theorem 5.1. We fix a prime λ above ℓ in a number field E which contains all Fourier coefficients of f. Assume that*

(1) *There exists a prime q satisfying $a(q)\not\equiv q^{k-1}+1 \mod \lambda$.*
(2) *There exists an unique eigenform $F\in S_2(\Gamma_1(\ell))$ such that $f\equiv F \mod \lambda$.*

Then there exists a complex number $\Omega_F^{\pm}$ such that for any Dirichlet character χ satisfying $(\mathrm{cond}\,\chi, p)=1$, we have

$$\frac{\tau(\chi^{-1})L(f\otimes\chi,1)}{(2\pi i)\Omega_f^{\pm}} \equiv \frac{\tau(\chi^{-1})L(F\otimes\chi,1)}{(2\pi i)\Omega_F^{\pm}} \mod \lambda.$$

Now we prove Theorem 5.1. By the Kohnen-Zagier formula [6], there exists an eigenform

$$g(z)=\sum_{n=1}^{\infty}b(n)q^n \in S_{3/2}(\Gamma_0(4\ell))$$

such that for any negative square-free integer D satisfying $\left(\frac{D}{\ell}\right)=-\varepsilon(F)$,

$$|b(|D|)|^2 = 2\cdot\frac{\sqrt{D}}{\pi}\cdot\frac{\langle g,g\rangle}{\langle F,F\rangle}L(F\otimes\chi_D,1),$$

where $\langle\cdot,\cdot\rangle$ is the Petersson inner product. We can normalize g by the relation $\frac{\langle F,F\rangle}{\langle g,g\rangle}=\Omega_f^{\pm}$. Taking a linear combination of twists of g, one may assume $b(|D|)=0$ if $\left(\frac{D}{\ell}\right)\neq-\varepsilon(F)$ and $D<0$. From the assumptions of the theorem, ℓ is not exceptional. This implies the existence of a prime q satisfying $a(q)\not\equiv q^{k-1}+1 \mod \lambda$, therefore the assumptions of Theorem 5.1 implies the assumptions of Theorem 5.2. Since $\tau(\chi_D)^{-1}=\pm 1/\sqrt{D}$, one can see that

$$\frac{L(f\otimes\chi,1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}} \equiv \frac{L(F\otimes\chi,1)\sqrt{D}}{(2\pi i)\Omega_F^{\pm}} = |b(|D|)|^2\cdot c \mod \lambda$$

with a λ-adic unit c. By the assumption (3), we have

$$\mathrm{ord}_\lambda\left(\frac{L(f\otimes\chi_{d_0},1)\sqrt{d_0}}{(2\pi i)\Omega_f^{\pm}}\right)=0,$$

therefore $\mathrm{ord}_\lambda(b(d_0))=\min\{\mathrm{ord}_\lambda(b(n))\,|\,n:\text{square-free},\ \chi_{d_0}(\ell)=-\varepsilon(f)\}$. Hence Lemma 4.5 implies

$$\#\big\{D\in S(X)\,\big|\,\chi_D(\ell)=-\varepsilon(f),\ \mathrm{ord}_\lambda(b(D))=s\big\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X},$$

thus we have

$$\#\left\{D\in S(X)\,\middle|\,\frac{L(f\otimes\chi_D,1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}}\not\equiv 0 \mod \lambda\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

This completes the proof.

Remark 5.3. *Lemma 4.5 states only for g given in Theorem 2.1, but one can show the similar result for any eigenform $g\in S_{k+1/2}(N,\chi)$ if $k\geq 2$ ($S'_{\frac{3}{2}}(N,\chi)$ if $k=1$) corresponding to some eigenform $f\in S_{2k}(\Gamma_0(M))$ under the Shimura correspondence.*

Example 5.4. *Let*

$$f = \Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} \in S_{12}(\Gamma_0(1))$$

and

$$F = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \in S_2(\Gamma_0(11)).$$

Then it is well-known that $f \equiv F \mod 11$, $\dim S_2(\Gamma_0(11)) = 1$ *and the* $\bmod 11$ *Galois representation associated to* f *is surjective. Moreover one can check that*

$$\frac{L(\Delta\otimes\chi_{-3},1)}{\Omega^{+}_{\Delta\otimes\chi_{-3}}} = 36741600 \not\equiv 0 \mod 11$$

by using MAGMA. So the assumptions of Theorem 5.1 are satisfied for $f=\Delta$. *Hence we have*

$$\#\left\{D\in S(X)\;\middle|\;\frac{L(\Delta\otimes\chi_D,1)\sqrt{D}}{(2\pi i)\Omega^{\pm}_{\Delta}}\not\equiv 0 \mod 11\right\} \gg \frac{\sqrt{X}}{\log X}.$$

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