Title	Trudinger–Moser inequality on the whole plane with the exact growth condition
Author(s)	Ibrahim, Slim; Masmoudi, Nader; Nakanishi, Kenji
Citation	Journal of the European Mathematical Society (2015), 17(4): 819-835
Issue Date	2015
URL	http://hdl.handle.net/2433/200756
Right	© 2015 EMS Publishing House.; This is not the published version. Please cite only the published version. この論文は出版社版でありません。引用の際には出版社版をご確認ご利用ください。
Type	Journal Article
Textversion	author

TRUDINGER-MOSER INEQUALITY ON THE WHOLE PLANE WITH THE EXACT GROWTH CONDITION

SLIM IBRAHIM, NADER MASMOUDI, AND KENJI NAKANISHI

Abstract. Trudinger-Moser inequality is a substitute to the (forbidden) critical Sobolev embedding, namely the case where the scaling corresponds to L^∞. It is well known that the original form of the inequality with the sharp exponent (proved by Moser) fails on the whole plane, but a few modified versions are available. We prove a precised version of the latter, giving *necessary and sufficient conditions* for the boundedness, as well as for the compactness, in terms of the growth and decay of the nonlinear function. It is tightly related to the ground state of the nonlinear Schrödinger equation (or the nonlinear Klein-Gordon equation), for which the range of the time phase (or the mass constant) as well as the energy is given by the best constant of the inequality.

1. Introduction

There are several extensions of the critical Sobolev embedding

$$\dot{H}^1(\mathbb{R}^d) \subset L^{2d/(d-2)}(\mathbb{R}^d) \tag{1.1}$$

from $d \geq 3$ to $d = 2$, where the simple limit estimate fails

$$H^1(\mathbb{R}^2) \not\subset L^\infty(\mathbb{R}^2). \tag{1.2}$$

One way is to replace the right hand side by BMO, Besov, Triebel-Lizorkin or Morrey-Campanato spaces of the same type (scaling). These are all taking account of possible oscillations of the functions in $H^1(\mathbb{R}^2)$. If one is interested more in possible growth, another substitute is given by the Trudinger-Moser inequality [38, 31, 32, 25]

Proposition 1.1. *For any open set $\Omega \subset \mathbb{R}^2$ with bounded measure $|\Omega| < \infty$, there exists a constant $\kappa(\Omega) > 0$ such that*

$$u \in H_0^1(\Omega),\ \|\nabla u\|_{L^2(\Omega)} \leq 1 \implies \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u|^2} - 1\right) dx \leq \kappa(\Omega). \tag{1.3}$$

Moreover, this fails if 4π is replaced with any $\alpha > 4\pi$. The constant $\kappa(\Omega)$ is bounded by $|\Omega|$, but in general unbounded as $|\Omega| \to \infty$.

The goal of this paper is to give a precised version of this inequality in the whole space $\mathbb{R}^2$, with necessary and sufficient conditions in terms of the growth of general nonlinear functionals (not only for $e^{\alpha|u|^2} - 1$). Before stating our result, let us first

2010 *Mathematics Subject Classification.* 35J20,46E35,35Q55.

Key words and phrases. Sobolev critical exponent, Trudinger-Moser inequality, concentration compactness, nonlinear Schrödinger equation, ground state.

S. I. is partially supported by NSERC# 371637-2009 grant and a start up fund from University of Victoria.

N. M is partially supported by an NSF Grant DMS-0703145.

recall the following two versions of the Trudinger-Moser inequality on $\mathbb{R}^2$. The first one is for smaller exponents.

Proposition 1.2. *For any $\alpha < 4\pi$, there exists a constant $c_\alpha > 0$ exists such that*

$$u \in H^1(\mathbb{R}^2),\ \|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1 \implies \int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha|u|^2} - 1\right) dx \le c_\alpha \|u\|_{L^2(\mathbb{R}^2)}^2. \tag{1.4}$$

Moreover, this fails if α is replaced with 4π.

One can normalize to $\|u\|_{L^2} = 1$ by scaling. This version was proved in [9], using the symmetric decreasing rearrangement as Moser did [31]. The necessity $\alpha < 4\pi$ was proved in [1], also using Moser's example.

The second one is to strengthen the condition on $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ to the whole $H^1(\mathbb{R}^2)$ norm. Then the value $\alpha = 4\pi$ becomes admissible.

Proposition 1.3. *There exists a constant $\kappa > 0$ such that*

$$u \in H^1(\mathbb{R}^2),\ \|u\|_{H^1(\mathbb{R}^2)} \le 1 \implies \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u|^2} - 1\right) dx \le \kappa. \tag{1.5}$$

Moreover, this fails if 4π is replaced with any $\alpha > 4\pi$.

This version was proved in [34], again by Moser's argument, while the failure for $\alpha > 4\pi$ is clear from the sharpness in the previous two propositions.

In short, the failure of the original Trudinger-Moser (1.3) on $\mathbb{R}^2$ can be recovered either by weakening the exponent $\alpha = 4\pi$ or by strengthening the norm $\|\nabla u\|_{L^2}$. It is worth noting, however, that proving these two estimates on $\mathbb{R}^2$ is considerably easier than the critical case 4π on Ω, which suggests that there is some room of improvement, even though they are "sharp" in their formulations.

Then a natural question arises,

$$\textit{What if we keep both the conditions } \alpha = 4\pi \textit{ and } \|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1\textit{?} \tag{1.6}$$

Our answer to this question is to weaken the exponential nonlinearity as follows:

Proposition 1.4. *There exists a constant $c > 0$ such that*

$$u \in H^1(\mathbb{R}^2),\ \|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1 \implies \int_{\mathbb{R}^2} \frac{\mathrm{e}^{4\pi|u|^2} - 1}{(1+|u|)^2}\, dx \le c\|u\|_{L^2(\mathbb{R}^2)}^2. \tag{1.7}$$

Moreover, this fails if the power 2 in the denominator is replaced with any $p < 2$.

Obviously this implies Proposition 1.2. It is less obvious that it also implies Proposition 1.3, but indeed this follows by Hölder only; see Section 6. Hence Proposition 1.4 can be regarded as a unified improvement of those two previous versions, while it can be easily deduced from the part (B) of our full Theorem 1.5 below, by taking

$$2\pi K = 1, \quad g(u) = \frac{(\mathrm{e}^{4\pi|u|^2} - 1)}{(1+|u|)^2}. \tag{1.8}$$

The following theorem completely determines the growth order, not only among exponentials and power functionals, but for general functions, in terms of necessary and sufficient conditions, both for the boundedness and for the compactness.

Theorem 1.5. *For any Borel function $g : \mathbb{R} \to [0, \infty)$, define the functional G by*

$$G(\varphi) = \int_{\mathbb{R}^2} g(\varphi(x))dx. \tag{1.9}$$

Then for any $K > 0$ we have the following **(B)** *and* **(C)**.
(B) *Boundedness: The following (1) and (2) are equivalent.*

(1) $\overline{\lim}_{|u|\to\infty} e^{-2|u|^2/K} |u|^2 g(u) < \infty$ *and* $\overline{\lim}_{u\to 0} |u|^{-2} g(u) < \infty$.
(2) *There exists a constant $C_{g,K} > 0$ such that*

$$\varphi \in H^1(\mathbb{R}^2),\ \|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)} \le 2\pi K \implies G(\varphi) \le C_{g,K} \|\varphi\|^2_{L^2}. \tag{1.10}$$

Moreover, if (1) fails then there exists a sequence $\varphi_n \in H^1(\mathbb{R}^2)$ satisfying

$$\|\nabla\varphi_n\|^2_{L^2} < 2\pi K, \quad \|\varphi_n\|_{L^2} \to 0, \quad G(\varphi_n) \to \infty \quad (n \to \infty). \tag{1.11}$$

(C) *Compactness: The following (3) and (4) are equivalent.*

(3) $\lim_{|u|\to\infty} e^{-2|u|^2/K} |u|^2 g(u) = 0$ *and* $\lim_{u\to 0} |u|^{-2} g(u) = 0$.
(4) *For any sequence of radial $\varphi_n \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla\varphi_n\|^2_{L^2} \le 2\pi K$ and weakly converging to some $\varphi \in H^1(\mathbb{R}^2)$, we have $G(\varphi_n) \to G(\varphi)$.*

Moreover, if (3) fails then there exists a sequence of radial $\varphi_n \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla\varphi_n\|^2_{L^2} < 2\pi K$, converging to 0 weakly in H^1, and $G(\varphi_n) > \delta$ for some $\delta > 0$.

Remark 1. 1) This theorem shows that the true threshold for the L^2 Trudinger-Moser inequality in the whole space under the condition $\|\nabla\varphi\|_{L^2(\mathbb{R}^2)} \le 1$ is given by the functional in (1.7), and loss of compactness happens only for it.

2) As we will see in the proof, the concentration sequence constructed in (3) is very different from those in the higher dimensional case—it must contain a nontrivial tail going to the spatial infinity which is the main contribution to L^2, even though the main contribution to G is the concentrating part. This is also different from the concentration in the original Trudinger-Moser inequality (1.3) (see [27]).

3) In the Orlicz space corresponding to $g(u) \sim e^{\alpha|u|^2}$, [5] gives a more precise description of concentration compactness, in terms of the profile decomposition. However, the above phenomena for $g(u) \sim e^{2|u|^2/K} |u|^{-2}$ do not seem to be observable in a linear setting such as in the Orlicz space.

4) For the original Trudinger-Moser inequality (1.3), the exponent 4π can be improved if it is allowed to depend on $\|u\|_{L^2}/\|\nabla u\|_{L^2}$. Precisely, it is proved in [2]

$$\sup_{u\in H^1_0(\Omega),\ \|\nabla u\|_{L^2}=1} \int_{\mathbb{R}^2} \left(e^{4\pi|u|^2(1+\alpha\|u\|^2_{L^2})} - 1 \right) dx < \infty, \tag{1.12}$$

if and only if α is less than the first Dirichlet eigenvalue of $-\Delta$ on Ω. Our inequality (1.7) does not admit such improvement because of the scaling invariance.

In Section 2, we prove the necessity of (1) and (3) in Theorem 1.5, by constructing sequences φ_n obtained from rescaling of Moser's example. In Section 3, we study the optimal growth of a function in the exterior of the ball when the L^2 and $\dot{H}^1$ norm are given. This is used in proving the sufficiency or the main part of Theorem 1.5 in Section 4.

2. Proof of the necessity of (1) and (3): Moser's example

First we consider the much easier case with the condition as $|u| \to 0$. Let $\varphi_n(x)$ be a sequence of radial functions in $H^1(\mathbb{R}^2)$ defined by

$$\varphi_n(x) = \begin{cases} a_n & (|x| < R_n), \\ a_n(1 - |x| + R_n) & (R_n < |x| < R_n + 1) \\ 0 & (|x| > R_n + 1) \end{cases} \tag{2.1}$$

for some sequences $a_n \to 0$ and $R_n \to \infty$ chosen later. We have

$$\|\varphi_n\|_{L^2}^2 \sim a_n^2 R_n^2, \quad \|\nabla\varphi_n\|_{L^2}^2 \sim a_n^2 R_n, \quad G(\varphi_n) \geq \pi R_n^2 g(a_n). \tag{2.2}$$

If (1) is violated by $\overline{\lim}_{|u|\to 0} |u|^{-2} g(u) = \infty$, then we can find a sequence $a_n \searrow 0$ such that $g(a_n) \geq n|a_n|^2$. Let $R_n = a_n^{-1/2} + a_n^{-1} n^{-1/4}$. Then $R_n \to \infty$, $a_n R_n \to 0$ and $G(\varphi_n) \geq n a_n^2 R_n^2 \to \infty$.

If (3) is violated by $\overline{\lim}_{|u|\to 0} |u|^{-2} g(u) > 0$, then we can find a sequence $a_n \searrow 0$ and $\delta > 0$ such that $g(a_n) \geq \delta |a_n|^2$. Let $R_n = 1/a_n$. Then $R_n \to \infty$, $a_n R_n = 1$, $a_n^2 R_n \to 0$ and $G(\varphi_n) \geq \delta a_n^2 R_n^2 \geq \delta$.

It remains to treat the case where the condition for $|u| \to \infty$ fails. First, we recall the following fundamental example of Moser: Let f_α be defined by:

$$f_\alpha(x) = \begin{cases} 0 & \text{if} \quad |x| \geq 1, \\ -\frac{\log|x|}{\sqrt{2\alpha\pi}} & \text{if} \quad \mathrm{e}^{-\alpha} \leq |x| \leq 1, \\ \sqrt{\frac{\alpha}{2\pi}} & \text{if} \quad |x| \leq \mathrm{e}^{-\alpha}, \end{cases}$$

where $\alpha > 0$. One can also write f_α as

$$f_\alpha(x) = \sqrt{\frac{\alpha}{2\pi}}\, \mathbf{L}\left(\frac{-\log|x|}{\alpha}\right), \tag{2.3}$$

where

$$\mathbf{L}(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ t & \text{if} \quad 0 \leq t \leq 1, \\ 1 & \text{if} \quad t \geq 1. \end{cases}$$

Straightforward computations show that $\|f_\alpha\|_{L^2(\mathbb{R}^2)}^2 = \frac{1}{4\alpha}(1 - \mathrm{e}^{-2\alpha}) - \frac{1}{2}\mathrm{e}^{-2\alpha}$ and $\|\nabla f_\alpha\|_{L^2(\mathbb{R}^2)} = 1$.

In order to fit this example in our estimate on $\mathbb{R}^2$, we need to make sure that the L^2 norm does not go to zero. This requires a rescaling of Moser's example. Choose sequences $1 \ll b_k \nearrow \infty$ and $K_k \nearrow K$ such that

$$c_k := \mathrm{e}^{-2b_k^2/K_k}\, b_k^2 g(b_k) \to \overline{\lim_{|u|\to\infty}}\, \mathrm{e}^{-2|u|^2/K}\, |u|^2 g(u), \tag{2.4}$$

and let $R_k = e^{-b_k^2/K_k}$. We define a radial function $\psi_k(r) \in H^1(\mathbb{R}^2)$ by

$$\psi_k(r) = \begin{cases} b_k & (r < R_k) \\ b_k \frac{|\log r|}{|\log R_k|} & (R_k \leq r < 1) \\ 0 & (r \geq 1) \end{cases} \tag{2.5}$$

Then we have

$$\begin{aligned} &\|\nabla\psi_k\|_{L^2}^2 = 2\pi b_k^2 \int_{R_k}^1 \frac{dr}{r|\log R_k|^2} = 2\pi K_k < 2\pi K, \\ &\|\psi_k\|_{L^2}^2 \sim \frac{2\pi b_k^2}{|\log R_k|^2} = \frac{2\pi K_k^2}{b_k^2}, \quad G(\psi_k) \ge 2\pi R_k^2 g(b_k) = \frac{2\pi c_k}{b_k^2}, \end{aligned} \tag{2.6}$$

and so $G(\psi_k)\|\psi_k\|_{L^2}^{-2} \ge c_k/K^2$.

Now let $\varphi_k = \psi_k(x/S_k)$, where we choose $S_k = b_k$ if (3) fails (i.e. if $c_k > 0$), and $S_k = o(b_k)$ such that $S_k^2 c_k/b_k^2 \to \infty$ if (1) fails (i.e. if $c_k = \infty$). In both cases φ_k is bounded in H^1 and satisfies

$$\begin{aligned} &\|\nabla\varphi_k\|_{L^2}^2 = \|\nabla\psi_k\|_{L^2}^2 = 2\pi K_k < 2\pi K, \\ &\|\varphi_k\|_{L^2}^2 = S_k^2\|\psi_k\|_{L^2}^2 \sim 2\pi K_k^2 \frac{S_k^2}{b_k^2}, \quad G(\varphi_k) \ge 2\pi R_k^2 g(b_k) = \frac{2\pi c_k S_k^2}{b_k^2} \end{aligned} \tag{2.7}$$

Moreover, $\varphi_k(x) \to 0$ for every $x \ne 0$, because $|\varphi_k(x)| \lesssim \varepsilon$ if $|x| \ge S_k\, \mathrm{e}^{-\varepsilon b_k} = o(1)$ for any $\varepsilon > 0$. This ends the proof for the necessity of (1) and (3).

3. Radial Trudinger-Moser and optimal descending

In this section, we prove the following theorem that will be used in the proof of point (2) of the main theorem 1.5. This can be regarded as the exponential version of the radial Sobolev inequality.

Theorem 3.1. *There exists a constant $C > 0$ such that for any radial $\varphi \in H^1$, any $K > 0$ and any $R > 0$,*

$$\|\nabla\varphi\|_{L^2(|x|>R)}^2 \le 2\pi K \implies \frac{\mathrm{e}^{2\varphi(R)^2/K}}{\varphi(R)^2/K^2} \le C\|\varphi/R\|_{L^2(|x|>R)}^2 \text{ or } |\varphi(R)|^2 \le K. \tag{3.1}$$

Note that the second case is as if the (false) critical radial Sobolev inequality

$$\varphi \in H^1_{\mathrm{radial}}(\mathbb{R}^2) \implies \|\varphi\|_{L^\infty(|x|>R)} \lesssim \|\nabla\varphi\|_{L^2(|x|>R)} \tag{3.2}$$

were recovered. Hence the first case is essential for the Trudinger-Moser type inequalities, for which the function $\mathrm{e}^{\varphi^2}/\varphi$ is optimal. More precisely, we have

Theorem 3.2. *Let*

$$\mu(h) := \inf\{\|\varphi\|_{L^2(|x|>1)} \mid \varphi \in H^1_r,\ \varphi(1) = h,\ \|\varphi_r\|_{L^2(|x|>1)}^2 \le 2\pi\}, \tag{3.3}$$

for $h > 1$. Then we have $\mu(h) \sim \mathrm{e}^{h^2}/h$ for $h > 1$.

Obviously, the first theorem follows from the second one, by rescaling. To prove the latter, we consider the discrete version:

$$\mu_d(h) := \inf\{\|a\|_{(e)} \mid \|a\|_1 = h,\ \|a\|_2 \le 1\}. \tag{3.4}$$

for $h > 1$, where the norms on any sequence $a = (a_n)_{n=0}^\infty$ are defined by

$$\|a\|_p^p = \sum_{n=0}^\infty |a_n|^p, \quad \|a\|_{(e)}^2 = \sum_{n=0}^\infty \mathrm{e}^{2n} a_n^2. \tag{3.5}$$

Lemma 3.3. *We have $\mu(h) \sim \mu_d(h)$ for $h > 1$.*

Proof. μ_d is naturally obtained by optimizing the energy for given values on the lattice. For $\mu(h)$, it suffices to consider radial $\varphi \in H^1$ satisfying $\varphi_r \le 0 \le \varphi$.

Let $h_k = \varphi(\mathrm{e}^k)$ and $a_k = h_k - h_{k+1}$ for $k = 0, 1, 2 \dots$. We can optimize the energy on each interval $[\mathrm{e}^k, \mathrm{e}^{k+1}]$ by replacing φ with

$$\psi(r) = a_k |\log(\mathrm{e}^{-k-1} r)| + h_{k+1} \quad (\mathrm{e}^k \le r \le \mathrm{e}^{k+1}). \tag{3.6}$$

Then we have $\psi(1) = h_0 = \varphi(1) = h$ and

$$\int_{\mathrm{e}^k}^{\mathrm{e}^{k+1}} \psi_r(r)^2 r dr = a_k^2 = (\varphi(\mathrm{e}^k) - \varphi(\mathrm{e}^{k+1}))^2 \le \int_{\mathrm{e}^k}^{\mathrm{e}^{k+1}} \varphi_r(r)^2 r dr, \tag{3.7}$$

where the last inequality follows from Schwarz. For the L^2 norm we have

$$\|\psi\|_{L^2(r>1)}^2 \lesssim \sum_{k=0}^{\infty} h_k^2 \,\mathrm{e}^{2k} \lesssim h_0^2 + \int_0^{\infty} \|\varphi\|_{L^2(r>1)}^2, \tag{3.8}$$

where h_0^2 is estimated by using the energy as follows. For $1 < r < \mathrm{e}^{1/4}$ we have

$$\varphi(1) - \varphi(r) \le \int_r^1 \varphi_r(s) ds \le \sqrt{\int_1^{\mathrm{e}^{1/4}} |\varphi_r|^2 r dr \int_1^{\mathrm{e}^{1/4}} \frac{dr}{r}} \le \frac{1}{2}, \tag{3.9}$$

which implies $\varphi(r) \ge h_0/2$ since $h_0 = h > 1$, and so

$$h_0^2 \lesssim \int_1^{\mathrm{e}^{1/4}} |\varphi|^2 r dr \lesssim \|\varphi\|_{L^2}^2. \tag{3.10}$$

Hence for $\mu(h)$ it suffices to consider such ψ. Moreover we have

$$\|a\|_2^2 \le \|\psi_r\|_{L^2(|x|>1)}^2/(2\pi) \le 1, \quad \|a\|_1 = h_0 = h, \tag{3.11}$$

and

$$\begin{aligned} \|\psi\|_{L^2(r>1)}^2 \sim \sum_{j\ge 0} h_j^2 \,\mathrm{e}^{2j} &= \sum_j \sum_{k,l \ge j} a_k a_l \,\mathrm{e}^{2j} \\ &= \sum_{k,l} a_k a_l \sum_{j \le \min(k,l)} \mathrm{e}^{2j} \sim \sum_{k \le l} a_k a_l \,\mathrm{e}^{2k} \sim \|a\|_{(e)}^2, \end{aligned} \tag{3.12}$$

where $\lesssim$ for the last equivalence follows from Young on $\mathbb{Z}$. □

Now, Theorem 3.2, and hence Theorem 3.1, follow from

Lemma 3.4. *For $h > 1$ we have*

$$\mu_d(h) \sim \frac{\mathrm{e}^{h^2}}{h}. \tag{3.13}$$

This is essentially achieved by constant sequences of finite length, corresponding to the Moser's function in the continuous version. The (e) norm determines the fall-off or the length of the sequence, and then optimization of the embedding $\ell^2 \subset \ell^1$ on the finite length forces it to be a constant.

Proof. Since $\mu_d(h)$ is increasing in h, it suffices to show $\mu_d(\sqrt{n}) \sim \mathrm{e}^n/\sqrt{n}$ for all integer n. $\lesssim$ is easily seen by choosing $a = (1, \dots, 1)/\sqrt{n}$, so we consider $\gtrsim$. Suppose by contradiction that for some $\varepsilon \ll 1$, $n \gg 1$ and sequence a we have

$$\|a\|_2 \le 1, \quad \|a\|_1 = \sqrt{n}, \quad \|a\|_{(e)}^2 \le \frac{\varepsilon^2 \mathrm{e}^{2n}}{n}. \tag{3.14}$$

From the last condition we get

$$n \le j \implies |a_j| \lesssim \frac{\varepsilon}{\sqrt{n}} \mathrm{e}^{n-j}, \tag{3.15}$$

and so letting $a'_j = a_j$ for $j \le n$ and $a'_j = 0$ for $j > n$, we get

$$\|a'\|_1 \ge \|a\|_1 - \sum_{j>n} |a_j| \ge \sqrt{n} - \frac{C\varepsilon}{\sqrt{n}}. \tag{3.16}$$

Then by the support of a' we have

$$n - C\varepsilon \le \|a'\|_1^2 = n\|a'\|_2^2 - \sum_{j,k\le n} (a_j - a_k)^2/2 \le n - \sum_{j,k\le n} (a_j - a_k)^2/2, \tag{3.17}$$

hence

$$\sum_{j,k\le n} (a_j - a_k)^2 \lesssim \varepsilon. \tag{3.18}$$

Choose $m \le n$ so that $\min_{j\le n} |a_j| = |a_m|$. Then we get from the above estimate

$$\|a'\|_1 - n|a_m| \le \|a_j - a_m\|_{\ell^1(j\le n)} \le \sqrt{n}\|a_j - a_m\|_{\ell^2(j\le n)} \lesssim \sqrt{n\varepsilon}. \tag{3.19}$$

Combining it with (3.16), we get

$$|a_m| \gtrsim \sqrt{n}/n = 1/\sqrt{n}, \tag{3.20}$$

provided that $\varepsilon > 0$ is small enough. Since $|a_n| \ge |a_m|$, we obtain $\|a\|_{(e)} \gtrsim \mathrm{e}^n/\sqrt{n}$, which yields a contradiction. Hence, we deduce that $\mu_d(h)^2 \sim \mathrm{e}^{2h^2}/h^2$. ☐

4. Proof of (2) and (4) of Theorem 1.5

Proof. To prove (2) of the theorem 1.5, it suffices to show that

$$G(\varphi) = \int_{\mathbb{R}^2} \min(|\varphi|^2, |\varphi|^{-2}) \mathrm{e}^{2|\varphi|^2} dx \lesssim \|\varphi\|_{L^2}^2, \tag{4.1}$$

for all non-negative, radially decreasing $\varphi \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla\varphi\|_{L^2}^2 = 2\pi$. Here we took $K = 1$. Fix such a radial function $\varphi(x) = \varphi(r)$ and let $g(s) = \min(|s|^2, |s|^{-2}) \mathrm{e}^{2|s|^2}$.

Choose $R_0 > 0$ such that $\|\nabla\varphi\|_{L^2(r>R_0)}^2 = 2\pi K_0$, where $K_0 = \kappa \in (2/3, 1)$ is a constant which will be determined later, below (4.25). It is obvious by the scaling invariance that R_0 depends on φ, but we do not need any bound on R_0. Let

$$R := \inf\{r > 0 \mid \varphi(r) \le 1\}, \tag{4.2}$$

then the desired estimate (4.1) in the region $\{r > R\}$ follows from $g(\varphi) \lesssim |\varphi|^2 \le 1$, which is enough if $R = 0$. Otherwise we have $R > 0$ and $\varphi(R) = 1$.

First we dispose of the easiest case $|\varphi(R_0)|^2 \le K_0$, namely the second case in Theorem 3.1. Then we have $R < R_0$ since $K_0 < 1$. Hence in the region $\{r < R\}$, we have by Schwarz,

$$\varphi(r) - 1 = \int_r^R |\varphi_r| dr \le \sqrt{(1-K_0)\log(R/r)}, \tag{4.3}$$

and so, putting $\hat{K} := 1 - K_0/2$,

$$\frac{\varphi(r)^2}{\hat{K}} \le \frac{(\varphi(r)-1)^2}{1-K_0} + \frac{1}{\hat{K} - (1-K_0)} \le \log(R/r) + \frac{2}{K_0}, \tag{4.4}$$

where we used the Young inequality $(a+b)^2/(\alpha+\beta) \le a^2/\alpha + b^2/\beta$. Thus we obtain

$$\int_0^R e^{2\varphi(r)^2} r dr \lesssim \int_0^R (R/r)^{2\hat{K}} r dr \lesssim R^2 \le 2\int_0^R |\varphi(r)|^2 r dr, \tag{4.5}$$

since $\varphi(r) \ge 1$ for $r \le R$. The implicit constants can be chosen respectively as

$$e^{2(2-\kappa)/\kappa}, \quad 1/\kappa. \tag{4.6}$$

Combining the above estimate with that on $r > R$, we conclude the desired (4.1) in the case $\varphi(R_0)^2 \le K_0$.

Therefore, in the rest of proof, we assume that $\varphi(R_0)^2 > K_0$, which is the main case. The region $r > R_0$ is subcritical and easily estimated by the same argument as above. Since the region $r > R$ is already estimated, it suffices to consider the case $R_0 < R$. Then in the region $\{R_0 < r < R\}$, we argue in the same way as (4.3)–(4.5), putting $K' := (1 + K_0)/2 < 1$,

$$\begin{aligned} &\varphi(r) - 1 = \int_r^R |\varphi_r| dr \le \sqrt{K_0 \log(R/r)}, \\ &\frac{\varphi(r)^2}{K'} \le \frac{(\varphi(r)-1)^2}{K_0} + \frac{1}{K'-K_0} \le \log(R/r) + \frac{2}{1-K_0}, \\ &\int_{R_0}^R \mathrm{e}^{2\varphi(r)^2} r dr \lesssim \int_{R_0}^R (R/r)^{2K'} r dr \lesssim R^2 - (R_0)^2 \le 2\int_{R_0}^R |\varphi(r)|^2 r dr. \end{aligned} \tag{4.7}$$

The above implicit constants can be chosen respectively as

$$e^{2(1+\kappa)/(1-\kappa)}, \quad \frac{1}{1-\kappa}. \tag{4.8}$$

Combining the above estimate with that for $r > R$, we obtain

$$\int_{|x|>R_0} g(\varphi(x)) dx \lesssim \int_{|x|>R_0} |\varphi(x)|^2 dx = 2\pi M_0. \tag{4.9}$$

Now we proceed to the main part $r < R_0$. Let

$$\begin{aligned} &R_j = R_0 \,\mathrm{e}^{-j}, \quad h_j = \varphi(R_j), \quad a_j = \sqrt{\int_{R_j}^{R_{j-1}} |\varphi_r|^2 r dr}, \\ &K_j = \int_{R_j}^\infty |\varphi_r|^2 r dr, \quad M_0 = \int_{R_0}^\infty |\varphi|^2 r dr. \end{aligned} \tag{4.10}$$

Then Theorem 3.1 gives

$$\frac{e^{2h_0^2/K_0}}{h_0^2} R_0^2 \lesssim M_0/K_0^2 \sim M_0. \tag{4.11}$$

By the monotone convergence theorem, we may assume that φ is constant on $|x| < R_N$ for some $N \in \mathbb{N}$. Then it suffices to show that

$$\sum_{j=0}^{N} \frac{e^{2h_j^2}}{h_j^2} R_j^2 \lesssim M_0. \tag{4.12}$$

First we derive a bound for each j. By Schwarz inequality, we have

$$h_j - h_{j-1} = \int_{R_j}^{R_{j-1}} |\varphi_r| dr \le a_j, \tag{4.13}$$

and so, using that $K_j = K_{j-1} + a_j^2$, we get that

$$h_j^2 \le (h_{j-1} + a_j)^2 = \frac{K_j h_{j-1}^2}{K_{j-1}} + K_j - \frac{(a_j h_{j-1} - K_{j-1})^2}{K_{j-1}} \le \frac{K_j h_{j-1}^2}{K_{j-1}} + K_j. \tag{4.14}$$

Hence,

$$\frac{h_j^2}{K_j} \le \frac{h_{j-1}^2}{K_{j-1}} + 1.$$

Now, we define H_j and ξ_j by

$$H_0 = h_0, \quad H_j = H_{j-1} + a_j, \quad \xi_j = H_j^2/K_j. \tag{4.15}$$

Then we have

$$h_j \le H_j, \quad \xi_j \le \xi_{j-1} + 1 \le \xi_0 + j, \tag{4.16}$$

which implies that

$$\eta_j := e^{2H_j^2/K_j} R_j^2 = e^{2(\xi_j - j)} R_0^2 \tag{4.17}$$

is monotone decreasing in j. This is not sufficient to sum over j, for which we have to sharpen the above estimate.

The idea is to exploit the room given by the factor $1/K_j$ in the exponential to show that the sum (4.12) is essentially dominated by the first term η_0. The possible growth of the denominator h_j^2 will not play any role for the summability and can be replaced by h_0^2.

Let $J = \{1, \dots, N\}$ and define the sets A and B by

$$A := \{j \in J \mid a_j H_{j-1} \le K_{j-1} - K_j/3\}, \quad B := J \setminus A. \tag{4.18}$$

On the region A, the sequence η_j decays fast enough to be summed without using the factor $1/K_j$, while on B, the decrease of $1/K_j$ is effective enough to supply the summability.

Indeed, for $j \in A$, we have by the same computation as in (4.14),

$$\xi_j \le \xi_{j-1} + 8/9, \tag{4.19}$$

whereas for $j \in B$ we have

$$a_j^2 \gtrsim \frac{K_{j-1}^2}{H_{j-1}^2} = \frac{K_{j-1}}{\xi_{j-1}} \gtrsim \frac{1}{\xi_0 + j - 1}. \tag{4.20}$$

For the sum over A we have

$$\sum_{j\in A} \eta_j \le \sum_{k=1}^{\#A} \mathrm{e}^{-2k/9}\, \eta_0 \lesssim \eta_0 = \mathrm{e}^{2h_0^2/K_0}\, R_0^2, \tag{4.21}$$

and so

$$\sum_{j\in A} \frac{\mathrm{e}^{2H_j^2/K_j}}{h_j^2} R_j^2 \lesssim \frac{\mathrm{e}^{2h_0^2/K_0}}{h_0^2} R_0^2 \lesssim M_0. \tag{4.22}$$

To bound the sum over B, let $a, a+1, \dots, b$ be any maximal consecutive sequence in B. Then for any $j \in \{a, \dots, b\}$ we have

$$\xi_j \le \xi_a + j - a, \quad a_{j+1}^2 \ge \frac{\delta}{\xi_a + j - a}, \tag{4.23}$$

for some fixed constant $\delta > 0$. Now let

$$\zeta_j := K_j(\xi_a + j - a) - j. \tag{4.24}$$

Then we have $H_j^2 - j \le \zeta_j$ and

$$\zeta_j - \zeta_{j-1} = a_j^2(\xi_a + j - 1 - a) + K_j - 1 \ge \delta + \kappa - 1. \tag{4.25}$$

Now we choose κ sufficiently close to 1 so that the right hand side is bigger than $\delta/2$. Then we have

$$\sum_{j=a}^{b} \mathrm{e}^{2H_j^2}\, R_j^2 \le \sum_{j=a}^{b} \mathrm{e}^{2\zeta_j}\, R_0^2 \le \sum_{k=0}^{b-a} \mathrm{e}^{2\zeta_b - \delta k}\, R_0^2 \lesssim \mathrm{e}^{2\zeta_b}\, R_0^2 \le \eta_a \le \eta_{a-1}, \tag{4.26}$$

and so

$$\sum_{j\in B} \mathrm{e}^{2H_j^2}\, R_j^2 \lesssim \eta_0 + \sum_{j\in A} \eta_j \lesssim \eta_0, \tag{4.27}$$

which implies, together with (4.22), the desired estimate in (2).

Finally we prove (4) from (3) using the boundedness (B) proved above. By the radial Sobolev inequality we have,

$$|\varphi(r)|^2 \lesssim \|\varphi\|_{L^2} \|\varphi_r\|_{L^2}/r, \tag{4.28}$$

and hence, $\varphi_n(r) \to 0$ as $r \to \infty$ uniformly in n. Moreover,

$$[\varphi_n]_{R_0}^{R_1} = \int_{R_0}^{R_1} \partial_r \varphi_n dr \tag{4.29}$$

converges as $n \to \infty$ for any $0 < R_0 < R_1 < \infty$. Hence $\varphi_n(r) \to \varphi(r)$ at every $r \in (0, \infty)$. By the radial Sobolev inequality and that $g(\varphi) = o(|\varphi|^2)$ as $\varphi \to 0$, for any $\varepsilon > 0$ there is $R > 0$ independent of n such that

$$\int_R^\infty g(\varphi_n) r dr \le \int_R^\infty \varepsilon |\varphi_n|^2 r dr \lesssim \varepsilon \|\varphi_n\|_{L^2}^2. \tag{4.30}$$

By (2) and the fact that $g(\varphi) = o(\exp(2|\varphi|^2)|\varphi|^{-2})$ as $|\varphi| \to \infty$, there is $L > 1$ independent of n such that

$$\int_{|\varphi_n|>L} g(\varphi_n)dx \le \int_{|\varphi_n|>L} \varepsilon \exp(2|\varphi_n|^2)|\varphi_n|^{-2}dx \lesssim \varepsilon\|\varphi_n\|_{L^2}^2. \tag{4.31}$$

Now define g^L by

$$g^L(u) = \begin{cases} g(u) & (|u| \le L) \\ g(Lu/|u|) & (|u| > L) \end{cases}. \tag{4.32}$$

Then we have

$$\overline{\lim_{n\to\infty}} |G(\varphi_n) - G(\varphi)| \lesssim \varepsilon + \overline{\lim_{n\to\infty}} \int_{|x|<R} |g^L(\varphi_n) - g^L(\varphi)|dx = \varepsilon, \tag{4.33}$$

by the dominated convergence theorem. This ends the proof of (4). □

5. Elliptic equation

Now we consider the existence of solutions for the nonlinear elliptic equation

$$-\Delta Q + cQ = f'(Q), \tag{5.1}$$

for $Q(x) : \mathbb{R}^2 \to \mathbb{R}$, $c > 0$, and exponential nonlinearity f. The existence of the ground state, namely the positive radial solution, with the least energy among all solutions, has been studied by the ODE technique in [3, 18] including supercritical nonlinearity, and by the variational technique in [6] for subcritical nonlinearity, and in [9] including the critical nonlinearity.

Combining the above precised Trudinger-Moser inequality with the argument in [22], we deduce the following. Let D be the operator defined by $Df(u) = uf'(u)$.

Theorem 5.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(u) = o(u^2)$ as $u \to 0$, $(D-2)f \ge \varepsilon f \ge 0$ for some $\varepsilon > 0$, and that there exists $\kappa_0 \ge 0$ such that for all $\kappa_+ > \kappa_0 > \kappa_-$,*

$$\lim_{|u|\to\infty} Df(u)\,\mathrm{e}^{-\kappa_+|u|^2} = 0, \quad \lim_{|u|\to\infty} f(u)\,\mathrm{e}^{-\kappa_-|u|^2} = \infty, \tag{5.2}$$

and $\lim_{|u|\to\infty} Df(u)/f(u) = \infty$. Then there exists $c_* \in (0, \infty]$ such that there is a positive radial solution Q_c of (5.1) for each $c \in (0, c_*]$, which has the least energy among all the solutions. Moreover, $c_* = C_{f,\kappa_0}$ in (2) of Theorem 1.5 when it is finite, while $c_* = \infty$ is equivalent to

$$\overline{\lim_{|u|\to\infty}} f(u)\,\mathrm{e}^{-\kappa_0|u|^2}\,|u|^2 = \infty. \tag{5.3}$$

In addition, we have

$$\kappa_0\|\nabla Q_c\|_{L^2}^2 \le 4\pi, \tag{5.4}$$

where the equality holds if and only if $c = c_*$.

The subcritical nonlinearity is covered by taking $\kappa_0 = 0$, while the supercritical nonlinearity such as those in [18, Theorem 6] is also covered by taking $\kappa_0 > 0$. Although it is not easy to fully compare the conditions in [3, 18] with our variational condition, there is certainly a new case, that is when

$$f(u) \gg e^{\kappa_0|u|^2}/|u|^2, \quad f'(u) = o(e^{\kappa_0|u|^2}) \quad (|u| \to \infty). \tag{5.5}$$

Such nonlinearity has been investigated for the Dirichlet problem

$$-\Delta u = f'(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{5.6}$$

on a bounded domain Ω, including the threshold case $f'(u) = e^{\kappa_0|u|^2}/u$, see [13]. The ground state for $c = 1$ was constructed in [9] under the conditions $|f'(u)| \lesssim e^{4\pi u^2}$, $(D-2)f \geq \varepsilon f$ and, for some $p \in (2, \infty)$,

$$Df(u) \geq \frac{p}{2}|u|^p \left(\frac{\varepsilon}{2+\varepsilon}\right)^{1-p/2} \inf_{0\neq u\in H^1(\mathbb{R}^2)} \frac{\|u\|_{H^1}^p}{\|u\|_{L^p(\mathbb{R}^2)}^p}. \tag{5.7}$$

It seems difficult to compare this and our condition $1 \leq C_{f,\kappa_0}$.

We omit a proof of the above theorem, for it was completely proved in [22] except for the necessary and sufficient condition for $c_* = \infty$, but it was stated in the form (2) in Theorem 1.5. Note that in [22] we chose $c = \min(1, c_*)$ (with $c_* = C^*_{TM}$ in the notation therein), simply because the mass coefficient of the Klein-Gordon equation, corresponding to c in the static equation (5.1), was fixed to 1, but there was no reason for choosing 1. In other words, we could obviously choose any $c \in (0, c_*)$.

It is worth noting that the compactness (4) of Theorem 1.5 does not seem useful for the above problem, but the compactness on a minimizing sequence comes from the superpower growth $Df(u)/f(u) \to \infty$ together with a variational constraint, see [22]. A related fact is that the best constants in (1.3) and in (1.5) are not attained by any concentrating sequences, see [10, 15, 34].

A natural question is what happens if $c > c_*$. We do not have any existence or non-existence result in this range, but the Pohozaev identity yields

$$\langle -\Delta Q + cQ + f'(Q) | x \cdot \nabla Q \rangle = \int_{\mathbb{R}^2} [c|Q|^2 - f(Q)]dx, \tag{5.8}$$

and so any solution of (5.1) must have super-critical kinetic energy $\kappa_0\|\nabla Q\|_2^2 > 4\pi$ for the Trudinger-Moser inequality. This implies that the constrained minimization in terms of the energy $E_c(\varphi) := \int_{\mathbb{R}^2} \frac{1}{2}[|\nabla\varphi|^2 + c|\varphi|^2] - f(\varphi)dx$

$$m_c := \inf\{E_c(\varphi) \mid 0 \neq \varphi \in H^1(\mathbb{R}^2),\ \langle E_c'(\varphi)|x\cdot\nabla\varphi\rangle = 0\} \tag{5.9}$$

does not have any minimizer, but $m_c = m_{c_*} = 2\pi/\kappa_0 = E_{c_*}(Q_{c_*})$ for any $c > c_*$. See [22] for more detail, including other constraints. This leads to a different dynamical picture on the threshold energy m_c for the nonlinear Klein-Gordon equation

$$\ddot{u} - \Delta u + cu = f'(u), \tag{5.10}$$

from the standard power nonlinearity $f'(u) = u^p$. See [23] for the detail.

6. From TM with the exact growth to TM with H^1

In this section we show how one can derive Proposition 1.3 from our inequality (1.7) using Hölder only. Let $u \in H^1(\mathbb{R}^2)$ satisfy $\|\nabla u\|_{L^2} \leq 1$. First observation is that by Taylor expansion of exp, there is a constant $C_0 > 0$ such that

$$\forall n \in \mathbb{N}, \quad \int_{\mathbb{R}^2} (4\pi|u|^2)^n dx \leq C_0(n+1)!\|u\|_{L^2}^2, \tag{6.1}$$

hence there is a constant $C_1 > 0$ such that for any $p \geq 1$

$$\||u|^2\|_{L^p} \leq C_1 p \|u\|_{L^2}^{2/p}, \tag{6.2}$$

which is extended to noninteger p by Hölder.

Next, if $\|u\|_{H^1}^2 \leq 1$ then for some $\theta \in (0,1)$ we have

$$\|u\|_{L^2}^2 \leq \theta, \quad \|\nabla u\|_{L^2}^2 \leq 1-\theta. \tag{6.3}$$

If we can take $\theta \geq 1/2$, then (1.5) follows from (1.4) applied to $\sqrt{2}u$ with $\alpha = 2\pi$. So we may assume that $\theta < 1/2$. Let $A := \{x \in \mathbb{R}^2 \mid |u(x)| \geq 1\}$. Then by Hölder

$$\int_A e^{4\pi|u|^2} dx \leq \left[\int_A \frac{e^{4\pi|u|^2/(1-\theta)}}{|u|^2/(1-\theta)} dx\right]^{1-\theta} \||u|^2/(1-\theta)\|_{L^{\frac{1-\theta}{\theta}}(A)}^{1-\theta}, \tag{6.4}$$

where the first term on the right is bounded by (1.7)

$$\int_A \frac{e^{4\pi|u|^2/(1-\theta)}}{|u|^2/(1-\theta)} dx \leq C_2 \frac{\|u\|_{L^2}^2}{1-\theta} \leq C_2 \frac{\theta}{1-\theta}, \tag{6.5}$$

while the second term is bounded by (6.1)

$$\left\|\frac{|u|^2}{1-\theta}\right\|_{L^{\frac{1-\theta}{\theta}}(A)} \leq C_1 \frac{1-\theta}{\theta} \left\|\frac{u}{\sqrt{1-\theta}}\right\|_{L^2}^{\frac{2\theta}{1-\theta}} \leq C_1 \left[\frac{1-\theta}{\theta}\right]^{\frac{1-2\theta}{1-\theta}}. \tag{6.6}$$

Putting them into (6.4), we obtain

$$\int_A e^{4\pi|u|^2} dx \lesssim \left[\frac{\theta}{1-\theta}\right]^{1-\theta} \left[\frac{1-\theta}{\theta}\right]^{1-2\theta} \leq \left[\frac{\theta}{1-\theta}\right]^{\theta}, \tag{6.7}$$

which is bounded for $0 < \theta < 1/2$. The estimate on $\mathbb{R}^2 \setminus A$ is trivial.

Acknowledgments

The first author is thankful to Professor Yoshio Tsutsumi and all members of the Math Department at Kyoto University for their very generous hospitality. The authors are grateful to the referees for their careful reading and pointing out a gap in the original version (the second case of (3.1) was missing).

References

[1] S. Adachi and K. Tanaka, *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 2051–2057.

[2] Adimurthi and O. Druet, *Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality*, Comm. Partial Differential Equations **29** (2004), no. 1-2, 295–322.

[3] F. V. Atkinson and L. A. Peletier, *Ground states and Dirichlet problems for $-\Delta u = f(u)$ in R^2*, Arch. Rational Mech. Anal. **96** (1986), 147–165.

[4] H. Bahouri and P. Gérard, *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math. **121** (1999), 131–175.

[5] H. Bahouri, M. Majdoub, and N. Masmoudi. On the lack of compactness in the 2D critical Sobolev embedding. *J. Funct. Anal.*, 260(1):208–252, 2011.

[6] H. Berestycki, T. Gallouet, and O. Kavian, *Équations de champs scalaires euclidiens non-linéaires dans le plan*, C. R. Acad. Sci. Paris **297** (1983), 307–310.

[7] H. Berestycki and P.L. Lions, *Nonlinear Scalar field equations, I. Existence of ground state*, Arch. Rat. Mech. Anmal., **82** (1983), 313–346.

[8] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations, **5** (1980), 773–789.

[9] D. M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in R^2*, Comm. Partial Differential Equations **17** (1992), no. 3-4, 407–435.

[10] L. Carleson and A. Chang, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sc. Math. **110** (1986), 113–127.

[11] A. Cianchi, *A sharp embedding theorem for Orlicz-Sobolev spaces*, Indiana Univ. Math. J., **45**(1996), 39–65.

[12] D. G. De Figueiredo and B. Ruf, *Existence and Non-Existence of Radial Solutions for Elliptic Equations with Critical Exponent in $\mathbb{R}^2$*. Comm. Pure and Appl. **48** (1995), 639–655.

[13] D. G. De Figueiredo, J. M. do Ó and B. Ruf, *On an inequality by N. Trudinger and J. Moser and related elliptic equation*, Comm. Pure Appl. Math. **55** (2002) 1–18.

[14] D. E. Edmunds, R. Kerman and L. Pick, *Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms*, J. Funct. Anal., **170** (2000), 307–355.

[15] M. Flucher, *Extremal functions for the Trudinger-Moser inequality in 2 dimensions*, Comm. Math. Helv. **67** (1992), 471–479.

[16] P. Gérard, *Oscillations and concentration effects in semilinear dispersive wave equations*, J. Funct. Anal. **133** (1996), 50–68.

[17] P. Gérard, *Description du défaut de compacité de l'injection de Sobolev*, ESAIM Control Optim. Calc. Var. 3 (1998), 213–233 (electronic, URL: http://www.emath.fr/cocv/).

[18] M. García-Huidobro, R. Manásevich, J. Serrin, M. Tang and C. S. Yarur, *Ground states and free boundary value problems for the n-Laplacian in n dimensional space.* J. Funct. Anal. **172** (2000), no. 1, 177–201.

[19] K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand., **45** (1979), 77–102.

[20] J. A. Hempel, G. R. Morris and N. S. Trudinger, *On the sharpness of a limiting case of the Sobolev imbedding theorem* , Bull. Austral. Math. Soc., **3** (1970), 369–373.

[21] S. Ibrahim, M. Majdoub and N. Masmoudi, *Double logarithmic inequality with a sharp constant*, Proc. Amer. Math. Soc. **135** (2007), no. 1, 87–97.

[22] S. Ibrahim, N. Masmoudi and K. Nakanishi, *Scattering threshold for the focusing nonlinear Klein-Gordon equation*, Analysis & PDE. **4** (2011), no. 3, 405–460.

[23] S. Ibrahim, N. Masmoudi and K. Nakanishi, *Threshold solutions in the case of mass-shift for the critical Klein-Gordon equation*, Trans. Amer. Math. Soc. **366** (2014), 5653–5669.

[24] S. Jaffard, *Analysis of the lack of compactness in the critical Sobolev embeddings*, J. Funct. Anal., **161** (1999), 384–396.

[25] V. I. Judovič. Some estimates connected with integral operators and with solutions of elliptic equations. *Dokl. Akad. Nauk SSSR*, 138:805–808, 1961.

[26] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana, **1** (1985), 45–121.

[27] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I.*, Rev. Mat. Iberoamericana 1 (1985), no. 1, 145–201.

[28] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, I*, Ann. Inst. H. Poincare Anal. Non Linéaire 1 (1984), 109–145.

[29] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **1** (1984), 223–283.

[30] J. Malý and L. Pick, *An elementary proof of sharp Sobolev embeddings*, Proc. Amer. Math. Soc., **130** (2002), 555–563.

[31] J. Moser, *A sharp form of an inequality of N. Trudinger*, Ind. Univ. Math. J. **20**(1971), pp. 1077-1092.

[32] S. I. Pohozhaev, *The Sobolev embedding in the case $pl = n$*, Proc. Tech. Sci. Conf. on Adv. Sci. Research 1964-1965, Mathematics Section, 158-170, Moskov. 'Energet. Inst. Moscow, 1965.

[33] M.-M. Rao and Z.-D. Ren, *Applications of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, **250**, Marcel Dekker Inc., 2002.

[34] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^2$, J. Funct. Anal. **219** (2005), no. 2, 340–367.

[35] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, Vol.3, Berlin, 1996.

[36] W.-A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., **55** (1977), 149–162.

[37] M. Struwe, *Critical points of embeddings of* $H_0^{1,n}$ *into Orlicz spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **5** (1988), 425–464.

[38] N.S. Trudinger, *On imbedding into Orlicz spaces and some applications*, J. Math. Mech. **17**(1967), pp. 473-484.

Department of Mathematics and Statistics, University of Victoria, PO Box 3060 STN CSC, Victoria, BC, V8P 5C3, Canada
E-mail address: `ibrahim@math.uvic.ca`

New York University, The Courant Institute for Mathematical Sciences.
E-mail address: `masmoudi@courant.nyu.edu`

Department of Mathematics, Kyoto University
E-mail address: `n-kenji@math.kyoto-u.ac.jp`