Appearance of Boulware-Deser ghost in bigravity with doubly coupled matter

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We discuss the ghost freeness in the case when we add matter coupled to two metrics to the ghost-free bigravity. In this paper we show that the Boulware-Deser ghost generally revives in the presence of doubly coupled matter and that ghost freeness strongly restricts the model of kinetically doubly coupled matter. This result may anticipate difficulties in the attempt to derive the ghost-free bigravity as a low-energy effective theory, starting with a model applicable at high energies.

I.   INTRODUCTION

Bigravity, the theory that has two dynamical metrics interacting with each other through non-derivative interaction terms, has been investigated since the proposal in Ref. [1]. In the presence of the mass of graviton which derives from the non-derivative interaction terms, however, it is known that an additional degree of freedom generally appears and that it has a wrong-sign kinetic term. Such degree of freedom causes instability at the quantum level and is called the Boulware-Deser (BD) ghost [2]. Recently, de Rham, Gabadadze and Tolley discovered the unique ghost-free[1] non-derivative interaction in the case with one non-dynamical fiducial metric, which is called dRGT massive gravity [3, 4] (see also [5]). References [6–8] showed that dRGT massive gravity is certainly ghost free and that the proof of ghost-freeness can be extended to the case where both metrics are dynamical. Reference [6] also argued that the case in which a matter field couples to both metrics is allowed, although such a matter field would in general violate the equivalence principle.

This finding makes it possible to study healthy bimetric theory [9]. Many works about the ghost-free bigravity have been done, for example, in the cosmological context for the case where each matter couples only to either metric [10–15] (see also [16]) and for the case where matter couples to both metrics [17–19]. In Refs. [11, 14, 15] they studied the nature of the cosmological solutions of the ghost-free bigravity and found that even in the healthy branch where Higuchi ghost is absent, a gradient instability appears in the high energy regime compared with the mass scale introduced in the Lagrangian in the matter-dominated or radiation-dominated era. Therefore, if the ghost-free bigravity can describe the real universe, it should exceed the range of validity in the high energy regime and it must be extended to some more fundamental theory there. Such an extension of the ghost-free bigravity is attempted in Ref. [20], where the ghost free multigravity is related to higher dimensional general relativity with a discrete extra dimension, and in Ref. [21], where the ghost-free bigravity is embedded in the DGP 2-brane model. In this context, we can naturally consider matter that couples to both metrics by introducing a five-dimensional matter field.

Despite the above discussion, in this paper we claim that the BD ghost generally appears when we add matter coupled to both metrics by hand to the ghost-free bigravity. This is because, when the matter has two kinetic terms which couple to respective metrics, the conjugate momentum of the matter field depends on both lapse functions and the kinetic term written by the conjugate momentum becomes nonlinear in two lapse functions. Considering perturbations on Friedmann and anisotropic background solutions, we investigate in which case an extra degree of freedom is present and whether the extra degree of freedom is a ghost mode or not. Consequently, when matter has two kinetic terms corresponding to two metrics, we find that an extra degree of freedom appears and that there is no mechanism to avoid the mode to become a ghost mode in the limit of Minkowski solution in the healthy branch.

This paper is organized as follows. In Sec. 2 we present a brief summary and a detailed follow-up calculation of the proof that has been given by Hassan and Rosen [6–8]. Here we consider only the gravitational sector and confirm that the ghost-free bigravity really has four Lagrange multipliers that remain unspecified by the consistency conditions of the time evolution of the constraints, corresponding to the gauge degrees of freedom,

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[1] Here and in the following "ghost-free" strictly means that the model does not have the BD ghost, without excluding the possible existence of another type of ghost.

which was not explicitly shown in Refs. [6, 8] by the calculation of the Poisson brackets among the Hamiltonian and momentum constraints. In Sec. 3, we consider the coupling to matter and investigate in which case the BD ghost appears in doubly coupled matter models. Section. 4 is devoted to the summary of this paper.

II. GRAVITATIONAL SECTOR

In this section we give detailed calculations necessary in prooving that the Boulware-Deser ghost is really absent in the so-called ghost-free bigravity. First, we introduce the setup of the model following the notation in Refs. [6–8] and give a brief summary of the proof given in the above References. The action that we consider is given by

$$S = \int d^4x \sqrt{-\det g} \left[\frac{M_g^2}{2} R^{(g)} + 2m^2 M_{\text{eff}}^2 \sum_{n=0}^{4} c_n e_n \left(\sqrt{g^{-1}f} \right) \right] + \frac{M_f^2}{2} \int d^4x \sqrt{-\det f} R^{(f)} , \qquad (\text{II.1})$$

where $g_{\mu\nu}$ and $f_{\mu\nu}$ are, respectively, the physical and the hidden metrics. M_g is the 4 dimensional Planck mass for $g_{\mu\nu}$, M_f is that for $f_{\mu\nu}$, and M_{eff} is defined as

$$M_{\text{eff}}^2 = \left(\frac{1}{M_g^2} + \frac{1}{M_f^2} \right)^{-1} . \qquad (\text{II.2})$$

We also introduce m and c_n (corresponding to β_n in Ref. [6–8]) as model parameters. The interaction terms between $g_{\mu\nu}$ and $f_{\mu\nu}$ are given by

$$e_0 = 1 , \quad e_1 = [Y] , \quad e_2 = [Y]^2 - [Y^2] , \quad e_3 = [Y]^3 - 3[Y][Y^2] + 2[Y^3] ,$$
$$e_4 = [Y]^4 - 6[Y]^2[Y^2] + 8[Y][Y^3] + 3[Y^2]^2 - 6[Y^4] , \qquad (\text{II.3})$$

where we have introduced $Y_\nu^\mu = \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$ and $[Y^n] = \text{Tr}(Y^n)$.

In this paper we count the number of degrees of freedom in the ghost-free bigravity defined by Eq. (II.1) using Hamiltonian formulation. To do Hamiltonian analysis, we decompose the metrics discriminating temporal and spatial components as

$$N^{-2} = -g^{00} , \quad N_i = g_{0i} , \quad \gamma_{ij} = g_{ij} ,$$
$$L^{-2} = -f^{00} , \quad L_i = f_{0i} , \quad {}^3f_{ij} = f_{ij} . \qquad (\text{II.4})$$

Using these variables, the Lagrangian of the ghost-free bigravity (II.1) becomes

$$\mathcal{L} = M_g^2 \left[\pi^{ij} \partial_t \gamma_{ij} + N R_{(g)}^0 + N_i R_{(g)}^i \right] + 2m^2 M_{\text{eff}}^2 \sqrt{\det\gamma} \, N \sum_{n=0}^{4} c_n e_n \left(\sqrt{g^{-1}f} \right)$$
$$+ M_f^2 \left[p^{ij} \partial_t \, {}^3f_{ij} + L R_{(f)}^0 + L_i R_{(f)}^i \right] , \qquad (\text{II.5})$$

up to irrelevant total derivative terms, where π^{ij} and p^{ij} are the momenta canonically conjugate to γ_{ij} and ${}^3f_{ij}$, respectively, and

$$R_0^{(g)} = \sqrt{\det\gamma} \, {}^3R_{(\gamma)} + \frac{1}{\sqrt{\det\gamma}} \left(\frac{1}{2} \pi_i^i \pi_j^j - \pi^{ij} \pi_{ij} \right) , \qquad (\text{II.6})$$

$$R_i^{(g)} = 2\sqrt{\det\gamma} \, \gamma_{ij} \nabla_k^{(g)} \left(\frac{\pi^{jk}}{\sqrt{\det\gamma}} \right) . \qquad (\text{II.7})$$

Here $\nabla_i^{(g)}$ and ${}^3R_{(\gamma)}$ are the covariant differenciation and the Ricci scalar defined by γ_{ij}, respectively. $R_0^{(f)}$ and $R_i^{(f)}$ are defined in the same way as $R_0^{(g)}$ and $R_i^{(g)}$ using ${}^3f_{ij}$ and p^{ij} instead of γ_{ij} and π^{ij}.

Obviously this Lagrangian is nonlinear in terms of the lapse functions N and L and the shift vectors N^i and L^i, and it is difficult to count the number of degrees of freedom using these variables. In order to make the Lagrangian linear in N, L and L^i, Hassan and Rosen [6] defined a new shift-like vector n^i as

$$N^i - L^i = \left(L\delta_j^i + ND_j^i \right) n^j \,, \tag{II.8}$$

where $N^i = \gamma^{ij} N_j$, $L^i = {}^3f^{ij} L_j$ and the matrix D is determined by the condition,

$$\sqrt{x} D = \sqrt{(\gamma^{-1} - Dnn^T D^T)\, {}^3f} \,, \tag{II.9}$$

where,

$$x := 1 - n^i\, {}^3f_{ij} n^j \,. \tag{II.10}$$

The condition that determines D imposes an important property,

$$f_{ik} D_j^k = f_{jk} D_i^k \,. \tag{II.11}$$

We often use this property in the following calculation without mentioning it.

Rewriting the shift vector N^i using n^i, the Lagrangian becomes

$$\mathcal{L} = M_g^2 \pi^{ij} \partial_t \gamma_{ij} + M_f^2 p^{ij} \partial_t\, {}^3f_{ij} - \mathcal{H}_0 \,, \tag{II.12}$$

where

$$\mathcal{H}_0 := - L^i \left(M_g^2 R_i^{(g)} + M_f^2 R_i^{(f)} \right) - L \left(M_f^2 R^{0(f)} + M_g^2 n^i R_i^{(g)} + 2m^2 M_{\text{eff}}^2 \sqrt{\det\gamma}\, U \right)$$
$$- N \left(M_g^2 R^{0(g)} + M_g^2 R_i^{(g)} D_j^i n^j + 2m^2 M_{\text{eff}}^2 \sqrt{\det\gamma}\, V \right) \,. \tag{II.13}$$

U and V are defined as

$$U := c_1 \sqrt{x} + c_2 \left[\sqrt{x}^2 D_i^i + n^i\, {}^3f_{ij} D_k^j n^k \right]$$
$$+ c_3 \left[\sqrt{x} \left(D_l^l n^i\, {}^3f_{ij} D_k^j n^k - D_l^i n^l\, {}^3f_{ij} D_k^j n^k \right) + \frac{1}{2}\sqrt{x}^3 \left(D_i^i D_j^j - D_j^i D_i^j \right) \right] + c_4 \frac{\sqrt{\det {}^3f}}{\sqrt{\det\gamma}} \,, \tag{II.14}$$

$$V := c_0 + c_1 \sqrt{x} D_i^i + \frac{1}{2} c_2 \sqrt{x}^2 \left[D_i^i D_j^j - D_j^i D_i^j \right] + \frac{1}{6} c_3 \sqrt{x}^3 \left[D_i^i D_j^j D_k^k - 3D_i^i D_k^j D_j^k + 2D_j^i D_k^j D_i^k \right] \,. \tag{II.15}$$

Variation of the Lagrangian (II.12) with respect to n^k leads to the equations of motion, which determines n^k. At this point the Lagrangian is already linear in N, L and L^i. Calculating the derivatives of $\mathcal{H}_0$ and $\mathcal{C}$ with respect to n^k, we obtain

$$\frac{\partial \mathcal{H}_0}{\partial n^k} = L \mathcal{C}_k \,, \tag{II.16}$$

$$\frac{\partial \mathcal{C}}{\partial n^k} = \mathcal{C}_i \frac{\partial \left(D_j^i n^j \right)}{\partial n^k} \,, \tag{II.17}$$

where

$$\mathcal{C}_i := R_i^{(g)} - 2\tilde{m}^2 \sqrt{\det\gamma}\, \frac{n^l\, {}^3f_{lj}}{\sqrt{x}} \tilde{V}_i^j \,, \tag{II.18}$$

$$\tilde{V}_i^j := \frac{1}{\sqrt{x}} \frac{\partial V}{\partial D_j^i}$$
$$= c_1 \delta_i^j + c_2 \sqrt{x} \left[\delta_i^j D_k^k - D_i^j \right] + c_3 \sqrt{x}^2 \left[\frac{1}{2} \left(D_m^m D_n^n - D_n^m D_m^n \right) \delta_i^j + D_k^j D_i^k - D_i^j D_l^l \right] \,, \tag{II.19}$$

with $M_g^2 \tilde{m}^2 := M_{\text{eff}}^2 m^2$ ($\tilde{V}_i^j$ is defined differently in [6]). Here we used

$$\frac{\partial \left(\sqrt{\det \gamma}\, V \right)}{\partial n^k} = -\frac{\sqrt{\det \gamma}}{\sqrt{x}} n^l \tilde{V}_l^m\, {}^3 f_{mn} \frac{\partial \left(D_j^n n^j \right)}{\partial n^k}\,, \qquad (\text{II.20})$$

$$\frac{\partial \left(\sqrt{\det \gamma}\, U \right)}{\partial n^k} = -\frac{\sqrt{\det \gamma}}{\sqrt{x}} \tilde{V}_k^i\, {}^3 f_{ij} n^j\,. \qquad (\text{II.21})$$

Therefore the equation of motion of n^k becomes

$$\frac{\partial \mathcal{L}}{\partial n^k} = \mathcal{C}_i \left[L \delta_k^i + N \frac{\partial \left(D_j^i n^j \right)}{\partial n^k} \right]\,. \qquad (\text{II.22})$$

The matrix in the square brackets in Eq. (II.22), which is the Jacobian of the transformation (II.8), is invertible in general. Therefore the equations of motion of n^i become

$$\mathcal{C}_i = 0\,. \qquad (\text{II.23})$$

These are independent of N, L and L^i (and p^{ij}), and hence we can solve these equations to describe n^i in terms of γ_{ij}, π^{ij} and ${}^3 f_{ij}$. Substituting $n^i \left(\gamma_{ij}, \pi^{ij}, {}^3 f_{ij} \right)$ to the Lagrangian (II.12), we find that the Lagrangian is still linear in N, L, and L^i. We obtain three primary constraints

$$\mathcal{C} = M_g^2 R^{0(g)} + M_g^2 R_i^{(g)} D_j^i n^j + 2 m^2 M_{\text{eff}}^2 \sqrt{\det \gamma}\, V\,, \qquad (\text{II.24})$$

$$C^L = M_f^2 R_0^{(f)} + M_g^2 n^k R_k^{(g)} + 2 m^2 M_{\text{eff}}^2 \sqrt{\det \gamma}\, U\,, \qquad (\text{II.25})$$

$$C_i^L = M_g^2 R_i^{(g)} + M_f^2 R_i^{(f)}\,, \qquad (\text{II.26})$$

by taking the variations of this Lagrangian with respect to N, L, and L^i. With Eqs (II.23), Eqs. (II.16) and (II.17) imply that the constraints $\mathcal{C}$ and C^L vanish under the differentiation with respect to n^i, and hence we can treat n^i as if it were fixed when we calculate the variations of the primary constraints.

In Ref. [6], it was claimed that in the case of dRGT massive gravity, where $f_{\mu\nu}$ is fixed and non-dynamical, the variation of the Lagrangian with respect to N gives a constraint $\mathcal{C} = 0$ and its consistency condition gives another constraint, whose consistency condition determines the Lagrange multiplier N. Hence, there are two constraints and Boulware-Deser ghost is absent. They also claimed that in the bigravity case this procedure can be simply extended and these two constraints, along with four constraints C^L and C_i^L produced by the variation with respect to L and L^i, respectively, and four gauge fixing conditions, reduce 24 variables, γ_{ij}, π^{ij}, ${}^3 f_{ij}$ and p^{ij}, to 14 degrees of freedom. This counting is correct, but it is not confirmed whether all the Poisson brackets of the primary constraints really vanish except for the one between $\mathcal{C}$ and C^L. Although this might be almost obvious due to the existence of general coordinate invariance in the ghost-free bigravity, we explicitly calculate the Poisson brackets between the primary constraints to confirm that the proof of the absence of ghost in dRGT massive gravity can be extended to bigravity.

In order to calculate the Poisson brackets, we have to calculate the partial differentiations of U and V with respect to γ_{ij} and ${}^3 f_{ij}$ as well as n^i. For this purpose, the following relations as given in Eqs. (II.20) and (II.21) for n^i are useful:

$$\frac{\partial \left(\sqrt{\det \gamma}\, V \right)}{\partial \gamma_{ab}} = -\frac{\sqrt{\det \gamma}}{\sqrt{x}} \left[n^l \tilde{V}_l^m\, {}^3 f_{mn} \frac{\partial \left(D_j^n n^j \right)}{\partial \gamma_{ab}} - \frac{1}{2} \left(\sqrt{x} V \gamma^{ab} - \gamma^{al} (D^{-1})_l^m \tilde{V}_m^n\, {}^3 f_{nk} \gamma^{kb} \right) \right]\,, \qquad (\text{II.27})$$

$$\frac{\partial \left(\sqrt{\det \gamma}\, V \right)}{\partial\, {}^3 f_{ab}} = -\frac{\sqrt{\det \gamma}}{\sqrt{x}} \left[n^l \tilde{V}_l^m\, {}^3 f_{mn} \frac{\partial \left(D_j^n n^j \right)}{\partial\, {}^3 f_{ab}} - \frac{1}{2} x \tilde{V}_l^a\, {}^3 f^{lm} D_m^b \right]\,, \qquad (\text{II.28})$$

$$n^k \gamma_{ik} \frac{\partial \left(\sqrt{\det \gamma}\, U \right)}{\partial \gamma_{ij}} = \frac{\sqrt{\det \gamma}}{2\sqrt{x}} n^l\, {}^3 f_{li} \tilde{V}_k^i \left({}^3 f^{kj} - n^k n^j \right)\,, \qquad (\text{II.29})$$

$$\frac{\partial U}{\partial \gamma_{ij}} \gamma_{jk} + \frac{\partial U}{\partial\, {}^3 f_{ij}}\, {}^3 f_{jk} = -\frac{1}{2\sqrt{x}} n^i \tilde{V}_k^m\, {}^3 f_{ml} n^l\,. \qquad (\text{II.30})$$

We also use the Poisson brackets between $R_0^{(g)}$ and $R_i^{(g)}$.

$$M_g^2\{R_0^{(g)}(x)\,,\,R_0^{(g)}(y)\} = -\left[R_{(g)}^i(x)\frac{\partial}{\partial x^i}\delta^3(x-y) - R_{(g)}^i(y)\frac{\partial}{\partial y^i}\delta^3(x-y)\right],\qquad\text{(II.31)}$$

$$M_g^2\{R_0^{(g)}(x)\,,\,R_i^{(g)}(y)\} = -R_{(g)}^0(y)\frac{\partial}{\partial x^i}\delta^3(x-y)\,,\qquad\text{(II.32)}$$

$$M_g^2\{R_i^{(g)}(x)\,,\,R_j^{(g)}(y)\} = -\left[R_j^{(g)}(x)\frac{\partial}{\partial x^i}\delta^3(x-y) - R_i^{(g)}(y)\frac{\partial}{\partial y^j}\delta^3(x-y)\right].\qquad\text{(II.33)}$$

The same relations hold for $R_0^{(f)}$ and $R_i^{(f)}$ with M_g^2 replaced with M_f^2.

Next, we present the calculations of Poisson brackets between three primary constraints, $\{C^L\,,\,C^L\}$, $\{C_i^L(x)\,,\,\mathcal{C}(y)\}$, $\{C_i^L\,,\,C^L\}$, and $\{\mathcal{C}\,,\,C^L\}$. We do not show the other Poisson brackets, $\{\mathcal{C}\,,\,\mathcal{C}\}$ and $\{C_i^L\,,\,C_j^L\}$, because $\{\mathcal{C}\,,\,\mathcal{C}\}$ has been already confirmed to be proportional to $\mathcal{C}$ with the aid of $\mathcal{C}_i = 0$ in Ref. [8] and $\{C_i^L\,,\,C_j^L\}$ is obviously proportional to the constraint C_i^L itself by use of Eq. (II.33). Notice that we do not use $C^L = 0$ to confirm $\{\mathcal{C}\,,\,\mathcal{C}\} = 0$ and $\{C_i^L\,,\,C_j^L\} = 0$.

- $\{C^L\,,\,C^L\}$

  Here, we should note that U depends on π^{ij} only through n^k. Since n^k can be treated as fixed in C^L owing to $\mathcal{C}_i = 0$, the contribution of $\partial\left(\sqrt{\det\gamma}\,U\right)/\partial\pi^{ij}$ vanishes. Therefore the term $\{R_0^{(g)}(x)\,,\,\sqrt{\det\gamma}\,U(y)\}$, which is proportional to $\delta^3(x-y)$ because $R_0^{(g)}$ and U do not contain derivatives of π^{ij} and γ_{ij}, is canceled by its conjugate $\{\sqrt{\det\gamma}\,U(x)\,,\,R_0^{(g)}(y)\}$. As a result, we have

  $$\{C^L(x)\,,\,C^L(y)\} = M_f^4\{R_0^{(f)}(x)\,,\,R_0^{(f)}(y)\} + M_g^4 n^i n^j\{R_i^{(g)}(x)\,,\,R_j^{(g)}(y)\}$$
  $$-\left[2m^2 M_{\text{eff}}^2\,n^i\frac{\partial R_i^{(g)}}{\partial\pi^{mn}}\frac{\partial\left(\sqrt{\det\gamma}\,U\right)}{\partial\gamma_{mn}} - (x\leftrightarrow y)\right].\qquad\text{(II.34)}$$

  To proceed the calculation, we introduce localized smoothing functions $F(x)$ and $G(x)$ and take inner products of the smoothing functions and C^L as

  $$\langle FC^L\rangle := \int d^3x\, F(x)C^L(x)\,,\quad \langle GC^L\rangle := \int d^3y\, G(y)C^L(y)\,.\qquad\text{(II.35)}$$

  Then we compute,

  $$\{\langle FC^L\rangle\,,\,\langle GC^L\rangle\} = \int d^3x \int d^3y\, F(x)G(y)\{C^L(x)\,,\,C^L(y)\}\,,\qquad\text{(II.36)}$$

  from which we can extract $\{C^L(x)\,,\,C^L(y)\}$. Using Eqs. (II.31) and (II.33), we find

  $$\{\langle FC^L\rangle\,,\,\langle GC^L\rangle\} = \left\langle\left(-M_f^2 R^{n\,(f)} - M_g^2 n^n n^j R_j^{(g)} - 4\tilde{m}^2 M_g^2 n^i \gamma_{im}\frac{\partial\left(\sqrt{\det\gamma}\,U\right)}{\partial\gamma_{mn}}\right)F\partial_n G - (F\leftrightarrow G)\right\rangle$$
  $$= M_g^2\left\langle\left({}^3f^{ij} - n^i n^j\right)\left(R_j^{(g)} - 2\tilde{m}^2\frac{\sqrt{\det\gamma}}{\sqrt{x}}\tilde{V}_j^k\,{}^3f_{kl}n^l\right)(F\partial_i G) - (F\leftrightarrow G)\right\rangle.\quad\text{(II.37)}$$

  Here we have used the constraint $C_i^L = 0$ and Eq. (II.29) in the second equality. Therefore we can confirm that under the constraint $\mathcal{C}_j = 0$, $\{C^L(x)\,,\,C^L(y)\}$ vanishes. Note that we have not used $C^L = 0$ here.

- $\{C_i^L\,,\,\mathcal{C}\}$

  We calculate

  $$\{C_i^L(x)\,,\,\mathcal{C}(y)\} = M_g^4\left[\{R_i^{(g)}(x)\,,\,R_0^{(g)}(y)\} + \{R_i^{(g)}(x)\,,\,R_k^{(g)}D_l^k n^l(y)\} + \{R_i^{(g)}(x)\,,\,D_l^k n^l(y)\}R_k^{(g)}(y)\right.$$
  $$\left. + 2\tilde{m}^2\{R_i^{(g)}(x)\,,\,\sqrt{\det\gamma}\,V(y)\} + 2\tilde{m}^2\{R_i^{(f)}(x)\,,\,\sqrt{\det\gamma}\,V(y)\}\right],$$
  $$\text{(II.38)}$$

which should vanishes because C_i^L is just the generator of the spatial translation and constraint $\mathcal{C}(y)$ is satisfied over the whole spatial hypersurface. Introducing localized smoothing functions $F^i(x)$ and $G(y)$, we compute

$$
\begin{aligned}
\{\langle F^i C_i^L \rangle\, ,\, \langle G\mathcal{C} \rangle\} = M_g^2 \Big\langle &- \left(F^j \partial_j G\right) R_0^{(g)} - \left(F^j \nabla_j^{(g)}\left(G D_l^m n^l\right) - \left(\nabla_i^{(g)} F^m\right) G D_l^i n^l\right) R_m^{(g)} \\
&+ 2G\left(\nabla_i^{(g)} F^m\right)\gamma_{mj}\left(R_k^{(g)}\frac{\partial\left(D_l^k n^l\right)}{\partial \gamma_{ij}} + 2\tilde{m}^2\frac{\partial\left(\sqrt{\det\gamma}\,V\right)}{\partial \gamma_{ij}}\right) \\
&+ 2G\left(\nabla_i^{(f)} F^m\right)\,{}^3f_{mj}\left(R_k^{(g)}\frac{\partial\left(D_l^k n^l\right)}{\partial\, {}^3f_{ij}} + 2\tilde{m}^2\frac{\partial\left(\sqrt{\det\gamma}\,V\right)}{\partial\, {}^3f_{ij}}\right)\Big\rangle \\
= M_g^2 \Big\langle &- \left(F^j \partial_j G\right)\left(R_0^{(g)} + D_k^l n^k R_l^{(g)}\right) - F^k G R_j^{(g)}\nabla_k^{(g)}\left(D_l^j n^l\right) \\
&+ G\left(\nabla_i^{(g)} F^m\right)\left[D_l^i n^l R_m^{(g)} + 2\tilde{m}^2\frac{\sqrt{\det\gamma}}{\sqrt{x}}\left(-\tilde{V}_m^n \left(D^{-1}\right)^l{}_n\, {}^3f_{lj}\gamma^{ij} + \sqrt{x}^2 \tilde{V}_m^l D_l^i\right)\right] \\
&+ G\left(\nabla_m^{(g)} F^m\right) 2\tilde{m}^2\sqrt{\det\gamma}\,V - 2\tilde{m}^2 G\,\delta\Gamma^m{}_{ik} F^k \sqrt{\det\gamma}\,\sqrt{x}\,\tilde{V}_m^l D_l^i\Big\rangle . \qquad \text{(II.39)}
\end{aligned}
$$

In the second equality, we have used the relations (II.27), (II.28) and the constraint $\mathcal{C}_k = 0$, and rewrote $\nabla^{(f)}$ to $\nabla^{(g)}$ using the difference of the connections associated with the two metrics,

$$
\delta\Gamma^m{}_{ik} := \Gamma_{(g)}^m{}_{ik} - \Gamma_{(f)}^m{}_{ik} . \qquad \text{(II.40)}
$$

The quantity in the square brackets in the final expression of Eq. (II.39) vanishes using the relation

$$
- \left(D^{-1}\right)^l{}_m\, {}^3f_{lj}\gamma^{ij} + \sqrt{x}^2 D_m^i = -{}^3f_{ml} n^l D_k^i n^k , \qquad \text{(II.41)}
$$

which is derived from the definition of the matrix D and x, with the aid of the constraint $\mathcal{C}_m = 0$. One can argue that the remainig terms also cancel with each other by the following discussion. Integrating by parts the fourth term in the final expression of Eq. (II.39) produces the term in which $\nabla_m^{(g)}$ operates on G, which cancels the first term with the aid of the constraint $\mathcal{C} = 0$, and the one in which $\nabla_m^{(g)}$ operates on V, which is rewritten as

$$
\begin{aligned}
\nabla_m^{(g)} V &= \frac{\partial V}{\partial\left(\sqrt{x}D_b^a\right)}\nabla_m^{(g)}\sqrt{x}D_b^a , \\
&= -\sqrt{x}\tilde{V}_j^i D_k^j \delta\Gamma^k{}_{im} - \frac{1}{\sqrt{x}}\tilde{V}_j^i\, {}^3f_{ik}n^k\nabla_m^{(g)}\left(D_l^j n^l\right) , \qquad \text{(II.42)}
\end{aligned}
$$

where we used

$$
\nabla_a^{(g)}\, {}^3f_{ij} = \left(\nabla_a^{(g)} - \nabla_a^{(f)}\right){}^3f_{ij} = -2\,{}^3f_{l(i}\delta\Gamma^l{}_{j)a} . \qquad \text{(II.43)}
$$

Substituting Eq. (II.42) into Eq. (II.39), the last term in Eq. (II.39) is canceled by the first term in Eq. (II.42), and the second term in Eq. (II.39) is combined with the second term in Eq. (II.42) to be zero owing to the constraint $\mathcal{C}_j = 0$. Therefore all terms in Eq. (II.39) are canceled and $\{C_i^L(x)\, ,\, \mathcal{C}(y)\} = 0$ is proved. Here we have not used $C^L = 0$ as in the case of $\{C^L\, ,\, C^L\}$.

- $\{C_i^L\, ,\, C^L\}$

Here we also introduce localized smoothing functions $F^i(x)$ and $G(y)$ and compute

$$\{\langle F^i C_i^L \rangle , \langle G C^L \rangle\} = \left\langle -M_f^2 \left(F^k \nabla_k^{(f)} G \right) R_0^{(f)} - M_g^2 \left(F^k \nabla_k^{(g)} \left(G n^j \right) - \left(\nabla_k^{(g)} F^j \right) G n^k \right) R_j^{(g)} \right.$$

$$\left. + 4\tilde{m}^2 M_g^2 \left[G \left(\nabla_i^{(g)} F^m \right) \gamma_{mj} \frac{\partial \left(\sqrt{\det \gamma} \, U \right)}{\partial \gamma_{ij}} + G \left(\nabla_i^{(f)} F^m \right) {}^3 f_{mj} \frac{\partial \left(\sqrt{\det \gamma} \, U \right)}{\partial {}^3 f_{ij}} \right] \right\rangle ,$$

$$= 2\tilde{m}^2 \left\langle \left(F^k \nabla_k^{(f)} G \right) \sqrt{\det \gamma} \, U - \left\{ F^k \nabla_k^{(g)} n^j - \left(\nabla_k^{(g)} F^j \right) n^k \right\} G \frac{\sqrt{\det \gamma}}{\sqrt{x}} \tilde{V}_j^n \, {}^3 f_{nl} n^l \right.$$

$$\left. + G \left(\nabla_i^{(g)} F^m \right) \sqrt{\det \gamma} \left[\delta_m^i U - \frac{1}{\sqrt{x}} n^i \tilde{V}_m^l \, {}^3 f_{lk} n^k \right] - 2G \delta \Gamma^m{}_{ik} F^k \, {}^3 f_{mj} \frac{\partial U}{\partial {}^3 f_{ij}} \right\rangle , \tag{II.44}$$

which is also expected to vanish as in the case of $\{C_i^L(x) , \mathcal{C}(y)\}$. Here we have used the constraints $\mathcal{C}_j = 0$ and $C^L = 0$ and the relation (II.30) in the second equality. We find that the second term in the braces and the second term in the square brackets cancel each other. The first term can be integrated by parts as

$$\left\langle \left(F^j \nabla_j^{(g)} G \right) \sqrt{\det \gamma} \, U \right\rangle$$

$$= -G \left(\nabla_j^{(g)} F^j \right) \sqrt{\det \gamma} \, U - G F^j \left(\nabla_j^{(g)} n^i \right) \frac{\partial \left(\sqrt{\det \gamma} \, U \right)}{\partial n^i} - G F^j \left(\nabla_j^{(g)3} f_{ab} \right) \frac{\partial \left(\sqrt{\det \gamma} \, U \right)}{\partial {}^3 f_{ab}}$$

$$= -G \left(\nabla_j^{(g)} F^j \right) \sqrt{\det \gamma} \, U + G F^j \left(\nabla_j^{(g)} n^i \right) \frac{\sqrt{\det \gamma}}{\sqrt{x}} \tilde{V}_i^k \, {}^3 f_{kl} n^l + 2 \sqrt{\det \gamma} \, G F^j \, {}^3 f_{la} \delta \Gamma^l{}_{bj} \frac{\partial U}{\partial {}^3 f_{ab}} , \tag{II.45}$$

where we used the relation (II.21) and the definition of the connection $\delta\Gamma$, (II.40). From this equation (II.44), we find that three terms in Eq. (II.45) are canceled by the first term in the square brackets, the terms proportional to $\tilde{V}$ and the last term in Eq. (II.44), respectively. Hence, the Poisson bracket between C_i^L and C^L is confirmed to vanish. Notice that we used $C^L = 0$ here, unlike the previous calculations.

- $\{\mathcal{C} , C^L\}$

  Finally we give an explicit expression for $\{\mathcal{C} , C^L\}$. As before, we compute

$$\{\langle F \mathcal{C} \rangle , \langle G C^L \rangle\}$$

$$= M_g^2 \left\langle n^j R_j^{(g)} G \nabla_i^{(g)} \left(D_k^i n^k F \right) + 4\tilde{m}^2 \frac{\partial \sqrt{\det \gamma} \, U}{\partial \gamma_{mn}} \gamma_{im} G \nabla_n^{(g)} \left(D_k^i n^k F \right) \right.$$

$$+ 2\tilde{m}^2 \frac{\sqrt{\det \gamma}}{\sqrt{x}} \left(D^{-1} \right)_l^k \tilde{V}_k^m \, {}^3 f_{mj} \gamma^{jn} F \nabla_n^{(g)} \left(n^l G \right)$$

$$\left. - \tilde{m}^2 \left[\frac{2}{\sqrt{\det \gamma}} \frac{\partial \sqrt{\det \gamma} \, U}{\partial \gamma_{mn}} \left(\pi \gamma_{mn} - 2\pi_{mn} \right) + \frac{\sqrt{\det \gamma}}{\sqrt{\det {}^3 f}} \sqrt{x} \tilde{V}_l^m \, {}^3 f^{lk} D_k^n \left(p \, {}^3 f_{mn} - 2 p_{mn} \right) \right] F G \right\rangle . \tag{II.46}$$

Here we have used the constraints $\mathcal{C}_i = 0$ and $\mathcal{C} = 0$. The last term includes no derivative and is linear in p^{mn}, which appears only in the squared or differentiated form in all constraints. Therefore this term cannot be removed and hence $\{\mathcal{C} , C^L\}$ never becomes zero.

Hence, we conclude that all the Poisson brackets among the primary constraints $\mathcal{C}$, C^L, and C_i^L are zero, except for $\{\mathcal{C} , C^L\}$, as is expected from the calculation in Ref. [6]. We stress that we used $C^L = 0$ only to show $\{C_i^L , C^L\} = 0$, which allows us to simplify the calculation in the presence of a doubly coupled matter, which will be discussed in Sec. 3.

III. DOUBLY COUPLED MATTER

Next, we examine the existence of the BD ghost in the presence of a doubly coupled matter field. The proof in Sec. 2 can be easily extended to the case in which there is matter that couples to either metric, but not to

both. This is because the matter contribution can be totally absorbed by the redefinition of $R_0^{(g)}$ and $R_i^{(g)}$ (or $R_0^{(f)}$ and $R_i^{(f)}$) in all constraints and the fundamental Poisson brackets (II.31)-(II.33) hold without any change.

If we consider the case in which there is a matter field which is coupled to both metrics, however, the proof will in general break down. In this case, it is impossible to define such a new shift-like vector which makes the Lagrangian linear in the both lapse functions. Hence, we will not have the constraint that corresponds to $\mathcal{C}$ in this case, and hence the model will possess the BD ghost. One particular example of a ghost-free theory with a matter field which couples to both metrics is

$$
\begin{aligned}
S \;=\; & \int d^4x \sqrt{-g}\left[\frac{M_g^2 R^{(g)}}{2} + 2m^2 M_{\text{eff}}^2 \sum_n c_n e_n\!\left(\sqrt{g^{\mu\nu}(f_{\mu\nu} + \alpha\,\partial_\mu\phi\,\partial_\nu\phi)}\right)\right] \\
& + \int d^4x \sqrt{-f}\left[\frac{M_f^2 R^{(f)}}{2} - \frac{1}{2}\,f^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right],
\end{aligned}
\tag{III.1}
$$

where α is a constant. This model is free from the BD ghost because we can rewrite the action into the one with no coupling to the metric $g_{\mu\nu}$, making a field redefinition as

$$
f_{\mu\nu} \to \tilde{f}_{\mu\nu} - \alpha\,\partial_\mu\phi\,\partial_\nu\phi\,.
\tag{III.2}
$$

In the remaining kinetic term, higher temporal derivative terms do not arise after the replacement (III.2), for the same reason why the temporal derivatives of the lapse and shift can be removed by integrating by parts in the Einstein-Hilbert action. This possibility seems the natural and ghost-free extension of the model proposed in [22] (see also [23]) to the bigravity case.

A. Seeking for models with doubly coupled matter which have no ghost

We have introduced some models that have no BD ghost. In this section we want to show that, as soon as we further extend the proposed ghost-free action, the ghost will appear. To prove the existence of the ghost, it will suffice to give a proof of its existence on some reasonable background.

Let us analyze then the Lagrangian

$$
\mathcal{L} \ni \sqrt{-g}\,P(X,\phi) + \sqrt{-f}\,\tilde{P}(\tilde{X},\phi)\,,
\tag{III.3}
$$

where with P and $\tilde{P}$ are arbitrary functions of $\{\phi, X\}$ and $\{\phi, \tilde{X}\}$, respectively, with

$$
X = -\frac{1}{2}\,g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\,, \qquad\qquad \tilde{X} = -\frac{1}{2}\,f^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\,.
\tag{III.4}
$$

1. The Friedmann background

As a first step, let us study scalar-type linear perturbations around the FLRW background. We fix the gauge so that the metrics are given by

$$
\begin{aligned}
ds^2 \;=\; & -a^2(1+2\alpha)\,dt^2 - 2\partial_i\chi\,dt dx^i + a^2\delta_{ij}dx^i dx^j\,, \\
d\tilde{s}^2 \;=\; & -\tilde{c}^2\tilde{a}^2(1+2\tilde{\alpha})\,dt^2 - 2\partial_i\tilde{\chi}\,dt dx^i + \tilde{a}^2[\delta_{ij}(1+2\tilde{\Phi}) + 2\partial_i\partial_j\tilde{\gamma}]\,dx^i dx^j\,,
\end{aligned}
$$

and the scalar field by

$$
\phi = \bar{\phi} + \delta\phi\,.
\tag{III.5}
$$

Therefore, in total, we have seven scalar variables describing the three-dimensional scalar-sector of the perturbation fields. Following the standard way of reducing the action for the perturbation variables, we first eliminate the lapse and shift perturbations α, χ, $\tilde{\alpha}$, and $\tilde{\chi}$, using the Hamiltonian and momentum constraints. If the BD

ghost is present, we will be left with three scalar variables with none of them being a Lagrange multiplier. In other words, we can see whether the BD ghost is present or not, by evaluating the determinant of the kinetic matrix $\mathcal{A}_{ij}$ (after integrating out the auxiliary fields α, χ, $\tilde{\alpha}$, and $\tilde{\chi}$).

We will find that this determinant does not vanish in general. If it does not vanish, we can conclude that the BD ghost is present. However, there might be a subclass of theories for which the ghost is absent: at least we should see the disappearance of the ghost in the case $\tilde{P}(\tilde{X}, \phi) = \tilde{P}(\phi)$.

After some algebra, we find

$$\det(\mathcal{A}_{ij}) \propto \dot{a}^2 \dot{\tilde{\phi}}^2 (6\,\xi^2 c_3 + 4\,\xi\,c_2 + c_1)\,(P_{,X} + 2X P_{,XX}) \left(\tilde{P}_{,\tilde{X}} + 2\tilde{X}\tilde{P}_{,\tilde{X}\tilde{X}}\right), \tag{III.6}$$

where $\xi := \tilde{a}/a$. First of all, $\det(\mathcal{A}_{ij})$ vanishes when the background solution satisfies $\Gamma \equiv c_1 + 4\,\xi\,c_2 + 6\,\xi^2 c_3 = 0$ [2], or for the following sub-classes of theories

$$P_{,X} + 2X P_{,XX} = 0, \qquad \text{or} \qquad \tilde{P}_{,\tilde{X}} + 2\tilde{X}\tilde{P}_{,\tilde{X}\tilde{X}} = 0. \tag{III.7}$$

One solution for Eq. (III.7) is provided by the case $\tilde{P}(\tilde{X}, \phi) = \tilde{P}(\phi)$. However, we realize that there could also be another possibility, that is

$$P = P(X, \phi), \qquad \tilde{P} = F_1(\phi)\sqrt{\tilde{X}} + F_2(\phi), \qquad \text{with} \qquad F_1 \neq 0. \tag{III.8}$$

Notice that the roles of P and $\tilde{P}$ can be exchanged.

This result implies that on the FLRW background, at least, at linear level in the perturbation fields, the BD ghost is absent not only for the known case $\tilde{P}(\tilde{X}, \phi) = \tilde{P}(\phi)$, but also for the more general class of models defined in (III.8). This result, at first sight, seems to contradict our previous calculations. However, even if the BD ghost is absent on the FLRW background (at linear level in the perturbation fields), we still need to check if the BD ghost remains absent for this new class of models (III.8) as we change the background. In fact, in the succeeding section we will discuss perturbation in an anisotropic background for the models (III.8) as a counterexample, for which the BD ghost is present.

2. The anisotropic background

As we have already seen, the study on the FLRW background has already sufficiently restricted the possibilities for the choice of function $\tilde{P}$ to the case $\tilde{P} = F_1(\phi)\sqrt{\tilde{X}} + F_2(\phi)$. In order to understand whether this model does possess the BD ghost (or not) on other general backgrounds, we found it convenient to focus on the simplest example of an anisotropic background.

Focusing on a particular anisotropic Bianchi-I type manifold with residual axial symmetry, we will consider a sub-case where we have $F_1 = \text{constant}$ and $F_2 = 0$ in (III.8). The field profile compatible with the Bianchi I symmetry can be written then as

$$\phi = v_0 \int^t \frac{\tilde{c}(\eta)}{\tilde{f}_1(\eta)}\,d\eta + v_1 x + \delta\phi, \tag{III.9}$$

where we have chosen the time-profile for the field ϕ, leaving $g_{00} = -N(t)^2$ as a free variable to be determined by the equations of motion. We have also introduced a possible non-trivial x-profile $v_1 x$ for the field, which is

---

[2] We will not discuss here this solution, for two reasons. First, it is a constraint on the choice of the background ξ, rather than a condition that is identically satisfied by a given model. This implies that we cannot be sure whether or not the BD ghost would be absent on different backgrounds even for the same model. Second, this choice of the background ξ is known to be pathological in the sense that many modes (not only the BD scalar) are absent in the quadratic action, and hence we do not consider this case in the following discussion.

still consistent with the symmetry of the background, as we have imposed $\partial \tilde{P}/\partial \phi = 0$. Indeed this background-profile for ϕ implies that

$$\tilde{X} = \frac{4v_1^2}{\tilde{f}_1^2}, \tag{III.10}$$

where, for simplicity, we have also fixed $v_0 = 3v_1$. Finally we further restrict our attention to the case with the simplest linear functions $P(X)$, so that, in total, the contribution of this scalar field to the Lagrangian reads

$$\mathcal{L} \ni \sqrt{-g}\, C_1 X + \sqrt{-\tilde{g}}\tilde{C}_1 \sqrt{\tilde{X}}, \tag{III.11}$$

where C_1 and $\tilde{C}_1$ are numerical constants. If the BD ghost is present for this simple example, it will also be present for the general class of theories defined in (III.8).

For ghost-free models on this axial-symmetric Bianchi-I background, we should have in total only 3 odd (vectors under 2d rotations) modes and 5 even (scalars under 2d rotations) modes. We consider the even modes. With an appropriate choice of gauge, the two metrics can be written as

$$
\begin{aligned}
ds^2 &= -N^2(1+\alpha_1)dt^2 + 2\partial_x\chi_1\, dtdx + 2\partial_i\eta_1 dtdy^i + f_1^2(1+\beta_1)dx^2 + f_2^2\delta_{ij}dy^idy^j, \tag{III.12}\\
d\tilde{s}^2 &= -\tilde{c}^2(1+\alpha_2)dt^2 + 2\partial_x\chi_2\, dtdx + 2\partial_i\eta_2 dtdy^i + 2(\partial_x\partial_i\zeta_2)\, dxdy^i \\
&\quad + \tilde{f}_1^2(1+\beta_2)dx^2 + \tilde{f}_2^2[\delta(1+\Phi_2) + \partial_i\partial_j\gamma_2]\, dy^idy^j, \tag{III.13}
\end{aligned}
$$

where $\{y^1, y^2\} \equiv \{y, z\}$.

Next, in order to reduce the action, we need to integrate out all the known auxiliary fields, which are the six lapse and shifts, $\alpha_1, \chi_1, \eta_1, \alpha_2, \chi_2, \eta_2$. After integrating out the auxiliary fields in the even modes, we are left with $12 - 6 = 6$ even modes (whereas, as for the odd sector, as already mentioned, the gauge fixing removes one mode and the momentum constraint other two, so that only 3 odd modes remain). In total (for the even and odd sectors) we are left with nine propagating modes. However, if the BD ghost were absent, we would find that the determinant of the kinetic matrix for the even modes should vanish, i.e. one of the six even modes should reduce to a Lagrange multiplier.

To simplify the analysis we focus on a background which is an exact solution of the equations of motion, defined by

$$
\begin{aligned}
\tilde{f}_2(t) &= (\xi + w\bar{\xi}_2)\, f_2(t), \tag{III.14}\\
\tilde{f}_1(t) &= \xi\, f_1(t), \tag{III.15}\\
\tilde{c}(t) &= (\xi + w\bar{\xi}_0), \tag{III.16}\\
N(t) &= 1, \tag{III.17}\\
f_1(t) &= 1, \tag{III.18}\\
f_2(t) &= e^{wt}, \tag{III.19}
\end{aligned}
$$

where ξ, $\bar{\xi}_0$, $\bar{\xi}_2$, and w are all constants. These constants, together with the parameters of the theory $c_0, c_1, c_2, c_3, c_4, C_1$, and $\tilde{C}_1$ are constrained to solve the six independent equations of motion. We solve the six independent equations of motion for the variables $\tilde{C}_1, c_0, c_1, c_3, c_4$, and C_1 in terms of $v_1, \xi, \bar{\xi}_0, \bar{\xi}_2, w$ and $\kappa := M_f^2/M_g^2$. Notice that the background has a Minkowski limit, that is $w \to 0$, in which the quantity

$$\Gamma = c_1 + 4c_2\xi + 6c_3\xi^3 \approx \frac{2\xi M_g^2(1 + \kappa\xi^2)(4\bar{\xi}_0 + 5\bar{\xi}_2)}{\bar{\xi}_0^2(\bar{\xi}_0 - \bar{\xi}_2)m^2 M_{\text{eff}}^2} + \mathcal{O}(w) \tag{III.20}$$

does not necessarily vanish. This shows that this Minkowski limit does not, in general, belong to the unhealthy branch of the FLRW background, defined by the condition $\Gamma = 0$.

Although it is not necessary in this context, we will fix the parameters imposing

$$(4\bar{\xi}_0 + 5\bar{\xi}_2)(\bar{\xi}_0 - \bar{\xi}_2) > 0, \tag{III.21}$$

(to have a positive Γ for the stability of the background). In this Minkowski limit, using the constraint equations, we also find

$$C_1 \approx \frac{\xi^3}{4v_1^2}\,|\tilde{C}_1 v_1|\,, \tag{III.22}$$

$$|\tilde{C}_1 v_1| \approx \frac{4M_g^2(1+\kappa\xi^2)}{\xi^2\bar{\xi}_0}\,w\,, \tag{III.23}$$

so that $w/\bar{\xi}_0 > 0$.

As already mentioned earlier, in order to see whether the BD ghost is absent or not, we need to study the six by six kinetic matrix for the even modes which remain after having integrated out the auxiliary variables. We study the eigenvalues of such a kinetic matrix in the large momenta limit (i.e. for large k and q), where all variables are supposed to be expanded by the Fourier mode $e^{-i(kx+q_1y+q_2z)}$ with $q = \sqrt{q_1^2 + q_2^2}$.

We find that the kinetic matrix $\mathbf{K}$ has in general a non-zero determinant, i.e. the BD ghost will be present in general. To make it evident, we expand the determinant (around $w \to 0$) to find

$$\det(\mathbf{K}) = -\frac{5}{4096}\frac{k^2 q^8 M_g^{12}(1+\kappa\xi^2)^2\kappa^3\xi^5 w}{4v_1^2(2k^2+q^2)^2\bar{\xi}_0^3} < 0\,. \tag{III.24}$$

To be more precise, we have studied the sign of all six eigenvalues and checked that only one of them is negative in this limit, which makes the determinant negative. Therefore, on axial symmetric anisotropic Bianchi-I manifolds, we can conclude that the case $\tilde{P} \propto \sqrt{\tilde{X}}$ has 1) one more degree of freedom than models which are free from the BD ghost, and 2) the extra mode is indeed a ghost, at least on the Minkowski limit of such an anisotropic solution.

Finally, from the analysis of perturbations around the FLRW and anisotropic backgrounds, we conclude that only $\tilde{P} = \tilde{P}(\phi)$ is the possibility to avoid the BD ghost in the class of models defined by (III.8).

IV. PROOF OF THE ABSENCE OF GHOST IN THE CASE $\tilde{P} = \tilde{P}(\phi)$

In this section we give a proof of the absence of ghost in the case $\tilde{P} = \tilde{P}(\phi)$. We consider a k-essence scalar field ϕ which couples to both metrics, but whose kinetic term contains only the physical metric $g_{\mu\nu}$. Namely, the Lagrangian of the scalar field is

$$\mathcal{L}_m = \sqrt{-g}\,P(X,\,\phi) + \sqrt{-f}\,\tilde{P}(\phi)\,. \tag{IV.1}$$

The case in which the kinetic term of matter contains only the hidden metric can be also treated in the same way. The conjugate momentum of ϕ is derived from this action as

$$\pi_\phi = N^{-1}\sqrt{\det\gamma}\left(\frac{\partial P}{\partial X}\right)_\phi\left(\partial_t\phi - N^i\partial_i\phi\right)\,, \tag{IV.2}$$

where the subscript ϕ indicates that the partial differentiation is taken with ϕ fixed. Using the definition of X, π_ϕ^2 is written as

$$\pi_\phi^2 = \det\gamma\left(\frac{\partial P}{\partial X}\right)_\phi^2\left[2X + \gamma^{ij}\partial_i\phi\partial_j\phi\right]\,. \tag{IV.3}$$

This implies that X can be expressed by γ^{ij}, ϕ and π_ϕ without using N, N^i and $\partial_t\phi$. Therefore we find that the matter Hamiltonian becomes

$$\mathcal{H}_m = NC_m + N^i\pi_\phi\partial_i\phi + L\sqrt{\det{}^3 f}\,\tilde{P}\,, \tag{IV.4}$$

where

$$C_m := \sqrt{\det \gamma} \left[\left(\frac{\partial P}{\partial X} \right)_\phi \left(2X + \gamma^{ij} \partial_i \phi \partial_j \phi \right) - P \right] . \tag{IV.5}$$

This matter Hamiltonian $\mathcal{H}_m$ is linear in N, N^i and L. C_m depends on γ^{ij}, ϕ and π_ϕ, but π_ϕ appears in C_m only through X, i.e. $C_m = C_m \left(\gamma^{ij}, \phi, X \left(\gamma^{ij}, \phi, \pi_\phi \right) \right)$. After rewriting the shift vector N^i using Eq. (II.8), the total Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_{total} = &\pi^{ij} \partial_t \gamma_{ij} + p^{ij} \partial_t {}^3 f_{ij} + \pi_\phi \partial_t \phi \\
&+ L^i \left[M_g^2 R_i^{(g)} - \pi_\phi \partial_i \phi + M_f^2 R_i^{(f)} \right] \\
&+ L \left[M_f^2 R^{0(f)} + n^i \left(M_g^2 R_i^{(g)} - \pi_\phi \partial_i \phi \right) + 2m^2 M_{\mathrm{eff}}^2 \sqrt{\det \gamma}\, U - \sqrt{\det {}^3 f}\, \tilde{P} \right] \\
&+ N \left[M_g^2 R^{0(g)} - C_m + D_j^i n^j \left(M_g^2 R_i^{(g)} - \pi_\phi \partial_i \phi \right) + 2m^2 M_{\mathrm{eff}}^2 \sqrt{\det \gamma}\, V \right] .
\end{aligned} \tag{IV.6}$$

We redefine the constraints $\mathcal{C}$, C^L and C_i^L as the coefficient of N, L and L^i in Eq. (IV.6), respectively. We also redefine the constraint $\mathcal{C}_i$ to the one obtained from the variation of $\mathcal{L}_{total}$ with respect to n^i as

$$\mathcal{C}_i := M_g^2 R_i^{(g)} - \pi_\phi \partial_i \phi - 2\tilde{m}^2 M_g^2 \sqrt{\det \gamma} \frac{n^l\, {}^3 f_{lj}}{\sqrt{x}} \tilde{V}_i^j . \tag{IV.7}$$

Let us confirm that all the Poisson brackets of three constraints $\mathcal{C}$, C^L and C_i^L vanish except $\{\mathcal{C}, C^L\}$.

We introduce $R_0^{(g)\prime} := R_0^{(g)} - M_g^{-2} C_m$ and $R_i^{(g)\prime} := R_i^{(g)} - M_g^{-2} \pi_\phi \partial_i \phi$, then the constraints $\mathcal{C}$, $\mathcal{C}_i$, C_i^L takes the same forms as (II.24), (II.18), (II.26) with $R_0^{(g)}$ and $R_i^{(g)}$ replaced with $R_0^{(g)\prime}$ and $R_i^{(g)\prime}$. We find that $R_0^{(g)\prime}$ and $R_i^{(g)\prime}$ satisfy the same relations as Eqs. (II.31), (II.32) and (II.33), as is shown below.

In order to show these relations, we have to evaluate the partial differentiations of C_m with respect to ϕ, π_ϕ, and γ_{mn}. First, we have

$$\left(\frac{\partial C_m}{\partial X} \right)_\phi = \sqrt{\det \gamma}\, A , \tag{IV.8}$$

where

$$A := \left(\frac{\partial^2 P}{\partial X^2} \right)_\phi \left(2X + \gamma^{ij} \partial_i \phi \partial_j \phi \right) + \left(\frac{\partial P}{\partial X} \right)_\phi . \tag{IV.9}$$

Using Eq. (IV.3), we find

$$\left(\frac{\partial X}{\partial \phi} \right)_{\pi_\phi, \gamma^{ij}} = -A^{-1} \left[\left(\frac{\partial P}{\partial X} \right)_\phi \partial^i \phi \frac{\partial (\partial_i \phi)}{\partial \phi} + \frac{\partial^2 P}{\partial X \partial \phi} \left(2X + \gamma^{ij} \partial_i \phi \partial_j \phi \right) \right] , \tag{IV.10}$$

$$\left(\frac{\partial X}{\partial \pi_\phi} \right)_{\phi, \gamma^{ij}} = \left(\det \gamma\, A \left(\frac{\partial P}{\partial X} \right)_\phi \right)^{-1} \pi_\phi . \tag{IV.11}$$

Solving Eq. (IV.3) for X, we find X depends on γ_{ij} only through the form $\frac{\pi_\phi}{\sqrt{\det \gamma}}$ and $\gamma^{ij} \partial_i \phi \partial_j \phi$. Therefore the partial differentiations of X with respect to γ_{mn} is

$$\left(\frac{\partial X}{\partial \gamma_{mn}} \right)_{\phi, \pi_\phi} = \frac{1}{2} A^{-1} \left[\left(\frac{\partial P}{\partial X} \right)_\phi \partial^m \phi \partial^n \phi - \frac{\pi_\phi^2}{\det \gamma} \left(\frac{\partial P}{\partial X} \right)_\phi^{-1} \gamma^{mn} \right] . \tag{IV.12}$$

Using the above relations, we obtain the partial differentiations of C_m with respect to ϕ as

$$
\left(\frac{\partial C_m}{\partial \phi}\right)_{\pi_\phi,\,\gamma^{ij}} = \left(\frac{\partial C_m}{\partial X}\right)_{\phi,\,\gamma^{ij}} \left(\frac{\partial X}{\partial \phi}\right)_{\pi_\phi,\,\gamma^{ij}} + \sqrt{\det\gamma}\left[2\left(\frac{\partial P}{\partial X}\right)_\phi \partial^i\phi\frac{\partial\left(\partial_i\phi\right)}{\partial\phi} + \frac{\partial^2 P}{\partial X\partial\phi}\left(2X+\gamma^{ij}\partial_i\phi\partial_j\phi\right) - \left(\frac{\partial P}{\partial \phi}\right)_X\right],
$$

$$
= \sqrt{\det\gamma}\left[\left(\frac{\partial P}{\partial X}\right)_\phi \partial^i\phi\frac{\partial\left(\partial_i\phi\right)}{\partial\phi} - \left(\frac{\partial P}{\partial \phi}\right)_X\right],
\tag{IV.13}
$$

and the one with respect to π_ϕ as

$$
\left(\frac{\partial C_m}{\partial \pi_\phi}\right)_{\phi,\,\gamma^{ij}} = \sqrt{\det\gamma}\left(\frac{\partial P}{\partial X}\right)_\phi^{-1}\pi_\phi .
\tag{IV.14}
$$

The partial differentiations of C_m with respect to γ_{mn} is written as

$$
\left(\frac{\partial C_m}{\partial \gamma_{mn}}\right)_{\phi,\,\pi_\phi} = \frac{C_m}{2}\gamma^{mn} - \sqrt{\det\gamma}\left(\frac{\partial P}{\partial X}\right)_\phi \partial^m\phi\partial^n\phi + \left(\frac{\partial C_m}{\partial X}\right)_\phi\left(\frac{\partial X}{\partial \gamma_{mn}}\right)_{\phi,\,\pi_\phi},
$$

$$
= -\left(\frac{\sqrt{\det\gamma}}{2}P\gamma^{mn} + \left(\frac{\partial P}{\partial X}\right)_\phi \partial^m\phi\partial^n\phi\right).
\tag{IV.15}
$$

Now, we confirm that the same relations as Eqs. (II.32) and (II.33) hold for $R_0^{(g)\prime}$ and $R_i^{(g)\prime}$. Using Eqs. (IV.13) and (IV.14), the Poisson bracket between $R_0^{(g)\prime}$ and itself becomes

$$
\left\{\left\langle FM_g^2 R_0^{(g)\prime}\right\rangle,\ \left\langle GM_g^2 R_0^{(g)\prime}\right\rangle\right\} = \left\langle\left[M_g^2 R_{(g)}^{i\prime}\left(G\partial_i F\right)\right] - [F\leftrightarrow G]\right\rangle .
\tag{IV.16}
$$

Here, among the contribution from C_m, only the term containing $\frac{\partial(\partial_i\phi)}{\partial\phi}$ in Eq. (IV.13) escapes from the cancelation with its conjugate.

Using Eq. (IV.13), (IV.14), (IV.15), we obtain the Poisson bracket between $R_0^{(g)\prime}$ and $R_i^{(g)\prime}$ as

$$
\left\{\left\langle F^i M_g^2 R_i^{(g)\prime}\right\rangle,\ \left\langle GM_g^2 R_0^{(g)\prime}\right\rangle\right\}
$$

$$
= \left\langle -M_g^2 R_0^{(g)}F^i\partial_i G - 2\gamma_{mi}\frac{\partial C_m}{\partial\gamma_{mn}}G\nabla_n^{(g)}F^i + C_m F^i\partial_i G - \sqrt{\det\gamma}\,PG\nabla_i^{(g)}F^i - \sqrt{\det\gamma}\,\partial_i\phi\partial^n\phi\left(\frac{\partial P}{\partial X}\right)_\phi G\nabla_n^{(g)}F^i\right\rangle,
$$

$$
= -\left\langle M_g^2 R_0^{(g)\prime}F^i\partial_i G\right\rangle .
\tag{IV.17}
$$

$\{R_i^{(g)\prime},\ R_j^{(g)\prime}\}$ is easily found to satisfy

$$
M_g^2\{R_i^{(g)\prime}(x),\ R_j^{(g)\prime}(y)\} = -\left[R_j^{(g)\prime}(x)\frac{\partial}{\partial x^i}\delta^3(x-y) - R_i^{(g)\prime}(y)\frac{\partial}{\partial y^j}\delta^3(x-y)\right].
\tag{IV.18}
$$

From the above calculations, we find that the same relations as Eqs. (II.31), (II.32) and (II.33) hold for $R_0^{(g)\prime}$ and $R_i^{(g)\prime}$. Recall that C^L was not used to discuss the Poisson brackets among the constraints $\mathcal{C}$ and C_i^L. Hence, the calculations about $\mathcal{C}$ and C_i^L in Sec. 2 do not change under the replacements $R_0^{(g)}\to R_0^{(g)\prime}$ and $R_i^{(g)}\to R_i^{(g)\prime}$. By contrast, when doubly coupled matter is present, C^L becomes different from the one in Sec. 2 by the term $\sqrt{\det{}^3\!f}\,\tilde{P}$ even after these replacements. This prohibits us simply applying the discussion in Sec. 2 to the Poisson brackets related to C^L.

Now, let us look at the calculations of the Poisson brackets between $\mathcal{C}$, C^L, C_i^L, one by one.

- $\{\mathcal{C},\,\mathcal{C}\}$, $\{C_i^L,\,\mathcal{C}\}$ and $\{C_i^L,\,C_j^L\}$

  In the calculation of these Poisson brackets in Sec. 2, we did not use the constraints $C^L=0$, hence these brackets are proved to be zero just by replacing $R_0^{(g)}$ and $R_i^{(g)}$ with $R_0^{(g)\prime}$ and $R_i^{(g)\prime}$, respectively.

- $\{C^L, C^L\}$

  We can easily find that all the contribution of $\sqrt{\det {}^3 f}\, \tilde{P}(\phi)$ cancels with its conjugate, because $\sqrt{\det {}^3 f}\, \tilde{P}(\phi)$ does not depend on the derivatives of the variables. Therefore $\sqrt{\det {}^3 f}\, \tilde{P}$ does not affect the calculation of $\{C^L, C^L\}$ and we obtain $\{C^L, C^L\} = 0$ in the same way as in Sec. 2, using the replacements $R_0^{(g)} \to R_0^{(g)\prime}$ and $R_i^{(g)} \to R_i^{(g)\prime}$.

- $\{C_i^L, C^L\}$

  Focusing on $\sqrt{\det {}^3 f}\, \tilde{P}(\phi)$ in C^L, we calculate

$$
\left\{ \left\langle F^i \left(M_f^2 R_i^{(f)} + M_g^2 R_i^{(g)\prime} \right) \right\rangle, \left\langle G \sqrt{\det {}^3 f}\, \tilde{P}(\phi) \right\rangle \right\}
$$
$$
= \left\langle \sqrt{\det {}^3 f}\, \tilde{P} G \nabla_i^{(f)} F^i + \partial_i \phi \sqrt{\det {}^3 f}\, \frac{\partial \tilde{P}}{\partial \phi} F^i G \right\rangle
$$
$$
= - \left\langle \sqrt{\det {}^3 f}\, \tilde{P} F^i \partial_i G \right\rangle .
\tag{IV.19}
$$

This compensates the modification in $C^L = 0$ used in the second equality of Eq. (II.44). The rest of calculations can be done in the same way as in Sec. 2 just by replacing $R_0^{(g)}$ and $R_i^{(g)}$ with $R_0^{(g)\prime}$ and $R_i^{(g)\prime}$, respectively, to obtain $\{C_i^L, C^L\} = 0$.

From the above calculations, we have proven that the bigravity model has no BD ghost even in the presence of a doubly-coupled field matter whose Lagrangian is given by Eq. (IV.1).

V. SUMMARY

In this paper, we have shown that doubly coupled matter generally brings the BD ghost. As a first step, we have presented the detailed proof of ghost-freeness using the Hamiltonian analysis in the bigravity model obtained as an extension of dRGT massive gravity. The proof has been already given in Refs. [6–8], but we have explicitly shown that four Lagrange multipliers remain unspecified after all the Poisson brackets among the constraints close, which is requested by the general covariance, using Hamiltonian formulation.

Next, we extended the above proof to the case with matter. The extension of the proof of ghost-freeness to the case with matter can follow in the same way as in GR as long as matter only couples to either metric. However, we found that the BD ghost is present when matter couples to both metrics through the kinetic terms of the matter. To prove the appearance of the BD ghost, we have considered linear perturbations on the Friedmann and Bianchi-I backgrounds. We found that there really exists an extra degree of freedom and, at least, one mode becomes ghost in the Minkowski limit of an anisotropic solution. Furthermore, we show that the BD ghost is absent when either of the two metrics is coupled only through the potential term using Hamiltonian analysis.

Therefore, the form of doubly coupled matter is considerably restricted not only by experiments of the equivalence principle, but also by the condition to avoid the BD ghost,. The result seems inconsistent with the idea to embed the ghost-free bigravity into compactified higher dimensional models [20, 21]. As long as we consider the healthy branch of bigravity, we cannot trace back the evolution of the universe beyond the energy scale determined by the bare graviton mass m [11, 14, 15]. Hence, in order to take a ghost-free bigravity model as a realistic model of our universe, we need to find a way to derive such a model at low energies starting with a more complete model which is valid even at high energies. One possibile completion might be obtained by considering braneworld models. If we can realize a situation in which only two gravitons among the modes in the Kaluza-Klein tower survive at low energies, bi-gravity models may arise. In the braneworld setup, it would be easy to add a bulk scalar field to the models in such a way that only a single scalar mode effectively remains at low energies. However, in this case, the kinetic term of this low-energy scalar field will necessarily contain both metrics corresponding to the two low-energy gravitons. By construction, the braneworld models seem to avoid the appearance of ghost. However, the result in the present paper tells that the generic form of the kinetic term of a scalar field leads to the appearance of the BD ghost. This may suggest that there is a crucial difficulty in realizing the idea that a completion of ghost-free bigravity models might be obtained by

extending it to the braneworld setup. However, the actual coupling between the low-energy scalar and gravitons in the model derived from the braneworld setup can be more complicated than the one we have discussed in this paper. Hence, we may find an alternative way of coupling between matter and two metrics that avoids the appearance of the BD ghost.

Acknowledgments

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