Substitutable Choice Functions and Convex Geometry

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Abstract

For a finite nonempty set $E$ we consider a choice function $C : 2^E \rightarrow 2^E$ (i.e., $C(X) \subseteq X$ ($X \subseteq E$)) that satisfies

(A1) For any $X$ with $\emptyset \neq X \subseteq E$, $C(X) \neq \emptyset$.

(A2) (Substitutability) For any $X \subseteq E$ and $e \in C(X)$, $C(X \setminus \{e\}) \subseteq C(X \setminus \{e\})$.

We call an ordering (or permutation) $(e_1, e_2, \ldots, e_n)$ of $E$ admissible if for each $i = 1, 2, \ldots, n$ we have $e_i \in C(E \setminus \{e_1, e_2, \ldots, e_{i-1}\})$.

We show that the collection of all the admissible orderings is an antimatroid. Equivalently, defining $\mathcal{F} = \{E \setminus \{e_1, e_2, \ldots, e_{i-1}\} \mid (e_1, e_2, \ldots, e_n) :$ admissible ordering, $i \in \{1, 2, \ldots, n\}\}$, we get a convex geometry $(E, \mathcal{F})$.

The present result reveals that a convex geometry (or an antimatroid) naturally arises from any substitutable choice function (satisfying (A1)).

Keywords:
Choice function, substitutability, convex geometry, antimatroid

1. Introduction

Koshevoy [3] and Johnson and Dean [4] pointed out the correspondence between path-independent choice functions ([6]) and convex geometries ([2]) (also see [1, 5]).

In the present note we consider choice functions that satisfy a substitutability condition, which is weaker than the path-independence condition. We will show that every substitutable choice function yields a convex geometry, which reveals a lattice structure and a closure space behind any substitutable choice function through the associated convex geometry.
2. Definitions and Assumptions

Let $E$ be a finite set with its cardinality $|E| = n$. We consider a choice function $C : 2^E \to 2^E$, i.e.,

$$\forall X \subseteq E : C(X) \subseteq X.$$  \hspace{1cm} (1)

We assume

(A1) For any $X$ with $\emptyset \neq X \subseteq E$ we have $C(X) \neq \emptyset$.

An ordering (or permutation) $(e_1, e_2, \cdots, e_n)$ of $E$ is called admissible if for each $i = 1, 2, \cdots, n$

$$e_i \in C(E \setminus \{e_1, e_2, \cdots, e_{i-1}\}).$$  \hspace{1cm} (2)

Here, note that at least one admissible ordering of $E$ exists due to Assumption (A1). (Imagine the repeating process of choosing an element from the set specified by the choice function for a current underlying set (initially $E$) and removing the chosen element from the current underlying set.) We call each initial segment $(e_1, e_2, \cdots, e_i)$ $(i = 1, 2, \cdots, n)$ of an admissible ordering $(e_1, e_2, \cdots, e_n)$ of $E$ an admissible sequence and a set $\{e_1, e_2, \cdots, e_i\}$ an admissible set. Let $\mathcal{F}$ be the collection of complements of admissible sets, i.e.,

$$\mathcal{F} = \{E \setminus \{e_1, e_2, \cdots, e_{i-1}\} \mid (e_1, e_2, \cdots, e_n) : \text{admissible ordering}, i \in \{1, 2, \cdots, n\}\}$$  \hspace{1cm} (3)

We also assume

(A2) For any $X \in \mathcal{F}$ and $e \in C(X)$ we have

$$C(X) \setminus \{e\} \subseteq C(X \setminus \{e\}).$$  \hspace{1cm} (4)

This is a substitutability condition for choice function $C$.

3. An Associated Convex Geometry

We show that the collection $\mathcal{F}$ given by (3) provides us with a convex geometry $(E, \mathcal{F})$ on $E$.

We first show the following basic lemma.
Lemma 3.1. For any admissible ordering \((e_1, e_2, \cdots, e_n)\), if we have
\[ e_j \in C(E \setminus \{e_1, e_2, \cdots, e_{i-1}\}) \] (5)
for some integers \(i\) and \(j\) with \(1 \leq i < j \leq n\), then for any integer \(k\) with \(i \leq k \leq j - 1\) a new ordering given by
\[ (e_1, \cdots, e_{k-1}, e_j, e_k, \cdots, e_{j-1}, e_{j+1}, \cdots, e_n) \] (6)
is also admissible. (The new ordering is obtained by shifting \(e_j\) to the position immediately before \(e_k\).)

(Proof) For any integer \(l\) with \(i \leq l \leq j - 1\) define \(X_l = E \setminus \{e_1, e_2, \cdots, e_{l-1}\}\). Because of the assumption we have \(e_i, e_j \in C(X_l)\). Hence it follows from Assumption (A2) that for any given \(k\) with \(i \leq k \leq j - 1\)
\[ e_i, e_j \in C(X_l) \quad (k \leq l \leq j - 1), \] (7)
which implies, due to (A2) again,
\[ e_i \in C(X_l) \setminus \{e_j\} \subseteq C(X_l \setminus \{e_j\}) \quad (k \leq l \leq j - 1). \] (8)
Hence the ordering given by (6) is admissible. \(\square\)

From this lemma we show the following.

Lemma 3.2. For any distinct \(X, Y \in \mathcal{F}\) we have \(X \cap Y \in \mathcal{F}\).

(Proof) Suppose that \(X = E \setminus \{e_1, e_2, \cdots, e_k\}\) and \(Y = E \setminus \{e'_1, e'_2, \cdots, e'_l\}\) for admissible sequences \(L_X = (e_1, e_2, \cdots, e_k)\) and \(L_Y = (e'_1, e'_2, \cdots, e'_l)\). Also suppose that
\[ e_1 = e'_1, \cdots, e_p = e'_p, \quad e_{p+1} \neq e'_{p+1} \] (9)
for some integer \(p\) with \(0 \leq p < \min\{k, l\}\).

Now there exist the following three cases I~III.

[Case I: \(e'_{p+1} \not\in X\)]

Since \(e'_{p+1} = e_q\) for some \(q\) with \(p + 1 < q \leq k\), from Lemma 3.1,
\[ (e_1, \cdots, e_p, e'_{p+1}, e_{p+1}, \cdots, e_{q-1}, e_{q+1}, \cdots, e_k) \] (10)
is also an admissible sequence that gives \(X\). Replacing \(L_X\) by sequence (10), we get a new admissible sequence \(L_X\) that has a longer common initial segment with \(L_Y\).
Similarly as in Case I, we can get a new admissible sequence $L_Y$ that has a longer common initial segment with $L_X$.

[Case III: $e'_{p+1} \in X \setminus Y$ and $e_{p+1} \in Y \setminus X$]

In the present case, due to Lemma 3.1 we get new admissible sequences (longer by one)

\begin{equation}
(e_1, \ldots, e_p, e'_{p+1}, e_{p+1}, \ldots, e_k),
\end{equation}

\begin{equation}
(e'_1, \ldots, e'_p, e'_{p+1}, e_{p+1}, e'_2, \ldots, e'_l),
\end{equation}

the length of whose common initial segment becomes larger by two than that of $L_X$ and $L_Y$. Also note that they give admissible sets $E \setminus (X \setminus \{e'_{p+1}\})$ and $E \setminus (Y \setminus \{e_{p+1}\})$. So we replace $X$ and $Y$ by $X \setminus \{e'_{p+1}\}$ and $Y \setminus \{e_{p+1}\}$, respectively, and new $L_X$ and $L_Y$ are given by (11) and (12), respectively.

While there exists $p$ such that (9) holds, update $X, Y, L_X, L_Y$ as shown in Cases I~III described above. We eventually obtain $X$ and $Y$ such that (i) $E \setminus X$ and $E \setminus Y$ are admissible sets and (ii) $X \subseteq Y$ or $Y \subseteq X$, and then $X$ or $Y$ is equal to the original $X \cap Y$. Hence the original $X \cap Y$ satisfies $X \cap Y \in F$.

From this we obtain our main result as follows.

**Theorem 3.3.** $(E, F)$ is a convex geometry.

(Proof) Since $\emptyset, E \in F$ and for any nonempty $X \in F$ there exists some $e \in X$ such that $X \setminus \{e\} \in F$ by (A1), it follows from Lemma 3.2 that $(E, F)$ is a convex geometry.

\[\square\]

4. Concluding Remarks

We have shown that a convex geometry arises from any substitutable choice function. It should be noted that our result depends only on Assumptions (A1) and (A2). Hence $C(X) \ (X \in 2^E \setminus F)$ do not affect the structure of the associated convex geometry. It should also be noted that defining a closure operator $\text{cl}$ by $\text{cl}(X) = \cap\{Y \mid X \subseteq Y \in F\} \ (X \subseteq E)$ and a new choice function $\hat{C}(X) = C(\text{cl}(X)) \ (X \subseteq E)$, we get a path-independent choice function $\hat{C}$ that gives the same associated convex geometry $(E, F)$.


