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Fundamental Problems on Douady space

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Let X be a compact complex space. Then Duady*) [1] proved the existence of a complex space $\mathcal{D}$ together with a commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "j"'] \arrow[dr, "\pi"'] & \mathcal{D} \times X \arrow[d, "\text{projection}"] \\ & \mathcal{D} \end{tikzcd}$$

where

(1) j is a closed imbedding and π is a flat morphism (morphism = complex-analytic map), and

(2) given any diagram (having the same property as in (1))

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook, "i"'] \arrow[dr, "p"'] & T \times X \arrow[d, "\text{projection}"] \\ & T \end{tikzcd}$$

*) 以下 Duady は Douady のミスプリント

there exists a unique morphism $h : T \to \mathcal{D}$ such that

$$\begin{array}{ccc} \mathcal{F} & = & T \times_{\mathcal{D}} \mathcal{S} \\ \downarrow i & & \downarrow j \\ T \times X & = & T \times_{\mathcal{D}} (\mathcal{D} \times X) \end{array}$$

We call $\mathcal{D}$ the <u>Douady space</u> of X. For each integer $m \geq 0$, we have an open and closed complex subspace $\mathcal{D}_m$ of $\mathcal{D}$ such that

$$d \in \mathcal{D}_m \Longleftrightarrow d \in \mathcal{D} \text{ and } \dim \pi^{-1}(d) = m$$

Thus

$$\mathcal{D} = \coprod_{m \geq 0} \mathcal{D}_m \quad (\text{disjoint})$$

<u>Remark 1.</u> Douady space in complex-analytic geometry as above is a generalization of what Grothendieck called <u>Hilbert scheme</u> in algebraic geometry. Namely, if an imbedding $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ is given, then

$$\mathcal{D} = \coprod_H \mathcal{D}_H$$

where $d \in \mathcal{D}_H \Longleftrightarrow d \in \mathcal{D}$ and H is the

Hilbert polynomial of $\pi^{-1}(d)$ with respect to the projective imbedding $\pi^{-1}(d) \subset X \subset \mathbb{P}^N_{\mathbb{C}}$. (Here $H(\nu) = \chi(\mathcal{O}_{\pi^{-1}(d)}(\nu))$, $\forall \nu \in \mathbb{Z}$, where (ν) denotes the twisting ν-times by the tautological line bundle on $\mathbb{P}^N_{\mathbb{C}}$.) In this case, moreover, $\mathcal{D}_m$ is the disjoint union of those $\mathcal{D}_H$ with $\deg H = m$, and for every H, $\mathcal{D}_H$ is projective (and hence compact).

As Douady extended the notion of Hilbert scheme to complex-analytic geometry, Barlet [2] did the notion of Chow variety. The group of <u>cycles</u> (complex-analytic), denoted by $C(X)$, for a complex-analytic space X (not necessarily compact), is the additive group freely generated by reduced irreducible closed complex subspaces of X. Each cycle $c \in C(X)$ is uniquely written as

$$c = \sum_i m_i Y_i$$

where Y_i are different complex subspaces and m_i are integers $\neq 0$. We say that c is a q-*cycle* if $\dim Y_i = q$ for all i. The union of those Y_i (or rather their point-sets) is called the *support* of c and is denoted by $|c|$. We say that c is *positive* if $m_i > 0$ for all i. We denote by $C_+^q(X)$ the set of all the positive q-cycles of X.

If f is a map (set-theoretical) from a complex space V (reduced) into $C_+^q(X)$, we ask whether f is *complex-analytic* or not. This complex-analyticity is defined as follows.

First of all, if $\mathbb{D}$ is a complex *manifold* (manifold = complex space which is smooth), then $C_+^0(\mathbb{D})$ is naturally identified with the point-set of the complex space

$$\mathrm{Sym}(\mathbb{D}) = \coprod_{\nu \geq 0} \mathrm{Sym}^\nu(\mathbb{D})$$

where $\mathrm{Sym}^\nu(\mathbb{D})$ denotes the ν-times symmetric product of $\mathbb{D}$, i.e., the quotient space of $\mathbb{D} \times \cdots \times \mathbb{D}$ (ν-times) by the permutations of the factors. This identification $C_+^0(\mathbb{D}) = \mathrm{Sym}(\mathbb{D})$ will play the key role.

For $c \in C_+^q(X)$, we consider various closed imbeddings $i : U \hookrightarrow \mathbb{D}^q \times \mathbb{D}^{N-q} \subset \mathbb{C}^N$, where U is an open subset of X and $\mathbb{D}^a$ denotes the unit polydisc in $\mathbb{C}^a$. Say $c = \sum_i m_i Y_i$ as before. We say that i is an <u>well-adjusted chart</u> for c if the projection $Y_i \cap U \longrightarrow \mathbb{D}^q$ is proper and finite-to-one everywhere for all i. If i is such, the intersection cycle $c \cdot (b \times \mathbb{D}^{N-q})$ is well defined in $\mathbb{D}^q \times \mathbb{D}^{N-q}$, which projects to a 0-cycle in $\mathbb{D}^{N-q}$, where b is any point in $\mathbb{D}^q$. This 0-cycle will be denoted by $s(c, b) \in C_+^0(\mathbb{D}^{N-q}) = \mathrm{Sym}(\mathbb{D}^{N-q})$.

We say that $f : V \longrightarrow C_+^q(X)$ is <u>complex-analytic</u> if for every $v \in V$ the following conditions are satisfied:

(a) for every open neighborhood A of $|f(v)|$ in X, there exists an open neighborhood B of

v in V such that if $u \in B$ then $|f(u)| \subset A$.

(b) for every $x \in |f(v)|$, there exists an open neighborhood B' of v in V and a closed imbedding $i : U \hookrightarrow \mathbb{D}^q \times \mathbb{D}^{N-q}$ which is well-adjusted for all $f(u)$ with $u \in B'$ and such that the map $B' \times \mathbb{D}^q \longrightarrow \mathrm{Sym}(\mathbb{D}^{N-q})$ by $(u, b) \longmapsto s(f(u), b)$ is a complex-analytic morphism.

Barlet proved the existence of a complex-analytic map $\beta : \mathcal{B}_q \longrightarrow C^q_+(X)$ where $\mathcal{B}_q$ is a reduced complex space and β is <u>bijective</u> (point-set-theoretically). We call $\mathcal{B}_q$ the <u>Barlet space</u> (in dimension q) of X.

<u>PROBLEM I</u>. For a compact complex space X, if m is an integer ≥ 0 and if every connected component of $\mathcal{B}_q$ for every $q \leq m$ is compact, then is it true that every connected component of $\mathcal{D}m$ is compact?

It is known by Fujiki [3] and others

that if X is a compact Kähler manifold, then every connected component of $\mathfrak{B}_q$ for every $q \geq 0$ is compact. Hence PROBLEM I leads us to:

PROBLEM I* Assume that X is a compact Kähler complex manifold. Is it then true that every connected component of $\mathfrak{D}_m$ for every $m \geq 0$ should be compact?

In order to give a good sense to the next problem, we need to extend the notion of "Kähler" to a general complex space (with singularities). Let Z be a complex space. We then consider various pairs (j, ω) where j is a closed imbedding of the restriction of Z to an open subset, say U, into a complex manifold W (e.g. an open polydisc in some $\mathbb{C}^N$) and ω is a Kähler form on W. For each point $u \in U$, ω defines a hermitian form on the tangent space of W at $j(u)$. Hence by j it induces a hermitian form on the Zariski tangent space of Z at u. This induced form will be denoted by $(j,\omega)_u$. We shall say that Z is <u>Kähler</u> if there exists an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of Z and a pair (j_α,ω_α) with $j_\alpha : U_\alpha \to W_\alpha$ for every $\alpha\in\Lambda$ such that for every $\alpha,\beta\in\Lambda$ and for every $u\in U_\alpha\cap U_\beta$ we have $(j_\alpha,\omega_\alpha)_u = (j_\beta,\omega_\beta)_u$.

<u>PROBLEM II</u> When X is a Kähler compact complex space, is it true that $\mathcal{D}$ is Kähler?

As a special case of this problem, we ask

<u>PROBLEM II*</u> Let $f: M \to N$ be a flat surjective morphism of compact complex manifolds. Is it true that if M is Kähler, so is N?

References

[1] Douady, A., "Le problème de modules pour les sous-espaces analytiques compacts d'un espace analytique donné", Ann. Inst. Fourier, Grenoble 16 (1966), pp. 1-95

[2] Barlet, D., "Espace analytique réduit des cycles analytiques complexes de dimension finie", Sem. F. Norguet, Lecture note in Math. 482 Springer (1975), pp. 1-158

[3] Fujiki, A., "Closedness of the Douady Spaces of Compact Kähler Spaces", Mimeographed note (RIMS, Kyoto Univ.).