Title	Conserved quantities of the integrable discrete hungry systems
Author(s)	Kakizaki, Sonomi; Fukuda, Akiko; Yamamoto, Yusaku; Iwasaki, Masashi; Ishiwata, Emiko; Nakamura, Yoshimasa
Citation	Discrete and Continuous Dynamical Systems - Series S (2015), 8(5): 889-899
Issue Date	2015-10
URL	http://hdl.handle.net/2433/202060
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Type	Journal Article
Textversion	author

Conserved Quantities of the Integrable Discrete Hungry Systems

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Abstract

In this paper, conserved quantities of the discrete hungry Lotka-Volterra (dhLV) system are derived. Our approach is based on the Lax representation of the dhLV system, which expresses the time evolution of the dhLV system as a similarity transformation on a certain square matrix. Thus, coefficients of the characteristic polynomial of this matrix constitute conserved quantities of the dhLV system. These coefficients are calculated explicitly through a recurrence relation among the characteristic polynomials of its leading principal submatrices. The conserved quantities of the discrete hungry Toda (dhToda) equation is also derived with the help of the Bäcklund transformation between the dhLV system and the dhToda equation.

keywords: Conserved quantities, discrete hungry Lotka-Volterra system, discrete hungry Toda equation, characteristic polynomial, leading principal submatrix.

2010 MSC: Primary: 37K10; Secondary: 39A99

This was partially supported by Grants-in-Aid for Young Scientists (B) No. 20740064, Scientific Research (C) No. 23540163 from the Japan Society for the Promotion of Science, and Core Research for Evolutional Science and Technology (CREST) Program "Highly Productive, High Performance Application Frameworks for Post Petascale Computing" of Japan Science and Technology Agency (JST)

1 Introduction

It is well known that integrable systems have conserved quantities. They play a key role in analysis of integrable systems and in construction of their analytical solutions. Conserved quantities are also useful in designing numerical algorithms based on integrable systems and in analyzing their convergence properties. For example, the famous integrable discrete Toda (dhToda) equation is shown to be applicable to computing the eigenvalues of a positive definite tridiagonal matrix [6]. The resulting algorithm is just the same as the quotient difference (qd) algorithm, which is employed in the worldwide famous linear algebra package [7]. Also, based on the integrable discrete Lotka-Volterra (dhLV) system, an efficient algorithm for computing the singular values of a bidiagonal matrix has been proposed [1, 2]. To establish the global convergence of these algorithms, the conserved quantities of the dhToda equation and the dhLV system are used effectively.

Recently, in [3, 5], some of the authors proposed two types of eigenvalue computation algorithms based on two integrable discrete hungry systems, namely, the discrete hungry Lotka-Volterra (dhLV) system and the discrete hungry Toda (dhToda) equation. The hungry Lotka-Volterra (hLV) system is a generalization of the well known Lotka-Volterra system and describes a predator-pray interaction between species, in which the kth species preys on the $(k+1)$th, $(k+2)$th, $\ldots$, $(k+M)$th ones. The dhLV system is an integrable time-discretization of the hLV system and is given as

$$\begin{cases} u_k^{(n+1)} = u_k^{(n)} \displaystyle\prod_{j=1}^{M} \frac{1+\delta^{(n)} u_{k+j}^{(n)}}{1+\delta^{(n+1)} u_{k-j}^{(n+1)}}, \quad k=1,2,\ldots,M_m, \quad n=0,1,\ldots, \\ u_{-M+1}^{(n)} \equiv 0,\ u_{-M+2}^{(n)} \equiv 0, \ldots, u_0^{(n)} \equiv 0, \\ u_{M_m+1}^{(n)} \equiv 0,\ u_{M_m+2}^{(n)} \equiv 0, \ldots, u_{M_m+M}^{(n)} \equiv 0, \end{cases} \tag{1}$$

where $u_k^{(n)}$ and $\delta^{(n)}$ correspond to the number of the kth species and the discretization parameter at the discrete time n, respectively, and $M_k := (M+1)k - M$. The dhLV system (1) has a matrix representation called the Lax representation, which expresses the time evolution from n to $n+1$ as a similarity transformation on an $(M_m+M) \times (M_m+M)$ matrix. The eigenvalue computation algorithm given in [3] is derived from this representation. In the derivation of the algorithm, it was shown that the following two quantities are conserved quantities of the dhLV system:

$$\sum_{i=1}^{M_m} \left[u_i^{(n)} \prod_{j=1}^{M} (1+\delta^{(n)} u_{i-j}^{(n)}) \right], \tag{2}$$

$$\prod_{i=1}^{m} \left[u_{M_i}^{(n)} \prod_{j=1}^{M} (1+\delta^{(n)} u_{M_i-j}^{(n)}) \right]. \tag{3}$$

The conserved quantities in (2) and (3) coincide with the coefficient of the $(M_m - 1)$th order term and the constant term in the characteristic polynomial of the Lax matrix, respectively. It is obvious that other conserved quantities of the dhLV system (1) can be obtained from other coefficients of the characteristic polynomial of the Lax matrix. However, it is not usually easy to explicitly get the coefficients of all the terms in the characteristic polynomial of a matrix. Thus the explicit form of them has not been completely reported yet. The first purpose of this paper is to get these coefficients, or the conserved quantities of the dhLV system (1), by using elementary linear algebraic techniques.

In the study of the numbered ball and box system [8], an inverse ultra-discretization leads to the dhToda equation

$$\begin{cases} Q_k^{(n+M)} = Q_k^{(n)} + E_k^{(n)} - E_{k-1}^{(n+1)}, \quad k = 1, 2, \ldots, m, \\ E_k^{(n+1)} = \dfrac{Q_{k+1}^{(n)}}{Q_k^{(n+M)}} E_k^{(n)}, \quad k = 1, 2, \ldots, m-1, \\ E_0^{(n)} \equiv 0, \end{cases} \tag{4}$$

where the subscript k and the superscript n correspond to spatial and discrete-time variables, respectively. The conserved quantities of the dhToda equation (4) are derived in [8] by a combinatorial approach. Since the dhToda equation (4) is related to the dhLV system (1) by a transformation of variables called the Bäcklund transformation, it is worth noting that we may get the conserved quantities of the dhToda equation (4) by combining the Bäcklund transformation with the conserved quantities of the dhLV system (1). The second purpose is to comprehend the conserved quantities of the dhToda equation (4) again from the viewpoint of basic linear algebra.

This paper is organized as follows. In §2, we first derive a recurrence relation among the characteristic polynomials of the leading principal submatrices of the Lax matrix of the dhLV system (1). By solving the recurrence relation, we obtain the explicit form of the characteristic polynomial of the Lax matrix. The coefficients of this polynomial give the conserved quantities of the dhLV system (1). In §3, with the help of the Bäcklund transformation between the dhLV system (1) and the dhToda equation (4), we also present the conserved quantities of the dhToda equation (4). Finally in §4, we give some conclusions.

2 Conserved quantities of the dhLV system

In this section, we derive the conserved quantities of the dhLV system (1). To this end, we focus on the Lax representation for the dhLV system (1) and compute the characteristic polynomial of a matrix appearing in it explicitly. It can be shown that the coefficients of the characteristic polynomial are the conserved quantities of the dhLV system (1).

2.1 Lax representations

As with many other integrable systems, the dhLV system (1) has the Lax representations. The Lax representation expresses the time evolution of the dhLV system (1) as a similarity transformation on a square matrix $L^{(n)}$. Thus the eigenvalues of $L^{(n)}$ are independent of n and so are the coefficients of its characteristic polynomial. Therefore, if we can compute these coefficients explicitly, they constitute the conserved quantities of the dhLV system (1).

According to [4], two kinds of the Lax representations for the dhLV system (1) have been known so far. In the 1st Lax representation, $L^{(n)}$ is an $(M_m + M) \times (M_m + M)$ upper Hessenberg matrix with the upper bandwidth M. In the 2nd Lax representation, the matrix, which we denote by $\mathcal{A}^{(n)}$, is a product of one $m \times m$ lower bidiagonal matrix and M upper bidiagonal matrices. These two representations are essentially equivalent, in the sense that the eigenvalues of $L^{(n)}$ are the $(M+1)$th roots of those of $\mathcal{A}^{(n)}$ [5]. Thus, the coefficients of the characteristic polynomials of these two matrices lead to the same set of conserved quantities. In this paper, we focus on the 1st Lax representation, because its characteristic polynomial can be computed using only elementary linear algebra. Note that these Lax representations play a key role in designing eigenvalue algorithms for some special type of band matrix or a totally nonnegative band matrix [3].

The 1st Lax representation of the dhLV system (1) is defined as

$$L^{(n+1)} R^{(n)} = R^{(n)} L^{(n)}, \tag{5}$$

where $L^{(n)}$ and $R^{(n)}$ are banded matrices

$$L^{(n)} = \begin{pmatrix} \overbrace{0 & \dots & 0}^{M} & U_1^{(n)} & & & & \\ 1 & 0 & \dots & 0 & U_2^{(n)} & & & \\ & \ddots & \ddots & & \ddots & \ddots & & \\ & & \ddots & \ddots & & \ddots & U_{M_m}^{(n)} & \\ & & & \ddots & \ddots & & 0 & \\ & & & & \ddots & \ddots & \vdots & \\ & & & & & 1 & 0 & \end{pmatrix}, \tag{6}$$

$$R^{(n)} = \begin{pmatrix} V_1^{(n)} & & & & & & \\ 0 & V_2^{(n)} & & & & & \\ \vdots & 0 & \ddots & & & & \\ 0 & \vdots & \ddots & \ddots & & & \\ \delta^{(n)} & 0 & & \ddots & \ddots & & \\ & \ddots & \ddots & & \ddots & \ddots & \\ & & \delta^{(n)} & \multicolumn{3}{c}{\underbrace{0 \quad \ldots \quad 0}_{M}} & V_{M_m+M}^{(n)} \end{pmatrix}, \tag{7}$$

with the entries $U_k^{(n)}$ and $V_k^{(n)}$ which are expressed by using the dhLV variables as

$$U_k^{(n)} := u_k^{(n)} \prod_{j=1}^{M} (1 + \delta^{(n)} u_{k-j}^{(n)}), \tag{8}$$

$$V_k^{(n)} := \prod_{j=0}^{M} (1 + \delta^{(n)} u_{k-j}^{(n)}). \tag{9}$$

The equalities in the entries of the Lax representation (5) are just equivalent to the dhLV system (1). Of course, the evolution from n to $n+1$ of $u_k^{(n)}$ in the dhLV system (1) corresponds to that in the Lax representation (5). It is obvious that if $u_1^{(0)} > 0, u_2^{(0)} > 0, \ldots, u_{M_m}^{(0)} > 0$, then $u_1^{(n)} > 0, u_2^{(n)} > 0, \ldots, u_{M_m}^{(n)} > 0$ and also $V_1^{(n)} > 0, V_2^{(n)} > 0, \ldots, V_{M_m+M}^{(n)} > 0$ for $n = 0, 1, \ldots$. So, $R^{(n)}$ is nonsingular and there exists the inverse $(R^{(n)})^{-1}$. Thus, we can rewrite the Lax representation (5) as

$$L^{(n+1)} = R^{(n)} L^{(n)} (R^{(n)})^{-1}, \tag{10}$$

which implies that $L^{(n)}$ and $L^{(n+1)}$ are similar to each other. In other words, the characteristic polynomials of $L^{(n)}$ and $L^{(n+1)}$ are not changed under the evolution from n to $n+1$ in the dhLV system (1). Therefore, the coefficients of the characteristic polynomial of $L^{(n)}$ are conserved quantities of the dhLV system (1).

2.2 Characteristic polynomial of the Lax matrix $L^{(n)}$

Hereinafter, let us consider obtaining an explicit form of the characteristic polynomial of $L^{(n)}$. For simplicity, let $\bar{L}^{(n)} := L^{(n)} - \lambda I$. Moreover, let $\bar{L}_k^{(n)}$ denote the leading principal submatrix of order k. Since $\bar{L}_1^{(n)} = -\lambda$ and $\bar{L}_2^{(n)}, \bar{L}_3^{(n)}, \ldots, \bar{L}_M^{(n)}$ are lower bidiagonal matrices with all the diagonal entries $-\lambda$, it immediately follows that

$$\det(\bar{L}_k^{(n)}) = (-\lambda)^k, \quad k = 1, 2, \ldots, M. \tag{11}$$

With respect to $\det(\bar{L}_k^{(n)})$ for $k = M+1, M+2, \ldots, M_m + M$, we derive a recurrence relation among them through considering cofactor expansions of determinants. In the following, we express k as $k = M + \ell + 1$, where $\ell \in \{0, 1, \ldots, M_m - 1\}$. The cofactor expansion of $\det(\bar{L}_{M+\ell+1}^{(n)})$ by the $(M+\ell+1)$th row gives

$$
\begin{aligned}
\det(\bar{L}_{M+\ell+1}^{(n)}) = (-1) \det &\begin{pmatrix}
-\lambda & & & & U_1^{(n)} & & & \\
1 & -\lambda & & & & \ddots & & \\
 & 1 & \ddots & & & & U_{\ell-1}^{(n)} & \\
 & & \ddots & \ddots & & & & 0 \\
 & & & \ddots & \ddots & & & U_{\ell+1}^{(n)} \\
 & & & & \ddots & -\lambda & & \\
 & & & & & 1 & -\lambda & \\
 & & & & & & 1 &
\end{pmatrix} \\
&+ (-\lambda) \det(\bar{L}_{M+\ell}^{(n)}).
\end{aligned}
\tag{12}
$$

By further expanding the 1st determinant in the right hand side of (12) with respect to the last column, we have

$$
\begin{aligned}
&\det(\bar{L}_{M+\ell+1}^{(n)}) \\
&\quad = (-1)^{M+2(\ell+1)} U_{\ell+1}^{(n)} \\
&\qquad \times \det \left(\begin{array}{ccccccc|cccc}
-\lambda & & & & & U_1^{(n)} & & & & & \\
1 & -\lambda & & & & & \ddots & & & & \\
 & 1 & \ddots & & & & & \ddots & & & \\
 & & \ddots & \ddots & & & & & \ddots & & \\
 & & & \ddots & \ddots & & & & & \ddots & \\
 & & & & \ddots & \ddots & & & & & U_{\ell-1}^{(n)} \\
 & & & & & 1 & -\lambda & & & & \\
\hline
 & & & & & & & 1 & -\lambda & & \\
 & & & & & & & & 1 & \ddots & \\
 & & & & & & & & & \ddots & -\lambda \\
 & & & & & & & & & & 1
\end{array}\right) \\
&\qquad + (-\lambda) \det(\bar{L}_{M+\ell}^{(n)}) \\
&\quad = (-1)^M U_{\ell+1}^{(n)} \det(\bar{L}_\ell^{(n)}) + (-\lambda) \det(\bar{L}_{M+\ell}^{(n)}).
\end{aligned}
\tag{13}
$$

From (11) and (13), it is concluded that the leading principal minors of $\bar{L}^{(n)}$ satisfy the recurrence relation

$$\begin{cases} \det(\bar{L}_k^{(n)}) = (-\lambda)^k, \quad k = 1, 2, \dots, M, \\ \det(\bar{L}_{M+\ell+1}^{(n)}) = (-1)^M U_{\ell+1}^{(n)} \det(\bar{L}_\ell^{(n)}) + (-\lambda) \det(\bar{L}_{M+\ell}^{(n)}), \quad \ell = 0, 1, \dots, M_m - 1, \end{cases} \tag{14}$$

where $\det(\bar{L}_0) \equiv 1$. The principal minors $\det(\bar{L}_{M+1}^{(n)}), \det(\bar{L}_{M+2}^{(n)}), \dots, \det(\bar{L}_{M_m+M}^{(n)})$ can be successively calculated by the 2nd equation of (14) under the initial condition given by the 1st one. Therefore, by taking into account that $\bar{L}_{M_m+M}^{(n)} = \bar{L}^{(n)}$, we see that the characteristic polynomial of $L^{(n)}$ is derived from (14). As an example of the characteristic polynomial of $L^{(n)}$, we here present the case where $M = 2$ and $m = 3$ for readers' understanding. The explicit form of the characteristic polynomial of $L^{(n)}$ is given as

$$\begin{aligned} &-\lambda^9 + (U_1^{(n)} + U_2^{(n)} + U_3^{(n)} + U_4^{(n)} + U_5^{(n)} + U_6^{(n)} + U_7^{(n)})\lambda^6 \\ &\quad - [U_1^{(n)}(U_4^{(n)} + U_5^{(n)} + U_6^{(n)} + U_7^{(n)}) + U_2^{(n)}(U_5^{(n)} + U_6^{(n)} + U_7^{(n)}) \\ &\quad + U_3^{(n)}(U_6^{(n)} + U_7^{(n)}) + U_4^{(n)}U_7^{(n)}]\lambda^3 + U_1^{(n)}U_4^{(n)}U_7^{(n)}, \end{aligned}$$

since it follows from (14) that

$$\begin{aligned} &\det(\bar{L}_1^{(n)}) = -\lambda, \quad \det(\bar{L}_2^{(n)}) = \lambda^2, \quad \det(\bar{L}_3^{(n)}) = U_1^{(n)} - \lambda^3, \\ &\det(\bar{L}_4^{(n)}) = \lambda^4 - (U_1^{(n)} + U_2^{(n)})\lambda, \quad \det(\bar{L}_5^{(n)}) = -\lambda^5 + (U_1^{(n)} + U_2^{(n)} + U_3^{(n)})\lambda^2, \\ &\det(\bar{L}_6^{(n)}) = \lambda^6 - (U_1^{(n)} + U_2^{(n)} + U_3^{(n)} + U_4^{(n)})\lambda^3 + U_1^{(n)} + U_4^{(n)}, \\ &\det(\bar{L}_7^{(n)}) = -\lambda^7 + (U_1^{(n)} + U_2^{(n)} + U_3^{(n)} + U_4^{(n)} + U_5^{(n)})\lambda^4 \\ &\qquad\qquad - [U_1^{(n)}(U_4^{(n)} + U_5^{(n)}) + U_2^{(n)}U_5^{(n)}]\lambda, \\ &\det(\bar{L}_8^{(n)}) = \lambda^8 - (U_1^{(n)} + U_2^{(n)} + U_3^{(n)} + U_4^{(n)} + U_5^{(n)} + U_6^{(n)})\lambda^5 \\ &\qquad\qquad + [U_1^{(n)}(U_4^{(n)} + U_5^{(n)} + U_6^{(n)}) + U_2^{(n)}(U_5^{(n)} + U_6^{(n)}) + U_3^{(n)}U_6^{(n)}]\lambda^2, \\ &\det(\bar{L}_9^{(n)}) = -\lambda^9 + (U_1^{(n)} + U_2^{(n)} + U_3^{(n)} + U_4^{(n)} + U_5^{(n)} + U_6^{(n)} + U_7^{(n)})\lambda^6 \\ &\qquad\qquad + [U_1^{(n)}(U_4^{(n)} + U_5^{(n)} + U_6^{(n)} + U_7^{(n)}) + U_2^{(n)}(U_5^{(n)} + U_6^{(n)} + U_7^{(n)}) \\ &\qquad\qquad + U_3^{(n)}(U_6^{(n)} + U_7^{(n)}) + U_4^{(n)}U_7^{(n)}]\lambda^3 + U_1^{(n)}U_4^{(n)}U_7^{(n)}. \end{aligned}$$

2.3 Conserved quantities

In this subsection, we solve the recurrence relation (14) and obtain the coefficients of the characteristic polynomial of $L^{(n)}$ explicitly. These coefficients give the conserved quantities of the dhLV system (1).

By explicitly calculating some of the leading principal minors of $\bar{L}^{(n)}$ in (14), we can conjecture that they are expressed as

$$\det(\bar{L}^{(n)}_{(M+1)p+q}) = (-\lambda)^{(M+1)p+q} + \sum_{i=1}^{p} \left[(-1)^{iM}(-\lambda)^{(M+1)(p-i)+q} \sum_{(j_1,j_2,\ldots,j_i)\in\Psi_{i,p,q}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i} \right], \tag{15}$$

where $\Psi_{i,p,q} = \{(j_1, j_2, \ldots, j_i) | 1 \le j_1,\ j_1 + M + 1 \le j_2,\ j_2 + M + 1 \le j_3,\ \cdots,\ j_{i-1} + M + 1 \le j_i \le M_p + q\}$. The following discussion proves (15) by induction on $k = (M+1)p + q$. For this purpose, we prepare a lemma on the summation of $U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i}$ over the index set $\Psi_{i,p,q}$.

Lemma 2.1. *For $i = 1, 2, \ldots, p$ and $q = 0, 1, \ldots, M$, it holds that*

$$\begin{aligned} \sum_{(j_1,j_2,\ldots,j_i)\in\Psi_{i,p,q}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i} &= \sum_{(j_1,j_2,\ldots,j_i)\in\Psi_{i,p,q-1}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i} \\ &\quad + U^{(n)}_{M_p+q} \sum_{(j_1,j_2,\ldots,j_{i-1})\in\Psi_{i-1,p-1,q}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_{i-1}}. \end{aligned} \tag{16}$$

In particular, for $i = p$ and $q = 0$, it holds that

$$\sum_{(j_1,j_2,\ldots,j_p)\in\Psi_{p,p,0}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_p} = U^{(n)}_{M_p} \sum_{(j_1,j_2,\ldots,j_{p-1})\in\Psi_{p-1,p-1,0}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_{p-1}}. \tag{17}$$

Proof. We divide the sum on the left hand side of (16) into two parts, the one not involving $U^{(n)}_{M_p+q}$ and the one involving $U^{(n)}_{M_p+q}$. It is obvious that the 1st part can be written as the 1st term on the right hand side of (16). As for the 2nd part, it holds that $j_i = M_p + q$. Since $j_{i-1} + M + 1 \le j_i$, we have $j_{i-1} \le M_{p-1} + q$. This leads to the 2nd term on the right hand side of (16). Now, consider the case where $i = p$ and $q = 0$. In this case, the 1st term on the right hand side of (16) vanishes since $\Psi_{p,p,-1}$ is the null set. Hence we have (17). □

With the help of Lemma 2.1, we get a theorem concerning the explicit form of the principal minors of $\bar{L}^{(n)}$.

Theorem 2.2. *For $p = 0, 1, \ldots, m$ and $q = 0, 1, \ldots, M$ such that $1 \le (M+1)p + q \le (M+1)m$, the principal minors of $\bar{L}^{(n)}$ has the explicit form as in* (15).

Proof. The proof is given by induction on $k = (M+1)p + q$.

For $k = 1, 2, \ldots, M$, namely, $p = 0$ and $q = 1, 2, \ldots, M$, it is obvious that the 2nd term on the right hand side of (15) becomes 0. Thus, in this case, (15) coincides with the 1st equation of (14). In the case where $k = M+1$, namely, $p = 1$ and $q = 0$, by taking into account that $i = 1$ and $\Psi_{1,1,0} = \{i_1 | i_1 = 1\}$ in (15), we get $\det(\bar{L}^{(n)}_{M+1}) = (-\lambda)^{M+1} + (-1)^M U^{(n)}_1$. This coincides with the result obtained by letting $\ell = 0$ in the 2nd equation of (14) and using the 1st equation of (14). Thus, (15) gives the correct result in the case where $k = M+1$.

Let us consider the case where $k = (M+1)p + q$ for $p = 1, 2, \ldots, m$ and $q = 1, 2, \ldots, M$. The replacement of $M + \ell + 1$ with $(M+1)p + q$ in the subscript in the 2nd equation of (14), immediately leads to

$$\begin{aligned}
&\det(\bar{L}^{(n)}_{(M+1)p+q}) \\
&\quad = (-1)^M U^{(n)}_{M_p+q} \det(\bar{L}^{(n)}_{(M+1)(p-1)+q}) + (-\lambda)\det(\bar{L}^{(n)}_{(M+1)p+q-1}) \\
&\quad = (-1)^M U^{(n)}_{M_p+q} \Bigg[(-\lambda)^{(M+1)(p-1)+q} + \sum_{i=1}^{p-1} (-1)^{iM} (-\lambda)^{(M+1)(p-1-i)+q} \\
&\qquad\quad \times \sum_{(j_1,j_2,\ldots,j_i)\in\Psi_{i,p-1,q}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i} \Bigg] + (-\lambda)^{(M+1)p+q} \\
&\qquad\quad + \sum_{i=1}^{p} (-1)^{iM} (-\lambda)^{(M+1)(p-i)+q} \sum_{(j_1,j_2,\ldots,j_i)\in\Psi_{i,p,q-1}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i}. \\
&\quad = (-\lambda)^{(M+1)p+q} + (-1)^M (-\lambda)^{(M+1)(p-1)+q} \sum_{j_1=1}^{M_p+q} U^{(n)}_{j_1} \\
&\qquad\quad + \sum_{i=2}^{p} (-1)^{iM} (-\lambda)^{(M+1)(p-i)+q} \Bigg(\sum_{(j_1,j_2,\ldots,j_i)\in\Psi_{i,p,q-1}} U^{(n)}_{j_1} U^{(n)}_{j_2} \ldots U^{(n)}_{j_i} \\
&\qquad\quad + U^{(n)}_{M_p+q} \sum_{(j_1,j_2,\ldots,j_{i-1})\in\Psi_{i-1,p-1,q}} U^{(n)}_{j_1} U^{(n)}_{j_2} \ldots U^{(n)}_{j_{i-1}} \Bigg).
\end{aligned} \tag{18}$$

By using Lemma 2.1 in (18), we can integrate the two summations with respect to $j_1, j_2, \ldots, j_{i-1}$ and $j_1, j_2, \ldots, j_i$ into $\sum_{(j_1,j_2,\ldots,j_i)\in\Psi_{i,p,q}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i}$. Moreover, by noting that the sum $\sum_{j_1=1}^{M_p+q} U^{(n)}_{j_1}$ in (18) can be rewritten as $\sum_{j_1\in\Psi_{1,p,q}} U^{(n)}_{j_1}$, we have (15).

The proof remains the case where $k = (M+1)p$ for $p = 2, 3, \ldots, m$. From (14), we have

$$\det(\bar{L}^{(n)}_{(M+1)p}) = (-1)^M U^{(n)}_{M_p} \det(\bar{L}^{(n)}_{(M+1)(p-1)}) - \lambda \det(\bar{L}^{(n)}_{(M+1)(p-1)+M}). \tag{19}$$

Under the assumption that $\det(\bar{L}^{(n)}_{(M+1)(p-1)})$ and $\det(\bar{L}^{(n)}_{(M+1)(p-1)+M})$ can be given as

(15), by letting $q=0$, we get

$$
\begin{aligned}
&\det(\bar{L}^{(n)}_{(M+1)p}) \\
&= (-1)^M U^{(n)}_{M_p} \left[(-\lambda)^{(M+1)(p-1)} + \sum_{i=1}^{p-1} (-1)^{iM} (-\lambda)^{(M+1)(p-1-i)} \right. \\
&\qquad \left. \times \sum_{(j_1,j_2,\dots,j_i)\in\Psi_{i,p-1,0}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i} \right] + (-\lambda)^{(M+1)(p-1)+M+1} \\
&\quad + \sum_{i=1}^{p-1} (-1)^{iM} (-\lambda)^{(M+1)(p-1-i)+M+1} \sum_{(j_1,j_2,\dots,j_i)\in\Psi_{i,p-1,M}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i} \\
&= (-\lambda)^{(M+1)p} + (-1)^M (-\lambda)^{(M+1)(p-1)} \sum_{j_1=1}^{M_p} U^{(n)}_{j_1} \\
&\quad + \sum_{i=2}^{p-1} (-1)^{iM} (-\lambda)^{(M+1)(p-i)} \left(\sum_{(j_1,j_2,\dots,j_i)\in\Psi_{i,p-1,M}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i} \right. \\
&\qquad \left. + U^{(n)}_{M_p} \sum_{(j_1,j_2,\dots,j_{i-1})\in\Psi_{i-1,p-1,0}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_{i-1}} \right) \\
&\quad + (-1)^{pM} U^{(n)}_{M_p} \sum_{(j_1,j_2,\dots,j_{p-1})\in\Psi_{p-1,p-1,0}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_{p-1}} \\
&= (-\lambda)^{(M+1)p} + (-1)^M (-\lambda)^{(M+1)(p-1)} \sum_{j_1\in\Psi_{1,p,0}} U^{(n)}_{j_1} \\
&\quad + \sum_{i=2}^{p-1} (-1)^{iM} (-\lambda)^{(M+1)(p-i)} \sum_{(j_1,j_2,\dots,j_i)\in\Psi_{i,p,0}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i} \\
&\quad + (-1)^{pM} \sum_{(j_1,j_2,\dots,j_p)\in\Psi_{p,p,0}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_p} \\
&= (-\lambda)^{(M+1)p} \\
&\quad + \sum_{i=1}^{p} (-1)^{iM} (-\lambda)^{(M+1)(p-i)} \sum_{(j_1,j_2,\dots,j_i)\in\Psi_{i,p,0}} U^{(n)}_{j_1} U^{(n)}_{j_2} \cdots U^{(n)}_{j_i},
\end{aligned}
\tag{20}
$$

where in the 3rd equality, we rewrote $\sum_{j_1=1}^{M_p} U^{(n)}_{j_1}$ as $\sum_{j_1\in\Psi_{1,p,0}} U^{(n)}_{j_1}$ to derive the 2nd term in the right-hand side. To derive the 3rd and 4th terms, we used (16) and (17) in Lemma 2.1, respectively. □

It is obvious that (15) with $p=m$ and $q=0$ is the characteristic polynomial of $L^{(n)}$. Thus, by giving our focus on the coefficients of it, we have a theorem for the conserved quantities of the dhLV system (1).

Theorem 2.3. *The conserved quantities of the dhLV system* (1) *is given as*

$$\sum_{(j_1,j_2,\ldots,j_i)\in\Psi_{i,m,0}} U_{j_1}^{(n)}U_{j_2}^{(n)}\cdots U_{j_i}^{(n)}, \quad i=1,2,\ldots,m. \tag{21}$$

The cases where $i=1$ and $i=m$ in (21) correspond to the conserved quantities given in (2) and (3), respectively.

3 Conserved quantities of the discrete hungry Toda equation

In this section, we derive conserved quantities of the dhToda equation (4) from those of the dhLV system (1) given in Theorem 2.3.

In [4], some of the authors found a transformation of variables, called the Bäcklund transformation, for relating the dhLV system (1) to the dhToda equation (4).

Theorem 3.1 ([4])**.** *As $\delta^{(n)}\to\infty$, the Bäcklund transformation between the dhLV system* (1) *and the dhToda equation* (4) *is given as*

$$U_{M_k}^{(n)} = Q_k^{(n)}Q_k^{(n+1)}\cdots Q_k^{(n+M-1)}, \quad k=1,2,\ldots,m, \tag{22}$$

$$\begin{aligned} U_{M_k+p}^{(n)} &= E_k^{(n+p-1)}Q_k^{(n+p)}Q_k^{(n+p+1)}\cdots Q_k^{(n+p+M-2)}, \\ &\qquad k=1,2,\ldots,m-1, \quad p=1,2,\ldots,M. \end{aligned} \tag{23}$$

Theorem 3.1 is built by relating the Lax matrices of the dhLV variables to those of the dhToda ones by using a simple linear algebraic argument. It is natural to expect that the conserved quantities of the dhToda equation (4) can be derived from those of the dhLV system (1) by combining Theorem 2.3 with Theorem 3.1. For notational convenience, let us rewrite the dhToda variables $Q_k^{(n)}$ and $E_k^{(n)}$ using a new variable $w_\ell^{(n)}$ as

$$w_{(k-1)(M+1)+j}^{(n)} = \begin{cases} E_{k-1}^{(n)}, & k=2,3,\ldots,m, \quad j=0, \\ Q_k^{(n+j-1)}, & k=1,2,\ldots,m, \quad j=1,2,\ldots,M. \end{cases} \tag{24}$$

According to the subscript of the new variable, the original dhToda variables are reordered as $Q_1^{(n)}$, $Q_1^{(n+1)}$, $\ldots$, $Q_1^{(n+M-1)}$, $E_1^{(n)}$, $Q_2^{(n)}$, $Q_2^{(n+1)}$, $\ldots$, $Q_2^{(n+M-1)}$, $E_2^{(n)}$, $\ldots$, $Q_{m-1}^{(n)}$, $Q_{m-1}^{(n+1)}$, $\ldots$, $Q_{m-1}^{(n+M-1)}$, $E_{m-1}^{(n)}$, $Q_m^{(n)}$, $Q_m^{(n+1)}$, $\ldots$, $Q_m^{(n+M-1)}$. Equation (22) is easily rewritten using the new variable $w_\ell^{(n)}$ as

$$U_{M_k}^{(n)} = w_{(k-1)(M+1)+1}^{(n)}w_{(k-1)(M+1)+2}^{(n)}\cdots w_{(k-1)(M+1)+M}^{(n)}. \tag{25}$$

To rewrite (23), we note that $Q_k^{(n+M)}E_k^{(n+1)} = E_k^{(n)}Q_{k+1}^{(n)}$ holds from the 2nd equation of the dhToda equation (4). By applying it to the right-hand side of (23) repeatedly and by using (24), we get

$$\begin{aligned} U_{M_k+p}^{(n)} &= Q_k^{(n+p)} Q_k^{(n+p+1)} \cdots Q_k^{(n+p+M-2)} E_k^{(n+p-1)} \\ &= Q_k^{(n+p)} Q_k^{(n+p+1)} \cdots Q_k^{(n+p+M-3)} E_k^{(n+p-2)} Q_{k+1}^{(n+p-2)} \\ &\vdots \\ &= Q_k^{(n+p)} Q_k^{(n+p+1)} \cdots Q_k^{(n+M-1)} E_k^{(n)} Q_{k+1}^{(n)} Q_{k+1}^{(n+1)} \cdots Q_{k+1}^{(n+p-2)} \\ &= w_{(k-1)(M+1)+p+1}^{(n)} w_{(k-1)(M+1)+p+2}^{(n)} \cdots w_{(k-1)(M+1)+p+M}^{(n)}. \end{aligned} \tag{26}$$

Equations (25) and (26) can be unified into the following simple relationship:

$$U_\ell^{(n)} = w_\ell^{(n)} w_{\ell+1}^{(n)} \dots w_{\ell+M-1}^{(n)}, \quad \ell = 1, 2, \dots, M_m. \tag{27}$$

Inserting (27) into (21) of Theorem 2.3 leads to a theorem on conserved quantities of the dhToda equation (4).

Theorem 3.2 (cf. [8]). *The conserved quantities of the dhToda equation* (4) *is given by*

$$\sum_{(j_1, j_2, \dots, j_i) \in \Psi_{i,m,0}} \left(\prod_{q=0}^{M-1} w_{j_1+q}^{(n)} \right) \left(\prod_{q=0}^{M-1} w_{j_2+q}^{(n)} \right) \cdots \left(\prod_{q=0}^{M-1} w_{j_i+q}^{(n)} \right), \quad i = 1, 2, \dots, m. \tag{28}$$

In fact, conserved quantities of the dhToda equation (4) have been derived in [8] by making use of a combinatorial approach. It is observed that the conserved quantities in Theorem 3.2 are equal to those in [8].

4 Conclusion

In this paper, we derived the conserved quantities of the dhLV system by computing the characteristic polynomial of a matrix appearing in its Lax representation. To this end, we first presented a recurrence relation among the characteristic polynomials of the leading principal submatrices of the Lax matrix. Then, by solving the recurrence relation, we obtained the coefficients of the characteristic polynomial of the Lax matrix, which give the conserved quantities of the dhLV system. As an application of this result, we showed that the conserved quantities of the dhToda equation can also be obtained, with the help of the Bäcklund transformation between the dhLV system and the dhToda equation. The conserved quantities of the dhToda equation derived in this way coincide with those given in [8]. It is to be noted that the derivation of the conserved quantities in this paper is achieved using only elementary linear algebraic techniques.

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