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Yokoyama, Naoto ...[et al]. Nonlinear energy in a wave turbulence system. Proceedings of the Estonian Academy of Sciences 2015, 64(3): 430-437

ISSUE DATE:
2015

URL:
http://hdl.handle.net/2433/202076

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Nonlinear energy in a wave turbulence system

Naoto Yokoyama\textsuperscript{a} and Masanori Takaoka\textsuperscript{b}

\textsuperscript{a} Department of Aeronautics and Astronautics, Kyoto University, Kyoto 615-8540, Japan
\textsuperscript{b} Department of Mechanical Engineering, Doshisha University, Kyotanabe 610-0394, Japan

Received 8 December 2014, accepted 20 May 2015, available online 28 August 2015

Abstract. Single-wavenumber representations of nonlinear energies are required to investigate energy budget due to nonlinear interactions among Fourier modes in wave turbulence. While we have reported in a previous paper that the single-wavenumber representations successfully works for the Föppl–von Kármán equation, we will show here that for the Majda–McLaughlin–Tabak model the single-wavenumber representations of nonlinear energies is not necessarily unique. Introducing auxiliary variables composed differently from complex amplitudes, two natural representations of the nonlinear energy are obtained. It is numerically observed that the two kinds of the nonlinear-energy spectra, based on these two representations, are qualitatively similar, but the energy budgets are clearly different. To select the appropriate single-wavenumber representation of the nonlinear energy, the properties which an eligible single-wavenumber representation should have are discussed.

Key words: wave turbulence, nonlinear energy, energy budget.

1. INTRODUCTION

In the pioneer work by Zakharov \cite{Zakharov1968}, it is shown that the dynamics of many wave turbulence systems are governed by Hamiltonians. Complex amplitudes are used as elementary waves in the weak turbulence theory, since they represent linear waves in a canonical representation. The selection of elementary waves determines the form of nonlinear terms in the Hamiltonian, and it reflects the wave interactions of elementary waves. The energy of a wavenumber has been represented as quadratic quantities of the complex amplitudes in studies of the weak turbulence systems. Namely, only the linear part of energy has been considered under the assumption of small nonlinearity. However, the integrated quadratic energy, which is obtained by the integration of the quadratic energy over the wavenumbers, is not conserved in general but conserved only in the weakly nonlinear limit.

In the finite nonlinear regime, the quadratic energy is inappropriate for the investigation of the energy budgets because of its nonconservativity. The nonequilibrium statistically-steady state of wave turbulence is realized in the scales called the inertial subrange between energy-input scales and energy-dissipation scales. Then, the energy fluxes estimated by the energy input and/or the energy dissipation \cite{McLaughlin1984,Tabak1984} have sometimes been used to evaluate the energy flux in the inertial subrange. However, such energy fluxes are dominated by the forcing-range statistics and the dissipation-range statistics. The energy flux as well as the energy transfer should be evaluated directly from the nonlinear term of the governing equation. The quadratic-energy flux obtained in first-principle direct numerical simulations, e.g. \cite{Jin1995}, is non-zero at the smallest or largest wavenumbers.

\textsuperscript{*} Corresponding author, yokoyama@kuaero.kyoto-u.ac.jp
A single-wavenumber representation of the nonlinear energy is indispensable to investigate the energy budget of a wavenumber mode. However, the nonlinear parts of the Hamiltonian consist of the convolution, and they cannot be represented by the nonlinear energy with a single wavenumber in a quadratic form of the complex amplitudes. Recently, a single-wavenumber representation of the nonlinear energy is found in elastic-wave turbulence governed by the Föppl–von Kármán equation [5]. The forward cascade of the energy is reported there in the well-defined manner for the first time, where the detailed energy balance is also confirmed. It is expected to examine how it works in other wave turbulence systems.

To be motivated by the success of the single-wavenumber representation of the nonlinear energy in the elastic-wave turbulence, the properties of such representation are examined here in the Majda–McLaughlin–Tabak (MMT) model [6] as an example of the simplest wave turbulence system. The MMT model is a one-dimensional wave turbulence system that has many interesting characteristics depending on its parameters. It was reported that four statistically-steady states consist with the weak turbulence theory and one statistically-steady state inconsistent with the theory [6–9]. Moreover, it was also reported that their coexistence and nonlocal interactions in the wavenumber space [7–9], and coherent structures such as solitons, quasisolitons, and quasibreathers [10,11].

Two natural single-wavenumber representations of the nonlinear energy are found in the MMT model, and we will report here numerical results obtained from the two representations. The nonuniqueness of the two single-wavenumber representations of the nonlinear energy in the MMT model results in the different appearances of energy spectra and energy budgets, which stem mainly from the nonlocal interactions among modes. From the viewpoint of the energy budgets, the appropriateness of the single-wavenumber representations is discussed.

2. MMT MODEL AND ITS ENERGY

The MMT model is a model for wave turbulence systems which have four-wave nonlinear interactions. In the MMT model, the Hamiltonian is given by the complex amplitudes $a_k$ as

$$H = \sum_k |k|^\alpha |a_k|^2 + \frac{\lambda}{2} \sum_{k+k_1-k_2-k_3=0} |k||k_1||k_2||k_3|^{-\sigma} a_k^* a_k^* a_k a_{k_1},$$

(1)

where the parameter $\alpha > 0$ determines the linear dispersion relation, and the parameters $\lambda = \pm 1$ and $\sigma$ determine the nonlinear interactions. Note that the MMT model can reproduce various kinds of wave turbulence by changing the parameter values. The governing equation of the complex amplitude for a wavenumber $k \in \mathbb{Z}$ is given as a canonical equation:

$$\frac{\partial a_k}{\partial t} = -i \delta H \delta a_k = -i|k|^\alpha a_k - i\lambda \sum_{-k_1+k_2+k_3=k} |k||k_1||k_2||k_3|^{-\sigma} a_k^* a_k a_{k_1} a_{k_2} a_{k_3}.$$  

(2)

Note that $a_k$ is not the complex conjugate of $a_{-k}$.

The MMT model has three conservatives: the total energy, the wave action, and the momentum. The total energy is the Hamiltonian of Eq. (1), and the wave action and the momentum are respectively defined as $\sum_k |a_k|^2$ and $\sum_k k |a_k|^2$. It should be emphasized that the total energy, i.e., the Hamiltonian, includes the quartic form of $a_k$, while the latter two conservatives are expressed in quadratic forms of the complex amplitudes. We here define the linear energy $H^L$ and the nonlinear energy $H^N$ respectively for the first and the second terms in the right-hand side of Eq. (1).

If we introduce two auxiliary variables,

$$b_k = \sum_{k_1+k_2=k} |k_1||k_2|^{-\sigma} a_{k_1} a_{k_2}, \text{ and } c_k = \sum_{k_1-k_2=k} |k_1||k_2|^{-\sigma} a_{k_1}^* a_{k_2},$$

(3)
then the nonlinear part of the Hamiltonian (1) can be rewritten respectively as
\[ \mathcal{H}^N = \sum_k \frac{\lambda}{2} |b_k|^2, \quad \text{and} \quad \mathcal{H}^N = \sum_k \frac{\lambda}{2} |c_k|^2. \] (4)

Namely, the nonlinear energy of \( k \) can be defined naturally as a single-wavenumber mode by the two auxiliary variables,
\[ E_k^b = \frac{\lambda}{2} |b_k|^2, \quad \text{and} \quad E_k^c = \frac{\lambda}{2} |c_k|^2. \] (5)

As the first step to investigate the energy budget, we focus our attention on the energy transfer and the energy flux. The energy transfer is defined as the time derivative of the energy. Thus, the linear-energy transfer is defined as
\[ T^L_k = \frac{\partial E^L_k}{\partial t} = |k|^a a_k^b \frac{\partial a_k^b}{\partial t} + \text{c.c.} = -i\lambda |k|^a \sum_{k_1+k_2+k_3=k} (|k||k_1||k_2||k_3|)^{\sigma} a_k^b a_{k_1}^c a_{k_2} a_{k_3} + \text{c.c.} \] (6)

The total-energy transfers are defined as
\[ T^{Tb}_k = \frac{\partial (E^L_k + E^b_k)}{\partial t} = T^L_k + \left( \frac{\lambda}{2} b_k^c \frac{\partial b_k^c}{\partial t} + \text{c.c.} \right), \quad \text{and} \quad T^{Te}_k = \frac{\partial (E^L_k + E^c_k)}{\partial t} = T^L_k + \left( \frac{\lambda}{2} c_k^c \frac{\partial c_k^c}{\partial t} + \text{c.c.} \right), \] (7)
where the time derivatives \( \partial b_k / \partial t \) and \( \partial c_k / \partial t \) are obtained by Eqs (2) and (3).

The total-energy fluxes are defined as
\[ \mathcal{P}^{Tb}_k = -\sum_{|k'| \leq k} T^{Tb}_{k'}, \quad \text{and} \quad \mathcal{P}^{Te}_k = -\sum_{|k'| > k} T^{Te}_{k'}. \] (8)

Because of the energy conservation, the fluxes at the maximal wavenumber are exactly 0, i.e., \( \mathcal{P}^{Tb}_\infty = \mathcal{P}^{Te}_\infty = 0 \). In other words, these fluxes satisfy \( \mathcal{P}^{Tb}_k = \sum_{|k'| \leq k} T^{Tb}_{k'} \) and \( \mathcal{P}^{Te}_k = \sum_{|k'| > k} T^{Te}_{k'} \). After the conventional flux, the linear-energy pseudo-flux can be defined as
\[ \mathcal{P}^L_k = -\sum_{|k'| \leq k} T^L_{k'}, \] (9)
though it is not an actual flux but a spurious flux since the linear energy is not conservative. It means that \( \mathcal{P}^L_\infty \neq 0 \). Moreover, \( \mathcal{P}^L_k \neq \sum_{|k'| > k} T^L_{k'} \).

3. NUMERICAL RESULTS

We performed numerical simulations of the MMT model, where external force \( F_k \), drag \( D_k^e \) and dissipation \( D_k^d \) are added to obtain a statistically-steady state as
\[ \frac{\partial a_k}{\partial t} = -i\delta \frac{\partial}{\partial a_k} + F_k - D_k^e - D_k^d. \] (10)

The periodic boundary condition with \( 2\pi \) is assumed. The pseudo-spectral method with the aliasing removal by the 4/2-law is used to obtain the convolution in the nonlinear term. The number of the aliasing-free modes is \( N_{\max} = 2^{14} \). The random external force \( F_k \) that is time-un correlated acts at small wavenumbers \( 3 \leq |k| \leq 8 \). The drag \( D_k^e \) is added at smaller wavenumbers as \( D_k^e = \gamma a_k \) for \( |k| \leq 2 \), and the dissipation
the nonlinear energies, based on $b_H$ is much larger than the nonlinear energies in the energy-containing range. Thus, the forcing range, the nonlinear energies have in the higher harmonic range $6$ spectra obtained in the numerical simulation are drawn in Fig. 1. While the linear energy has large values in that the nonlinear energies come from $a_L$, the linear energy is much smaller than the nonlinear energies in the drag range. It is due to the fact for the modes in the larger wavenumber range $2$ power law consistent with the very small nonlinearity there. The modes in this range play the energy source $a$ wavenumber values. The nonlinear spectra in the inertial subrange are almost parallel, because the nonlinear energies for $E$ is close to the MMT spectrum, while the nonlinear-energy spectrum based on $b_L$ have qualitatively similar forms, they have different exponents of the MMT spectrum is $k^{s_L}$. Though the two nonlinear energies $E_{cb}^L$ and $E_{cc}^L$ in the inertial subrange $10 \leq k \leq 2 \times 10^3$, cannot be represented by a single power law. These energy spectra in the smaller wavenumber range $10 \leq k \leq 30$ show a very steep power law consistent with the very small nonlinearity there. The modes in this range play the energy source for the modes in the larger wavenumber range $2 \times 10^3 \leq k \leq 2 \times 10^3$, where the linear-energy spectrum $E_k^{L}$ is close to the MMT spectrum, while the nonlinear-energy spectrum $\delta_k^{L}$ has a less steep slope. Note that the exponent of the MMT spectrum is $\alpha + 2(\sigma - 5/4) = -9/4$ [8].

Though the two nonlinear energies $E_{cb}^L$ and $E_{cc}^L$ have qualitatively similar forms, they have different values. The nonlinear spectra in the inertial subrange are almost parallel, because the nonlinear energies for a wavenumber $k$ are written respectively as

$$
\frac{1}{2} \langle |b_k|^2 \rangle \approx \sum_{k_1+k_2=k} \langle |k_1||k_2| \rangle \langle |a_{k_1}|^2 \rangle \langle |a_{k_2}|^2 \rangle^{2\sigma},
$$

$$
\frac{1}{2} \langle |c_k|^2 \rangle \approx \frac{1}{2} \sum_{k_1+k_2=k} \langle |k_1||k_2| \rangle \langle |a_{k_1}|^2 \rangle \langle |a_{k_2}|^2 \rangle^{2\sigma} + \frac{1}{2} \delta_{k,0} \left( \sum_{k_1} |k_1|^{2\sigma} \langle |a_{k_1}|^2 \rangle \right),
$$

if the random phase approximation can be applied over all the wavenumbers and the system is statistically isotropic. In fact, as shown in Fig. 1, $\delta_k^{L} \approx \delta_k^{cb}/2$ in the range $10 < k < 30$, where the nonlinearity is weak, while $\delta_k^{c} \ll \delta_k^{cb}$ in the range $k > 10^2$, where the nonlinearity is relatively strong. This result suggests the

Fig. 1. Linear- and nonlinear-energy spectra. The abscissa is linearly scaled for $k \leq 1$ and logarithmically scaled for $k \geq 1$. Each subrange is shown on the top axis: “H.H.”, “W.T.” and “E.F.” stand respectively for the higher harmonics, weak turbulence, and the external force.

$D_k^L$ is added at large wavenumbers as $D_k^L = \nu (|k| - k_d)^4 a_k$ for $|k| > k_d = N_{\max} / 8 = 2048$ where $\gamma$ and $\nu$ are constant parameters. These ranges do not overlap each other, and the dynamics of the wavenumber modes in the range $8 < |k| \leq 2048$ is affected by none of external force, drag, and dissipation. The parameters of the MMT model are selected as $\alpha = 1/2$, $\sigma = -3/4$, and $\lambda = 1$ to mimic the deep-water gravity waves [11]. The fourth-order Runge–Kutta method is used for the time integration.

Because of the statistical isotropy, we define the linear-energy spectrum as $\delta_k^{L} = E_k^{L} + E^{-L}_{-k}$, the nonlinear-energy spectrum based on $b_k$ as $\delta_k^{b} = E_k^{b} + E^{-b}_{-k}$, and that based on $c_k$ as $\delta_k^{c} = E_k^{c} + E^{-c}_{-k}$. Such spectra obtained in the numerical simulation are drawn in Fig. 1. While the linear energy has large values in the forcing range, the nonlinear energies have in the higher harmonic range $6 \leq k \leq 16$. The linear energy is much larger than the nonlinear energies in the energy-containing range. Thus, $\mathcal{H}_k^L \gg \mathcal{H}_k^N$. On the other hand, the linear energy is much smaller than the nonlinear energies in the drag range. It is due to the fact that the nonlinear energies come from $a_L$’s in other ranges. In the dissipation range $k > k_d = 2048$, both of the nonlinear energies, based on $b_k$ and that based on $c_k$ are larger than the linear energy.

The energy spectra $\delta_k^{L}$, $\delta_k^{b}$, and $\delta_k^{c}$ in the inertial subrange $10 \leq k \leq 2 \times 10^3$, cannot be represented by a single power law. These energy spectra in the smaller wavenumber range $10 \leq k \leq 30$ show a very steep power law consistent with the very small nonlinearity there. The modes in this range play the energy source for the modes in the larger wavenumber range $2 \times 10^3 \leq k \leq 2 \times 10^3$, where the linear-energy spectrum $E_k^{L}$ is close to the MMT spectrum, while the nonlinear-energy spectrum $\delta_k^{L}$ has a less steep slope. Note that the exponent of the MMT spectrum is $\alpha + 2(\sigma - 5/4) = -9/4$ [8].

Though the two nonlinear energies $E_{cb}^L$ and $E_{cc}^L$ have qualitatively similar forms, they have different values. The nonlinear spectra in the inertial subrange are almost parallel, because the nonlinear energies for a wavenumber $k$ are written respectively as

$$
\frac{1}{2} \langle |b_k|^2 \rangle \approx \sum_{k_1+k_2=k} \langle |k_1||k_2| \rangle \langle |a_{k_1}|^2 \rangle \langle |a_{k_2}|^2 \rangle^{2\sigma},
$$

$$
\frac{1}{2} \langle |c_k|^2 \rangle \approx \frac{1}{2} \sum_{k_1+k_2=k} \langle |k_1||k_2| \rangle \langle |a_{k_1}|^2 \rangle \langle |a_{k_2}|^2 \rangle^{2\sigma} + \frac{1}{2} \delta_{k,0} \left( \sum_{k_1} |k_1|^{2\sigma} \langle |a_{k_1}|^2 \rangle \right),
$$

if the random phase approximation can be applied over all the wavenumbers and the system is statistically isotropic. In fact, as shown in Fig. 1, $\delta_k^{L} \approx \delta_k^{cb}/2$ in the range $10 < k < 30$, where the nonlinearity is weak, while $\delta_k^{c} \ll \delta_k^{cb}$ in the range $k > 10^2$, where the nonlinearity is relatively strong. This result suggests the
inconsistent with Eq. (11), i.e., the weak turbulence theory. Our numerical simulation shows that the energy budget is not so simple even in the forcing range. The modes in the dissipation range, and the former is much larger than the latter. The transfer \( T_k \) cannot by definition. For \( T_k \), however, changes its sign from negative to positive in the dissipation range. In the inertial subrange, the values of \( T_k \) fluctuate around 0. In this sense, there exists no range where the energy cascades if we use \( c_k \) as the auxiliary variable.

The linear-energy pseudo-flux and the total-energy fluxes are drawn in Fig. 3. The linear-energy pseudo-flux and the total-energy fluxes are drawn in Fig. 2. The inset shows the enlargement of the dissipation range. The abscissa is linearly scaled for \( k \leq 1 \) and logarithmically scaled for \( k \geq 1 \).

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**Fig. 2.** Linear-energy transfer and total-energy transfers. The inset shows the enlargement of the dissipation range. The abscissa is linearly scaled for \( k \leq 1 \) and logarithmically scaled for \( k \geq 1 \).
Figure 3 indicates that the total-energy fluxes depend on which mode is selected as an elementary wave. When one uses $b_k$ as an elementary wave, we observe that the wave field receives the energy from the external force, the energy cascades in the inertial subrange, and the energy is dissipated in the dissipation range. When one uses $c_k$ as an elementary wave, on the other hand, the large contribution of the total-energy transfer at $k = 0$ and the non-constancy of the total-energy flux in the inertial subrange suggest that the energy is not transferred as step-by-step cascades, but by nonlocal interactions in the wavenumber space. These differences appear also in Fig. 1: the power spectrum of $\mathcal{E}_k^b$ around $k \approx 10^3$ is shallower and wider than that of $\mathcal{E}_k^c$. In the range $20 < k < 200$, both total-energy fluxes $P_{Tb}$ and $P_{Tc}$, similarly to the nonlinear-energy spectra $\mathcal{E}_k^b$ and $\mathcal{E}_k^c$, are almost parallel, but their values and even their signs are different, as seen in Fig. 3.

4. CONCLUDING REMARKS

The single-wavenumber representations of the nonlinear energy in the wave turbulence systems is examined in the MMT model in order to take our understanding in [5] one step further. The two natural single-wavenumber representations of the nonlinear energy are found in the MMT model. One is based on an auxiliary variable $b_k = \sum_{k_1+k_2=k} |k_1| |k_2| \bar{\sigma} a_{k_1} a_{k_2}$, and the other on the other auxiliary variable $c_k = \sum_{k_1-k_2=k} |k_1| |k_2| \bar{\sigma} a_{k_1} a_{k_2}$. The two natural representations indicate that the representations are not necessarily unique in the MMT model. It is in contrast with the unique representation in the elastic-wave turbulence, reported in [5].

Our numerical simulation of the MMT model reveals the following facts. While the two kinds of the nonlinear-energy spectra are qualitatively similar, the two total-energy transfers as well as the two total-energy fluxes are different at the zero mode and in the inertial subrange. The most noticeable difference is that the total-energy flux based on $c_k$ is not constant in the inertial subrange while that based on $b_k$ is constant. Furthermore, even their signs are different from each other. The differences come mainly from the interactions between the zero mode and other modes. The zero mode of $b_k$, i.e., $b_0 = \sum_k |k|^{-2\sigma} a_k a_{-k}$ comes from the interaction between $a_k$ and $a_{-k}$. On the other hand, the zero mode of $c_k$, i.e., $c_0 = \sum_k |k|^{-2\sigma} |a_k|^2$ comes from the selfinteraction of $a_k$. The apparent nonlocal interactions between $c_0$ and $a_k$ (and $c_k$) where the wavenumber $k$ is in the inertial subrange make the nonconstant energy flux of $P_{Tc}$ in the inertial subrange. If we believe the energy cascade, the auxiliary variable $b_k$ is preferable to $c_k$. However, the nonlocal interactions are reported in the focusing MMT model with $\lambda = -1$ [8], and the nonlocal interactions might play a dominant role in the present system. Therefore, the appropriateness of the representation including the possibility of a more appropriate auxiliary variable should be judged carefully.
Lastly, it may be interesting to point out that if \( a_k \) were complex conjugates of \( a_{-k} \) such as the Fourier coefficient of a real variable, the auxiliary variables \( b_k \) and \( c_k \) would become complex conjugates of each other as known from the definition (3). In this sense, the differences observed above stem from the non-conjugacy of \( a_k \) with \( a_{-k} \). The MMT model as well as the auxiliary variables \( b_k \) and \( c_k \) is merely mathematical and arbitrary to some extent. The nonlinear energy in the MMT model is in contrast to that in the elastic-wave turbulence [5], which gives the stretching energy and has a physical meaning itself. If we restrict the initial conditions of \( a_k \) and the external force \( F_k \) to have this complex conjugacy in the MMT model, the nonlinear energies based on the two auxiliary variables become identical, and we may be able to investigate the energy budget in more detail. Furthermore, we can develop the MMT model by considering the physically plausible characteristics. The study along this line is currently in progress and will be reported elsewhere.

ACKNOWLEDGEMENTS

Numerical computation in this work was carried out at the Yukawa Institute Computer Facility. This work was partially supported by JSPS KAKENHI grant No. 25400412.

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Mittelineaarse setet lainete energia turbulentsii tüüpi lainesüsteemides

Naoto Yokoyama ja Masanori Takaoka

Turbulentsi tüüpi mittelineaarseset lainesüsteemide üksikute komponentide energia bilansi ja mittelineaarsest interaktsooniast põhjustatud energiavahetuse arvutamisel on tarvis klassikaline (lainekõruse ruuduga võrdeline) energia kontseptsioon asendada keerukama mittelineaarse funktsiooniga. On näidatud, et selline üldistus pole ühene ja vöivad eksisteerida erinevad, kuigi teatas võrdsed ekvivalentsed võimalused. Näitena on vaadeldud Majda-McLaughlini-Tabaki mudelit. Üksikute lainekomponentide kompleksseid amplituude