RATIONALITY OF W-ALGEBRAS: PRINCIPAL NILPOTENT CASES

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Abstract. We prove the rationality of all the minimal series principal W-algebras discovered by Frenkel, Kac and Wakimoto [FKW], thereby giving a new family of rational and C_2-cofinite vertex operator algebras. A key ingredient in our proof is the study of Zhu's algebra of simple W-algebras via the quantized Drinfeld-Sokolov reduction. We show that the functor of taking Zhu's algebra commutes with the reduction functor. Using this general fact we determine the maximal spectrums of the associated graded of Zhu's algebras of vertex operator algebras associated with admissible representations of affine Kac-Moody algebras as well.

1. Introduction

Let $\mathcal{W}^k(\mathfrak{g}) = \mathcal{W}^k(\mathfrak{g}, f_{prin})$ be the W-algebra associated with a complex finite-dimensional simple Lie algebra $\mathfrak{g}$ and a principal nilpotent element f_{prin} of $\mathfrak{g}$ at level k [FL, LF, FF]. In [A2] we have confirmed the conjecture of Frenkel, Kac and Wakimoto [FKW] on the existence of modular invariant representations of $\mathcal{W}^k(\mathfrak{g})$ for an appropriate level k. These representations are called the *minimal series representations* of $\mathcal{W}^k(\mathfrak{g})$ since in the case that $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ they are precisely the minimal series representations [BPZ] of the Virasoro algebra. It has been expected [FKW] and widely believed that these representations of $\mathcal{W}^k(\mathfrak{g})$ form a minimal model of the corresponding conformal field theory in the sense of [BPZ] as in the case that $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. In the language of vertex operator algebras this amounts to showing that the vertex operator algebras associated with minimal series representations of W-algebras are *rational* and *C_2-cofinite*. We have established the C_2-cofiniteness property previously in [A5]. The main purpose of this paper is to resolve the remaining rationality problem.

Denote by $\mathcal{W}_k(\mathfrak{g})$ the unique simple quotient of $\mathcal{W}^k(\mathfrak{g})$ at a non-critical level k. The vertex operator algebra $\mathcal{W}_k(\mathfrak{g})$ is isomorphic to a minimal series representation as a module over $\mathcal{W}^k(\mathfrak{g})$ if and only if

$$k + h_{\mathfrak{g}}^{\vee} = p/q \in \mathbb{Q}_{>0},\ p,q \in \mathbb{N},\ (p,q) = 1, \tag{1}$$
$$\text{and } \begin{cases} p \geq h_{\mathfrak{g}}^{\vee},\ q \geq h_{\mathfrak{g}} & \text{if } (q, r^{\vee}) = 1, \\ p \geq h_{\mathfrak{g}},\ q \geq r^{\vee} h_{{}^L\mathfrak{g}}^{\vee} & \text{if } (q, r^{\vee}) = r^{\vee}, \end{cases}$$

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where $h_{\mathfrak{g}}$ is the Coxeter number of $\mathfrak{g}$, $h_{\mathfrak{g}}^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$, ${}^{L}\mathfrak{g}$ is the Langlands dual Lie algebra of $\mathfrak{g}$, and $r^{\vee}$ is the maximal number of the edges in the Dynkin diagram of $\mathfrak{g}$. The central charge $c(k)$ of $\mathcal{W}_k(\mathfrak{g})$ is given by the formula

$$c(p/q-h_{\mathfrak{g}}^{\vee})=l-12\frac{|q\rho-p\rho^{\vee}|^2}{pq}=-\frac{l((h_{\mathfrak{g}}+1)p-h_{\mathfrak{g}}^{\vee}q)(r^{\vee}h_{{}^L\mathfrak{g}}^{\vee}p-(h_{\mathfrak{g}}+1)q)}{pq},$$

where l is the rank of $\mathfrak{g}$, ρ is the half sum of positive roots of $\mathfrak{g}$ and $\rho^{\vee}$ is the half sum of positive coroots of $\mathfrak{g}$.

Main Theorem. *Let k be as in (1). The vertex operator algebra $\mathcal{W}_k(\mathfrak{g})$ is rational (and C_2-cofinite [A5]). The set of isomorphism classes of minimal series representations of $\mathcal{W}^k(\mathfrak{g})$ forms the complete set of the isomorphism classes of simple modules over $\mathcal{W}_k(\mathfrak{g})$.*

Main Theorem has been proved in [BFM, Wan] in the case that $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$ and in [DLTY] in the case that $\mathfrak{g}=\mathfrak{sl}_3(\mathbb{C})$ and $k=5/4-3$ (or[1] $4/5-3$).

Let us explain the outline of the proof of Main Theorem briefly. A crucial step in the proof is the classification of the simple modules over the simple quotient $\mathcal{W}_k(\mathfrak{g})$. For this purpose it is sufficient [Zhu] to determine Zhu's algebra of $\mathcal{W}_k(\mathfrak{g})$. We carry out this by studying Zhu's algebra of W-algebras via the quantized Drinfeld-Sokolov reduction. Since this is a general argument we work in a more general setting: Let f be any nilpotent element of $\mathfrak{g}$, $\mathcal{W}^k(\mathfrak{g},f)$ the (universal) W-algebra associated with $(\mathfrak{g},f)$ at level k. By definition [FF, KRW] we have

$$\mathcal{W}^k(\mathfrak{g},f)=H_f^0(V^k(\mathfrak{g})),$$

where $V^k(\mathfrak{g})$ is the universal affine vertex algebra associated with $\mathfrak{g}$ at level k and $H_f^{\bullet}(M)$ denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction [KRW] associated with $(\mathfrak{g},f)$ with coefficient in a $V^k(\mathfrak{g})$-module M. We show that

$$A(H_f^0(L))\cong H_f^0(A(L)) \tag{2}$$

for any quotient L of $V^k(\mathfrak{g})$ at any level k (in fact we prove a stronger assertion, see Theorem 8.1). Here, for a conformal vertex algebra V, $A(V)$ denotes Zhu's algebra[2] of V, and $H_f^0(A(L))$ denotes the (finite-dimensional analogue of) BRST cohomology associated with $(\mathfrak{g},f)$ with coefficient in $A(L)$, which is identical to $A(L)_{\dagger}$ in Losev's notation [Los], see Section 3.

In the case that $f=f_{prin}$ the classification problem is relatively simple since $A(\mathcal{W}^k(\mathfrak{g},f_{prin}))\cong\mathcal{Z}(\mathfrak{g})$ ([A2]), where $\mathcal{Z}(\mathfrak{g})$ is the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, and hence, $A(\mathcal{W}_k(\mathfrak{g}))$ is a quotient of the commutative algebra $\mathcal{Z}(\mathfrak{g})$. Moreover under the assumption of Main Theorem we have shown in [A2] that

$$\mathcal{W}_k(\mathfrak{g})\cong H_{f_{prin}}^0(L(k\Lambda_0))$$

as conjectured in [FKW], where $L(k\Lambda_0)$ is the unique simple quotient vertex algebra of $V^k(\mathfrak{g})$ which is an admissible representation [KW1] as a $\widehat{\mathfrak{g}}$-module. It follows from

[1] There is the Feigin-Frenkel duality $\mathcal{W}_{p/q-h_{\mathfrak{g}}^{\vee}}(\mathfrak{g})\cong\mathcal{W}_{q/r^{\vee}p-h_{{}^L\mathfrak{g}}^{\vee}}({}^L\mathfrak{g})$ for all $p,q\in\mathbb{C}^*$. (The details will be explained elsewhere.)

[2] More precisely, $A(V)$ is the L_0-twisted Zhu's algebra in the sense of [DSK] since $\mathcal{W}^k(\mathfrak{g},f)$ is $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded in general. It is the usual Zhu's algebra for $f=f_{prin}$.

(2) that Zhu's algebra $A(\mathcal{W}_k(\mathfrak{g}))$ of $\mathcal{W}_k(\mathfrak{g})$ is completely determined by $A(L(k\Lambda_0))$. We deduce the classification result in Main Theorem from that of admissible affine vertex algebras $L(k\Lambda_0)$ recently obtained by the author in [A8].

Once the classification of simple modules is established it is straightforward to see that there is no extension between two distinct simple $\mathcal{W}_k(\mathfrak{g})$-modules from the general result on the representation theory of $\mathcal{W}^k(\mathfrak{g})$ achieved in [A2]. Finally the fact that simple $\mathcal{W}_k(\mathfrak{g})$-modules do not admit non-trivial self-extensions follows from the result of Gorelik and Kac [GK] who established the complete reducibility of admissible representations of $\widehat{\mathfrak{g}}$.

The isomorphism (2) has an application to affine vertex algebras as well: It enables us to determine the variety $VarA(L(k\Lambda_0))$ associated with Zhu's algebra of any admissible affine vertex algebra $L(k\Lambda_0)$ (Theorem 9.3). This result was announced in [A8].

The assertion of Main Theorem is a special case of the conjecture of Kac and Wakimoto [KW2] on the rationality of *exceptional W-algebras*. In subsequent papers we prove the rationality of a large family of W-algebras, including all the exceptional W-algebras of type A, generalizing the result of [A6].

This paper is organized as follows. In Section 2 and Section 3 we reformulate some results of Ginzburg [Gin] and Losev [Los] in terms of BRST reduction for later purposes. In Section 4 we fix some notations for vertex algebras and clarify the relationship between Frenkel-Zhu's bimodules and Zhu's C_2-modules associated with vertex algebras. In Section 5 we discuss the effect of shifts of conformal vector to Frenkel-Zhu's bimodules, which is needed to describe Frenkel-Zhu's bimodules associated with W-algebras. In Section 6 we collect some basic facts about affine vertex algebras and study Zhu's C_2-modules and Frenkel-Zhu's bimodules associated with objects in the the Kazhdan-Lusztig parabolic full subcategory KL_k of $\mathcal{O}$ of $\widehat{\mathfrak{g}}$. In Section 7 we recall the definition of W-algebras and some results from [A5]. In Section 8 we show that the functor of taking Frenkel-Zhu's bimodules commutes with the reduction functor on the category KL_k. This result in particular proves (2). In Section 9 we recall the main result of [A8] and determine varieties $VarA(L(k\Lambda_0))$ associated with Zhu's algebras of admissible affine vertex algebras. Finally we prove Main Theorem in Section 10.

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Notation Throughout this paper the ground field is the complex number $\mathbb{C}$ and tensor products are meant to be as vector spaces over $\mathbb{C}$ if not otherwise stated.

2. The Slodowy slice and classical BRST reduction

Let R be a Poisson algebra. Recall that a *Poisson module* M over R is a R-module M in the usual associative sense equipped with a bilinear map

$$R \times M \to M, \quad (r, m) \mapsto \operatorname{ad} r(m) = \{r, m\},$$

which makes M a Lie algebra module over R satisfying

$$\{r_1, r_2 m\} = \{r_1, r_2\}m + r_2\{r_1, m\}, \quad \{r_1 r_2, m\} = r_1\{r_2, m\} + r_2\{r_1, m\}$$

for $r_1, r_2 \in R$, $m \in M$. Let R-PMod be the category of Poisson modules over R.

For any finite-dimensional Lie algebra $\mathfrak{a}$ the space $\mathbb{C}[\mathfrak{a}^*] = S(\mathfrak{a})$ is a Poisson algebra by the Kirillov-Kostant Poisson bracket. A Poisson module over $\mathbb{C}[\mathfrak{a}^*]$ is the same as a $\mathbb{C}[\mathfrak{a}^*]$-module M in the usual associative sense equipped with a Lie algebra module structure $\mathfrak{a} \to \operatorname{End} M$, $x \mapsto \operatorname{ad}(x)$, over $\mathfrak{a}$ such that $\operatorname{ad}(x)(fm) = \{x, f\}m + f \operatorname{ad}(x)(m)$ for $x \in \mathfrak{a}$, $f \in \mathbb{C}[\mathfrak{a}^*]$, $m \in M$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra as in Introduction, $(\,|\,)$ the normalized invariant inner product of $\mathfrak{g}$, that is, $1/2h_{\mathfrak{g}}^{\vee} \times$ the killing form of $\mathfrak{g}$. Let $\nu : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ be the isomorphism defined by the form $(\,|\,)$.

Let f be a nilpotent element of $\mathfrak{g}$, $\{e, f, h\}$ an $\mathfrak{sl}_2$-triple associated with f:

$$[h, e] = 2e, \quad [e, f] = h, \quad [h, f] = -2f.$$

Set

$$\chi = \nu(f) \in \mathfrak{g}^*.$$

The affine space

$$\mathcal{S}_f = \nu(f + \mathfrak{g}^e) \subset \mathfrak{g}^*$$

is called the *Slodowy slice* at χ to $\operatorname{Ad} G.\chi$, where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$ and G is the adjoint group of $\mathfrak{g}$. It is known [GG] that the Kirillov-Kostant Poisson structure of $\mathfrak{g}^*$ restricts to $\mathcal{S}_f$. Hence $\mathbb{C}[\mathcal{S}_f]$ is a Poisson algebra.

We have

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{x \in \mathfrak{g} | \operatorname{ad} h(x) = 2jx\}. \tag{3}$$

Put

$$\mathfrak{g}_{\geq 1} = \bigoplus_{j \geq 1} \mathfrak{g}_j \subset \mathfrak{g}_{>0} = \bigoplus_{j > 0} \mathfrak{g}_j = \mathfrak{g}_{1/2} \oplus \mathfrak{g}_{\geq 1}.$$

Denote by $G_{>0}$ the unipotent subgroup of G whose Lie algebra is $\mathfrak{g}_{>0}$. By [GG, Lemma 2.1] the coadjoint action gives the isomorphism

$$G_{>0} \times \mathcal{S}_f \xrightarrow{\sim} \chi + \mathfrak{g}_{\geq 1}^{\perp} \tag{4}$$

of affine varieties, where $\mathfrak{g}_{\geq 1}^{\perp}$ is the annihilator of $\mathfrak{g}_{\geq 1}$ in $\mathfrak{g}^*$.

Consider the affine subspace $\chi + \nu(\mathfrak{g}_{-1/2})$ of $\mathfrak{g}_{>0}^*$. We have

$$\mathbb{C}[\chi + \nu(\mathfrak{g}_{-1/2})] = \mathbb{C}[\mathfrak{g}_{>0}^*]/\bar{I}_{>0,\chi},$$

where $\bar{I}_{>0,\chi}$ is the Poisson ideal of $\mathbb{C}[\mathfrak{g}_{>0}^*]$ generated by $x - \chi(x)$ with $x \in \mathfrak{g}_{\geq 1}$. The Poisson bracket of the quotient algebra is given by

$$\{x, y\} = \chi([x, y]) \quad \text{for } x, y \in \mathfrak{g}_{1/2}$$

under the identification $\mathbb{C}[\chi+\nu(\mathfrak{g}_{-1/2})]\cong\mathbb{C}[\mathfrak{g}_{1/2}^*]=S(\mathfrak{g}_{1/2})$. As

$$\mathfrak{g}_{1/2}\times\mathfrak{g}_{1/2}\to\mathbb{C},\quad (x,y)\mapsto\chi([x,y]),\tag{5}$$

is a symplectic form, it follows that $\chi+\nu(\mathfrak{g}_{-1/2})$ is isomorphic to $T^*\mathbb{C}^{\dim\mathfrak{g}_{1/2}/2}$ as Poisson varieties.

Let

$$\mu:\mathfrak{g}^*\to\mathfrak{g}_{\geq 1}^*$$

be the restriction map. Then μ is the moment map for the action of the unipotent subgroup $G_{\geq 1}$ of G whose Lie algebra if $\mathfrak{g}_{\geq 0}$. We have

$$\mu^{-1}(\chi+\nu(\mathfrak{g}_{-1/2}))=\chi+\mathfrak{g}_{\geq 1}^{\perp}.\tag{6}$$

Let $\{x_i|i=1,\dots,\dim\mathfrak{g}_{>0}\}$ be a homogeneous basis of $\mathfrak{g}_{>0}$ with respect to the grading (3) such that the first $\dim\mathfrak{g}_{1/2}$-elements $\{x_i|i=1,\dots,\dim\mathfrak{g}_{1/2}\}$ forms a basis of $\mathfrak{g}_{1/2}$, and let $\{c_{ij}^k\}$ be the structure constant: $[x_i,x_j]=\sum_k c_{ij}^k x_k$. For $i=1,\dots,\dim\mathfrak{g}_{>0}$ let $\bar{\phi}_i$ denote the image of x_i under the natural Poisson algebra homomorphism $\mathbb{C}[\mathfrak{g}_{>0}^*]\twoheadrightarrow\mathbb{C}[\chi+\mathfrak{g}_{1/2}^*]$. By definition

$$\{\bar{\phi}_i,\bar{\phi}_j\}=\chi([x_i,x_j])\quad\text{for } i=1,\dots,\dim\mathfrak{g}_{1/2},$$

and $\bar{\phi}_i=\chi(x_i)$ for $i>\dim\mathfrak{g}_{1/2}$.

Let $\Pi\mathfrak{g}_{>0}^*$ denote the space $\mathfrak{g}_{>0}^*$ considered as a purely odd vector space, $T^*\Pi\mathfrak{g}_{>0}^*$ the tangent bundle of $\Pi\mathfrak{g}_{>0}^*$ which is a symplectic supermanifold. Then $\mathbb{C}[T^*\Pi\mathfrak{g}_{>0}^*]$ is a Poisson superalgebra, which is nothing but the exterior algebra $\bigwedge^\bullet(\mathfrak{g}_{>0}^*\oplus\mathfrak{g}_{>0})=\bigwedge^\bullet(\mathfrak{g}_{>0}^*)\otimes\bigwedge^\bullet(\mathfrak{g}_{>0})$ (with an obvious Poisson superbracket).

For a Poisson module M over $\mathbb{C}[\mathfrak{g}^*]$ set

$$\bar{C}(M)=M\otimes\mathbb{C}[\chi+\nu(\mathfrak{g}_{-1/2})]\otimes\mathbb{C}[T^*\Pi\mathfrak{g}_{>0}^*]=\bigoplus_{p\in\mathbb{Z}}\bar{C}^p(M),$$

$$\bar{C}^p(M)=\bigoplus_{i-j=p}M\otimes\mathbb{C}[\chi+\nu(\mathfrak{g}_{-1/2})]\otimes\bigwedge^i(\mathfrak{g}_{>0}^*)\otimes\bigwedge^j(\mathfrak{g}_{>0}).$$

Then $\bar{C}(\mathbb{C}[\mathfrak{g}^*])$ is naturally a graded Poisson superalgebra, and $\bar{C}(M)$ is a Poisson module over $\bar{C}(\mathbb{C}[\mathfrak{g}^*])$ (in an obvious "super" sense). Set

$$\bar{d}=\sum_{i=1}^{\dim\mathfrak{g}_{>0}}(x_i\otimes 1+1\otimes\bar{\phi}_i)\otimes x_i^*-1\otimes 1\otimes\frac{1}{2}\sum_{1\leq i,j,k\leq\dim\mathfrak{g}_{>0}}c_{ij}^k x_i^*x_j^*x_k\in\bar{C}^1(\mathbb{C}[\mathfrak{g}^*]),$$

where $\{x_i^*\}\subset\mathfrak{g}_{>0}^*\subset\mathbb{C}[T^*\Pi\mathfrak{g}_{>0}^*]$ is the dual basis of $\{x_i\}$.

Lemma 2.1. $\bar{d}^2=0$.

Since $\bar{d}$ is an odd element it follows from Lemma 2.1 that $(\operatorname{ad}\bar{d})^2=0$ on any Poisson module over $\bar{C}(\mathbb{C}[\mathfrak{g}^*])$. It follows that $(\bar{C}(\mathbb{C}[\mathfrak{g}^*]),\operatorname{ad}\bar{d})$ is a differential graded superalgebra and $(\bar{C}(M),\operatorname{ad}\bar{d})$ is a module over the differential graded algebra $(\bar{C}(\mathbb{C}[\mathfrak{g}^*]),\operatorname{ad}\bar{d})$. Let $H_f^\bullet(M)$ be the cohomology of the cochain complex $(\bar{C}(M),\operatorname{ad}\bar{d})$. The space $H_f^\bullet(\bar{\mathbb{C}}[\mathfrak{g}^*])$ inherits the $\mathbb{Z}$-graded Poisson superalgebra structure from $\bar{C}(\mathbb{C}[\mathfrak{g}^*])$ and $H_f^0(M)$ is naturally a module over $H_f^0(\mathbb{C}[\mathfrak{g}^*])$.

Theorem 2.2 ([KS87, DSK], see also Theorem 2.3 below)**.** *We have $H_f^i(\mathbb{C}[\mathfrak{g}^*])=0$ for $i\neq 0$ and $H_f^0(\mathbb{C}[\mathfrak{g}^*])\cong\mathbb{C}[\mathcal{S}_f]$ as Poisson algebras.*

Let $\overline{\mathcal{HC}}$ be the full subcategory of the category of $\mathbb{C}[\mathfrak{g}^*]$-PMod consisting of modules on which the Lie algebra action of $\mathfrak{g}$ is locally finite. Denote by $\bar{I}_\chi$ the ideal of $\mathbb{C}[\mathfrak{g}^*]$ generated by $y-\chi(y)$ with $y\in\mathfrak{g}_{\geq 1}$. Then, for $M\in\overline{\mathcal{HC}}$, $\bar{I}_\chi M$ is a Poisson submodule of M over $\mathbb{C}[\mathfrak{g}^*_{>0}]$.

The following assertion is a reformulation of a result of [Gin].

Theorem 2.3. *For $M\in\overline{\mathcal{HC}}$, we have*

$$H^i_f(M)\cong\begin{cases}(M/\bar{I}_\chi M)^{\operatorname{ad}\mathfrak{g}_{>0}} & \text{for } i=0,\\ 0 & \text{otherwise}\end{cases}$$

In particular the functor

$$\overline{\mathcal{HC}}\to\mathbb{C}[\mathcal{S}_f]\text{-PMod},\quad M\mapsto H^0_f(M),$$

is exact, and

$$\operatorname{supp}_{\mathbb{C}[\mathcal{S}_f]}H^0_f(M)=\mathcal{S}_f\cap\operatorname{supp}_{\mathbb{C}[\mathfrak{g}^*]}(M)$$

for a finitely generated object M of $\overline{\mathcal{HC}}$.

Proof. Since a cohomology functor commutes with injective limits we may assume that M is finitely generated. Set $\bar{C}=\bar{C}(M)$, $\bar{C}^p=\bar{C}^p(M)$, $\bar{C}^{ij}=M\otimes\mathbb{C}[\chi+\mathfrak{g}^*_{1/2}]\otimes\bigwedge^i(\mathfrak{g}^*_{>0})\otimes\bigwedge^{-j}(\mathfrak{g}_{>0})\subset\bar{C}$, so that $\bar{C}^p=\bigoplus_{\substack{i\geq 0,\ j\leq 0\\ i+j=p}}\bar{C}^{i,j}$. The differential $\operatorname{ad}\bar{d}:\bar{C}^p\to\bar{C}^{p+1}$ decomposes as

$$\operatorname{ad}\bar{d}=\bar{d}_-\oplus\bar{d}_+,$$

where

$$\bar{d}_-=\sum_i(x_i\otimes\mathrm{id}+\mathrm{id}\otimes\bar{\phi}_i)\otimes\operatorname{ad}x_i^*,\tag{7}$$

$$\bar{d}_+=\sum_i(\operatorname{ad}x_i\otimes\mathrm{id}+\mathrm{id}\otimes\operatorname{ad}\bar{\phi}_i)\otimes x_i^*+\sum_{i,j,k}\mathrm{id}\otimes\mathrm{id}\otimes c_{ij}^k x_k x_j^*\operatorname{ad}x_i^*\\ -\mathrm{id}\otimes\mathrm{id}\otimes\frac{1}{2}\sum_{i,j,k}c_{ij}^k x_i^*x_j^*\operatorname{ad}x_k.\tag{8}$$

Since $\bar{d}_-\bar{C}^{i,j}\subset\bar{C}^{i,j+1}$, $\bar{d}_+\bar{C}^{i,j}\subset\bar{C}^{i+1,j}$, it follows that

$$\{\bar{d}_-,\bar{d}_+\}=0,\quad \bar{d}_-^2=\bar{d}_+^2=0.$$

Consider the spectral sequence $E_r\Rightarrow H^\bullet_f(M)$ with

$$E_1^{p,q}=H^q(\bar{C}^{p,\bullet},\bar{d}_-),\quad E_2^{p,q}=H^p(E_1^{\bullet,q},\bar{d}_+).$$

By (7), $H^\bullet(\bar{C}^{p,\bullet},\bar{d}_-)$ is the homology of the Koszul complex of the $\mathbb{C}[\mathfrak{g}^*_{>0}]$-module $M\otimes\mathbb{C}[\chi+\nu(\mathfrak{g}_{-1/2})]\otimes\bigwedge^p(\mathfrak{g}^*_{>0})$ associated with the sequence $x_1,x_2,\dots,x_{\dim\mathfrak{g}_{>0}}$, where $\mathbb{C}[\mathfrak{g}^*_{>0}]$ acts only on the first two factors. Since $\mathbb{C}[\chi+\nu(\mathfrak{g}_{-1/2})]$ is a free $\mathbb{C}[\mathfrak{g}^*_{1/2}]$-module of rank 1 it follows that $H^\bullet(\bar{C}^{p,\bullet},\bar{d}_-)$ is isomorphic to the homology of the Koszul complex of the $\mathbb{C}[\mathfrak{g}^*_{\geq 1}]$-module $M\otimes\bigwedge^p(\mathfrak{g}^*_{>0})$ associated with the sequence $x_{\dim\mathfrak{g}_{1/2}+1}-\chi(x_{\dim\mathfrak{g}_{1/2}+1}),\dots,x_{\dim\mathfrak{g}_{>1}}-\chi(x_{\dim\mathfrak{g}_{>1}})$. Hence thanks to [Gin, Corollary 1.3.8] we have

$$E_1^{\bullet,q}\cong\begin{cases}(M/\bar{I}_\chi M)\otimes\bigwedge^\bullet(\mathfrak{g}^*_{>0}) & \text{for } q=0,\\ 0 & \text{for } q\neq 0.\end{cases}\tag{9}$$

Hence from (9) we see that $E_2^{\bullet,0}$ is isomorphic to the Lie algebra cohomology $H^{\bullet}(\mathfrak{g}_{>0}, M/\bar{I}_\chi M)$.

Now first consider the case that $M = \mathbb{C}[\mathfrak{g}^*]$. Since $\mathbb{C}[\mathfrak{g}^*]/\bar{I}_\chi \cong \mathbb{C}[\chi + \mathfrak{g}_{\geq 1}^{\perp}]$ we have $\mathbb{C}[\chi + \mathfrak{g}_{\geq 1}^{\perp}] = \mathbb{C}[G_{>0}] \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{S}_f]$ by (4), and thus,

$$H^i(\mathfrak{g}_{>0}, \mathbb{C}[\chi + \mathfrak{g}_{\geq 1}^{\perp}]) \cong \begin{cases} \mathbb{C}[\mathcal{S}_f] & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases} \tag{10}$$

For a general module M the argument of [GG, 6.2] shows that the multiplication map

$$\varphi : \mathbb{C}[\chi + \mathfrak{g}_{\geq 1}^{\perp}] \otimes_{\mathbb{C}[\mathcal{S}_f]} (M/\bar{I}_\chi M)^{\operatorname{ad} \mathfrak{g}_{>0}} \to M/\bar{I}_\chi M$$

is an isomorphism of $\mathfrak{g}_{>0}$-module, where $\mathfrak{g}_{>0}$ acts only on the first factor $\mathbb{C}[\chi + \mathfrak{g}_{\geq 1}^{\perp}]$ of $\mathbb{C}[\chi + \mathfrak{g}_{\geq 1}^{\perp}] \otimes_{\mathbb{C}[\mathcal{S}_f]} (M/\bar{I}_\chi M)^{\operatorname{ad} \mathfrak{g}_{>0}}$ and $\mathbb{C}[\mathcal{S}_f]$ acts on $(M/\bar{I}_\chi M)^{\operatorname{ad} \mathfrak{g}_{>0}}$ by the identification $\mathbb{C}[\mathcal{S}_f] = (\mathbb{C}[\mathfrak{g}^*]/\bar{I}_\chi \mathbb{C}[\mathfrak{g}^*])^{\operatorname{ad} \mathfrak{g}_{>0}}$. Therefore (10) gives that

$$E_2^{p,q} \cong \begin{cases} (M/\bar{I}_\chi M)^{\operatorname{ad} \mathfrak{g}_{>0}} & \text{for } p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that the spectral sequence collapses at $E_2 = E_\infty$, and the assertion follows. □

3. Finite W-algebras and equivalences of categories via BRST reduction

Let A be an associative algebra over $\mathbb{C}$ equipped with an increasing $\frac{1}{2}\mathbb{Z}$-filtration $F_\bullet A$ such that

$$F_p A \cdot F_q A \subset F_{p+q} A, \quad [F_p A, F_q A] \subset F_{p+q-1} A. \tag{11}$$

Then the associated graded space $\operatorname{gr}_F A = \bigoplus_{p \in \frac{1}{2}\mathbb{Z}} F_p A / F_{p-1/2} A$ is naturally a Poisson algebra. We assume that $\operatorname{gr}_F A$ is finitely generated as a ring.

Denote by A-biMod the category of A-bimodules. Let M be an object of A-biMod equipped with an increasing filtration $F_\bullet M$ compatible with the one on A, that is,

$$F_p A \cdot F_q M \cdot F_r A \subset F_{p+q+r} M, \quad [F_p A, F_q M] \subset F_{p+q-1} M.$$

Then $\operatorname{gr}_F M = \bigoplus_p F_p M / F_{p-1/2} M$ is naturally a Poisson module over $\operatorname{gr}_F A$. The filtration $F_\bullet M$ is called *good* if $\operatorname{gr}_F M$ is finitely generated over $\operatorname{gr}_F A$ in a usual associative sense. If this is the case we set

$$\mathit{Var} M = \operatorname{supp}(\operatorname{gr}_F M) \subset \operatorname{Spec}(\operatorname{gr}_F A),$$

equipped with the reduced scheme structure. It is well-known that $\mathit{Var} M$ is independent of the choice of a good filtration.

Let $F_\bullet U(\mathfrak{g})$ be the standard PBW filtration of $U(\mathfrak{g})$:

$$F_{-1} U(\mathfrak{g}) = 0, \quad F_0 U(\mathfrak{g}) = \mathbb{C}, \quad F_p U(\mathfrak{g}) = \mathfrak{g} F_{p-1} U(\mathfrak{g}) + F_{p-1} U(\mathfrak{g}).$$

Set $F_p U(\mathfrak{g})[j] = \{u \in U_p(\mathfrak{g}) | \operatorname{ad} h(u) = 2ju\}$, where, recall, h is defined in Section 2. Let

$$K_p U(\mathfrak{g}) = \sum_{i-j \leq p} F_i U(\mathfrak{g})[j].$$

Then $K_\bullet U(\mathfrak{g})$ is an increasing, exhaustive, separated filtration of $U(\mathfrak{g})$ that satisfies (11). The filtration $\{K_p U(\mathfrak{g})\}$ is called the *Kazhdan filtration*. The associated graded Poisson algebra $\mathrm{gr}_K U(\mathfrak{g})$ is naturally isomorphic to $\mathbb{C}[\mathfrak{g}^*]$.

Let M be a $U(\mathfrak{g})$-bimodule. A Kazhdan filtration of M is an increasing, exhaustive, separated, filtration $K_\bullet M$ which is compatible with the Kazhdan filtration of $U(\mathfrak{g})$.

Define

$$I_{>0,\chi} = \sum_{x \in \mathfrak{g}_{\geq 1}} U(\mathfrak{g}_{>0})(x - \chi(x)).$$

Then $I_{>0,\chi}$ is a two-sided ideal of $U(\mathfrak{g}_{>0})$. Set

$$\mathcal{D} = U(\mathfrak{g}_{>0})/I_{>0,\chi},$$

and let

$$\phi : U(\mathfrak{g}_{>0}) \twoheadrightarrow \mathcal{D}$$

be the natural surjective algebra homomorphism, $\phi_i = \phi(x_i)$, where $\{x_i\}$ is defined in section 2. Then

$$[\phi_i, \phi_j] = \chi([x_i, x_j]) \quad \text{for } i = 1, \dots, \dim \mathfrak{g}_{1/2},$$

and $\bar{\phi}_i = \chi(x_i)$ for $i > \dim \mathfrak{g}_{1/2}$. It follows that $\mathcal{D}$ is isomorphic to the Weyl algebra of rank $\dim \mathfrak{g}_{1/2}/2$. Let $K_\bullet \mathcal{D}$ be the filtration of $\mathcal{D}$ induced by $K_\bullet U(\mathfrak{g})$, that is, $K_p \mathcal{D}$ the image of $K_p U(\mathfrak{g}) \cap U(\mathfrak{g}_{>0})$ in $\mathcal{D}$. The associated graded Poisson algebra $\mathrm{gr}_K \mathcal{D}$ is isomorphic to $\mathbb{C}[\chi + \nu(\mathfrak{g}_{-1/2})]$ which appeared in Section 2.

Denote by $\mathcal{C}l$ the Clifford algebra associated with $\mathfrak{g}_{>0} \oplus \mathfrak{g}_{>0}^*$ and the bilinear form $\mathfrak{g}_{>0} \oplus \mathfrak{g}_{>0}^* \times \mathfrak{g}_{>0} \oplus \mathfrak{g}_{>0}^* \to \mathbb{C}$, $(x + f, x' + f') \mapsto f(x') + f'(x)$. The algebra $\mathcal{C}l$ contains $\bigwedge^\bullet(\mathfrak{g}_{>0}^*)$ and $\bigwedge^\bullet(\mathfrak{g}_{>0})$ as its subalgebras and the multiplication map $\bigwedge^\bullet(\mathfrak{g}_{>0}^*) \otimes \bigwedge^\bullet(\mathfrak{g}_{>0}) \to \mathcal{C}l$ is a linear isomorphism. Let $F_\bullet \mathcal{C}l$ be the increasing filtration of $\mathcal{C}l$ defined by $F_p \mathcal{C}l = \bigoplus_{j \leq p} \bigwedge^\bullet(\mathfrak{g}_{>0}^*) \otimes \bigwedge^j(\mathfrak{g}_{>0})$. Set $F_p \mathcal{C}l[j] = \{\omega \in F_p \mathcal{C}l \mid \mathrm{ad}\, h(\omega) = 2j\omega\}$, and define the filtration $K_\bullet \mathcal{C}l$ by

$$K_p \mathcal{C}l = \sum_{i-j \leq p} F_i \mathcal{C}l[j].$$

We have $\mathrm{gr}_K \mathcal{C}l \cong \mathbb{C}[T^* \Pi \mathfrak{g}_{>0}^*]$ as Poisson superalgebras.

Let $\mathcal{HC}$ be the full subcategory of $U(\mathfrak{g})$-biMod consisting of modules on which the adjoint $\mathfrak{g}$-action is locally finite.

For $M \in \mathcal{HC}$, let

$$C(M) = M \otimes \mathcal{D} \otimes \mathcal{C}l = \bigoplus_{p \in \mathbb{Z}} C^p(M),$$

$$C^p(M) = \bigoplus_{i-j=p} M \otimes \mathcal{D} \otimes \bigwedge^i(\mathfrak{g}_{>0}^*) \otimes \bigwedge^j(\mathfrak{g}_{>0}).$$

Here we have used the linear isomorphism $\mathcal{C}l \cong \bigwedge^\bullet(\mathfrak{g}_{>0}^*) \otimes \bigwedge^\bullet(\mathfrak{g}_{>0})$. The space $C(M)$ is naturally a $\mathbb{Z}$-graded bimodule over the $\mathbb{Z}$-graded superalgebra $C(U(\mathfrak{g}))$.

Set

$$d = \sum_i (x_i \otimes 1 + 1 \otimes \phi_i) \otimes x_i^* - 1 \otimes 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{ij}^k x_i^* x_j^* x_k \in C^1(U(\mathfrak{g})).$$

Lemma 3.1. *$d^2 = 0$ in $C(U(\mathfrak{g}))$.*

Since d is an odd element it follows from Lemma 3.1 that $(\operatorname{ad} d)^2 = 0$ on $C(M)$. By abuse of notation we denote by $H_f^{\bullet}(M)$ the cohomology of the cochain complex $(C(M), \operatorname{ad} d)$. Since $(C(U(\mathfrak{g})), \operatorname{ad} d)$ is a differential graded algebra $H_f^{\bullet}(U(\mathfrak{g}))$ is naturally a $\mathbb{Z}$-graded superalgebra and $H_f^{\bullet}(M)$ is naturally a bimodule over $H_f^{\bullet}(U(\mathfrak{g}))$.

The *finite W-algebra* [Pre] associated with $(\mathfrak{g}, f)$ may be defined as the associative algebra

$$U(\mathfrak{g}, f) := H_f^0(U(\mathfrak{g}))$$

([D³HK], see (14) below).

Let $K_{\bullet}M$ be a Kazhdan filtration of $M \in \mathcal{HC}$. Set

$$K_pC(M) = \sum_{p_1+p_2+p_3 \le p} K_{p_1}M \otimes K_{p_2}D \otimes K_{p_3}\mathcal{C}l.$$

When this is applied to $M = U(\mathfrak{g})$, $K_{\bullet}C(U(\mathfrak{g}))$ defines an increasing, exhaustive, separated filtration of $C(U(\mathfrak{g}))$ satisfying (11). Note that $d \in K_1C(U(\mathfrak{g}))$, and thus, $\operatorname{ad} d \cdot K_pC(U(\mathfrak{g})) \subset K_pC(U(\mathfrak{g}))$ and $\operatorname{ad} d$ defines a derivation of $\operatorname{gr}_K C(U(\mathfrak{g}))$. By definition the differential graded algebra $(\operatorname{gr}_K C(U(\mathfrak{g})), \operatorname{ad} d)$ is isomorphic to $(\bar{C}(\mathbb{C}[\mathfrak{g}^*]), \operatorname{ad} \bar{d})$, and $\operatorname{gr}_K C(U(\mathfrak{g}^*))$ is isomorphic to $\bar{C}(\operatorname{gr}_K M)$ as Poisson modules over $\bar{C}(\mathbb{C}[\mathfrak{g}^*])$, where $\bar{C}(\operatorname{gr}_K M)$ is the complex considered in Section 2.

Let $K_{\bullet}H_f^{\bullet}(M)$ be the filtration of $H_f^{\bullet}(M)$ induced from the filtration $K_{\bullet}C(M)$. We have

$$\operatorname{gr}_K H_f^0(U(\mathfrak{g})) \cong H_f^0(\operatorname{gr}_K U(\mathfrak{g})) \cong \mathbb{C}[\mathcal{S}_f]$$

as Poisson algebra ([GG, DSK]). In fact, we have the following more general assertion.

Theorem 3.2. (i) *Let M be an finitely generated object of $\mathcal{HC}$, $K_{\bullet}M$ a good Kazhdan-filtration of M. Then*

$$\operatorname{gr}_K H_f^i(M) \cong H_f^i(\operatorname{gr}_K M) \cong \begin{cases} (\operatorname{gr}_K M/\bar{I}_\chi \operatorname{gr}_K M)^{\operatorname{ad} \mathfrak{g}_{>0}} & \text{for } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

as Poisson modules over $\mathbb{C}[\mathcal{S}_f]$. In particular

$$\mathit{Var} H_f^0(M) = \mathit{Var} M \cap \mathcal{S}_f.$$

(ii) *We have $H_f^i(M) = 0$ for $i \neq 0$, $M \in \mathcal{HC}$. In particular the functor*

$$\mathcal{HC} \to U(\mathfrak{g}, f)\text{-biMod}, \quad M \mapsto H_f^0(M), \tag{12}$$

is exact.

Proof. (i) By the assumption $\operatorname{gr}_K M$ is an object of $\overline{\mathcal{HC}}$. Moreover, thanks to (the proof of) [Gin, Lemma 4.3.3], the filtration $K_{\bullet}C(M)$ is convergent in the sense of [CE]. Hence the assertion follows immediately from Theorem 2.3. (ii) Suppose that M is finitely generated. Then M admits a good Kazhdan filtration, and hence, $H_f^i(M) = 0$ for $i \neq 0$. But this prove the vanishing of all $M \in \mathcal{HC}$ since the cohomology functor commutes with injective limits. □

We shall now give yet another description of the functor (12), and show that (12) is equivalent to the functor constructed by Ginzburg [Gin] and Losev [Los], independently.

Choose a Lagrangian subspace l of $\mathfrak{g}_{1/2}$ with respect to the symplectic form (5), and let

$$\mathfrak{m} = l \oplus \mathfrak{g}_{\geq 1}.$$

Then $\mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}_{>0}$ and the restriction of χ to $\mathfrak{m}$ is a character, that is, $\chi([x,y]) = 0$ for $x, y \in \mathfrak{m}$. Let $\{x_i' | i = 1, \dots, \dim \mathfrak{m}\}$ be a basis of $\mathfrak{m}$, $\{x_i'^* | i = 1, \dots, \dim \mathfrak{m}\}$ the dual basis of $\mathfrak{m}^*$, $c_{ij}^k{}'$ the structure constants of $\mathfrak{m}$.

Let $\mathcal{C}l_{\mathfrak{m}}$ Clifford algebra associated with $\mathfrak{m} \oplus \mathfrak{m}^*$ and the natural bilinear form on it. For $M \in \mathcal{HC}$ set

$$C(M)' = M \otimes \mathcal{C}l_{\mathfrak{m}},$$

$$d' = \sum_{i=1}^{\dim \mathfrak{m}} (x_i' + \chi(x_i')) \otimes x_i'^* - 1 \otimes 1 \otimes \frac{1}{2} \sum_{1 \leq i,j,k \leq \dim \mathfrak{m}} c_{ij}^k{}' x_i'^* x_j'^* x_k' \in C(U(\mathfrak{g}))'.$$

Then we have $(d')^2 = 0$ and $(C(M'), \operatorname{ad} d')$ is a cochain complex as well. Denote by $H_f^\bullet(M)'$ the corresponding cohomology.

Proposition 3.3.

(i) *We have an algebra isomorphism $H_f^0(U(\mathfrak{g}))' \cong U(\mathfrak{g}, f)$.*

(ii) *For $M \in \mathcal{HC}$ we have $H_f^i(M)' = 0$ for $i \neq 0$ and $H_f^0(M)' \cong H_f^0(M)$ as modules over $U(\mathfrak{g}, f)$.*

Proof. We may assume that M be a finitely generated as in the proof of Theorem 3.2. Let $K_\bullet M$ be a good Kazhdan filtration. In the same manner as Theorem 3.2 one can show that

$$\operatorname{gr}_K H_f^i(M)' \cong \begin{cases} (\operatorname{gr}_K M / \mathfrak{m}_\chi \operatorname{gr}_K M)^{\operatorname{ad} \mathfrak{m}} & \text{for } i = 0, \\ 0 & \text{for } i \neq 0, \end{cases}$$

where $\mathfrak{m}_\chi$ is the ideal generated by $x - \chi(x)$ with $x \in \mathfrak{m}$. Since the natural map $(\operatorname{gr}_K M / \bar{I}_\chi \operatorname{gr}_K M)^{\operatorname{ad} \mathfrak{g}_{>0}} \to (\operatorname{gr}_K M / \mathfrak{m}_\chi \operatorname{gr}_K M)^{\operatorname{ad} \mathfrak{m}}$ is an isomorphism by the argument of [GG, 5.5] we have

$$\operatorname{gr}_K H_f^0(M) \xrightarrow{\sim} \operatorname{gr}_K H_f^0(M)' \tag{13}$$

as modules over $\mathbb{C}[\mathcal{S}_f]$.

Now in the same manner as in [AKM, 3.2.5] one can construct a map $H_f^0(M) \to H_f^0(M)'$, which induces the map (13), and hence must be an isomorphism. For $M = U(\mathfrak{g})$ this gives an algebra isomorphism $H_f^0(U(\mathfrak{g})) \xrightarrow{\sim} H_f^0(U(\mathfrak{g}))'$ and for a general M this gives the assertion (ii). □

Let $\mathbb{C}_\chi$ be the one-dimensional representation of $\mathfrak{m}$ defined by the character χ. For $M \in \mathcal{HC}$ the space

$$\mathrm{Wh}_{\mathfrak{m}}(M) := M \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$$

is equipped with a $(U(\mathfrak{g}), U(\mathfrak{g}, f))$-bimodule structure. Indeed, there is an obvious left $U(\mathfrak{g})$-module structure on $\mathrm{Wh}_{\mathfrak{m}}(M)$. To see the right $U(\mathfrak{g}, f)$-module structure consider the space $M \otimes \bigwedge^\bullet(\mathfrak{m})$, which is naturally a *right* module over $C(U(\mathfrak{g}))' = U(\mathfrak{g}) \otimes \mathcal{C}l_{\mathfrak{m}}$. Under this right module structure the element $d' \in C(U(\mathfrak{g}))'$ gives $M \otimes \bigwedge^\bullet(\mathfrak{m})$ the chain complex structure, and this complex is identical to the Chevalley complex for calculating the Lie algebra $\mathfrak{m}$-homology $H_\bullet(\mathfrak{m}, M \otimes \mathbb{C}_\chi)$ with coefficient in the diagonal $\mathfrak{m}$-module $M \otimes \mathbb{C}_\chi$, where $\mathfrak{m}$ acts on M by $xm = -mx$. The right $C(U(\mathfrak{g}))'$-action on $M\otimes\bigwedge^{\bullet}(\mathfrak{m})$ gives the right $U(\mathfrak{g},f)$-action on $H_{\bullet}(\mathfrak{m},M\otimes\mathbb{C}_\chi)$, in particular on $H_0(\mathfrak{m},M\otimes\mathbb{C}_\chi)=\mathrm{Wh}_{\mathfrak{m}}(M)$. This action obviously commutes with the left $U(\mathfrak{g})$-action.

By [Gin], we have $H_i(\mathfrak{m},M\otimes\mathbb{C}_\chi)=0$ for $i\neq 0$, $M\in\mathcal{HC}$, and hence, the functor

$$\mathrm{Wh}_{\mathfrak{m}}:\mathcal{HC}\rightarrow (U(\mathfrak{g}),U(\mathfrak{g},f))\text{-biMod},\quad M\mapsto \mathrm{Wh}_{\mathfrak{m}}(M)$$

is exact.

Let $\mathcal{C}$ be the full subcategory of $\mathfrak{g}$-Mod consisting of objects on which $x-\chi(x)$ acts locally nilpotently for all $x\in\mathfrak{m}$. Here, for any algebra A, A-Mod denotes the category of left A-modules. Note that $\mathrm{Wh}_{\mathfrak{m}}(M)$ with $M\in\mathcal{HC}$ belongs to $\mathcal{C}$ when it is considered as a left $\mathfrak{g}$-module.

For an object M of $\mathcal{C}$ consider the space $M\otimes\bigwedge^{\bullet}(\mathfrak{m}^*)$ as a (left) $C(U(\mathfrak{g}))'$-module. The cochain complex $(M\otimes\bigwedge^{\bullet}(\mathfrak{m}^*),d')$ is identical to the Chevalley complex for calculating Lie algebra $\mathfrak{m}$-cohomology $H^{\bullet}(\mathfrak{m},M\otimes\mathbb{C}_{-\chi})$ with coefficient in the diagonal $\mathfrak{m}$-module $M\otimes\mathbb{C}_{-\chi}$. It follows that $H^{\bullet}(\mathfrak{m},M\otimes\mathbb{C}_{-\chi})$ is a module over $U(\mathfrak{g},f)$, and we have a functor

$$\mathrm{Wh}^{\mathfrak{m}}:\mathcal{C}\rightarrow U(\mathfrak{g},f)\text{-Mod},\quad M\mapsto H^0(\mathfrak{m},M\otimes\mathbb{C}_{-\chi}).$$

By [Skr], one knows that $H^i(\mathfrak{m},M\otimes\mathbb{C}_{-\chi})=0$ for $i>0$, $M\in\mathcal{C}$, and $\mathrm{Wh}^{\mathfrak{m}}$ defines an *equivalence of categories.*

The following assertion can be proved in the same way as [A2, Theorem 2.4.2] using Proposition 3.3.

Proposition 3.4. *For $M\in\mathcal{HC}$ we have $H^0_f(M)\cong \mathrm{Wh}^{\mathfrak{m}}(\mathrm{Wh}_{\mathfrak{m}}(M)))$ as $U(\mathfrak{g},f)$-bimodules.*

Let

$$Y=\mathrm{Wh}_{\mathfrak{m}}(U(\mathfrak{g}))=U(\mathfrak{g})\otimes_{U(\mathfrak{m})}\mathbb{C}_\chi.$$

Then by Proposition 3.4 we obtain the usual realization of $U(\mathfrak{g},f)$:

$$U(\mathfrak{g},f)\cong \mathrm{Wh}^{\mathfrak{m}}(Y)\cong \mathrm{End}_{U(\mathfrak{g})}(Y)^{op}. \tag{14}$$

The assignment $U(\mathfrak{g},f)$-Mod $\rightarrow\mathcal{C}$, $E\mapsto Y\otimes_{U(\mathfrak{g},f)}E$, gives a functor which is quasi-inverse to Wh^m ([Skr]).

Remark 3.5. By Proposition 3.4 and [Los, 3.5], it follows that the functor $\mathcal{HC}\rightarrow U(\mathfrak{g},f)$-biMod, $M\mapsto H^0_f(M)$, coincides with the functor $\bullet_\dagger$ constructed by Losev [Los]. This observation enables us to improve the main result of [A3]; The details will appear elsewhere.

Let I be a two-sided ideal of $U(\mathfrak{g})$. Then $U(\mathfrak{g})/I$ is a quotient algebra, and thus, $H^0_f(U(\mathfrak{g})/I)$ inherits the algebra structure from $C(U(\mathfrak{g})/I)$. On the other hand, the exact sequence $0\rightarrow I\rightarrow U(\mathfrak{g})\rightarrow U(\mathfrak{g})/I\rightarrow 0$ induces the exact sequence

$$0\rightarrow H^0_f(I)\rightarrow U(\mathfrak{g},f)\rightarrow H^0_f(U(\mathfrak{g})/I)\rightarrow 0$$

by Theorem 3.2. Hence we have the algebra isomorphism

$$H^0_f(U(\mathfrak{g})/I)\cong U(\mathfrak{g},f)/H^0_f(I). \tag{15}$$

Let $\mathcal{C}^I$ denote the full subcategory of $\mathcal{C}$ consisting of objects which are annihilated by I.

Theorem 3.6. *For a two-sided ideal I of $U(\mathfrak{g})$ we have an equivalence of categories*

$$\mathcal{C}^I \cong H^0_f(U(\mathfrak{g})/I)\text{-Mod}, \quad M \mapsto \mathrm{Wh}^{\mathfrak{m}}(M).$$

Proof. By (15), $H^0_f(U(\mathfrak{g})/I)$-Mod can be identified by (15) with the full subcategory of $U(\mathfrak{g},f)$-Mod consisting objects M which are annihilated by $H^0_f(I)$. Therefore, thanks to Skryabin's equivalence, it is enough to check that $\mathrm{Wh}^{\mathfrak{m}}(M) \in H^0_f(U(\mathfrak{g})/I)$-Mod for $M \in \mathcal{C}^I$, and that $Y \otimes_{U(\mathfrak{g},f)} E \in \mathcal{C}^I$ for $E \in H^0_f(U(\mathfrak{g})/I)$-Mod. The former is easy to see. The latter follows from the proof of [Gin, Theorem 4.5.2]. □

4. Frenkel-Zhu's bimodules and Zhu's C_2-modules

Recall that a vertex algebra is a vector space V equipped with an element $\mathbf{1} \in V$ called the vacuum, $T \in \mathrm{End}(V)$, and a linear map

$$Y(?,z) : V \to (\mathrm{End}\, V)[[z,z^{-1}]], \quad a \mapsto Y(a,z) = a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

such that

(i) $\mathbf{1}(z) = \mathrm{id}_V$,
(ii) $a_{(n)}b = 0$ for $n \gg 0$, $a,b \in V$, and $a_{(-1)}\mathbf{1} = a$,
(iii) $(Ta)(z) = [T, a(z)] = \frac{d}{dz}a(z)$ for $a \in V$,
(iv) $(z-w)^n [a(z), b(w)] = 0$ in $\mathrm{End}(V)$ for $n \gg 0$, $a,b \in V$.

For a vertex algebra V we have the *Borcherds identity*

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)} = \sum_{i=1}^{\infty} (-1)^i \binom{r}{i} (a_{(p+r-i)}b_{(q+i)} - (-1)^r b_{(q+r-i)}a_{(p+i)})$$

in $\mathrm{End}\, V$ for all $p,q,r \in \mathbb{Z}$, $a,b,c \in V$.

A module over a vertex algebra V is a vector space M equipped with a linear map

$$Y^M(?,z) : V \to (\mathrm{End}\, M)[[z,z^{-1}]], \quad a \mapsto a^M(z) = \sum_{n \in \mathbb{Z}} a^M_{(n)} z^{-n-1},$$

such that $Y^M(\mathbf{1},z) = \mathrm{id}_M$, $a^M_{(n)}m = 0$ for $n \gg 0$, $a \in V$, $m \in M$, and

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)^M_{(p+q-i)} = \sum_{i=1}^{\infty} (-1)^i \binom{r}{i} (a^M_{(p+r-i)}b^M_{(q+i)} - (-1)^r b^M_{(q+r-i)}a^M_{(p+i)})$$

in $\mathrm{End}\, M$ for all $p,q,r \in \mathbb{Z}$, $a,b,c \in V$. In particular V itself if a module over V called the *adjoint module*. Let V-Mod be the abelian category of V-modules. Below if no confusion arises we write $a_{(n)}$ for $a^M_{(n)}$.

For a V-module M set

$$C_2(M) := \mathrm{span}_{\mathbb{C}}\{a_{(-2)}m | a \in V, m \in M\}.$$

Zhu's C_2-algebra [Zhu] of V is by definition the space

$$R_V =: V/C_2(V)$$

equipped with the Poisson algebra structure given by

$$\bar{a} \cdot \bar{b} = \overline{a_{(-1)}b}, \quad \{\bar{a}, \bar{b}\} = \overline{a_{(0)}b} \quad \text{for } a,b \in V,$$

where $\bar{a} = a + C_2(V)$. *Zhu's C_2-module* of M is the space $M/C_2(M)$ equipped with the Poisson module structure over R_V given by

$$\bar{a} \cdot \bar{m} = \overline{a_{(-1)}m}, \quad \{\bar{a}, \bar{m}\} = \overline{a_{(0)}m} \quad \text{for } a \in V,\ m \in M.$$

A vertex algebra V is called *finitely strongly generated* if R_V is finitely generated as a ring; it is called *rational* if any V-module is completely reducible; it is called *C_2-cofinite* if Zhu's C_2-algebra R_V is finite-dimensional. The C_2-cofiniteness condition is equivalent to the lisse condition in the sense of [BFM] ([A4]).

A vertex algebra V is called *conformal* if it is equipped with a vector $\omega \in V$, called the *conformal vector*, such that the corresponding field $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies the relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{(m^3 - m)\delta_{m+n,0}}{12} c_V \quad \text{for some } c_V \in \mathbb{C},$$
$$L_{-1} = T,$$
$$L_0 \text{ is diagonalizable on } V.$$

The number c_V is called the *central charge* of V.

In this paper we assume that a vertex algebra V is conformal and $\frac{1}{2}\mathbb{Z}$-graded[3] with respect to L_0:

$$V = \bigoplus_{d \in \frac{1}{2}\mathbb{Z}} V_d, \quad V_d = \{a \in V | L_0 a = da\}.$$

For a homogeneous elements $a \in V$ we denote by $\mathrm{wt}(a)$ the eigenvalue of L_0 on a.

A V-module M is called *graded* if

$$M = \bigoplus_{d \in \mathbb{C}} M_d, \quad M_d = \{m \in M | (L_0 - d)^r m = 0,\ r \gg 0\};$$

it is called *positively graded* if in addition there exists a finite set $\{d_1, \dots, d_r\} \subset \mathbb{C}$ such that $M_d = 0$ unless $d \in \bigcup_{i=1}^r (d_i + \frac{1}{2}\mathbb{Z}_{\geq 0})$. If V is C_2-cofinite any finitely generated V-module is positively graded ([ABD]). Let V-gMod be the abelian full subcategory of V-Mod consisting of positively graded V-modules, $\mathrm{Irr}(V)$ the set of isomorphism classes of simple objects of V-gMod.

Let $A(V)$ be the (L_0-twisted) *Zhu's algebra* of V ([FZ, DSK]). By definition

$$A(V) = V/O(V),$$

where $O(V)$ is the subspace of V spanned by the vectors

$$a \circ b := \sum_{i \geq 0} \binom{\mathrm{wt}(a)}{i} a_{(i-2)} b$$

with homogeneous vectors $a, b \in V$. The multiplication $*$ of $A(V)$ given by

$$a * b = \sum_{i \geq 0} \binom{\mathrm{wt}(a)}{i} a_{(i-1)} b.$$

Let M be a V-module. *Frenkel-Zhu's bimodule* [FZ] associated to M is the bimodule $A(M)$ over $A(V)$ defined by

$$A(M) = M/O(M),$$

[3] This is because W-algebras are $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded in general. However since principal W-algebras are $\mathbb{Z}_{\geq 0}$-graded it is enough to consider the $\mathbb{Z}$-graded case in order to prove Main Theorem.

where $O(M)$ is the subspace of M spanned by the elements

$$a \circ m := \sum_{i \geq 0} \binom{\mathrm{wt}(a)}{i} a_{(i-2)}m$$

with homogeneous vectors $a \in V$ and $m \in M$. The bimodule structure of $A(M)$ is given by

$$a * m = \sum_{i \geq 0} \binom{\mathrm{wt}(a)}{i} a_{(i-1)}m, \quad m * a = \sum_{i \geq 0} \binom{\mathrm{wt}(a)-1}{i} a_{(i-1)}m. \tag{16}$$

Note that

$$a * m - m * a = \sum_{i \geq 0} \binom{\mathrm{wt}(a)-1}{i} a_{(i)}m. \tag{17}$$

Lemma 4.1 ([FZ, Proposition 1.5.4])**.** *The assignment $M \mapsto A(M)$ defines a right exact functor from V-*Mod *to $A(V)$-*biMod.

Zhu's C_2-algebra R_V and Zhu's algebra $A(V)$ are related as follows: Set

$$V_{\leq p} = \bigoplus_{d \leq p} V_d$$

and let $F_pA(V)$ be the image of $V_{\leq p}$ in $A(V)$. Then $F_\bullet A(V)$ defines an increasing, exhaustive $\frac{1}{2}\mathbb{Z}$-filtration of $A(V)$ satisfying (11) ([Zhu]). (In the cases that we will consider in this paper the filtration $F_\bullet A(V)$ will be separated as well; this is true, for instance, if V is positively graded.) On the other hand the grading of V induces the grading of R_V: $R_V = \bigoplus_{p \in \frac{1}{2}\mathbb{Z}} (R_V)_p$, where $(R_V)_p$ is the image of V_p in R_V: $(R_V)_p \cong V_p/C_2(V)_p$, $C_2(V)_p = C_2(V) \cap V_p$. The linear map

$$(R_V)_p \to F_pA(V)/F_{p-1/2}A(V), \quad a + C_2(V)_p \mapsto a + O(V) \cap V_{\leq p} + V_{\leq p-1/2}$$

defines a surjective homomorphism

$$\pi_V : R_V \twoheadrightarrow \mathrm{gr}_F\, A(V) \tag{18}$$

of graded Poisson algebras ([ALY, Proposition 3.2]). It follows that $A(V)$ is finite-dimensional if V is C_2-cofinite.

For a graded V-module $M = \bigoplus_{d \in \mathbb{C}} M_d$, there is a similar relation between $M/C_2(M)$ and $A(M)$ as well: Set

$$M_{\leq p} = \bigoplus_{d \in p - \frac{1}{2}\mathbb{Z}_{\geq 0}} M_d,$$

and let $F_pA(M)$ be the image of $M_{\leq p}$ in $A(M)$. Then the space $\mathrm{gr}_F\, A(M) = \bigoplus_{p \in \mathbb{C}} F_pA(M)/F_{p-1/2}A(M)$ is a graded Poisson module over $\mathrm{gr}_F\, A(V)$, and hence over R_V by (18).

The following assertion can be proved in the same manner as [ALY, Proposition 3.2].

Lemma 4.2. *Let M be a graded V-module. The linear map $M_p/C_2(M)_p \to F_pA(M)/F_{p-1/2}A(M)$, $m + C_2(M)_p \mapsto m + O(M) \cap M_{\leq p} + M_{\leq p-1/2}$, defines a surjective homomorphism*

$$\pi_M : M/C_2(M) \twoheadrightarrow \mathrm{gr}_F\, A(M)$$

of Poisson modules over R_V. Here $C_2(M)_p = C_2(M) \cap M_p$.

Now assume for a moment that V is $\mathbb{Z}_{\geq 0}$-graded with respect to L_0. Let $\mathcal{U}(V)=\bigoplus_{d\in\mathbb{Z}}\mathcal{U}(V)_d$ be the current algebra [FZ, MNT] of V, which is a degreewise complete graded topological algebra. Then a V-module is the same as a continuous representation of $\mathcal{U}(V)$. Since

$$A(V)\cong\mathcal{U}(V)_0/\overline{\sum_{p>0}\mathcal{U}(V)_p\mathcal{U}(V)_{-p}} \tag{19}$$

([NT]), where $\overline{U}$ denotes the degreewise closure of U, an $A(V)$-module E can be regarded as a module over $\mathcal{U}(V)_{\leq 0}:=\bigoplus_{p\leq 0}\mathcal{U}(V)_p$ on which $\mathcal{U}(V)_p$, $p<0$, acts trivially. Set

$$M_V(E):=\mathcal{U}(V)\otimes_{\mathcal{U}(V)_{\leq 0}}E\in V\text{-gMod}, \tag{20}$$

and let $L_V(E)$ be the unique simple quotient of $M_V(E)$. By Zhu's theorem [Zhu] we have

$$\operatorname{Irr}(V)=\{L_V(E)|E\in\operatorname{Irr}(A(V))\}, \tag{21}$$

where, for any algebra A, $\operatorname{Irr}(A)$ denotes the set of isomorphism classes of simple objects of A-Mod.

5. The effect of shifts of conformal vector to Frenkel-Zhu's bimodules

Let V be a $\frac{1}{2}\mathbb{Z}$-graded conformal vertex algebra with conformal vector ω. Suppose that there exists an element $\xi\in V$ that satisfies the conditions

$$L_n\xi=\delta_{n,0}\xi,\quad \xi_{(n)}\xi=\kappa\delta_{n,1}\mathbf{1}\quad\text{for } n\in\mathbb{Z}_{\geq 0}$$

with some $\kappa\in\mathbb{C}$, and that $\xi_{(0)}$ acts semisimply on V with eigenvalues in $\mathbb{Z}$. Then one can "shift" the conformal vector ω by $\frac{1}{2}L_{-1}\xi$ to obtain a new conformal vector. Namely

$$\omega_\xi:=\omega+\frac{1}{2}\xi_{(-2)}\mathbf{1}$$

also defines a conformal vector of V, with central charge $c_{new}=c_{old}-3\kappa$, where $c_{\rm old}$ is the central charge of V with respect to ω.

Although the definition of Zhu's algebra and Frenkel-Zhu's bimodules depend on the choice of a conformal vector, the above shift of a conformal vector does not change the structure of Zhu's algebra nor Frenkel-Zhu's bimodules as we show below: for a V-module M let $A^{new}(M)$ (temporary) denote Frenkel-Zhu's bimodule of M with respect to the conformal vector ω_ξ and let $A^{\rm old}(M)$ (temporary) denote Frenkel-Zhu's bimodule with respect to the conformal vector ω.

Let $\Delta(z)$ be Li's Δ-operator [Li] associated with ξ:

$$\Delta(z)=z^{\frac{\xi_{(0)}}{2}}\exp\Big(\sum_{n\geq 1}\frac{\xi_{(n)}}{-2n}(-z)^n\Big).$$

Proposition 5.1.

(i) *The map $V\to V$, $a\mapsto\Delta(1)a$, induces an algebra isomorphism*

$$A^{old}(V)\xrightarrow{\sim}A^{new}(V).$$

(ii) *Let M be a V-module on which $\xi_{(0)}$ acts semisimply. Then the map $M \to M$, $m \mapsto \Delta(1)m$, induces an $A^{old}(V)(\cong A^{new}(V))$-bimodule isomorphism*

$$A^{old}(M) \xrightarrow{\sim} A^{new}(M).$$

Proposition 5.1 follows from the following lemma.

Lemma 5.2. *Let M be a V-module on which $\xi_{(0)}$ acts semisimply. Then*

$$\Delta(1)(a \circ_{old} m) = (\Delta(1)a) \circ_{new} (\Delta(1)m),$$
$$\Delta(1)(a *_{old} m) = (\Delta(1)a) *_{new} (\Delta(1)m),$$
$$\Delta(1)(m *_{old} a) = (\Delta(1)m) *_{new} (\Delta(1)a)$$

for $a \in V$, $m \in M$. Here $\circ_{old}$ and $*_{old}$ (respectively, $\circ_{new}$ and $*_{new}$) are operations (16) with respect to the grading defined by $L_{0,old}$ (respectively, $L_{0,new}$). Here $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_{n,old} z^{-n-2}$, $Y(\omega_\xi, z) = \sum_{n\in\mathbb{Z}} L_{n,new} z^{-n-2}$.

Proof. Let m be a homogeneous vector of M such that $\xi_{(0)} m = 2\lambda m$. Then $\mathrm{wt}(m)_{new} = \mathrm{wt}(m)_{old} - \lambda$, where $\mathrm{wt}(m)_{new}$ and $\mathrm{wt}(m)_{old}$ denote the eigenvalue of $L_{0,new}$ and $L_{0,old}$ on m, respectively. Write

$$\exp(\sum_{n\geq 1} \frac{\xi_{(n)}}{-2n}(-z)^{-n}) = \sum_{n\geq 0} u_n z^{-n}$$

with $u_n \in \mathbb{C}[\xi_{(1)}, \xi_{(2)}, \dots,]$. Since we have

$$\Delta(1)Y(a,z) = Y(\Delta(z+1)a, z)\Delta(1) \tag{22}$$

for any $a \in V$ by [Li, Proposition 3.2], we have

$$\begin{aligned}
\Delta(1)(a \circ_{old} m) &= \Delta(1)\,\mathrm{Res}_{z=0}(Y(a,z)\frac{(z+1)^{\mathrm{wt}(a)_{old}}}{z^2} m) \\
&= \mathrm{Res}_z(\Delta(1)Y(a,z)\frac{(z+1)^{\mathrm{wt}(a)_{old}}}{z^2} m) \\
&= \mathrm{Res}_{z=0}(Y(\Delta(z+1)a,z)\frac{(z+1)^{\mathrm{wt}(a)_{old}}}{z^2} \Delta(1)m) \\
&= \sum_{n\geq 0} \mathrm{Res}_{z=0}(Y(u_n a,z)\frac{(z+1)^{\mathrm{wt}(a)_{old}+\lambda-n}}{z^2} \Delta(1)m) \\
&= \sum_{n\geq 0} \mathrm{Res}_{z=0}(Y(u_n a,z)\frac{(z+1)^{\mathrm{wt}(u_n a)_{new}}}{z^2} \Delta(1)m) = (\Delta(1)a) \circ_{new} (\Delta(1)m).
\end{aligned}$$

The proof of the other equalities is similar. □

6. Affine vertex algebras

Let $\widehat{\mathfrak{g}}$ be the non-twisted affine Kac-Moody algebra associated with $\mathfrak{g}$ and $(\ |\ )$:

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K.$$

The commutation relations of $\widehat{\mathfrak{g}}$ are given by

$$[xt^m, yt^n] = [x,y]t^{m+n} + m\delta_{m+n,0}(x|y)K \quad \text{for } x, y \in \mathfrak{g}, m, n \in \mathbb{Z},$$
$$[K, \widehat{\mathfrak{g}}] = 0.$$

We consider $\mathfrak{g}$ as a subalgebra of $\widehat{\mathfrak{g}}$ by the embedding $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$, $x \mapsto xt^0$.

For $k \in \mathbb{C}$ define

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as a multiplication by k. There is a unique vertex algebra structure on $V^k(\mathfrak{g})$ such that $\mathbf{1} := 1 \otimes 1$ is the vacuum and

$$Y(xt^{-1}\mathbf{1}, z) = x(z) := \sum_{n \in \mathbb{Z}} (xt^n) z^{-n-1}$$

for $x \in \mathfrak{g}$. The vertex algebra $V(\mathfrak{g})$ is called the *universal affine vertex algebra* associated with $\mathfrak{g}$ at level k.

A $V^k(\mathfrak{g})$-module is the same as a smooth $\widehat{\mathfrak{g}}$-module of level k, where by a smooth $\widehat{\mathfrak{g}}$-module M we mean a $\widehat{\mathfrak{g}}$-module M such that $(xt^n)m =$ for $n \gg 0$, $x \in \mathfrak{g}$, $m \in M$.

We have

$$C_2(M) = \mathfrak{g}[t^{-1}]t^{-2}M \tag{23}$$

for a $V^k(\mathfrak{g})$-module M. It follows that the assignment $x \mapsto \overline{(xt^{-1})\mathbf{1}}$, $x \in \mathfrak{g}$, gives the isomorphism of Poisson algebras

$$\mathbb{C}[\mathfrak{g}^*] \xrightarrow{\sim} R_{V^k(\mathfrak{g})} = V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}). \tag{24}$$

We will identify $R_{V^k(\mathfrak{g})}$ with $\mathbb{C}[\mathfrak{g}^*]$ through the above isomorphism. The Poisson module structure of $M/C_2(M) = M/\mathfrak{g}[t^{-1}]t^{-2}M$ over $\mathbb{C}[\mathfrak{g}^*]$ is then given by

$$x \cdot \bar{m} = \overline{(xt^{-1})m}, \quad \{x, \bar{m}\} = \overline{(xt^0)m}$$

for $x \in \mathfrak{g}$, $m \in M$.

We will assume that k is non-critical, that is, $k \neq -h_{\mathfrak{g}}^{\vee}$, unless otherwise stated, although this condition is not essential. The standard conformal vector $\omega_{\mathfrak{g}}$ of $V^k(\mathfrak{g})$ is given by the Sugawara construction:

$$\omega_{\mathfrak{g}} = \frac{1}{2(k + h_{\mathfrak{g}}^{\vee})} \sum_i (X_i t^{-1})(X^i t^{-1})\mathbf{1},$$

where $\{X_i\}$ is a basis of $\mathfrak{g}$, $\{X^i\}$ the dual bases with respect to $(\ |\ )$. This gives a $\mathbb{Z}_{\geq 0}$-grading on $V^k(\mathfrak{g})$.

We have [FZ] the natural isomorphism of algebras

$$A(V^k(\mathfrak{g})) \cong U(\mathfrak{g}). \tag{25}$$

This can be also seen using (19) from the fact that the current algebra of $V^k(\mathfrak{g})$ is isomorphic to the standard degreewise completion [MNT] $\widetilde{U_k(\widehat{\mathfrak{g}})}$ of $U_k(\widehat{\mathfrak{g}}) := U(\widehat{\mathfrak{g}})/(K - k\,\mathrm{id})$. For a $\mathfrak{g}$-module E, we have

$$M_{V^k(\mathfrak{g})}(E) \cong U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} E, \tag{26}$$

where E is considered as a $\mathfrak{g}[t] \oplus \mathbb{C}K$-modules on which K acts as the multiplication by k and $\mathfrak{g}[t]t$ acts trivially.

Let $N_k(\mathfrak{g})$ be the unique maximal ideal of $V^k(\mathfrak{g})$. Then

$$L(k\Lambda_0) := V^k(\mathfrak{g})/N_k(\mathfrak{g})$$

is a simple vertex algebra called the *(simple) affine vertex algebra* associated with $\mathfrak{g}$ at level k.

Let KL_k be the full subcategory of the category of $V^k(\mathfrak{g})$-gMod consisting of objects M on which $\mathfrak{g} \subset \widehat{\mathfrak{g}}$ acts locally finitely. By (26), $M_{V^k(\mathfrak{g})}(E)$ is an object of KL_k for a finite-dimensional $\mathfrak{g}$-module E.

The following assertion is clear.

Lemma 6.1. (i) *The assignment $M \mapsto M/C_2(M)$ defines a right exact functor from KL_k to $\overline{\mathcal{HC}}$.*
(ii) *The assignment $M \mapsto A(M)$ defines a right exact functor from KL_k to $\mathcal{HC}$.*

Let KL_k^{Δ} be the full subcategory of KL consisting of modules which admit a finite filtration $0 = M_0 \subset M_1 \subset \dots M_r = M$ such that $M_i/M_{i+1} \cong M_{V^k(\mathfrak{g})}(E)$ for some finite-dimensional representation E_i for each i. Note that the adjoint module $V^k(\mathfrak{g})$ is an object of KL_k^{Δ} and that $M \in \mathsf{KL}_k$ belongs to KL_k^{Δ} if and only if it is a free $U(\mathfrak{g}[t^{-1}]t^{-1})$-module of finite rank.

Lemma 6.2. (i) *Let M be an object of KL_k^{Δ}. Then $\pi_M : M/C_2(M) \to \operatorname{gr}_F A(M)$ is an isomorphism.*
(ii) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in KL_k^{Δ}. Then the induced sequence $0 \to A(M_1) \to A(M_2) \to A(M_3) \to 0$ is exact as well.*
(iii) *Let M be a finitely generated object of KL_k. Then $A(M)$ is finitely generated as a left (or a right) $U(\mathfrak{g})$-module.*

Proof. (i) Let $F_\bullet O(M)$ be the filtration of $O(M)$ induced by the filtration $\{M_{\leq p}\}$ of M, $\operatorname{gr}_F O(M) = \bigoplus_p F_p O(M)/F_{p-1/2}O(M)$. The freeness of M over $U(\mathfrak{g}[t^{-1}]t^{-1})$ implies that $a_{(-2)}m \neq 0$ for any nonzero elements $a \in V^k(\mathfrak{g})$, $m \in M$. Hence $\operatorname{gr}_F O(M) = C_2(M) \subset M = \operatorname{gr}_F M$ and the assertion follows. (ii) It is sufficient to show that the induced sequence

$$0 \to \operatorname{gr}_F A(M_1) \to \operatorname{gr}_F A(M_2) \to \operatorname{gr}_F A(M_3) \to 0 \tag{27}$$

is exact. Since $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of free $U(\mathfrak{g}[t^{-1}]t^{-1})$-modules it induces an exact sequence

$$0 \to M_1/C_2(M_1) \to M_2/C_2(M_2) \to M_3/C_2(M_3) \to 0$$

by (23). By (i), this prove the exactness of (27). (iii) Since it is finitely generated, M is a quotient of an object of KL^{Δ}. By the right exactness of the functor $A(?)$ it is enough to show the assertion for objects of KL^{Δ}. By (ii) it then suffices to show the assertion for the modules of the form $M = M_{V^k(\mathfrak{g})}(E)$. But this follows from [FZ, Theorem 3.2.1]. $\square$

Let $\{e, f, h\}$ be the $\mathfrak{sl}_2$-triple defined in Section 2. In the definition of W-algebras $\mathcal{W}^k(\mathfrak{g}, f)$ below we shift the conformal vector $\omega_{\mathfrak{g}}$ of $V^k(\mathfrak{g})$ to the conformal vector

$$\omega_{\mathfrak{g},h} = \omega_{\mathfrak{g}} + \frac{1}{2}(ht^{-2})\mathbf{1} \tag{28}$$

to give a well-defined conformal vector of $\mathcal{W}^k(\mathfrak{g}, f)$. We will identify Frenkel-Zhu's bimodules of $M \in \mathsf{KL}_k$ with respect to $\omega_{\mathfrak{g},h}$ with Frenkel-Zhu's bimodules with respect to $\omega_{\mathfrak{g}}$ through Proposition 5.1 and denote both of them by $A(M)$.

7. W-algebras and Poisson modules over Slodowy slices

For a $V^k(\mathfrak{g})$-module M, let $(C^{\mathrm{ch}}(M), Q_{(0)})$ be the BRST complex of the (generalized) *quantized Drinfeld-Sokolov reduction* associated with $(\mathfrak{g}, f)$ defined in [FF, KRW]. We have

$$C^{\mathrm{ch}}(M) = M \otimes \mathcal{D}^{\mathrm{ch}} \otimes \bigwedge\nolimits^{\frac{\infty}{2}+\bullet},$$

where $\mathcal{D}^{\mathrm{ch}}$ is the $\beta\gamma$-system of rank $\frac{1}{2}\dim \mathfrak{g}_{1/2}$, $\bigwedge^{\frac{\infty}{2}+\bullet}$ is the space of semi-infinite forms associated with $\mathfrak{g}_{>0} \oplus \mathfrak{g}_{>0}^*$. The vertex algebra $\mathcal{D}^{\mathrm{ch}}$ is freely generated by the fields $\phi_i(z)$ with $i = 1, \dots, \dim \mathfrak{g}_{1/2}$ (corresponding to the basis $\{x_i\}$ of $\mathfrak{g}_{1/2}$) satisfying the OPE's

$$\phi_i(z)\phi_j(w) \sim \frac{\chi([x_i, x_j])}{z-w}.$$

The space $\bigwedge^{\frac{\infty}{2}+\bullet}$ of semi-infinite forms is a vertex superalgebra freely generated by the odd fields $\psi_1(z), \dots, \psi_{\dim \mathfrak{g}_{>0}}(z)$ (corresponding to the basis $\{x_i\}$ of $\mathfrak{g}_{>0}$) and $\psi_1^*(z), \dots, \psi^*_{\dim \mathfrak{g}_{>0}}(z)$ (corresponding to the dual basis $\{x_i^*\}$ of $\mathfrak{g}_{>0}^*$) satisfying the OPE's

$$\psi_i(z)\psi_j^*(w) \sim \frac{\delta_{ij}}{z-w}, \quad \psi_i(z)\psi_j(w) \sim \psi_i^*(z)\psi_j^*(w) \sim 0.$$

The differential $Q_{(0)}$ is the zero-mode of the fields

$$\begin{aligned} Q(z) &= \sum_{n\in\mathbb{Z}} Q_{(n)} z^{-n-1} \\ &:= \sum_{i=1}^{\dim \mathfrak{g}_{>0}} (x_i(z) + \phi_i(z))\psi_i^*(z) - \frac{1}{2} \sum_{1\le i,j,k\le \dim \mathfrak{g}_{>0}} c_{ij}^k \psi_i^*(z)\psi_j^*(z)\psi_k(w). \end{aligned}$$

Here we have omitted the tensor product symbol and have put $\phi_i(z) = \chi(x_i)$ for $i > \dim \mathfrak{g}_{1/2}$. (Note that in the formula of $Q(z)$ above there is no need to take the normal ordering because of the existence of the structure constant c_{ij}^k.)

By abuse of notation we denote also by $H_f^0(M)$ the cohomology of the complex $(C^{\mathrm{ch}}(M), Q_{(0)})$.

The W*-algebra associated with* $(\mathfrak{g}, f)$ at level k is by definition

$$\mathcal{W}^k(\mathfrak{g}, f) = H_f^0(V^k(\mathfrak{g})). \tag{29}$$

The space $\mathcal{W}^k(\mathfrak{g}, f)$ inherits the vertex algebra structure from $C^{\mathrm{ch}}(V^k(\mathfrak{g}))$. The vertex algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is conformal with the conformal vector $\omega_{\mathcal{W}}$ defined by

$$\omega_{\mathcal{W}} = \omega_{\mathfrak{g},h} + \omega_{\mathcal{D}} + \omega_{\bigwedge^{\frac{\infty}{2}+\bullet}},$$

where

$$\begin{aligned} Y(\omega_{\mathcal{D}}, z) &= \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}_{1/2}} :\partial_z \phi^i(z)\phi_i(z), \\ Y(\omega_{\bigwedge^{\frac{\infty}{2}+\bullet}}, z) &= -\sum_{i=1}^{\dim \mathfrak{g}_{>0}} m_i :\psi_i^*(z)\partial_z\psi_i(z): + \sum_{i=1}^{\dim \mathfrak{g}_{>0}} m_i :\partial_z\psi_i^*(z)\psi_i(z):. \end{aligned}$$

Here $\phi^i(z)$ is the field of $\mathcal{D}$ corresponding to the vector $x^i \in \mathfrak{g}_{1/2}$ such that $\chi([x^i, x_j]) = \delta_{ij}$, $m_i = j$ if $x_i \in \mathfrak{g}_j$, and we have used the state-field correspondence. Here the conformal vector $\omega_{\mathfrak{g}}$ of $V^k(\mathfrak{g})$ has been shifted to $\omega_{\mathfrak{g},h}$ so that $Q_{(0)}\omega_{\mathcal{W}} = 0$.

By definition the assignment $M \mapsto H^0_f(M)$ defines a functor from $V^k(\mathfrak{g})$-Mod to $\mathcal{W}^k(\mathfrak{g}, f)$-Mod.

For a $V^k(\mathfrak{g})$-module M, consider Zhu's C_2-module $C^{\mathrm{ch}}(M)/C_2C^{\mathrm{ch}}(M)$ over the Poisson superalgebra $R_{C^{\mathrm{ch}}(V^k(\mathfrak{g}))}$. Since we have $Q_{(0)}C_2C^{\mathrm{ch}}(M) \subset C_2C^{\mathrm{ch}}(M)$, $C^{\mathrm{ch}}(M)/C_2C^{\mathrm{ch}}(M)$ is a quotient complex, which is by definition isomorphic to the complex $(\bar{C}(M/C_2(M)), \operatorname{ad}\bar{d})$ studied in Section 2. We have the obvious map

$$\bar{\eta}_M : H^0_f(M)/C_2H^0_f(M) \to H^0_f(M/C_2(M)).$$

For the adjoint module $M = V^k(\mathfrak{g})$, $\bar{\eta}_{V^k(\mathfrak{g})}$ gives the isomorphism

$$\eta_{V^k(\mathfrak{g})} : R_{\mathcal{W}^k(\mathfrak{g},f)} \xrightarrow{\sim} \mathbb{C}[\mathcal{S}_f]$$

([DSK]). It follows that $\bar{\eta}_M$ is a homomorphism of Poisson modules over $\mathbb{C}[\mathcal{S}_f]$.

Theorem 7.1 ([A5]).

(i) *We have $H^i_f(M) = 0$ for $i \neq 0$, $M \in \mathsf{KL}_k$. In particular the functor $\mathsf{KL}_k \to \mathcal{W}^k(\mathfrak{g}, f)$-Mod, $M \mapsto H^0_f(M)$, is exact.*

(ii) *For $M \in \mathsf{KL}_k$, $\bar{\eta}_M$ gives the isomorphism*

$$H^0_f(M)/C_2(H^0_f(M)) \cong H^0_f(M/C_2(M))$$

of Poisson modules over $\mathbb{C}[\mathcal{S}_f]$.

Let N be an ideal of $V^k(\mathfrak{g})$. By Theorem 7.1 (i) $H^0_f(N)$ embeds into $\mathcal{W}^k(\mathfrak{g}, f)$, and we have the isomorphism

$$H^0_f(V^k(\mathfrak{g})/N) \cong \mathcal{W}^k(\mathfrak{g}, f)/H^0_f(N) \tag{30}$$

of vertex algebras. In particular,

$$H^0_f(L(k\Lambda_0)) \cong \mathcal{W}^k(\mathfrak{g}, f)/H^0_f(N_k(\mathfrak{g})).$$

8. Quantized Drinfeld-Sokolov reduction and Frenkel-Zhu's bimodules associated with W-algebras

For a $V^k(\mathfrak{g})$-module M, consider the bimodule $A(C^{ch}(M))$ over $A(C^{\mathrm{ch}}(V^k(\mathfrak{g})))$. Since we have $Q_{(0)}O(C^{ch}(M)) \subset O(C^{ch}(M))$, $(A(C^{ch}(M)), Q_{(0)})$ is a quotient complex, which is isomorphic to the complex $(C(A(M)), \operatorname{ad} d)$ studied in Section 2, where, throughout this section, $A(M)$ denotes Frenkel-Zhu's bimodule associated with M with respect to the conformal vector (28). Consider the map

$$\eta_M : \begin{array}{ccc} A(H^0_f(M)) & \to & H^0_f(A(M)), \\ [c] + O(H^0_f(M)) & \mapsto & [c + O(C(M))]. \end{array}$$

For the adjoint module $M = V^k(\mathfrak{g})$, $\eta_{V^k(\mathfrak{g})}$ gives the isomorphism

$$A(\mathcal{W}^k(\mathfrak{g}, f)) \xrightarrow{\sim} U(\mathfrak{g}, f) \tag{31}$$

of algebras ([A2, DSK], or see Proposition 8.4 (ii) below). It follows that η_M is a homomorphism of $U(\mathfrak{g}, f)$-bimodules.

We can now state the main result of this section:

Theorem 8.1. *For any object M of KL_k, η_M gives the isomorphism*
$$A(H_f^0(M))) \cong H_f^0(A(M))$$
of $U(\mathfrak{g}, f)$-bimodules.

Remark 8.2. Theorem 8.1 holds at the critical level $k = -h_{\mathfrak{g}}^{\vee}$ as well by considering the outer grading as in [A1, A2].

To avoid confusion we denote by $K_\bullet A(M)$ (instead by $F_\bullet A(M)$) the filtration of $A(M)$ with respect to the grading defined by the conformal vector (28) for $M \in \mathsf{KL}_k$.

Lemma 8.3. (i) *The filtration $K_\bullet A(V^k(\mathfrak{g}))$ coincides with the Kazhdan filtration of $U(\mathfrak{g}) = A(V^k(\mathfrak{g}))$.*
(ii) *Let M be an object of KL_k, Then $K_\bullet A(M)$ is a Kazhdan filtration of $A(M)$. It is good if M is finitely generated.*

Proof. (i) and the first assertion of (ii) is easily seen from the definition. To see the second assertion of (ii) observe that $M/C_2(M)$ is a finitely generated $\mathbb{C}[\mathfrak{g}^*]$-module for a finitely generated object M of KL_k. Hence so is $\operatorname{gr}_K A(M)$ by Lemma 4.2. □

Proposition 8.4.
(i) *For an object M of KL_k, $\eta_M : A(H_f^0(M)) \to H_f^0(A(M))$ is surjective.*
(ii) *For an object M of KL_k^Δ, $\eta_M : A(H_f^0(M)) \to H_f^0(A(M))$ is an isomorphism.*

Proof. (i) First, suppose that M is finitely generated. By Lemma 8.3, $K_\bullet A(M)$ is a good Kazhdan filtration of $A(M)$. Hence we have
$$\operatorname{gr}_K H_f^0(A(M)) \cong H_f^0(\operatorname{gr}_K A(M)) \tag{32}$$
by Theorem 3.2. Here $\operatorname{gr}_K H_f^0(A(M))$ is the associated graded with respect to the induced filtration $K_p H_f^0(A(M)) = \operatorname{Im}(H_f^0(K_p A(M)) \to H_f^0(A(M)))$. Since $\eta_M(K_p A(H_f^0(M))) \subset K_p H_f^0(A(M))$, η_M induces a homomorphism
$$\operatorname{gr}_K \eta_M : \operatorname{gr}_K A(H_f^0(M)) \to \operatorname{gr}_K H_f^0(A(M)).$$
It is enough to show that $\operatorname{gr} \eta_M$ is surjective.

Consider the surjection
$$\pi_M : M/C_2(M) \twoheadrightarrow \operatorname{gr}_K A(M).$$
Since both $M/C_2(M)$ and $\operatorname{gr}_K A(M)$ are objects of $\overline{\mathcal{HC}}$, this induces the surjection
$$H_f^0(\pi_M) : H_f^0(M/C_2(M)) \twoheadrightarrow H_f^0(\operatorname{gr}_K A(M)) \cong \operatorname{gr}_K H_f^0(A(M))$$
by Theorem 2.3.

Now we have the following commutative diagram:
$$\begin{array}{ccc} H_f^0(M)/C_2(H_f^0(M)) & \xrightarrow{\pi_{H_f^0(M)}} & \operatorname{gr}_K A(H_f^0(M)) \\ {\scriptstyle \bar{\eta}_M}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{gr} \eta_M} \\ H_f^0(M/C_2(M)) & \xrightarrow{H_f^0(\pi_M)} & \operatorname{gr}_K H_f^0(A(M)). \end{array} \tag{33}$$
Since $\bar{\eta}_M$ is an isomorphism by Theorem 7.1 (ii), it follows that $\operatorname{gr} \eta_M$ is surjective as required.

Next, let M be an arbitrary object of KL_k. There exists a sequence of finitely generated objects $M_0 \subset M_1 \subset M_2 \subset \dots$ in KL_k such that $M = \bigcup_i M_i$. Since (co)homology functor commutes with injective limits, $A(M) = \varinjlim_i A(M_i)$, $H_f^0(M) = \varinjlim_i H_f^0(M_i)$, $A(H_f^0(M)) = \varinjlim_i A(H_f^0(M_i))$, and $H_f^0(A(M)) = \varinjlim_i H_f^0(A(M_i))$. This proves the assertion.

(ii) By Lemma 6.2 (i) $H_f^0(\pi_M)$ is an isomorphism. Hence the commutativity of (33) implies that $\pi_{H_f^0(M)}$ and $\operatorname{gr}\eta_M$ are isomorphisms, and hence, so is η_M. □

Proof of Theorem 8.1. As in the proof of Proposition 8.4 it is sufficient to show the case that M is finitely generated. Then there exists an exact sequence

$$0 \to N \to V \to M \to 0 \tag{34}$$

in the category KL_k with $V \in \mathsf{KL}_k^{\Delta}$. By the right exactness of the functor $A(?)$ this yields an exact sequence

$$A(N) \to A(V) \to A(M) \to 0$$

in the category $\mathcal{HC}$. Applying the exact functor $H_f^0(?) : \mathcal{HC} \to U(\mathfrak{g}, f)$-biMod (Theorem 3.2) to the above sequence we obtain an exact sequence

$$H_f^0(A(N)) \to H_f^0(A(V)) \to H_f^0(A(M)) \to 0.$$

On the other hand by applying the exact functor $H_f^0(?) : \mathsf{KL}_k \to \mathcal{W}^k(\mathfrak{g}, f)$-Mod (Theorem 7.1) to (34) we obtain the exact sequence

$$0 \to H_f^0(N) \to H_f^0(V) \to H_f^0(M) \to 0.$$

This yields an exact sequence

$$A(H_f^0(N)) \to A(H_f^0(V)) \to A(H_f^0(M)) \to 0. \tag{35}$$

Now we have the following commutative diagram:

$$\begin{tikzcd} A(H_f^0(N)) \arrow[r] \arrow[d, "\eta_N"'] & A(H_f^0(V)) \arrow[r] \arrow[d, "\eta_V"'] & A(H_f^0(M)) \arrow[r] \arrow[d, "\eta_M"'] & 0 \\ H_f^0(A(N)) \arrow[r] & H_f^0(A(V)) \arrow[r] & H_f^0(A(M)) \arrow[r] & 0. \end{tikzcd} \tag{36}$$

By Proposition 8.4 η_N and η_M are surjective and η_V is an isomorphism. As the horizontal sequences are exact it follows that η_M is an isomorphism. This completes the proof. □

For an ideal N of $V^k(\mathfrak{g})$, let J_N denote the image of $A(N)$ in $A(V^k(\mathfrak{g})) = U(\mathfrak{g})$, so that

$$A(V^k(\mathfrak{g})/N) = U(\mathfrak{g})/J_N. \tag{37}$$

Note that $H_f^0(V^k(\mathfrak{g})/N)$ is a quotient vertex algebra of $\mathcal{W}^k(\mathfrak{g}, f)$ provided it is nonzero (see (30)).

Theorem 8.5. *For any ideal N of $V^k(\mathfrak{g})$, we have the isomorphism of algebras*

$$A(H_f^0(V^k(\mathfrak{g})/N)) \cong U(\mathfrak{g}, f)/H_f^0(J_N).$$

Proof. Set $L = V^k(\mathfrak{g})/N$. By Theorem 8.1,

$$A(H_f^0(L)) \cong H_f^0(A(L)),$$

and by Theorem 3.2 the exact sequence $0 \to J_N \to U(\mathfrak{g}) \to A(L) \to 0$ induces the exact sequence

$$0 \to H_f^0(J_N) \to U(\mathfrak{g}, f) \to H_f^0(A(L)) \to 0.$$

This completes the proof. □

The following assertion follows immediately from Theorems 3.6 and 8.5.

Theorem 8.6. *For any ideal N of $V^k(\mathfrak{g})$ we have the equivalence of categories*

$$\mathcal{C}^{J_N} \xrightarrow{\sim} A(H_f^0(V^k(\mathfrak{g})/N))\text{-Mod}, \quad M \mapsto \mathrm{Wh}^{\mathrm{m}}(M).$$

A quasi-inverse functor is given by $E \mapsto Y \otimes_{U(\mathfrak{g},f)} E$.

9. Varieties associated with Zhu's algebras of admissible affine vertex algebras

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, Δ the set of roots of $\mathfrak{g}$, Δ_+ the set of positive roots of $\mathfrak{g}$, W the Weyl group of $\mathfrak{g}$, $Q^\vee \subset \mathfrak{h}$ the coroot lattice of $\mathfrak{g}$, $P^\vee \subset \mathfrak{h}$ the coweight lattice of $\mathfrak{g}$, ρ the half sum of positive roots of $\mathfrak{g}$, $\rho^\vee$ the half sum of positive coroots of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$, let $M_{\mathfrak{g}}(\lambda)$ be the Verma module of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$, $L_{\mathfrak{g}}(\lambda)$ the unique simple quotient of $M_{\mathfrak{g}}(\lambda)$.

Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0$ the dual of $\widehat{\mathfrak{h}}$, where $\Lambda_0(K) = 1$, $\Lambda_0(\mathfrak{h}) = 0$. Let $\widehat{\Delta}^{re}$ be the set of real roots in the dual $\tilde{\mathfrak{h}}^*$ of the extended Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$, $\widehat{\Delta}^{re}_+$ the set of positive real roots, $\widehat{W} = W \ltimes Q^\vee$ the Weyl group of $\widehat{\mathfrak{g}}$, $\widetilde{W} = W \ltimes P^\vee$ the extended Weyl group of $\widehat{\mathfrak{g}}$, $\widehat{\rho} = \rho + h^\vee \Lambda_0$. For $\lambda \in \widehat{\mathfrak{h}}^*$, let $\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{re} | \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\}$, the set of integral roots of λ, $\widehat{W}(\lambda) = \langle s_\alpha | \alpha \in \widehat{\Delta}(\lambda) \rangle \subset \widehat{W}$ the integral Weyl group of λ, where s_α is the reflection with respect to α. Denote by $\bar{\lambda}$ the restriction of $\lambda \in \widehat{\mathfrak{h}}^*$ to $\mathfrak{h}$.

Set

$$\widehat{\mathfrak{h}}^*_k = \{\lambda \in \widehat{\mathfrak{h}}^* | \lambda(K) = k\},$$

the set of weights of $\widehat{\mathfrak{g}}$ of level k. For $\lambda \in \widehat{\mathfrak{h}}^*_k$, let $L(\lambda)$ be the irreducible representation of $\widehat{\mathfrak{g}}$ with highest weight λ. Clearly, $L(\lambda)$ is irreducible as a $V^k(\mathfrak{g})$-module.

A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called *admissible* if (1) λ is regular dominant, that is, $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin \{0, -1, -2, -3, \dots\}$ for all $\alpha \in \widehat{\Delta}^{re}_+$, and (2) $\mathbb{Q}\widehat{\Delta}(\lambda) = \mathbb{Q}\widehat{\Delta}^{re}$. The admissible weights of $\widehat{\mathfrak{g}}$ were classified in [KW1]. The module $L(\lambda)$ is called admissible if λ is admissible. Admissible representations are (conjecturally all) modular invariant representations of $\widehat{\mathfrak{g}}$ ([KW1]).

A number k is called *admissible for* $\widehat{\mathfrak{g}}$ if $k\Lambda_0$ is an admissible weight. By [KW2, Proposition 1.2], k is an admissible number for $\widehat{\mathfrak{g}}$ if and only if

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{N}, \quad (p, q) = 1, \quad p \geq \begin{cases} h^\vee_{\mathfrak{g}} & \text{if } (r^\vee, q) = 1, \\ h_{\mathfrak{g}} & \text{if } (r^\vee, q) = r^\vee. \end{cases} \tag{38}$$

A number k of the form (38) is called an *admissible number with denominator* q.

For an admissible number k of $\widehat{\mathfrak{g}}$, let Pr^k be the set of admissible weights λ of level k such that $\widehat{\Delta}(\lambda) \cong \widehat{\Delta}(k\Lambda_0)$ as root systems.

Theorem 9.1 ([A8]). *Let k be an admissible number for $\widehat{\mathfrak{g}}$, $\lambda \in \widehat{\mathfrak{h}}_k^*$. Then $L(\lambda)$ is a module over the vertex algebra $L(k\lambda_0)$ if and only if $\lambda \in Pr^k$. In particular the vertex operator algebra $L(k\Lambda_0)$ is rational in the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ as conjectured in* [AM].

By Zhu's theorem, the first statement of Theorem 9.1 is equivalent to that $L_{\mathfrak{g}}(\lambda)$ with $\lambda \in \mathfrak{h}^*$ is a module over $A(L(k\Lambda_0))$ if and only if $\lambda + k\Lambda_0 \in Pr^k$. On the other hand by Duflo's theorem [Duf] any primitive ideal of $U(\mathfrak{g})$ is the annihilating ideal of some irreducible highest weight module $L_{\mathfrak{g}}(\lambda)$. Hence Theorem 9.1 implies the following.

Corollary 9.2. *Let k be an admissible number for $\widehat{\mathfrak{g}}$. A simple $U(\mathfrak{g})$-module M is an $A(L(k\Lambda_0))$-module if and only if $\operatorname{Ann}_{U(\mathfrak{g})} M = \operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\bar{\lambda})$ for some $\lambda \in Pr^k$.*

Let k be an admissible number for $\widehat{\mathfrak{g}}$. We shall determine

$$VarA(L(k\Lambda_0)) := \operatorname{Specm}(\operatorname{gr}_F A(L(k\Lambda_0)))(\cong \operatorname{Specm}(\operatorname{gr}_K A(L(k\Lambda_0)))),$$

which is a G-invariant, conic, Poisson subvariety of $\mathfrak{g}^*$.

Recall [A4] that the associated variety X_V of a finitely strongly generated vertex algebra V is defined as

$$X_V = \operatorname{Specm}(R_V).$$

Note that V is C_2-cofinite if and only if X_V is zero-dimensional.

By (18), $VarA(L(k\Lambda_0))$ is a subvariety of $X_{L(k\Lambda_0)}$, which is also a G-invariant, conic, Poisson subvariety of $\mathfrak{g}^*$.

Let us identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through ν, and let $\mathcal{N} \subset \mathfrak{g}^* = \mathfrak{g}$ be the nilpotent cone.

By a conjecture of Feigin and Frenkel proved in [A5] we have

$$X_{L(k\Lambda_0)} \subset \mathcal{N} \quad \text{for an admissible number } k \text{ for } \widehat{\mathfrak{g}}.$$

In fact the following holds:

Theorem 9.3 ([A5]). *Let k be an admissible number for $\widehat{\mathfrak{g}}$. Then $X_{L(k\Lambda_0)}$ is an irreducible subvariety of $\mathcal{N}$ which depends only on the denominator q of k, that is, there exist a nilpotent element f_q of $\mathfrak{g}$ such that*

$$X_{L(k\Lambda_0)} = \overline{\operatorname{Ad} G.f_q}.$$

More explicitly, we have

$$X_{L(k\Lambda_0)} = \begin{cases} \{x \in \mathfrak{g} | (\operatorname{ad} x)^{2q} = 0\} & \text{if } (q, r^\vee) = 1, \\ \{x \in \mathfrak{g} | \pi_{\theta_s}(x)^{2q/r^\vee} = 0\} & \text{if } (q, r^\vee) = r^\vee, \end{cases}$$

where θ_s is the highest short root of $\mathfrak{g}$ and $\pi_{\theta_s} : \mathfrak{g} \to \operatorname{End}_{\mathbb{C}}(L_{\mathfrak{g}}(\theta_s))$ is the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight θ_s.

Theorem 9.3 has the following important consequence [A5]: By Theorems 2.3 and 7.1 we have

$$X_{H^0_f(L(k\Lambda_0))} \cong X_{L(k\Lambda_0)} \cap \mathcal{S}_f. \tag{39}$$

Hence the transversality of $\mathcal{S}_f$ with G-orbits (see [GG]) implies the following:

Theorem 9.4 ([A5]). *Let k be an admissible number with denominator q. Then the vertex algebra $H^0_{f_q}(L(k\Lambda_0))$ is a non-zero C_2-cofinite quotient of $\mathcal{W}^k(\mathfrak{g}, f)$.*

Now we are in a position to state the main result of this section.

Theorem 9.5. *Let k be an admissible number for $\widehat{\mathfrak{g}}$ with denominator q . We have an isomorphism of affine varieties*

$$VarA(L(k\Lambda_0)) \cong X_{L(k\Lambda_0)}.$$

Proof. By Theorem 9.3, it is sufficient to show the following assertion.

Proposition 9.6. *Let f be any nilpotent element of $\mathfrak{g}$, and let k be any complex number. The following conditions are equivalent:*

(i) $X_{L(k\Lambda_0)} \supset \overline{\mathrm{Ad}\, G.f}$.

(ii) $Var(A(L(k\Lambda_0))) \supset \overline{\mathrm{Ad}\, G.f}$.

Proof. Clearly (ii) implies (i) as $VarA(L(k\Lambda_0))) \subset X_{L(k\Lambda_0)}$. Conversely, suppose that $X_{L(k\Lambda_0)} \supset \overline{\mathrm{Ad}\, G.f}$. Since $VarA(L(k\Lambda_0))$ is G-invariant and closed it is sufficient to show that the point $f \in \mathfrak{g} = \mathfrak{g}^*$ is contained in $VarA(L(k\Lambda_0))$. By (39), $X_{H_f^0(L(k\Lambda_0))}$ contains f, and hence, $H_f^0(L(k\Lambda_0))$ is nonzero. It follows that $A(H_f^0(L(k\Lambda_0))) = H_f^0(A(L(k\Lambda_0)))$ is nonzero as well. Since $VarH_f^0(A(L(k\Lambda_0))) = VarA(L(k\Lambda_0)) \cap \mathcal{S}_f$ by Theorem 3.2 (i), $VarA(L(k\Lambda_0))$ intersects $\mathcal{S}_f$ non-trivially. As $VarH_f^0(A(L(k\Lambda_0)))$ is invariant under the natural $\mathbb{C}^*$-action on $\mathcal{S}_f$ which is contracting to f (see [Gin]), $VarA(L(k\Lambda_0))$ must contain the point f as required. □

□

Conjecture 1. For a finitely strongly generated simple vertex operator algebra V of CFT type we have $VarA(V)(:= \operatorname{Specm} \operatorname{gr}_F(A(V))) \cong X_V$.

Note that Conjecture 1 in particular implies the widely believed fact that a finitely strongly generated rational vertex operator algebra of CFT type must be C_2-cofinite.

10. Proof of Main Theorem

In this section we let $f = f_{prin}$, a principal nilpotent element of $\mathfrak{g}$,

$$\mathcal{W}^k(\mathfrak{g}) = \mathcal{W}^k(\mathfrak{g}, f_{prin}) = H^0_{f_{prin}}(V^k(\mathfrak{g})),$$
$$\text{and } \mathcal{W}_k(\mathfrak{g}) = \text{ the unique simple quotient of } \mathcal{W}^k(\mathfrak{g})$$

as in Introduction. The vertex algebra $\mathcal{W}^k(\mathfrak{g})$ is $\mathbb{Z}_{\geq 0}$-graded by L_0, where

$$Y(\omega_{\mathcal{W}}, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

The central charge $c(k)$ of $\mathcal{W}^k(\mathfrak{g})$ is given in Introduction. We have the isomorphisms

$$\mathbb{C}[\mathfrak{g}^*]^G \xrightarrow{\sim} \mathbb{C}[\mathcal{S}_f] = H^0(\bar{C}(\mathbb{C}[\mathfrak{g}^*]), \operatorname{ad} \bar{d}) \cong R_{\mathcal{W}^k(\mathfrak{g})}, \quad p \mapsto p \otimes 1,$$
$$\mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g}, f_{prin}) = H^0(C(U(\mathfrak{g})), \operatorname{ad} d) \cong A(\mathcal{W}^k(\mathfrak{g})), \quad z \mapsto z \otimes 1$$

([Kos], see also [A2]), where $\mathcal{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$. We will identity $A(\mathcal{W}^k(\mathfrak{g}))$ with $\mathcal{Z}(\mathfrak{g})$ through the above isomorphism.

For a central character $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$, let $\mathbb{C}_\gamma$ be the one-dimensional representation of $\mathcal{Z}(\mathfrak{g})$ defined by γ. Put

$$\mathbf{M}_{\mathcal{W}}(\gamma) = M_{\mathcal{W}^k(\mathfrak{g})}(\mathbb{C}_\gamma), \quad \mathbf{L}_{\mathcal{W}}(\gamma) = L_{\mathcal{W}^k(\mathfrak{g})}(\mathbb{C}_\gamma)$$

(see section 4). We have

$$\mathrm{Irr}(\mathcal{W}^k(\mathfrak{g}))=\{\mathbf{L}_{\mathcal{W}}(\gamma_\lambda)|\lambda\in\mathfrak{h}^*/W-\rho\},$$

where $\gamma_\lambda:\mathcal{Z}(\mathfrak{g})\rightarrow\mathbb{C}$ is the evaluation at $M_{\mathfrak{g}}(\lambda)$. Note that

$$\mathcal{W}_k(\mathfrak{g})\cong\mathbf{L}_{\mathcal{W}}(\gamma_{-(k+h_{\mathfrak{g}}^\vee)\rho^\vee}),$$

see [A2, 5.4].

Theorem 10.1. *Let N be an ideal of $V^k(\mathfrak{g})$ and suppose that $H^0_{f_{prin}}(V^k(\mathfrak{g})/N)\neq 0$, so that $H^0_{f_{prin}}(V^k(\mathfrak{g})/N)$ is a quotient vertex algebra of $\mathcal{W}^k(\mathfrak{g})$ (see (30)). We have*

$$\mathrm{Irr}(H^0_{f_{prin}}(V^k(\mathfrak{g})/N))=\{\mathbf{L}_{\mathcal{W}}(\gamma)|U(\mathfrak{g})\ker\gamma\supset J_N\}$$

(Here J_N is defined in Section 8, see (37)).

Proof. Recall Skryabin's equivalence for $f=f_{prin}$ in Section 3:

$$\mathcal{Z}(\mathfrak{g})\text{-Mod}\xrightarrow{\sim}\mathcal{C},\quad E\mapsto Y\otimes_{\mathcal{Z}(\mathfrak{g})}E,$$

which goes back to Kostant [Kos]. In particular, $\{Y_\gamma|\gamma\in\mathfrak{h}^*/W-\rho\}$ gives the complete set of isomorphism classes of simple object of $\mathcal{C}$, where $Y_\gamma=Y\otimes_{\mathcal{Z}(\mathfrak{g})}\mathbb{C}_\gamma$. We have [Kos]

$$\mathrm{Ann}_{U(\mathfrak{g})}Y_\gamma=U(\mathfrak{g})\ker\gamma.$$

Therefore Y_γ is annihilated by J_N if and only if $J_N\subset U(\mathfrak{g})\ker\gamma$. In other words $\{Y_\gamma|U(\mathfrak{g})\ker\gamma\supset J_N\}$ gives the complete set of isomorphism classes of simple objects of $\mathcal{C}^{J_N}$. By Theorem 8.6 this is equivalent to that

$$\mathrm{Irr}(A(H^0_f(V^k(\mathfrak{g})/N)))=\{\mathbb{C}_\gamma|U(\mathfrak{g})\ker\gamma\supset J_N\}.$$

This completes the proof. □

Recall that $X_{L(k\Lambda_0)}\subset\mathcal{N}$ for an admissible number k for $\widehat{\mathfrak{g}}$ (Theorem 9.3). An admissible number k is called *non-degenerate* if

$$X_{L(k\Lambda_0)}=\mathcal{N}=\overline{\mathrm{Ad}\,G.f_{prin}}.$$

From Theorem 9.3 and the fact that

$$(\theta|\rho^\vee)=h_{\mathfrak{g}}-1,\quad(\theta_s|\rho^\vee)=h^\vee_{{}^L\mathfrak{g}}-1,\tag{40}$$

where θ is the highest root of $\mathfrak{g}$, it follows that an admissible number k is non-degenerate if and only if k satisfies

$$q\geq\begin{cases}h_{\mathfrak{g}} & \text{if } (q,r^\vee)=1,\\ r^\vee h^\vee_{{}^L\mathfrak{g}} & \text{if } (q,r^\vee)=r^\vee,\end{cases}$$

where q is the denominator of k, that is, k is of the form (1).

Theorem 10.2. *Let k be an admissible number for $\widehat{\mathfrak{g}}$. Then $H^0_{f_{prin}}(L(k\Lambda_0))\neq 0$ if and only if k is non-degenerate. If this is the case then*

$$H^0_{f_{prin}}(L(k\Lambda_0))\cong\mathcal{W}_k(\mathfrak{g}).$$

Moreover, $\mathcal{W}_k(\mathfrak{g})$ is C_2-cofinite.

Proof. The fact that $H^0_{f_{prin}}(L(k\Lambda_0))\cong\mathcal{W}_k(\mathfrak{g})$ for a non-degenerate admissible number k was proved in [A2, Theorem 9.1.4]. The rest of the assertion is the special case of Theorem 9.4. □

Let

$$Pr^k_{non-deg} = \{\lambda \in Pr^k | \langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z} \text{ for all } \alpha \in \Delta\},$$

the set of *non-degenerate admissible weights* [FKW, Lemma 1.5] of level k. It is known [FKW] that $Pr^k_{non-deg}$ is non-empty if and only if k is non-degenerate. Put

$$Pr^k_{\mathcal{W}} = \{\gamma_{\bar{\lambda}} | \lambda \in Pr^k_{non-deg}\}.$$

Then $\sharp Pr^k_{\mathcal{W}} = \sharp Pr^k_{non-deg} / \sharp W$ since W acts on $Pr^k_{non-deg}$ freely (by the dot action).

The irreducible representations $\{\mathbf{L}_{\mathcal{W}}(\gamma) | \gamma \in Pr^k_W\}$ are called *minimal series representations* of $\mathcal{W}^k(\mathfrak{g})$. In [A2] we have verified the conjectural character formula of minimal series representations of $\mathcal{W}_k(\mathfrak{g})$ given by Frenkel-Kac-Wakimoto [FKW]. (In fact the main result of [A2] gives the character of all $\mathbf{L}_{\mathcal{W}}(\gamma)$, see Theorem 10.8 and Corollary 10.9 below.)

Remark 10.3. The module $\mathbf{L}_{\mathcal{W}}(\gamma)$ with $\gamma \in Pr^k_{\mathcal{W}}$ admits a two-sided resolution in terms of free field realizations [A7]. However we do not need this result.

Theorem 10.4. *Let k be a non-degenerate admissible number for $\widehat{\mathfrak{g}}$, γ a central character of $\mathcal{Z}(\mathfrak{g})$. Then $\mathbf{L}_{\mathcal{W}}(\gamma)$ is a module over $\mathcal{W}_k(\mathfrak{g})$ if and only if it is a minimal series representation of $\mathcal{W}^k(\mathfrak{g})$, that is,*

$$\operatorname{Irr}(\mathcal{W}_k(\mathfrak{g})) = \{\mathbf{L}_{\mathcal{W}}(\gamma) | \gamma \in Pr^k_{\mathcal{W}}\}.$$

Proof. Set $J_k = J_{N_k(\mathfrak{g})}$, so that

$$A(L(k\Lambda_0)) = U(\mathfrak{g})/J_k.$$

By Theorem 10.2, we have $\mathcal{W}_k(\mathfrak{g}) = H^0_{f_{prin}}(V^k(\mathfrak{g})/N_k(\mathfrak{g}))$. Hence Theorem 10.1 gives that

$$\operatorname{Irr}(\mathcal{W}_k(\mathfrak{g})) = \{\mathbf{L}_{\mathcal{W}}(\gamma) | U(\mathfrak{g}) \ker \gamma \supset J_k\}.$$

Now recall that $\bar{\lambda} \in \mathfrak{h}^*$ is called *anti-dominant* if $\langle \bar{\lambda} + \rho, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in \Delta_+$. Clearly, for any central character $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ there exists an anti-dominant $\bar{\lambda} \in \mathfrak{h}^*$ such that $\gamma = \gamma_{\bar{\lambda}}$. It is well-known that $L_{\mathfrak{g}}(\bar{\lambda}) = M_{\mathfrak{g}}(\bar{\lambda})$ for an anti-dominant $\bar{\lambda}$ and that

$$\operatorname{Ann}_{U(\mathfrak{g})} M_{\mathfrak{g}}(\bar{\lambda}) = U(\mathfrak{g}) \ker \chi_{\bar{\lambda}}.$$

We have

$$\begin{aligned}
&\{\mathbf{L}_{\mathcal{W}}(\gamma) | U(\mathfrak{g}) \ker \gamma \supset J_k\}\\
=&\{\mathbf{L}_{\mathcal{W}}(\gamma_{\bar{\lambda}}) | \bar{\lambda} \in \mathfrak{h}^*,\ \bar{\lambda} \text{ is anti-dominant},\ \operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\bar{\lambda}) \supset J_k\} \quad \text{(by the above)}\\
=&\{\mathbf{L}_{\mathcal{W}}(\gamma_{\bar{\lambda}}) | \bar{\lambda} \in \mathfrak{h}^*,\ \bar{\lambda} \text{ is anti-dominant},\ L_{\mathfrak{g}}(\bar{\lambda}) \text{ is an } A(L(k\Lambda_0))\text{-module}\}\\
=&\{\mathbf{L}_{\mathcal{W}}(\gamma_{\bar{\lambda}}) | \lambda \in \widehat{\mathfrak{h}}^*,\ \bar{\lambda} \text{ is anti-dominant},\ L(\lambda) \text{ is an } L(k\Lambda_0)\text{-module}\}\\
=&\{\mathbf{L}_{\mathcal{W}}(\gamma_{\bar{\lambda}}) | \lambda \in Pr^k,\ \bar{\lambda} \text{ is anti-dominant}\} \quad \text{(by Theorem 9.1)}\\
=&\{\mathbf{L}_{\mathcal{W}}(\gamma_{\bar{\lambda}}) | \lambda \in Pr^k_{non-deg}\} = \{\mathbf{L}_{\mathcal{W}}(\gamma) | \gamma \in Pr^k_{\mathcal{W}}\}.
\end{aligned}$$

This completes the proof. □

Theorem 10.5. *For a non-degenerate admissible number k for $\widehat{\mathfrak{g}}$, Zhu's algebra $A(\mathcal{W}_k(\mathfrak{g}))$ is semisimple.*

In order to prove Theorem 10.5, we consider the Lie algebra *homology* functor

$$\mathfrak{g}\text{-Mod} \to \mathcal{Z}(\mathfrak{g})\text{-Mod}, \quad M \mapsto H_0(\mathfrak{n}_-, M).$$

Since $M_{\mathfrak{g}}(\lambda)$ is free over $U(\mathfrak{n}_-)$,

$$H_i(\mathfrak{n}_-, M_{\mathfrak{g}}(\lambda)) \cong \begin{cases} \mathbb{C}_{\gamma_\lambda} & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases} \tag{41}$$

Lemma 10.6. *Let $\lambda \in \mathfrak{h}^*$ be regular, that is, $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta$. Then for an exact sequence $0 \to \mathbb{C}_{\gamma_\lambda} \overset{\phi_1}{\to} E \overset{\phi_2}{\to} \mathbb{C}_{\gamma_\lambda} \to 0$ of $\mathcal{Z}(\mathfrak{g})$-modules, there exists an exact sequence $0 \to M_{\mathfrak{g}}(\lambda) \to N \to M_{\mathfrak{g}}(\lambda) \to 0$ of $\mathfrak{g}$-modules such $E \cong H_0(\mathfrak{n}_-, N)$ as $\mathcal{Z}(\mathfrak{g})$-modules.*

Proof. Choose homogeneous generators $p_1, \dots, p_{\operatorname{rk}\mathfrak{g}}$ of of $\mathcal{Z}(\mathfrak{g})$. Let

$$\Upsilon : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^W$$

be the Harish-Chandra isomorphism, so that $zv_\lambda = \Upsilon(z)(\lambda + \rho)v_\lambda$ for $z \in \mathcal{Z}(\mathfrak{g})$, where v_λ is the highest weight vector of $M_{\mathfrak{g}}(\lambda)$. Set $v = \phi_1(1)$ and fix $v' \in E$ such that $\phi_2(v') = 1$. Then there exists $d_1, \dots, d_{\operatorname{rk}\mathfrak{g}} \in \mathbb{C}$ such that

$$p_i v' = \Upsilon(p_i)(\lambda + \rho)v' + d_i v.$$

Let us identify $S(\mathfrak{h})$ with $\mathbb{C}[\alpha_1^\vee, \dots, \alpha_{\operatorname{rk}\mathfrak{g}}^\vee]$. It is well-known that

$$\det\left(\frac{\partial \Upsilon(p_i)}{\partial \alpha_j^\vee}\right)_{1 \leq i,j \leq \operatorname{rk}\mathfrak{g}} = C \prod_{\alpha \in \Delta_+} \alpha^\vee, \tag{42}$$

where C is some nonzero constant. The hypothesis on λ implies that the value of (42) at $\lambda + \rho$ is non-zero. It follows that there exists some $\mu \in \mathfrak{h}^*$ such that

$$\Upsilon(p_i)(\lambda + t\mu + \rho) = \Upsilon(p_i)(\lambda + \rho) + td_i + O(t^2) \tag{43}$$

for all $i = 1, \dots, \operatorname{rk}\mathfrak{g}$.

Let $A = \mathbb{C}[t]$, $\mathfrak{h}_A = \mathfrak{h} \otimes_{\mathbb{C}} A$. Denote by $A_{\lambda + t\mu}$ the $\mathfrak{h}_A$-module that is a rank one free A-module on which $h \in \mathfrak{h}$ acts as multiplication by the scalar $\lambda(h) + t\mu(h)$. Set $M = A_{\lambda+t\mu}/t^2 A_{\lambda+t\mu}$ and view M as an $\mathfrak{h}$-module. Observe that $tM \cong \mathbb{C}_\lambda$ and we have the exact sequence

$$0 \to tM \to M \to \mathbb{C}_\lambda \to 0 \tag{44}$$

of $\mathfrak{h}$-modules. Set

$$N = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} M,$$

where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and M is regarded as a $\mathfrak{b}$-module via the natural surjection $\mathfrak{b} \to \mathfrak{h}$. Applying the induction functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ?$ to (44) we obtain the exact sequence

$$0 \to M_{\mathfrak{g}}(\lambda) \to N \to M_{\mathfrak{g}}(\lambda) \to 0 \tag{45}$$

of $\mathfrak{g}$-modules. Next applying the functor $H_0(\mathfrak{n}, ?)$ we get the exact sequence

$$0 \to \mathbb{C}_{\gamma_\lambda} \to H_0(\mathfrak{n}_-, N) \to \mathbb{C}_{\gamma_\lambda} \to 0$$

of $\mathcal{Z}(\mathfrak{g})$-modules by (41). By construction, $H_0(\mathfrak{n}_-, N) \cong E$ as required. $\square$

Proposition 10.7. *For $\lambda \in Pr^k$ we have $L(\lambda) \cong M_{L(k\Lambda_0)}(L_{\mathfrak{g}}(\bar{\lambda}))$ (see (20)).*

Proof. We have a surjective map

$$M_{V^k(\mathfrak{g})}(L_{\mathfrak{g}}(\bar{\lambda})) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} L_{\mathfrak{g}}(\bar{\lambda}) \twoheadrightarrow M_{L(k\Lambda_0)}(L_{\mathfrak{g}}(\bar{\lambda}))$$

of $\widehat{\mathfrak{g}}$-modules. It follows that $M_{L(k\Lambda_0)}(L_{\mathfrak{g}}(\bar{\lambda}))$ is an object of $\mathcal{O}$ of $\widehat{\mathfrak{g}}$. Being a $L(k\Lambda_0)$-module, $M_{L(k\Lambda_0)}(L_{\mathfrak{g}}(\bar{\lambda}))$ decomposes into a direct sum of admissible representations by Theorem 9.1. Since it is generated by the highest weight vector of $L_{\mathfrak{g}}(\bar{\lambda})$, $M_{L(k\Lambda_0)}(L_{\mathfrak{g}}(\bar{\lambda}))$ must be isomorphic to $L(\lambda)$. $\square$

Proof of Theorem 10.5. Since $\mathcal{W}_k(\mathfrak{g}) = H^0_{f_{prin}}(L(k\Lambda_0))$ is C_2-cofinite by Theorem 9.4, Zhu's algebra $A(\mathcal{W}_k(\mathfrak{g}))$ is finite-dimensional. Also, we have shown that $\mathrm{Irr}(A(\mathcal{W}_k(\mathfrak{g}))) = \{\mathbb{C}_\gamma | \gamma \in Pr^k_{\mathcal{W}}\}$ in Theorem 10.4.

Let $\lambda \in Pr^k_{non-deg}$, and let

$$0 \to \mathbb{C}_{\gamma_{\bar{\lambda}}} \to E \to \mathbb{C}_{\gamma_{\bar{\lambda}}} \to 0 \tag{46}$$

be an exact sequence of $A(\mathcal{W}_k(\mathfrak{g}))$-modules. We need to show that this sequence splits.

Recall that $L_{\mathfrak{g}}(\bar{\lambda}) = M_{\mathfrak{g}}(\bar{\lambda})$ for $\lambda \in Pr^k_{non-deg}$. By Lemma 10.6 there exists an exact sequence

$$0 \to L_{\mathfrak{g}}(\bar{\lambda}) \to N \to L_{\mathfrak{g}}(\bar{\lambda}) \to 0 \tag{47}$$

of $\mathfrak{g}$-modules that gives the exact sequence (46) by applying the functor $H_0(\mathfrak{n}_-, ?)$. Since $\mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\bar{\lambda}) = U(\mathfrak{g}) \ker \gamma_\lambda$ we have

$$\mathrm{Ann}_{U(\mathfrak{g})} N = U(\mathfrak{g}) \, \mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})} E. \tag{48}$$

On the other hand, by applying the exact functor $Y \otimes_{\mathcal{Z}(\mathfrak{g})} ?$ to (46) we obtain the exact sequence of $A(L(k\Lambda_0))$-modules

$$0 \to Y_{\gamma_{\bar{\lambda}}} \to Y \otimes_{\mathcal{Z}(\mathfrak{g})} E \to Y_{\gamma_{\bar{\lambda}}} \to 0$$

by Theorem 8.6. It follows similarly that

$$\mathrm{Ann}_{U(\mathfrak{g})}(Y \otimes_{\mathcal{Z}(\mathfrak{g})} E) = U(\mathfrak{g}) \, \mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})} E. \tag{49}$$

From (48) and (49), it follows that N is a module over $A(L(k\Lambda_0))$ as well, and (47) is an exact sequence of $A(L(k\Lambda_0))$-modules. Therefore by applying the functor $\mathcal{U}(L(k\Lambda_0)) \otimes_{\mathcal{U}(L(k\Lambda_0))_{\leq 0}} ?$ to (47) we obtain an exact sequence

$$0 \to L(\lambda) \to M_{L(k\Lambda_0)}(N) \to L(\lambda) \to 0 \tag{50}$$

of $L(k\Lambda_0)$-modules by Proposition 10.7. Here the map $L(\lambda) \to M_{L(k\Lambda_0)}(N)$ is injective since $L(\lambda)$ is simple. Now, thanks to Gorelik and Kac [GK], an admissible $\widehat{\mathfrak{g}}$-module does not admit a non-trivial self-extension. Therefore (50) must split. Restricting (50) we see that (47) splits, and therefore, (46) splits as well. This completes the proof. $\square$

Let $\mathcal{O}_k$ be the full subcategory of category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ consisting of modules of level k, which can be regarded as a full subcategory of $V^k(\mathfrak{g})$-Mod. Let $H^0_-(?) : \mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g})$-Mod be the quantized Drinfeld-Sokolov "$-$"-reduction functor [FKW].

Recall the following result.

Theorem 10.8 ([A2])**.** *Let k be any complex number.*

(i) *The functor $H^0_-(?) : \mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g})$-Mod is exact.*

(ii) *For $\lambda \in \widehat{\mathfrak{h}}^*_k$, $H^0_-(M(\lambda)) \cong \mathbf{M}_{\mathcal{W}}(\gamma_{\bar{\lambda}})$.*

(iii) *For* $\lambda \in \widehat{\mathfrak{h}}_k^*$, $H_-^0(L(\lambda)) \cong \begin{cases} \mathbf{L}_{\mathcal{W}}(\gamma_{\bar{\lambda}}) & \text{if } \bar{\lambda} \text{ is anti-dominant,} \\ 0 & \text{otherwise.} \end{cases}$

Let $[M(\lambda):L(\mu)]$ (resp. $[\mathbf{M}_{\mathcal{W}}(\gamma):\mathbf{L}_{\mathcal{W}}(\gamma')]$) be the multiplicity of $L(\mu)$ (resp. $\mathbf{L}_{\mathcal{W}}(\gamma')$) in the local composition factor of $M(\lambda)$ (resp. in the local composition factor of $\mathbf{M}_{\mathcal{W}}(\gamma)$).

Corollary 10.9. *Let* $\lambda, \mu \in \widehat{\mathfrak{h}}_k^*$ *and suppose that* $\bar{\mu}$ *is anti-dominant. Then*

$$[\mathbf{M}_{\mathcal{W}}(\gamma_{\bar{\lambda}}):\mathbf{L}_{\mathcal{W}}(\gamma_{\bar{\mu}})] = [M(\lambda):L(\mu)].$$

Proof. Since $\operatorname{ch} M(\lambda) = \sum_\mu [M(\lambda):L(\mu)] \operatorname{ch} L(\mu)$ we have

$$\operatorname{ch} \mathbf{M}_{\mathcal{W}}(\gamma_{\bar{\lambda}}) = \sum_{\substack{\mu \in \widehat{W}(\lambda)\circ\lambda \\ \bar{\mu} \text{ is anti-dominant}}} [M(\lambda):L(\mu))] \operatorname{ch} \mathbf{L}_{\mathcal{W}}(\gamma_{\bar{\mu}}).$$

It remains to observe that if $\mu, \mu' \in \widehat{W}(\lambda)\circ\lambda$, $\gamma_{\bar{\mu}} = \gamma_{\bar{\mu}'}$, and $\bar{\mu}$ and $\bar{\mu}'$ are both anti-dominant then $\mu = \mu'$. □

Theorem 10.10. *Let k be a non-degenerate admissible number for $\widehat{\mathfrak{g}}$. The simple vertex operator algebra $\mathcal{W}_k(\mathfrak{g})$ is rational.*

Proof. By Theorem 10.2, it is sufficient to show that

$$\operatorname{Ext}^1_{\mathcal{W}_k(\mathfrak{g})\text{-Mod}}(\mathbf{L}_{\mathcal{W}}(\gamma), \mathbf{L}_{\mathcal{W}}(\gamma')) = 0 \quad \text{for } \mathbf{L}_{\mathcal{W}}(\gamma), \mathbf{L}_{\mathcal{W}}(\gamma') \in \operatorname{Irr}(\mathcal{W}_k(\mathfrak{g})).$$

By Theorem 10.4 we can write $\gamma = \gamma_{\bar{\lambda}}$, $\gamma = \gamma_{\bar{\lambda}'}$ with $\lambda, \lambda' \in Pr^k_{non-deg}$. Let

$$0 \to \mathbf{L}_{\mathcal{W}}(\gamma') \to N \to \mathbf{L}_{\mathcal{W}}(\gamma) \to 0 \tag{51}$$

be an exact sequence of $\mathcal{W}_k(\mathfrak{g})$-modules.

Let Δ_γ be the L_0-eigenvalue of the lowest weight vector v_γ of $\mathbf{L}_{\mathcal{W}}(\gamma)$, which is a rational number. Suppose that $\Delta_\gamma < \Delta_{\gamma'}$, and choose a vector $v \in N_{\Delta_\gamma}$ such that the image of v in $\mathbf{L}_{\mathcal{W}}(\gamma)$ is v_γ. Then there is a $\mathcal{W}^k(\mathfrak{g})$-module homomorphism $\mathbf{M}_{\mathcal{W}}(\gamma) \to N$ that sends the highest weight vector of $\mathbf{M}_{\mathcal{W}}(\gamma)$ to v. If (51) is non-splitting, N must coincide with the image of $\mathbf{M}_{\mathcal{W}}(\gamma)$. In particular, $[\mathbf{M}_{\mathcal{W}}(\gamma):\mathbf{L}_{\mathcal{W}}(\gamma')] \neq 0$. By Corollary 10.9, this is equivalent to $[M(\lambda):L(\lambda')] \neq 0$. This forces that $\lambda = \lambda'$ since both λ and λ' are dominant weighs of $\widehat{\mathfrak{g}}$. This contradicts the assumption that $\Delta_\gamma < \Delta_{\gamma'}$.

By applying the duality functor $D(?)$ to (51), we see that the same argument applies to show that $\operatorname{Ext}^1_{\mathcal{W}_k(\mathfrak{g})\text{-Mod}}(\mathbf{L}_{\mathcal{W}}(\gamma), \mathbf{L}_{\mathcal{W}}(\gamma')) = 0$ in the case $\Delta_\gamma > \Delta_{\gamma'}$.

Finally, suppose that $\Delta_\gamma = \Delta_{\gamma'} =: \Delta$. Then we have the exact sequence

$$0 \to \mathbf{L}_{\mathcal{W}}(\gamma')_\Delta \to N_\Delta \to \mathbf{L}_{\mathcal{W}}(\gamma)_\Delta \to 0.$$

The semisimplicity of $A(\mathcal{W}_k(\mathfrak{g}))$ (Theorem 10.5) implies that the above sequence splits. Therefore (51) splits as well. This completes the proof. □

Main Theorem follows immediately from Theorems 10.4, 10.5 and 10.10. □

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