AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **56**, 081502 (2015); doi: 10.1063/1.4927689
View online: http://dx.doi.org/10.1063/1.4927689
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/56/8?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **56**, 081502 (2015)

On the neighborhood of an inhomogeneous stable stationary solution of the Vlasov equation—Case of an attractive cosine potential

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(Received 7 November 2014; accepted 21 July 2015; published online 6 August 2015)

We consider the one-dimensional Vlasov equation with an attractive cosine potential, and its non-homogeneous stable stationary states that are decreasing functions of the energy. We show that in the Sobolev space $W^{1,p}$ ($p > 2$) neighborhood of such a state, all stationary states that are decreasing functions of the energy are stable. This is in sharp contrast with the situation for homogeneous stationary states of a Vlasov equation, where a control over strictly more than one derivative is needed to ensure the absence of unstable stationary states in a neighborhood of a reference stationary state [Z. Lin and C. Zeng, Commun. Math. Phys. **306**, 291-331 (2011)]. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4927689]

I. INTRODUCTION

Vlasov equation is central in different areas of physics, notably plasma physics, where it is used with the Coulomb potential, and astrophysics, where the Newton potential is used instead. In this latter context, it is usually called "collisionless Boltzmann equation." Understanding the asymptotic behavior of a perturbation to a stationary state of the Vlasov equation is an old problem. A huge literature is devoted to the linearized dynamics, starting with the pioneering work of Landau.[1] The full non-linear problem, despite a large literature (for instance Refs. 2–6), is still not fully understood.

The subject has witnessed spectacular mathematical progresses recently. Mouhot and Villani[7,8] showed that if the initial condition is close, in some analytical norm, to a stable homogeneous stationary state, then the dynamics is an exponential relaxation towards another nearby stable homogeneous stationary state. Lin and Zeng in Ref. 9 investigated weaker norms of the Sobolev space $W^{s,p}$ with $p > 1$. They showed among other results that if the norm is weak enough (precisely, $s < 1 + 1/p$), any neighborhood of a stable homogeneous stationary state also contains unstable homogeneous stationary states, as well as small Bernstein-Greene-Kruskal waves.[2] In particular, complete damping for any initial condition, as in Mouhot-Villani's setting, is excluded. Conversely, if $s > 1 + 1/p$, there is a neighborhood of the reference stable state that contains no unstable stationary states. In a recent preprint, Faou and Rousset have extended stability results in the spirit of Mouhot and Villani to strong enough Sobolev norms.[10] All these impressive results hold for homogeneous stationary states: this is unfortunately a severe limitation, since it excludes all situations of interest for self-gravitating systems.

A natural question is then what could it be possible to show in the context of non-homogeneous stationary states? First, any asymptotically exponential relaxation as in Refs. 7 and 8 is impossible, since one always expects an algebraic relaxation, already for the linearized problem.[11,12] One may then conjecture an algebraic relaxation at the non-linear level (in the context of the 2D Euler equation, see Ref. 13 for such a conjecture, and Ref. 14 for a proof), but it seems difficult to prove. Now, is an analysis in the spirit of Ref. 9 possible? Again, the complexity of the linearized problem is a serious obstacle (see, for instance, Ref. 15 for a textbook account of the study of the linearized

Vlasov equation in astrophysics). However, a simple criterion for the stability of a large class of non-homogeneous stationary states has been found recently,[16] in the context of a simple toy model, called the Hamiltonian Mean-Field (HMF) model. There are of course many Hamiltonian systems with mean-field interaction, but conventionally the HMF model refers to a specific model with cosine interaction,[17] and we follow this conventional naming.

The purpose of this paper is to take advantage of this simple formulation to investigate the neighborhood of inhomogeneous stable stationary solutions in the case of the HMF model. This is a first partial advance, in the spirit of Lin and Zeng, for inhomogeneous stationary states. We will show that the results differ significantly from the homogeneous case: it is actually easier to rule out the presence of unstable states in a neighborhood of the reference stable state, since a $W^{1,p}$ norm is in some cases sufficient.

We state precisely our results in Section II, emphasizing the important difference with the homogeneous case and postpone the proofs to Section III. Section IV presents some numerical illustration of our findings.

II. STATEMENT OF THE RESULTS

The Vlasov equation associated to the HMF model is

$$\frac{\partial f}{\partial t} + \{h, f\} = 0, \quad \text{with} \quad \{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} \tag{1}$$

and the one-body Hamiltonian of the HMF model is

$$h(q,p,t) = \frac{p^2}{2} - M[f](t)\cos(q - \varphi(t)), \quad M[f](t)e^{i\varphi(t)} = \iint_\mu f(q,p,t)(\cos q + i \sin q)dqdp, \tag{2}$$

where μ represents the phase space of the one-body system.

Remark. Note that $0 \le M[f] \le 1$. Furthermore, thanks to the rotational symmetry of the HMF model, we may set the magnetization's phase φ to zero without loss of generality if f is stationary. We will always do so in the following.

Notation: If f is stationary, the one-body Hamiltonian h is integrable, and we can introduce the angle-action variables (Θ, J). An integrable Hamiltonian $h(q,p)$ can be expressed as a function of J only. We denote such a Hamiltonian as $H(J)$. We also write $\Omega(J) = \partial_J H(J)$.

Remark. The phase space of Hamiltonian (2) presents a separatrix, at energy M. Strictly speaking, one must then define the angle-action variables separately in the different regions delimited by the separatrix. This is technical and a little bit cumbersome, so we postpone it to Section III B.

Clearly, any function f that depends on (q,p) through the Hamiltonian h only is a stationary solution to (1). In this paper, we concentrate on the following special class of stationary solutions:

Definition. A function f is called a monotonous stationary solution if it can be written as

$$f(q,p) = F(h(q,p)),$$

with F a C^1, real, strictly decreasing function, and if it is normalized: $\iint_\mu f dqdp = 1$.

Note that a monotonous stationary solution f is non-homogeneous in space if and only if $M[f] \neq 0$. We further note that $M[f] = 1$ is excluded, since $M[f] = 1$ implies that f is concentrated on the p-axis and hence f is not C^1. As will be clear in the following, these stationary solutions may be stable or unstable. This is a difference with 3D self-gravitating systems, where stationary solutions that are strictly decreasing functions of the energy are always stable (see Ref. 18 for the most recent results in this direction).

To measure the distance between two stationary solutions, we will use the fractional Sobolev spaces $W^{s,a}$. In addition, we require that to be close to each other and two stationary solutions must not differ too much in their magnetization, which is rather natural. In the whole paper, we will use "stable" to mean "formally stable." We can now state our main result.

Theorem 1. *Let f be a non-homogeneous stable monotonous stationary state, such that $f \in W^{1,a}$ with $a > 2$. Let $\tilde{f}$ be another monotonous stationary state such that $\tilde{f} \in W^{1,a}$. Then, there exists $\varepsilon > 0$ such that*

$$\|f - \tilde{f}\|_{W^{1,a}} < \varepsilon \text{ and } |M[f] - M[\tilde{f}]| < \varepsilon \text{ imply that } \tilde{f} \text{ is stable}.$$

In other words, there exists a neighborhood of f in the $W^{1,a}(a > 2)$ norm that does not contain any unstable monotonous stationary state, with magnetization close to $M[f]$.

This is to be contrasted with the following statement concerning homogeneous stationary states:

Theorem 2. *Let f be a homogeneous stable monotonous stationary state, such that $f \in W^{s,a}$, with $a > 1$ and $s < 1 + 1/a$. Any neighborhood of f in the $W^{s,a}$ norm contains an unstable monotonous homogeneous stationary state.*

From Theorem 1, we see that using a norm that controls only one derivative of the distribution function is enough to ensure that a neighborhood of f is "simple," in the sense that it does not contain any unstable monotonous stationary state. By contrast, in the homogeneous case, even requiring more regularity (with $s > 1$) may not be enough.

We have stated Theorem 2 in this way to emphasize the contrast with the non-homogeneous case. It is actually a much weaker and less general statement of the results in Ref. 9. We will give a proof of it, because it is instructive and for self-consistency of the paper.

Idea of the proof of Theorem 1: The proof relies on the analysis of the simple formal stability criterion obtained for non-homogeneous monotonous stationary states in Ref. 16. From the condition $|M[f] - M[\tilde{f}]| < \varepsilon$, we may choose a small enough ε such that $M[\tilde{f}]$ is not zero, and hence we will assume that $\tilde{f}$ is non-homogeneous in the proof.

Notation: We need to define the average over the angle Θ variable, at fixed action J; for a function $A(\Theta, J)$, we denote it as (see Sec. III B for a more precise definition of the integrals over Θ and/or J)

$$\langle A \rangle_J = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\Theta, J) d\Theta. \tag{3}$$

Following Ref. 16, we now introduce the functional $I[f]$,

$$I[f] = 1 + \int \cos^2 q dq \int \frac{1}{p} \frac{\partial f}{\partial p} dp - 2\pi \int \frac{1}{\Omega(J)} \frac{\mathrm{d}(F \circ H)}{\mathrm{d}J} \langle \cos Q \rangle_J^2 \, dJ, \tag{4}$$

where $Q(\Theta, J)$ is the position variable q in angle-action coordinates. From Ref. 16, we have the convenient formal stability criterion.

Proposition 3. Let f be a monotonous stationary solution. Then, $I[f] > 0$ if and only if f is formally stable.

If $M = 0$, f is homogeneous, and action/angle variables coincides with the (q,p) variables. Hence, the average over Θ coincides with the average over the spatial variable q, and I is simplified to

$$I[f] = 1 + \pi \int \frac{1}{p} \frac{\partial f}{\partial p} dp. \tag{5}$$

With criterion (4) in hand, we only have to show that if f is a non-homogeneous monotonous stationary state such that $I[f] > 0$, and $\tilde{f}$ and f are close in $W^{1,a}$ norm and their magnetization are close, then $|I[\tilde{f}] - I[f]|$ is small.

To prove Theorem 2, it is enough to construct, for each f homogeneous monotonous stable stationary state, a nearby (in $W^{s,a}$ norm) homogeneous monotonous stationary state $\tilde{f}$, such that $I[\tilde{f}] < 0$.

Discussion: Clearly, the proof of Theorem 1 heavily relies on the stability criterion obtained in Ref. 16. Thus, it is not clear how to generalize this result to more general models than HMF. Indeed, in general, the linear stability analysis of a non-homogeneous stationary state is complicated; see, for instance, Ref. 15 for the three dimensional gravitational case, but note that the situation is not much better for general one dimensional systems.

III. PROOFS

A. Proof of Theorem 2

We first give for consistency a proof of Theorem 2, although it is contained (in a much more general form) in Ref. 9. The proof relies on stability functional (5): given a stable monotonous stationary state f_0 (hence $I[f_0] > 0$), we have to find a modification f_1, small in $W^{s,a}$ norm, such that $I[f_0 + f_1] < 0$. Following the strategy of Ref. 9, we introduce $g(p) = e^{-p^2/2}/(2\pi)^{3/2}$ and $g_{\varepsilon,\alpha}(p) = \varepsilon g(p/\varepsilon^{\alpha})$. Note that $\int g_{\varepsilon,\alpha} dq dp = \varepsilon^{1+\alpha}$. It is easy to see that

$$\int \frac{g'_{\varepsilon,\alpha}(p)}{p} dp = -\frac{1}{2\pi}\varepsilon^{1-\alpha}. \tag{6}$$

Furthermore, for small ε and $1 - \alpha + \alpha/a > 0$, we have the estimate[9]

$$\|g_{\varepsilon,\alpha}\|_{W^{s,a}} = O(\varepsilon^{1-s\alpha+\alpha/a}). \tag{7}$$

We now choose a modified state as

$$f_0(p) + f_1(p) = \frac{1}{1+\varepsilon^{1+\alpha}}\left(f_0(p) + g_{\varepsilon,\alpha}(p)\right), \tag{8}$$

which corresponds to a modification

$$f_1(p) = \frac{1}{1+\varepsilon^{1+\alpha}} g_{\varepsilon,\alpha}(p) - \frac{\varepsilon^{1+\alpha}}{1+\varepsilon^{1+\alpha}} f_0(p). \tag{9}$$

From (5) and (6), it is clear that $g_{\varepsilon,\alpha}$ induces a large negative variation of the stability functional as soon as $\alpha > 1$. Hence in this case, $I[f_0 + f_1] < 0$, and $f_0 + f_1$ is unstable.

From the expression of f_1 (9), we see that the only way $\|f_1\|_{W^{s,a}}$ could be large is if $\|g_{\varepsilon,\alpha}\|_{W^{s,a}}$ itself is large. Now, from (7), $g_{\varepsilon,\alpha}$ is small in $W^{s,a}$ norm if $1 - s\alpha + \alpha/a > 0$. We see that it is possible to choose α such that the two conditions $\alpha > 1$ and $1 - s\alpha + \alpha/a > 0$ are satisfied, as soon as $s < 1 + 1/a$. Remembering that (7) is valid for $1 - \alpha + \alpha/a > 0$, $\alpha > 1$ implies $a > 1$.

This completes the proof of Theorem 2. ■

B. Angle-action variables

We need to define a bijection between position/momentum (q, p) and angle/action (Θ, J) coordinates. We will repeatedly use this change of variable in both directions. To keep notations as understandable as possible, we will use the following convention: functions of (q, p) will be denoted with small letters (for instance, $q, p, h(q, p), j(q, p) \ldots$), and functions of (Θ, J) with capital letters (for instance $\Theta, J, H(J), Q(\Theta, J) \ldots$).

As a further difficulty, the presence of a separatrix in phase space imposes us to divide the phase space in three regions, in order to properly define the change of variables: above separatrix (U_1), inside the separatrix (U_2) and below separatrix (U_3), see Fig. 1. In equations

$$U_1 = \{(q, p) \mid h(q, p) > M,\ p > 0\},$$
$$U_2 = \{(q, p) \mid -M < h(q, p) < M\},$$
$$U_3 = \{(q, p) \mid h(q, p) > M,\ p < 0\}.$$

Over each of these three regions, it is possible to define a bijective change of variables

$$U_i \to V_i,$$
$$(q, p) \mapsto (\theta_i(q, p), j_i(q, p)),$$

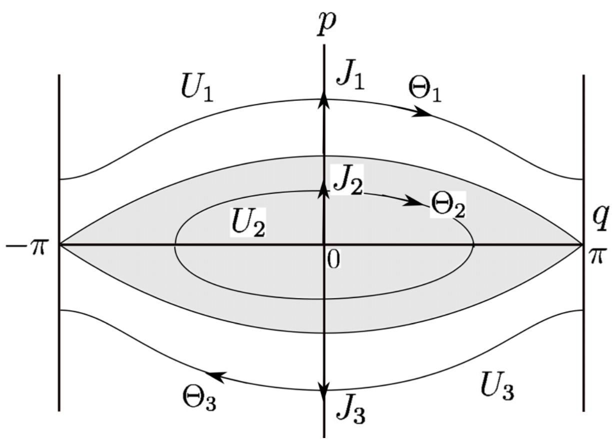

FIG. 1. The one particle phase space μ divided in the three regions U_1, U_2, and U_3.

with

$$\begin{aligned} V_1 &= \{(\Theta_1, J_1) \mid \Theta_1 \in]-\pi, \pi],\ J_1 > 4\sqrt{M}/\pi\}, \\ V_2 &= \{(\Theta_2, J_2) \mid \Theta_2 \in]-\pi, \pi],\ 0 < J_2 < 8\sqrt{M}/\pi\}, \\ V_3 &= \{(\Theta_3, J_3) \mid \Theta_3 \in]-\pi, \pi],\ J_3 > 4\sqrt{M}/\pi\}. \end{aligned} \tag{10}$$

The inverse change of variables reads

$$\begin{aligned} &V_i \to U_i, \\ &(\Theta_i, J_i) \mapsto (Q_i(\Theta_i, J_i), P_i(\Theta_i, J_i)). \end{aligned}$$

To keep notations simple, we will however use a single notation for each of these functions, $\theta(q,p), j(q,p), Q(\Theta, J), P(\theta, J)$. Similarly, any real function G of the angle-action variables is thus actually made of three distinct functions

$$G_i : V_i \to \mathbb{R} \quad i = 1,2,3.$$

We will however use for such a function a single notation $G(\Theta, J)$. The integrals over $d\Theta\ dJ$ are thus to be understood as the sum of three integrals over V_1, V_2, and V_3,

$$\iint_\mu G(\Theta, J) d\Theta dJ = \sum_{i=1}^{3} \iint_{V_i} G_i(\Theta_i, J_i) d\Theta_i dJ_i.$$

The average over Θ defined in (3) also yields three functions of the action, which we do not write explicitly.

C. General strategy

For later use, we rewrite stability functional (4) to make it easier to analyze.

Lemma 1. Let f be a monotonous stationary solution. Stability functional $I[f]$ (4) can be rewritten as

$$I[f] = 1 + \iint_\mu F'(h(q,p)) w(q,p) dq dp, \tag{11}$$

with

$$w(q,p) = \left\langle \cos^2 Q \right\rangle_{j(q,p)} - \left\langle \cos Q \right\rangle_{j(q,p)}^2. \tag{12}$$

Note that the function w implicitly depends on f through the definitions of the functions Q and j.

Proof. Remembering $F'(h(q,p)) = F'(H(J))$, the second term of (4) is

$$\iint_\mu F'(h(q,p))\cos^2 q\,dq\,dp = \int dJ\, F'(H(J)) \int \cos^2 Q(\Theta,J)\,d\Theta$$
$$= \iint_\mu F'(H(J))\left\langle \cos^2 Q\right\rangle_J d\Theta\, dJ = \iint_\mu F'(h(q,p))\left\langle \cos^2 Q\right\rangle_{j(q,p)} dq\,dp. \tag{13}$$

Similarly, the third term is

$$-2\pi \int F'(H(J))\left\langle \cos Q\right\rangle_J^2 dJ = -\iint_\mu F'(H(J))\left\langle \cos Q\right\rangle_J^2 d\Theta\, dJ = -\iint_\mu F'(h(q,p))\left\langle \cos Q\right\rangle_{j(q,p)}^2 dq\,dp. \quad \blacksquare \tag{14}$$

Remark. Looking back at (13) and using the fact $Q(\theta(q,p), j(q,p)) = q$, we may replace the function $w(q,p)$ defined in (12) with

$$w_1(q,p) = \cos^2 q - \left\langle \cos Q\right\rangle_{j(q,p)}^2. \tag{15}$$

We consider $f = F \circ h$ a stable non-homogeneous monotonous stationary state and $\tilde{f} = \tilde{F} \circ \tilde{h}$ another monotonous stationary state. $\tilde{h}$ is the Hamiltonian corresponding to $\tilde{f}$,

$$\tilde{h}(q,p) = \frac{p^2}{2} - \tilde{M}\cos q, \quad \tilde{M} = \iint_\mu \tilde{f}(q,p)\cos q\,dq\,dp, \tag{16}$$

and the angle-action variables associated to $\tilde{h}$ are written $(\tilde{\Theta}, \tilde{J})$. The change of variable is $(\tilde{\theta}(q,p), \tilde{j}(q,p))$, and the inverse change is $(\tilde{Q}(\tilde{\Theta},\tilde{J}), \tilde{P}(\tilde{\Theta},\tilde{J}))$. The stability functional for $\tilde{f}$ is

$$I[\tilde{f}] = 1 + \iint_\mu \tilde{F}'(\tilde{h}(q,p))\left[\left\langle \cos^2\tilde{Q}\right\rangle_{\tilde{j}(q,p)} - \left\langle \cos\tilde{Q}\right\rangle_{\tilde{j}(q,p)}^2\right] dq\,dp. \tag{17}$$

We write

$$\tilde{w}(q,p) = \left\langle \cos^2\tilde{Q}\right\rangle_{\tilde{j}(q,p)} - \left\langle \cos\tilde{Q}\right\rangle_{\tilde{j}(q,p)}^2.$$

Since f is stable, $I[f] > 0$. Thus, to prove Theorem 1, it is enough to show that if

H1. $\|\tilde{f} - f\|_{W^{1,a}}$ is small and

H2. $|\tilde{M} - M|$ is small, then

$$\left|I[\tilde{f}] - I[f]\right| \text{ is small.} \tag{18}$$

For convenience, we denote the discrepancies by

$$\Delta M = \tilde{M} - M \quad , \quad \Delta I = I[\tilde{f}] - I[f]. \tag{19}$$

ΔI can be rewritten as

$$\Delta I = \iint_\mu \left[(\tilde{F}'\circ\tilde{h})\tilde{w} - (F'\circ h)w\right] dq\,dp = \Delta I_1 - \Delta I_2, \tag{20}$$

where

$$\Delta I_1 = \iint_\mu \left[\tilde{F}'\circ\tilde{h} - F'\circ h\right]\tilde{w}\,dq\,dp \tag{21}$$

and

$$\Delta I_2 = \iint_\mu (F'\circ h)\left[\left\langle \cos\tilde{Q}\right\rangle_{\tilde{j}(q,p)}^2 - \left\langle \cos Q\right\rangle_{j(q,p)}^2\right] dq\,dp. \tag{22}$$

We have used here the remark after Lemma 1. We have

$$|\Delta I| \le |\Delta I_1| + |\Delta I_2| \tag{23}$$

and will show smallness of $|\Delta I_1|$ and $|\Delta I_2|$ in Secs. III D and III E, respectively.

D. $|\Delta I_1|$ is small

In this section, the hypothesis **H1** on $\|\tilde{f}-f\|_{W^{1,a}}$ will be crucial; we will also use **H2**. We begin with some Lemmas.

Lemma 2. Let m be a positive constant and the function u_a be defined by

$$u_a(q,p;m) = (|p|^a + |m\sin q|^a)^{1/a}. \tag{24}$$

Then, $||1/u_a||_{L^b}$ is finite for any $a>0$ and $1<b<2$. Moreover, when m is small, the leading order is $O(m^{-1/b})$.

Proof. The considered norm is

$$\left\|\frac{1}{u_a}\right\|_{L^b}^b = \iint_\mu \frac{dqdp}{(|p|^a + |m\sin q|^a)^{b/a}}. \tag{25}$$

We have to check the convergence of the integral at the points where the integrand diverges, which are $(q,p)=(0,0)$, $(\pm\pi,0)$, and when $|p|\to\infty$.

- Around $(0,0)$ and $(\pi,0)$:
  Let $(q_0,0)$ be the point we are considering. Using polar coordinates, $q-q_0=\frac{r}{m}\cos\rho$, $p=r\sin\rho$, we have $dqdp=\frac{r}{m}drd\rho$ and
  $$\frac{dqdp}{(|p|^a + |m\sin q|^a)^{b/a}} = \frac{1}{m}\frac{d\rho}{(|\sin\rho|^a + |\cos\rho|^a)^{b/a}} r^{1-b}dr. \tag{26}$$
  The integral over ρ is finite, and the integral over r converges for $b<2$.
- $|p|\to\infty$:
  $$\frac{dqdp}{(|p|^a + |m\sin q|^a)^{b/a}} \le \frac{dqdp}{|p|^b}. \tag{27}$$
  Thus, the integral converges for $1<b$.

Putting all together, we conclude that the integral converges for $1<b<2$. Moreover, if m is small, the leading order of $||1/u_a||_{L^b}$ is $O(1/m^{1/b})$ from the estimations around $(0,0)$ and $(\pi,0)$. ■

Remark. That m is non-zero is important to ensure convergence for $1<b<2$. If $m=0$, then the integrand diverges on the line $p=0$. As a result, around the line $p=0$,

$$\iint \frac{dqdp}{(|p|^a)^{b/a}} \simeq \int \frac{dp}{|p|^b}, \tag{28}$$

which converges for $b<1$. Considering the estimation for $|p|\to\infty$, which requires $1<b$, the interval of b for convergence is empty.

Since $|\cos Q|\le 1$, $|\tilde{w}|\le 1$. Hence,

$$|\Delta I_1| \le \iint_\mu \left|\tilde{F}'\circ\tilde{h} - F'\circ h\right| dqdp = \left\|\tilde{F}'\circ\tilde{h} - F'\circ h\right\|_{L^1}. \tag{29}$$

Since $M>0$, we introduce $u_a(q,p;M)$, and the Hölder inequality leads to

$$|\Delta I_1| \le \left\|(\tilde{F}'\circ\tilde{h} - F'\circ h)u_a\right\|_{L^a} \left\|\frac{1}{u_a}\right\|_{L^b}, \tag{30}$$

with a and b non-negative real numbers such that $1/a+1/b=1$. The norm $||1/u_a||_{L^b}$ is finite for $1<b<2$ by Lemma 2, and our job is to show that

$$\left\|(\tilde{F}'\circ\tilde{h} - F'\circ h)u_a\right\|_{L^a}^a = \iint_\mu \left|\tilde{F}'\circ\tilde{h} - F'\circ h\right|^a (|p|^a + |M\sin q|^a)\, dqdp \tag{31}$$

is small. The first term is rewritten as

$$\iint_\mu \left|\tilde{F}' \circ \tilde{h} - F' \circ h\right|^a |p|^a dqdp = \iint_\mu \left|\partial_p \left(\tilde{f} - f\right)\right|^a dqdp \le \left\|\tilde{f} - f\right\|_{W^{1,a}}^a \tag{32}$$

and is small by the hypothesis $\left\|\tilde{f} - f\right\|_{W^{1,a}}$ small. Using the trick,

$$M \sin q \left(\tilde{F}' \circ \tilde{h} - F' \circ h\right) = \partial_q \left(\tilde{f} - f\right) - \Delta M \sin q \tilde{F}' \circ \tilde{h}, \tag{33}$$

we rewrite the second term as

$$\begin{aligned} &\iint_\mu \left|\tilde{F}' \circ \tilde{h} - F' \circ h\right|^a |M \sin q|^a dqdp = \iint_\mu \left|\partial_q \left(\tilde{f} - f\right) - \Delta M \sin q \tilde{F}' \circ \tilde{h}\right|^a dqdp \\ &\le 2^a \max\left\{\iint \left|\partial_q \left(\tilde{f} - f\right)\right|^a dqdp,\ |\Delta M|^a \iint \left|\sin q \tilde{F}' \circ \tilde{h}\right|^a dqdp\right\} \\ &\le 2^a \max\left\{\left\|\tilde{f} - f\right\|_{W^{1,a}}^a,\ \frac{|\Delta M|^a}{M^a} \iint \left|\partial_q \tilde{f}\right|^a dqdp\right\}. \end{aligned} \tag{34}$$

Thus, the second term is also small by the hypothesis $\left\|\tilde{f} - f\right\|_{W^{1,a}}$ small, $|\Delta M|$ small, and $\tilde{f} \in W^{1,a}$. We have therefore proven that $|\Delta I_1|$ is small using the main hypotheses **H1** and **H2**. ∎

E. $|\Delta I_2|$ is small

In this section, the crucial hypothesis is **H2**, i.e., $|\Delta M|$ is small. If $\Delta M = 0$, then

$$\tilde{M} = M \implies \tilde{h} = h \implies (\tilde{\theta}, \tilde{j}) = (\theta, j) \text{ and } (\tilde{Q}, \tilde{P}) = (Q, P) \implies \Delta I_2 = 0, \tag{35}$$

so that ΔI_2 is trivially small. We therefore consider the case $\Delta M \neq 0$. We may choose $\Delta M > 0$ without loss of generality, since we can exchange the roles of f and $\tilde{f}$ in order to estimate $|\Delta I_2|$ when $\Delta M < 0$.

The quantity ΔI_2, which reads

$$\Delta I_2 = \iint_\mu (F' \circ h) \left[\langle \cos \tilde{Q} \rangle_{\tilde{j}(q,p)}^2 - \langle \cos Q \rangle_{j(q,p)}^2\right] dqdp, \tag{36}$$

depends on $\tilde{f}$ only through the Hamiltonian and magnetization, while ΔI_1 directly depends on the derivative of $\tilde{F}$. Thus, it is rather natural to expect that ΔI_2 is small if $\tilde{M}$ is close to M. A technical problem is that the separatrix changes as the magnetization changes, so that a direct comparison between $\langle \cos \tilde{Q} \rangle_{\tilde{j}(q,p)}$ and $\langle \cos Q \rangle_{j(q,p)}$ becomes difficult around the separatrix. To solve this problem, we divide the μ space into three regions:

1. Inside the separatrix.
2. Close to the separatrix.
3. Outside the separatrix.

For this purpose, we introduce M_1 and M_2 as

$$M_1 = M - 2\Delta M, \qquad M_2 = M + 2\Delta M. \tag{37}$$

From **H2**, we may consider a small ΔM which makes M_1 positive. Thus, we also have $\tilde{M} > 0$.

We now use the following strategy. ΔI_2 is divided in three parts, according to the division of the μ space detailed below. Secs. III E 2–III E 4 show the smallness of the contribution to ΔI_2 of the region close to, inside, and outside the separatrix, respectively.

1. Division of μ space

Using the Hamiltonian,

$$h(q,p) = \frac{p^2}{2} - M \cos q, \tag{38}$$

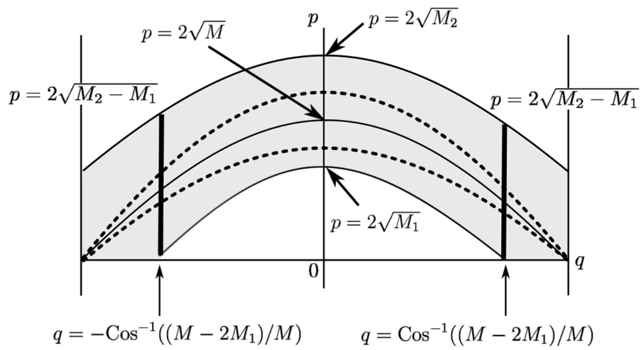

FIG. 2. Division of μ space. This figure describes the upper half μ space and the lower half μ is similarly divided thanks to the symmetry $p \to -p$. The shaded area is μ_2. The solid curves represent $h(q,p) = 2M_2 - M, M$ and $2M_1 - M$ from top to bottom, and the dashed curves are separatrices for $M + |\Delta M|$ and $M - |\Delta M|$ from top to bottom. The vertical solid lines at $q = \pm\mathrm{Cos}^{-1}((M - 2M_1)/M)$ are used to further divide μ_2.

and M_1, M_2 defined in (37), we divide μ into three μ_j as

$$\mu = \mu_1 \cup \mu_2 \cup \mu_3, \tag{39}$$

where

$$\begin{aligned} \mu_1 &= \{(q,p) \in \mu \mid h(q,p) < 2M_1 - M\}, \\ \mu_2 &= \{(q,p) \in \mu \mid 2M_1 - M \le h(q,p) \le 2M_2 - M\}, \\ \mu_3 &= \{(q,p) \in \mu \mid 2M_2 - M < h(q,p)\}. \end{aligned} \tag{40}$$

See Fig. 2 for a schematic picture. Accordingly, the second term ΔI_2 is divided as

$$\Delta I_2 = \Delta I_{21} + \Delta I_{22} + \Delta I_{23}, \tag{41}$$

where ΔI_{2j} corresponds to the integral over μ_j,

$$\Delta I_{2j} = \iint_{\mu_j} (F' \circ h) \left[\langle \cos \tilde{Q} \rangle^2_{\tilde{j}(q,p)} - \langle \cos Q \rangle^2_{j(q,p)} \right] dqdp. \tag{42}$$

2. Near the separatrix: μ_2

Proposition 4. Under the hypotheses of Theorem 1, $|\Delta I_{22}|$ *is small.*

We first show that the area of μ_2 is small.

Lemma 3. Let $A = \iint_{\mu_2} dqdp$. *Then, A is estimated as* $A \le 16\pi\sqrt{\Delta M}$.

Proof. Introducing $q_{\max} = \mathrm{Cos}^{-1}((M - 2M_1)/M)$, we further divide μ_2 into two parts,

$$\begin{aligned} \mu_{21} &= \{(q,p) \in \mu_2 \mid |q| \le q_{\max}\}, \\ \mu_{22} &= \{(q,p) \in \mu_2 \mid |q| > q_{\max}\}, \end{aligned} \tag{43}$$

and denote the areas of μ_{21} and μ_{22} as A_1 and A_2, respectively.

The upper and the lower bound of the upper half of μ_{21} is expressed as

$$p_{\mathrm{u}} = \sqrt{2(2M_2 - M + M\cos q)}, \quad p_{\mathrm{l}} = \sqrt{2(2M_1 - M + M\cos q)}, \tag{44}$$

for $|q| < q_{\max}$. The height of μ_{21} for a fixed q is estimated as

$$\begin{aligned} p_{\mathrm{u}} - p_{\mathrm{l}} = \frac{p_{\mathrm{u}}^2 - p_{\mathrm{l}}^2}{p_{\mathrm{u}} + p_{\mathrm{l}}} &= \frac{4(M_2 - M_1)}{\sqrt{2(2M_2 - M + M\cos q)} + \sqrt{2(2M_1 - M + M\cos q)}} \\ &\le \frac{4(M_2 - M_1)}{\sqrt{2(2M_2 - M + M - 2M_1)} + 0} = 2\sqrt{M_2 - M_1}, \end{aligned} \tag{45}$$

and hence the area A_1 is

$$A_1 = |\mu_{21}| \le 2 \cdot 2\sqrt{M_2 - M_1}\, 2q_{\max} = 8\sqrt{M_2 - M_1}\, q_{\max}. \tag{46}$$

In μ_{22}, one peace of the region is smaller than the rectangle whose vertices are $(q_{\max}, 0), (\pi, 0)$, $(\pi, 2\sqrt{M_2 - M_1})$, and $(q_{\max}, 2\sqrt{M_2 - M_1})$. Thus, A_2 is bounded as

$$A_2 = |\mu_{22}| \le 4 \cdot 2\sqrt{M_2 - M_1}(\pi - q_{\max}) = 8\sqrt{M_2 - M_1}(\pi - q_{\max}). \tag{47}$$

The total area A is therefore

$$A \le 8\pi\sqrt{M_2 - M_1} \le 16\pi\sqrt{\Delta M}. \quad \blacksquare \tag{48}$$

Proof of Proposition 4. From the fact $|\cos q| \le 1$, we have $\left|\langle\cos\tilde{Q}\rangle_{\tilde{j}}^2 - \langle\cos Q\rangle_j^2\right| \le 1$. F' is continuous, so that it is bounded by $F'_{\max} < +\infty$ in a neighborhood of the separatrix, containing μ_2 for ΔM small. Thus, we have

$$|\Delta I_{22}| \le \iint_{\mu_2} |F'(h(q,p))| \left|\langle\cos\tilde{Q}\rangle_{\tilde{j}(q,p)}^2 - \langle\cos Q\rangle_{j(q,p)}^2\right| dqdp \le F'_{\max} A. \tag{49}$$

Using Lemma 3 and **H2**, we conclude that $|\Delta I_{22}|$ is small.

3. Inside the separatrix: μ_1

Proposition 5. Under the hypotheses of Theorem 1, $|\Delta I_{21}|$ is small.

Proof. From $\left|\langle\cos\tilde{Q}\rangle_{\tilde{j}(q,p)} + \langle\cos Q\rangle_{j(q,p)}\right| \le 2$, we estimate ΔI_{21} as

$$\begin{aligned} |\Delta I_{21}| &\le 2\left\|(F' \circ h)\left(\phi_{\mathrm{in}}(q,p;\tilde{M}) - \phi_{\mathrm{in}}(q,p;M)\right)\right\|_{L^1(\mu_1)} \\ &\le 2\|F' \circ h\|_{L^1(\mu_1)} \sup_{(q,p)\in\mu_1} \left|\phi_{\mathrm{in}}(q,p;\tilde{M}) - \phi_{\mathrm{in}}(q,p;M)\right|. \end{aligned} \tag{50}$$

Here, we have introduced the following functions to simplify the notations:

$$\langle\cos Q\rangle_{j(q,p)} = \phi_{\mathrm{in}}(q,p;M), \qquad \langle\cos\tilde{Q}\rangle_{\tilde{j}(q,p)} = \phi_{\mathrm{in}}(q,p;\tilde{M}), \tag{51}$$

where

$$\phi_{\mathrm{in}}(q,p;M) = \varphi_{\mathrm{in}}(\psi(q,p;M)), \tag{52}$$

$$\varphi_{\mathrm{in}}(k) = \frac{2E(k)}{K(k)} - 1, \tag{53}$$

and

$$k = \psi(q,p;M) = \sqrt{\frac{p^2/2 + M(1 - \cos q)}{2M}}. \tag{54}$$

The functions $K(k)$ and $E(k)$ are the complete elliptic integrals of the 1st and the 2nd kinds, respectively.

The proof is done by the following three steps:

1. We show that $\|F' \circ h\|_{L^1(\mu_1)}$ is finite. [Lemma 4]
2. We extract the small ΔM from $\phi_{\mathrm{in}}(q,p;\tilde{M}) - \phi_{\mathrm{in}}(q,p;M)$. [Lemma 5]
3. We show that the remaining supremum part is finite. [Lemma 6]

The three following Lemmas prove the Proposition 5.

Lemma 4. $F' \circ h \in L^1$ (this implies of course that $F' \circ h \in L^1(\mu_1)$).

Proof. Using the function $u_a(q,p;M)$ and the Hölder inequality, we have

$$||F' \circ h||_{L^1} \le ||(F' \circ h)u_a||_{L^a} \left\| \frac{1}{u_a} \right\|_{L^b}, \tag{55}$$

where $1 \le a,b \le \infty$ and $1/a + 1/b = 1$. By Lemma 2, the factor $||1/u_a||_{L^b}$ converges for $1 < b < 2$, which corresponds to $2 < a < \infty$. On the other hand, we have

$$\begin{aligned} ||(F' \circ h)u_a||_{L^a} &= \left(\iint |F' \circ h|^a (|p|^a| + |M \sin q|^a) dq dp \right)^{1/a} \\ &= \left(\|\partial_p f\|_{L^a}^a + \|\partial_q f\|_{L^a}^a \right)^{1/a} \le ||f||_{W^{1,a}}. \end{aligned} \tag{56}$$

Thus, we have

$$||F' \circ h||_{L^1} \le ||f||_{W^{1,a}} \left\| \frac{1}{u_a} \right\|_{L^b} < +\infty. \quad \blacksquare \tag{57}$$

Our next job is to extract the small ΔM from the supremum part.

Lemma 5. For each point (q,p), there exists $M_* \in [M, \tilde{M}]$ such that

$$|\phi_{\text{in}}(q,p;\tilde{M}) - \phi_{\text{in}}(q,p;M)| = \Delta M \left| \frac{\partial \phi_{\text{in}}}{\partial M}(q,p;M_*) \right|. \tag{58}$$

Proof. We first remember that $M, \tilde{M} \in (0,1)$. The function $\psi(q,p;M)$ is C^1 with respect to M for $M \in (0,1)$, and $\varphi_{\text{in}}(k)$ is C^1 in $k \in [0,1)$. $\phi_{\text{in}}(q,p;M)$ is hence C^1 for $M \in (0,1)$. Thus, Taylor theorem proves the lemma. $\blacksquare$

Lemma 5 gives

$$\Delta I_{21} \le 2\Delta M ||F' \circ h||_{L^1(\mu_1)} \sup_{(q,p)\in\mu_1} \left| \frac{\partial \phi_{\text{in}}}{\partial M}(q,p;M_*(q,p)) \right|. \tag{59}$$

The last job is to show that the supremum is finite.

Lemma 6. $\sup_{(q,p)\in\mu_1} \left| \frac{\partial \phi_{\text{in}}}{\partial M}(q,p;M_*(q,p)) \right| < \infty.$

Proof. The concrete form of $\partial_M \phi_{\text{in}}$ is

$$\frac{\partial \phi_{\text{in}}}{\partial M}(q,p;M_*) = \frac{\partial \varphi_{\text{in}}}{\partial k}(\psi(q,p;M_*)) \frac{\partial \psi}{\partial M}(q,p;M_*). \tag{60}$$

The derivatives of φ_{in} and ψ are

$$\begin{aligned} \frac{\partial \varphi_{\text{in}}}{\partial k}(k) &= \frac{2}{K(k)^2}[E'(k)K(k) - E(K)K'(k)] \\ &= \frac{-2}{k} \left[\left(\frac{E(k)}{K(k)} - 1 \right)^2 + \frac{k^2}{1-k^2} \left(\frac{E(k)}{K(k)} \right)^2 \right] \end{aligned} \tag{61}$$

and

$$\frac{\partial \psi}{\partial M}(q,p;M_*) = \frac{-p^2}{4M_*\sqrt{2M_*}\sqrt{p^2/2 + M_*(1-\cos q)}} = \frac{-p^2}{8M_*^2 k}, \tag{62}$$

where k must be evaluated at $\psi(q,p;M_*(q,p))$. The derivative $\partial_M \phi_{\text{in}}$ is hence

$$\frac{\partial \phi_{\text{in}}}{\partial M} = \frac{p^2}{4M_*^2} \frac{1}{k^2} \left[\left(\frac{E(k)}{K(k)} - 1 \right)^2 + \frac{k^2}{1-k^2} \left(\frac{E(k)}{K(k)} \right)^2 \right]. \tag{63}$$

The functions $E(k)$ and $1/K(k)$ are finite in the interval $k \in [0,1]$. Therefore, remembering $M_*(q,p) \in [M, \tilde{M}]$ and is positive, it is enough to show

- no divergence at $k = 0$,
- no appearance of $k = 1$.

No divergence at $k = 0$: Around $k = 0$, from the Taylor expansions of $K(k)$, (B3), and $E(k)$, (B4), we have

$$\frac{E(k)}{K(k)} = 1 - \frac{k^2}{2} + O(k^4). \tag{64}$$

Thus, we have

$$\begin{aligned}\frac{\partial \phi_{\rm in}}{\partial M} &= \frac{p^2}{4M_*^2}\frac{1}{k^2}\left[\left(\frac{E(k)}{K(k)} - 1\right)^2 + \frac{k^2}{1-k^2}\left(\frac{E(k)}{K(k)}\right)^2\right] \\ &= \frac{p^2}{4M_*^2}\frac{1}{k^2}\left[O(k^4) + \frac{k^2}{1-k^2}(1 + O(k^2))\right] \\ &\to \frac{p^2}{4M_*^2} \quad (k \to 0).\end{aligned} \tag{65}$$

Actually, $k \to 0$ implies $(q,p) \to (0,0)$ and hence $\partial_M \phi_{\rm in} \to 0$.
No appearance of $k = 1$: $k = \psi(q,p;M_*)$ is an increasing function of p for a fixed q, thus it is enough to investigate the upper value of k on the upper boundary of μ_1,

$$p = \sqrt{2(2M_1 - M + M\cos q)}. \tag{66}$$

Substituting this p into $\psi(q,p;M_*)$, we have

$$k^2 = \frac{2M_1 - M + M_* + (M - M_*)\cos q}{2M_*} \le \frac{2M_1 - M + M_* + M_* - M}{2M_*} = 1 - 2\frac{\Delta M}{M_*} \le 1 - 2\frac{\Delta M}{\tilde{M}}. \tag{67}$$

Thus, k is bounded by a positive number which is smaller than 1,

$$k < \sqrt{1 - 2\frac{\Delta M}{\tilde{M}}} < 1. \quad \blacksquare \tag{68}$$

4. Outside the separatrix: μ_3

Proposition 6. Under the hypotheses of Theorem 1, $|\Delta I_{23}|$ *is small.*

Proof. The strategy of the proof is almost the same as for Proposition 5, but we replace $\varphi_{\rm in}(k)$, introduced in (53), by

$$\varphi_{\rm out}(k) = \frac{2k^2 E(1/k)}{K(1/k)} - 2k^2 + 1. \tag{69}$$

We show the following Lemma which corresponds to Lemma 6:

Lemma 7. $\sup_{(q,p)\in\mu_3}\left|\frac{\partial\phi_{\rm out}}{\partial M}(q,p;M_*(q,p))\right| < \infty.$

Proof. The derivative of $\varphi_{\rm out}$ is

$$\begin{aligned}\frac{\partial \varphi_{\rm out}}{\partial k} &= 4k\left(\frac{E(1/k)}{K(1/k)} - 1\right) - \frac{2}{K(1/k)^2}\left[E'(1/k)K(1/k) - E(1/k)K'(1/k)\right] \\ &= 4k\left(\frac{E(1/k)}{K(1/k)} - 1\right) - 2k\left[\left(\frac{E(1/k)}{K(1/k)} - 1\right)^2 + \frac{1}{1-k^2}\left(\frac{E(1/k)}{K(1/k)}\right)^2\right].\end{aligned} \tag{70}$$

The value of k must be evaluated at $k = \psi(q,p;M_*)$. The derivative $\partial_M \phi_{\rm out}$ is hence

$$\frac{\partial \phi_{\rm out}}{\partial M} = \frac{p^2}{4M_*^2}\left[\left(\frac{E(1/k)}{K(1/k)} - 1\right)^2 + \frac{1}{1-k^2}\left(\frac{E(1/k)}{K(1/k)}\right)^2 - 2\left(\frac{E(1/k)}{K(1/k)} - 1\right)\right]. \tag{71}$$

The functions $E(1/k)$ and $1/K(1/k)$ are finite in the interval $k \in [1,\infty]$, and hence it is enough to show

- no appearance of $k = 1$,
- no divergence at $k = \infty$.

No appearance of $k = 1$: As commented previously, ψ is an increasing function of p, and hence a lower bound for k is given by considering the lower boundary of μ_3,

$$p = \sqrt{2(2M_2 - M + M\cos q)}. \tag{72}$$

Substituting this p into $\psi(q,p;M_*)$, we have

$$k^2 = \frac{2M_2 - M + M_* + (M - M_*)\cos q}{2M_*} \geq \frac{2M_2 - M + M_* - |M - M_*|}{2M_*} = \frac{M + 2\Delta M}{M_*} \geq 1 + \frac{\Delta M}{\tilde{M}}. \tag{73}$$

Thus, we have proved

$$k > \sqrt{1 + \frac{\Delta M}{\tilde{M}}} > 1. \tag{74}$$

No divergence at $k = \infty$: Estimation (64) gives, in the limit $k \to \infty$,

$$\frac{E(1/k)}{K(1/k)} = 1 - \frac{1}{2k^2} + O(1/k^4), \tag{75}$$

and $p^2 \leq 4\tilde{M}k^2$. Thus, we have

$$\left|\frac{\partial \phi_{\text{out}}}{\partial M}\right| \leq \frac{\tilde{M}k^2}{M_*^2}\left|O(1/k^4) + \frac{1 + O(1/k^2)}{1 - k^2} + \frac{1}{k^2} + O(1/k^4)\right| \to 0, \quad (k \to \infty). \quad \blacksquare \tag{76}$$

IV. NUMERICAL TESTS

In this section, we present some numerical simulations of the Vlasov equation for the HMF model, using a semi-Lagrangian code.[20] The purpose is twofold:

(i) Illustrate numerically Theorems 1 and 2. We will show that a modification of the distribution function small in $W^{s,a}$ with $1 \leq s < 1 + 1/a$ can destabilize a homogeneous stable stationary state. By contrast, we never observe the destabilization of an inhomogeneous stationary state by such perturbations.

(ii) Perform a few numerical tests in a case not covered by Theorem 1, where modifications are in spaces rougher than $W^{1,a}$.

A. Set up

In this section, we concentrate on $a = 2$ and denote $W^{s,2}$ by H^s following the conventional notation. Let us consider the following modification of the reference state f_0:

$$g_{\varepsilon,\delta}(q,p) = \varepsilon^{\delta} e^{-(h(q,p)-h(0,0))/T\varepsilon^2}, \tag{77}$$

where $\delta > 0$ and

$$h(q,p) = \frac{p^2}{2} - M\cos q \tag{78}$$

is the one-body Hamiltonian. The modification we actually use is slightly different from (77) to ensure the normalization of the modified reference stationary state. The function $g_{\varepsilon,\delta}(q,p)$ is almost zero except for a neighborhood of the origin, thus we may approximate $h(q,p)$ in the inhomogeneous case as $h(q,p) \simeq (p^2 + q^2)/2$. The exponent δ controls the H^s norm of $g_{\varepsilon,\delta}$,

$$\|g_{\varepsilon,\delta}\|_{H^s} \simeq \begin{cases} \varepsilon^{1/2+\delta-s}, & \text{homogeneous case,} \\ \varepsilon^{1+\delta-s}, & \text{inhomogeneous case.} \end{cases} \tag{79}$$

Since $H^s \subset H^t$ for $t \leq s$, we have

$$g_{\varepsilon,\delta} \in \begin{cases} H^s(s < 1/2 + \delta), & \text{homogeneous case}, \\ H^s(s < 1 + \delta), & \text{inhomogeneous case}. \end{cases} \tag{80}$$

Let us estimate the contribution of $g_{\varepsilon,\delta}$ to the stability functional, that is, $I[g_{\varepsilon,\delta}] - 1$. The homogeneous case is straightforward. For the inhomogeneous case, we expand $\cos q$ in the Taylor series, and using the angle-action variables, $q = \sqrt{2J}\sin\Theta$ and $p = \sqrt{2J}\cos\Theta$, we obtain the following approximation of the function $w(q,p)$:

$$w(q,p) \simeq (p^2 + q^2)^2 + O((q,p)^6). \tag{81}$$

From the above approximation, we have the estimates of $I[g_{\varepsilon,\delta}] - 1$ both for the homogeneous and the inhomogeneous cases,

$$I[g_{\varepsilon,\delta}] - 1 = \begin{cases} \varepsilon^{\delta-1}, & \text{homogeneous case}, \\ \varepsilon^{\delta+4}, & \text{inhomogeneous case}. \end{cases} \tag{82}$$

These estimations imply the following: (i) In homogeneous case, the modification $g_{\varepsilon,\delta}$ may change the sign of the stability functional even in the limit $\varepsilon \to 0$ for $\delta \leq 1$, that is, when $g_{\varepsilon,\delta}$ is small in $H^s(s < 1/2 + \delta)$. (ii) In inhomogeneous case, no δ can change the sign of the stability functional in the limit $\varepsilon \to 0$ contrasting with the homogeneous case.

Based on the above considerations, we prepare a perturbed initial distribution

$$f_{\varepsilon,\delta,\mu}(q,p) = A\left[e^{-h(q,p)/T}(1 + \mu\cos q) + \varepsilon^{\delta} e^{-(h(q,p)-h(0,0))/T\varepsilon^2}\right], \tag{83}$$

where A is the normalization factor, the first term corresponds to the distribution in thermal equilibrium, the third term corresponds to the modification $g_{\varepsilon,\delta}$, and the second term proportional to μ is a perturbation to check the stability of the stationary state $f_{\varepsilon,\delta,0}(q,p)$. The magnetization M in the one-body Hamiltonian $h(q,p)$ must satisfies the self-consistent equation

$$M = \int f_{\varepsilon,\delta,0}(q,p)\cos q\, dq dp. \tag{84}$$

The critical temperature in thermal equilibrium states is $T_c = 0.5$ in the HMF model, and therefore, we will set $T = 0.6$ for homogeneous case and $T = 0.4$ for inhomogeneous case.

We perform numerical integration of the Vlasov equation by using the semi-Lagrangian code[20] with the time step $\Delta t = 0.05$. We introduce a mesh on the truncated phase space $(q,p) \in]-\pi,\pi] \times [-3,3]$, and the mesh size is 512×512 unless otherwise specified.

B. Homogeneous case

We set the magnetization as zero in the one-body Hamiltonian $h(q,p)$, (78). Typical temporal evolution of magnetization is exhibited in Fig. 3(a). In a short time interval, $M(t)$ decreases, and then it increases if the considered stationary state is unstable. The instability gets weaker as δ approaches the threshold value, which can be computed by the stability functional

$$I[f_{\varepsilon,\delta,0}] = 1 - \frac{1 + \varepsilon^{\delta-1}}{2T(1 + \varepsilon^{\delta+1})}. \tag{85}$$

We remark that for the thermal equilibrium with $T = 0.6$, $I[f_{0,\delta,0}] = 1/6 > 0$ and hence the unmodified distribution $f_{0,\delta,0}$ is stable. The strange looking discontinuity around $t = 512$ is an artifact due to the mesh size; indeed, it disappears when a finer 1024×1024 mesh is used. Nevertheless, in most cases, a 512×512 mesh is sufficient to judge the stability of the modified state.

We use the perturbation level $\mu = 10^{-4}$. Varying ε and δ, we compute values of magnetization at $t = 1000$ and judge the stability of the modified states $f_{\varepsilon,\delta,0}$. From the typical temporal evolutions of $M(t)$, we use the criterion that the state $f_{\varepsilon,\delta,0}$ is unstable if the final magnetization $M_{\mathrm{f}} = M(1000)$ is larger than $M_{\mathrm{f}}^{\mathrm{th}} = \mu/2 = 5.10^{-5}$, which is slightly larger than the initial value

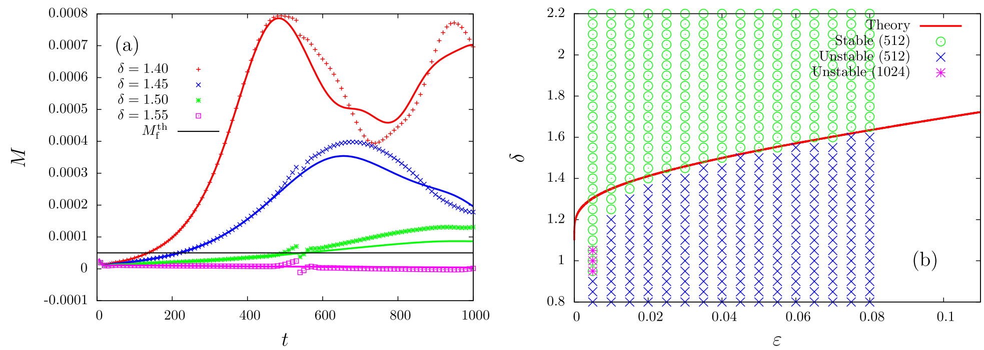

FIG. 3. (a) Typical temporal evolutions of magnetization $M(t)$. $\varepsilon = 0.05$, which gives the stability threshold value as $\delta_c \simeq 1.536$. The values of δ are $\delta = 1.40$ (red plus), 1.45 (blue crosses), 1.50 (green stars), and 1.55 (purple boxes), with which the solid lines are computed by using a finer mesh size 1024×1024 from top to bottom. The black straight line marks the level $M_f^{\rm th} = 5.10^{-5}$, which is used in the panel (b). (b) Phase diagram on the (ε, δ) plane. Solid red line represent the boundary of stability defined by $1 - (1+\varepsilon^{\delta-1})/2T(1+\varepsilon^{\delta+1}) = 0$. Green circles and blue crosses represent $M_f < M_f^{\rm th}$ and $M_f > M_f^{\rm th}$, respectively. Purple stars, overwritten on green circles, are for $M_f > M_f^{\rm th}$, but with a finer mesh size 1024×1024.

$M_{\rm i} = \mu/[2(1+\varepsilon^{\delta+1})]$. The phase diagram on the (ε, δ) plane is reported in Fig. 3(b), together with the theoretical threshold line defined by $I[f_{\varepsilon,\delta,0}] = 0$.

Numerical results are not in perfect agreement with the theoretical prediction. There are three numerical reasons. (1) Mesh size: A smaller ε implies that the modification is strongly concentrated around the line $p = 0$. As a result, we need a finer mesh to capture the modification for a smaller ε. Indeed, using the mesh size 1024×1024, three points on the line $\varepsilon = 0.005$ are found unstable, whereas the 512×512 mesh judged the same states stable. (2) Computational time: If δ goes up to the theoretical line with a fixed ε, the strength of instability gets weaker. Thus, a longer time computation is required to observe instability, since typical $M(t)$ curves decrease in a short time interval. (3) Weak instability: in relation with the point (2), if the instability is very weak, then the magnetization saturates at a lower level than the threshold $M_f^{\rm th}$.

Summarizing, a large δ, corresponding to a "smooth" space $H^{1/2+\delta}$, keeps the modified state stable, but a small δ, corresponding to a "rough" space changes the stability and the modified state becomes unstable. We stress that, no matter how small ε is, there is a modification which makes the state unstable.

C. Inhomogeneous case

The estimation of stability functional (82) suggests that a small modification by $g_{\varepsilon,\delta}$ cannot change the stability of the inhomogeneous stationary state $f_{0,\delta,0}$. We numerically confirm this suggestion.

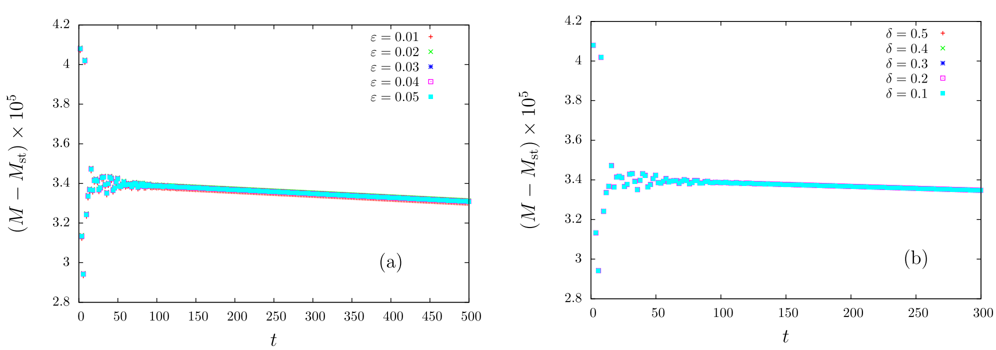

FIG. 4. Temporal evolutions of magnetization $M(t) - M_{\rm st}$, where $M_{\rm st}$ is the value satisfying the self-consistent equation. (a) $\delta = 0.5$, $\varepsilon = 0.01$–0.05. (b) $\varepsilon = 0.05$, $\delta = 0.5$–0.1. In both panels, $\mu = 10^{-4}$ and five types of points corresponding to five values of ε or δ almost collapse.

Computations are performed along two lines: (i) $\delta = 0.5$. (ii) $\varepsilon = 0.05$. We choose the value $\delta = 0.5$ since it gives the same threshold $s = 3/2$ as the homogeneous case with $\delta = 1$. Then, we examine stability by decreasing δ, which means that the modification $g_{\varepsilon,\delta}$ becomes "rough."

Along the two lines, no instability is observed (Fig. 4). Thus, the stability of the inhomogeneous stationary state does not seem to change even when the modification is "rough" enough. We have used a perturbation level $\mu = 10^{-4}$; changing it does not significantly affect the results.

V. CONCLUSION AND DISCUSSIONS

We have shown that, in the 1D Vlasov equation with a cosine potential (corresponding to the HMF model), any non-homogeneous stable monotonous stationary has a neighborhood in the $W^{1,a}(a > 2)$ norm that does not contain any unstable monotonous stationary states with nearby magnetization. This is in striking contrast with the homogeneous case, where all neighborhoods of a reference stable state in norms controlling only one derivative do contain unstable stationary states. These results are illustrated with direct simulations of the Vlasov equation, using a reference stationary state and controlling the norm of a modification of this reference state in various Sobolev spaces.

Theorem 1 points to an important difference in the mathematical structure of the neighborhoods of homogeneous and inhomogeneous stationary states of Vlasov equation. Understanding the physical consequences of this fact, especially with respect to the non-linear evolution of a perturbation, remains an open question.

Finally, we stress that the proof of Theorem 1 relies on the knowledge of a stability functional which is rather simple in the HMF model. Extending the theorem to other models having long-range interactions, where such a simple stability functional is not available, is another open problem.

ACKNOWLEDGMENTS

Y.Y.Y. acknowledges the support of JSPS KAKENHI Grant Number 23560069.

APPENDIX A: $W^{1,p}$ SPACES

For $X \subset \mathbb{R}^n$, the Sobolev space $W^{1,p}(X)$ is defined by

$$W^{1,p} = \{ f : X \to \mathbb{R} \mid ||f||_{L^p} + ||\nabla f||_{L^p} < \infty \} . \tag{A1}$$

The norm on $W^{1,p}$ is

$$||f||_{W^{1,p}} = ||f||_{L^p} + ||\nabla f||_{L^p}, \tag{A2}$$

where we recall

$$||f||_{L^p} = \begin{cases} \left(\displaystyle\int_X |f(x)|^p dx \right)^{1/p} & (1 \le p < \infty), \\ \sup\limits_{x \in X} |f(x)| & (p = \infty). \end{cases} \tag{A3}$$

We refer the reader to Ref. 19 for fractional Sobolev spaces, needed for Theorem 2.

APPENDIX B: SOME USEFUL PROPERTIES OF THE COMPLETE ELLIPTIC INTEGRALS

We list a few useful properties of the complete elliptic integrals.

1. K is monotonically increasing, and E is monotonically decreasing.
2. $K(0) = E(0) = \pi/2$.
3. $K(k) \to \infty$ $(k \to 1)$.
4. $E(1) = 1$.

5. The derivatives of K and E are

$$\frac{\mathrm{d}K}{\mathrm{d}k}(k) = \frac{E(k) - (1-k^2)K(k)}{k(1-k^2)} \tag{B1}$$

and

$$\frac{\mathrm{d}E}{\mathrm{d}k}(k) = \frac{E(k) - K(k)}{k}. \tag{B2}$$

6. Taylor expansions of K and E around $k = 0$ are

$$K(k) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9}{64}k^4 + \cdots\right) \tag{B3}$$

and

$$E(K) = \frac{\pi}{2}\left(1 - \frac{k^2}{4} - \frac{3}{64}k^4 - \cdots\right). \tag{B4}$$

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