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Relations into Algebras of Probabilistic Distributions

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Abstract

The paper proposes two types of convex relations into algebras of probabilistic distributions as a relational algebraic foundation of semantic domains of probabilistic systems \cite{4, 7, 8}. Following previous results by Tsumagari \cite{16}, we particularly focus on the associative law for the convex compositions defined via bounded combinations of probabilistic distributions, and prove that the convex compositions are associative for convex relations.

Keywords: algebras of probabilistic distributions, convex relations, associativity and distributivity of convex composition, relational calculus

1. Introduction

The concept of rings is basic in mathematics as a framework of numbers. Recently from a view point on algebraic study \cite{1} of semantic domains for distributed algorithms, the importance of variants of rings, such as Kleene algebras \cite{5} and idempotent semirings, has been increased. It is well-known that the set of all binary relations on a set forms a typical example of complete idempotent semirings.

When constructing a concrete model of semirings with preferable properties, we have to first focus our attention on the associativity of possible composition. For relations $\alpha : X \rightarrow Y$ and $\beta : Y \rightarrow Z$ the ordinary composite $\alpha \beta : X \rightarrow Y$ is defined as

$$(x, z) \in \alpha \beta \iff \exists y \in Y. (x, y) \in \alpha \land (y, z) \in \beta.$$  

Of course the ordinary composition of relations is associative. A multirelation is a relation of a form $\alpha : X \rightarrow \wp(Y)$, where $\wp(X)$ denotes the power set of $X$. Depending on applications, two definitions of composition of multirelations are known. One of them is called the \textit{reachability composition} studied by Peleg \cite{13} and Goldblatt \cite{3} for concurrent dynamic logic. The reachability composition $\alpha \cdot \beta$ of multirelations $\alpha : X \rightarrow \wp(Y)$ and $\beta : Y \rightarrow \wp(Z)$ is defined by

$$(x, T) \in \alpha \cdot \beta \iff \exists U \in \wp(Y). [(x, U) \in \alpha \land \exists(T_y)_{y \in U} \subseteq \wp(Z).$$

Another composition of multirelations is given by Parikh and Rewitzky. Their composition $\alpha ; \beta : X \rightarrow \wp(Z)$ of multirelations $\alpha : X \rightarrow \wp(Y)$ and $\beta : Y \rightarrow \wp(Z)$ is defined by

$$(x, T) \in \alpha ; \beta \iff \exists U \in \wp(Y). [(x, U) \in \alpha \land \forall y \in U. (y, T) \in \beta.$$  

It is readily seen that the definition of $\alpha ; \beta$ is making use of the membership relation and residual composition. For the associativity of this composition we need a condition called \textit{up-closed}. Up-closed multirelations

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provide a model of Parikh’s game logic [11, 12]. Rewitzky [6, 14] studied them as a semantic domain of predicate transformer semantics of nondeterministic programming language. Further Nishizawa, Tsumagari and Furusawa [10] demonstrated that the set of all up-closed multirelations forms a complete idempotent left semi-ring (complete IL-semiring) introduced by Möller [9].

On the other hand, McIver et al. [4, 7, 8] introduced a semantic domain of probabilistic programs and probabilistic Kleene algebra, and indicated that probabilistic Kleene algebras are useful to simplify a model of probabilistic distributed systems. Based on their works, Tsumagari [16] initially introduced two probabilistic (non-numerical) models of complete IL-semirings with the set of maps from a set into the unit interval [0, 1], and studied \textit{probabilistic multirelations} and the point-wise convexity of them. The point-wise convexity plays an important rôle for both models to satisfy the associativity of composition.

The aim of the paper is to expand Tsumagari’s work [16] and to give a relational foundation for relations into algebras of probabilistic distributions. Following his work we will reformulate probabilistic multirelations as certain convex relations, together with stepwise refinement. Then we will clarify how the convexity works into algebras of probabilistic distributions. Following his work we will reformulate probabilistic multirelations and the point-wise convexity of them. The point-wise convexity plays an important rôle for both models to satisfy the associativity of composition.

In section 2 we review the basic properties of algebras consist of maps from a set into the unit interval [0, 1] together with scalar products, multiplications and bounded sums. Section 3 studies convex combinations of probabilistic distributions. In section 4 we introduce convex composition of relations into algebras of probabilistic distributions by using convex combinations. In section 5 we show the associative law of convex combinations. In section 6 we introduce convex composition into algebras of probabilistic distributions. Section 7 summarizes this work.

\textbf{Notation.} In the paper we will denote by $I$, a singleton set. A (binary) relation $\alpha$ from a set $X$ to a set $Y$, written $\alpha : X \to Y$, is a subset $\alpha \subseteq X \times Y$. The empty relation $\emptyset_{XY} : X \to Y$ and the universal relation $\nabla_{XY} : X \to Y$ are defined by $\emptyset_{XY} = \emptyset$ and $\nabla_{XY} = X \times Y$ respectively. The converse of a relation $\alpha : X \to Y$ is denoted by $\alpha^\circ$. The identity relation $\{(x, x) \mid x \in X\}$ over $X$ is denoted by $\text{id}_X$. The ordinary composition of relations (which include functions) will be denoted by juxtaposition. For example, the composite of a relation $\alpha : X \to Y$ followed by $\beta : Y \to Z$ is denoted by $\alpha \beta$, and of course the composition of functions $f : X \to Y$ and $g : Y \to Z$ by $f g$. Also the traditional notation $f(x)$ will be written by $xf$ as a composite of functions $x : I \to X$ and $f : X \to Y$. However, the evaluation of a map $p : X \to [0, 1]$ at $x \in X$ will be expressed by $p(x) \in [0, 1]$. Note that the symbols of multiplication for reals and the ordinary composition of relations are omitted. Some proofs refer the point axiom (PA) and the Dedekind formula (DF), i.e.

\begin{align*}
\text{(PA)} & \quad \cup_{x \in X} x = \nabla_{IX}, \\
\text{(DF)} & \quad \alpha \beta \cap \gamma \subseteq (\alpha \cap \gamma^\circ)(\beta \cap \alpha^\circ \gamma),
\end{align*}

where $x \in X$ is identified as a function $x : I \to X$. Note that (PA) is equivalent to $\text{id}_X = \cup_{x \in X} x^2 x$ and that so is (DF) to $\alpha \beta \cap \gamma \subseteq (\alpha \cap \gamma \beta^\circ ) (\beta \cap \alpha^\circ \gamma)$. See [15] for more details on basic properties of relations.

\section{Maps to the unit interval}

We consider maps from a set $X$ to the unit interval $[0, 1]$. Such a map $p : X \to [0, 1]$ is often called a fuzzy set. The support $[p]$ of a map $p$ is the subset of $X$ defined by $[p] = \{x \in X \mid p(x) > 0\}$. The set of all maps from $X$ to $[0, 1]$ will be denoted by $Q(X)$. As we will discuss later, maps in $Q(X)$ will be restricted as to be probabilistic (sub-)distributions. The point-wise order $\leq$ on $Q(X)$ is a binary relation such that

\[ p \leq q \iff \forall x \in X. \ p(x) \leq q(x) \]

for $p, q \in Q(X)$. For a real $k \in [0, 1]$ a map $k_X : X \to [0, 1]$ such that $k_X(x) = k$ for all $x \in X$ is called the \textit{constant map over $X$} with value $k$. For a $a \in X$ define a map $\hat{a} : X \to [0, 1]$ by

\[ \hat{a}(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise}. \end{cases} \]
The constant maps $0_X$ and $1_X$ over $X$ are the least and the greatest elements of $Q(X)$, respectively.

We introduce the following operators to discuss algebras of probabilistic distributions. For a real $k \in [0, 1]$ and maps $p, q \in Q(X)$ we define maps $k \cdot p$, $p \ast q$, $p \oplus q \in Q(X)$ by

$$(k \cdot p)[x] = kp[x], \quad (p \ast q)[x] = p[x]q[x]$$

and

$$(p \oplus q)[x] = \min\{p[x] + q[x], 1\}$$

for all $x \in X$, respectively. The set $Q(X)$ forms an algebra called a prering.

**Proposition 1.** Let $p, q \in Q(X)$ and $k, k' \in [0, 1]$. Then the following hold:

(a) $(p \ast q) \ast r = p \ast (q \ast r)$, $p \ast q = q \ast p$,
(b) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$, $p \oplus q = q \oplus p$,
(c) $k \cdot p = k_X \cdot p$, $0 \cdot p = 0_X$, $1 \cdot p = p$, $(k k') \cdot p = k \cdot (k' \cdot p)$,
(d) $p \oplus 0_X = p$, $p \oplus 1_X = 1_X$,
(e) If $q[x] + r[x] \leq 1$ for all $x \in X$, then $p \ast (q \oplus r) = (p \ast q) \oplus (p \ast r)$.
(f) If $k + k' \leq 1$, then $(k + k') \cdot p = (k \cdot p) \oplus (k' \cdot p)$.
(g) $p \leq p' \land q \leq q' \rightarrow p \oplus q \leq p' \oplus q' \land p \ast q \leq p' \ast q'$.

Proof is omitted.

In general the distributive laws $p \ast (q \oplus r) = (p \ast q) \oplus (p \ast r)$ and $k \cdot (q \oplus r) = (k \cdot q) \oplus (k \cdot r)$ do not always hold.

The following proposition shows the basic properties about the support of maps in $Q(X)$.

**Proposition 2.** Let $p, q, r \in Q(X)$ and $k \in [0, 1]$. Then the following hold:

(a) $[0_X] = \emptyset$, and $[k_X] = X$ if $k > 0$,
(b) $[a] = \{a\}$,
(c) $[p \ast q] = [p] \cap [q]$,
(d) $[p \oplus q] = [p] \cup [q]$.

Proof is omitted.

**Proposition 3.** Let $p, q \in Q(X)$. Then

$$q \leq p \iff \exists t \in Q(X). q = p \ast t.$$  

Proof. ($\leftarrow$) $q[x] = p[x]t[x] \leq p[x]$ since $t[x], p[x] \in [0, 1]$. 

($\rightarrow$) Assume $q \leq p$. Define a map $t : X \rightarrow [0, 1]$ by

$$\forall x \in X. t[x] = \begin{cases} 
\frac{q[x]}{p[x]} & \text{if } p[x] > 0, \\
0 & \text{otherwise}.
\end{cases}$$

Then it is trivial that $q = p \ast t$.

The *sum* of a map $p \in Q(X)$ is the least upper bound of the set $\{\sum_{x \in F} p[x] \mid F : \text{finite subset of } X\}$, which is denoted by $||p||$. It is well-known that the sum $||p||$ of $p$ exists iff the above set is bounded. Also $[p]$ is a countable subset of $X$ if $||p||$ exists. (For each positive integer $n$ define a subset $[p]_n$ of $X$ by $[p]_n = \{x \in X \mid 1/n < p[x]\}$. Then each $[p]_n$ has finite (at most $(n-1)\cdot|n|$) members and so $[p] = \cup_{n>0} [p]_n$ is countable.)

We define three types of maps in $Q(X)$ which are used in this paper.
Definition 1. Three subsets of $Q(X)$ are defined as follows:

(a) $p \in Q_0(X) \iff p \in Q(X) \land ||p|| \leq 1$,
(b) $p \in Q_1(X) \iff p \in Q(X) \land ||p|| = 1$,
(c) $p \in Q_\ast(X) \iff p \in Q_0(X) \land [p]$ : finite subset of $X$.

Note that $0_X \in Q_\ast(X)$, $\hat{x} \in Q_\ast(X) \cap Q_1(X)$ and $p \ast q \in Q_\ast(X)$ for all $x \in X$, $p$, $q \in Q_\ast(X)$. Elements of $Q_\ast(X)$ are probabilistic sub-distributions, and those of $Q_\ast(X) \cap Q_1(X)$ are probabilistic distributions. Essentially, McIver et al. [4, 7, 8] have studied either the case of $Q_\ast(X)$ or the case of finite set $X$ in order to develop models of probabilistic systems.

The restriction of the point-wise order $\leq : Q(X) \to Q(X)$ onto $Q_\tau(X)$ is denoted by $\xi_\tau : Q_\tau(X) \to Q_\tau(X)$, that is,

$$\forall p, q \in Q_\tau(X). \ (p, q) \in \xi_\tau \iff p \leq q,$$

where the subscript/superscript $\tau$ is one of $0$, $1$, and $\ast$. Remark that the restricted order $\xi_\tau$ on $Q_1(X)$ is discrete, that is, for $\xi_\tau = id_{Q_1(X)}$. Thus the order on $Q_1(X)$ will not be used in the rest of the paper.

For $\tau \in \{0, \ast\}$ every map $t \in Q(X)$ yields a map $t_\tau : Q_\tau(X) \to Q_\tau(X)$ by

$$\forall p \in Q_\tau(X). \ p t_\tau = p \ast t.$$

Corollary 1. $(\xi_\tau)^t = \sqcup_{t \in Q(X)} t_\tau$, for $\tau \in \{0, \ast\}$.

Proof.

\[
(p, q) \in (\xi_\tau)^t \iff \qquad q \leq p \\
\iff \exists t \in Q(X), \ q = p \ast t = p t_\tau \quad \{ \text{Prop.3} \} \\
\iff \exists t \in Q(X), \ (p, q) \in t_\tau \\
\iff (p, q) \in \sqcup_{t \in Q(X)} t_\tau. \quad \square
\]

A map $e_X : X \to Q_\tau(X)$ is defined by $xe_X = \hat{x}$ for each $x \in X$, where $\tau \in \{0, 1, \ast\}$. Also, for $\tau \in \{0, \ast\}$

we define a relation $\varepsilon_\tau : X \to Q_\tau(X)$ by $\varepsilon_\tau = e_X(\xi_\tau)^t$. As discussed in detail later, $e_X$ and $\varepsilon_\tau$ are the
units of convex composition over certain convex relations, respectively.

3. Convex combinations

Extending finite bounded sums

$$\bigoplus_{j=1}^n q_j = q_1 \oplus q_2 \oplus \cdots \oplus q_n$$

of maps $q_1, q_2, \ldots, q_n \in Q(X)$, we will define the bounded sum of an arbitrary set of maps in $Q(X)$. For a set $\{q_j \mid j \in J\}$ of maps in $Q(Y)$ define a map $\bigoplus_{j \in J} q_j$ in $Q(Y)$ by

$$\forall y \in Y. \ (\bigoplus_{j \in J} q_j)[y] = \begin{cases} \sum_{j \in J}(q_j)[y] & \text{if } \sum_{j \in J}(q_j)[y] \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Of course, we mean $(\bigoplus_{j \in J} q_j)[y] = 1$ even if the sum $\sum_{j \in J}(q_j)[y]$ diverges.

The support of bounded sums of a set of maps in $Q(X)$ is given by the union of supports of their maps contained in the set.

Proposition 4. For all subsets $\{q_j \mid j \in J\} \subseteq Q(X)$ the following holds:

$$\left( \bigoplus_{j \in J} q_j \right) = \bigcup_{j \in J}[q_j].$$
Proof. Let $p \notin \bigoplus_{j \in J} q_j$, \quad \iff \quad (\bigoplus_{j \in J} q_j)_{[y]} = 0
\iff \quad \sum_{j \in J} (q_j)_{[y]} = 0
\iff \quad \forall j \in J, (q_j)_{[y]} = 0
\iff \quad \forall j \in J, y \notin [q_j]
\iff \quad y \notin \bigcup_{j \in J} [q_j]. \quad \square$

Let $f : X \to \mathbb{Q}(Y)$ be a map. Define a map $f_o : \mathbb{Q}(X) \to \mathbb{Q}(Y)$ by

$$p f_o = \bigoplus_{x \in X} p_{[x]} (xf),$$

where $p \in \mathbb{Q}(X)$. The map $p f_o$ is called a convex combination of $p$ and $f$. We need this notion to raise the composition of convex relations.

Example 1. Set $X = \{x, y\}$ and define maps $f$, $f'$, $g$, $h : X \to \mathbb{Q}(X)$ by $xf = yf = \hat{x}$, $xf' = yf' = \hat{y}$, $xg = \hat{x}$, $yg = 0_X$, $xh = \hat{y}$ and $yh = 0_X$. We have $f_o$, $f'_o$, $g_o$ and $h_o$ such that

$$p f_o = p_{[x]} \hat{x} \oplus p_{[y]} \hat{x}, \quad p f'_o = p_{[x]} \hat{y} \oplus p_{[y]} \hat{y}, \quad p g_o = p_{[x]} \hat{x}, \quad p h_o = p_{[x]} \hat{y},$$

for all $p \in \mathbb{Q}(X)$. Especially for $p' \in Q_1(\mathcal{X})$, $p' f_o = \hat{x}$ and $p' f'_o = \hat{y}$ hold. \quad \square

The basic properties of convex combinations are listed below.

Proposition 5. Let $k \in [0, 1]$, $p \in \mathbb{Q}_0(X)$ and $f : X \to \mathbb{Q}(Y)$. Then the followings hold:

(a) $(p f_o)_{[y]} = \sum_{x \in X} p_{[x]} (xf)_{[y]}$,  
(b) $\|p f_o\| = \sum_{x \in X} p_{[x]} \|xf\|$,  
(c) $[p f_o] = \bigcup_{x \in [p]} [xf]$,  
(d) $0_X f_o = 0_Y$,  
(e) $xf_o = xf$,  
(f) $p(e_X) o = p$,  
(g) $p(\nabla f Y) o = (k\|p\|) Y$,  
(h) $k \cdot (p f_o) = (k \cdot p) f_o$.

Proof. (a) $(p f_o)_{[y]} = \sum_{x \in X} p_{[x]} (xf)_{[y]}$:

$$\sum_{x \in X} p_{[x]} (xf)_{[y]} \leq \sum_{x \in X} p_{[x]} \{ x \in \mathbb{Q}(Y) \} \{ p \in \mathbb{Q}_0(X) \} \leq 1.$$

$$(p f_o)_{[y]} = \min \{ \sum_{x \in X} p_{[x]} (xf)_{[y]}, 1 \} = \sum_{x \in X} p_{[x]} (xf)_{[y]}.$$

(b) $\|p f_o\| = \sum_{x \in X} p_{[x]} \|xf\|$:

$\|p f_o\| = \sum_{y \in Y} (p f_o)_{[y]} = \sum_{y \in Y} \sum_{x \in X} p_{[x]} (xf)_{[y]} \{ (a) \} = \sum_{x \in X} p_{[x]} \|xf\| = \sum_{x \in X} p_{[x]} \|xf\|.$

(c) $[p f_o] = \bigcup_{x \in [p]} [xf]$:

$\quad [p f_o] = \bigcup_{x \in X} [p_{[x]} (xf)] = \bigcup_{x \in [p]} [xf].$
Proposition 6. For $\tau \in \{0,1,*\}$ the following hold:

(a) If $p \in \mathcal{Q}_\tau(X)$ and $f : X \to \mathcal{Q}_\tau(Y)$, then $p f_o \in \mathcal{Q}_\tau(Y)$.

(b) If $p \in \mathcal{Q}_\tau(X)$ and $f : X \to \mathcal{Q}_\tau(Y)$, then there exist $p' \in \mathcal{Q}_\tau(N)$ and $f' : N \to \mathcal{Q}_\tau(Y)$ such that $p f_o = p' f'_o$.

Proof. (a0) $[p \in \mathcal{Q}_0(X) \land f : X \to \mathcal{Q}_0(X)] \to p f_o \in \mathcal{Q}_0(Y)$:

$$||p f_o|| = \sum_{x \in X} P_{[x]} ||x f|| \quad \{ \text{Prop.5 (b)} \}$$
$$\leq \sum_{x \in X} P_{[x]} \quad \{ x f \in \mathcal{Q}_0(Y) \}$$
$$= ||p|| \quad \{ p \in \mathcal{Q}_0(X) \}$$

(a1) $[p \in \mathcal{Q}_1(X) \land f : X \to \mathcal{Q}_1(X)] \to p f_o \in \mathcal{Q}_1(Y)$:

$$||p f_o|| = \sum_{x \in [p]} P_{[x]} ||x f|| \quad \{ \text{Prop.5 (b)} \}$$
$$= \sum_{x \in [p]} P_{[x]} \quad \{ x f \in \mathcal{Q}_1(Y) \}$$
$$= 1. \quad \{ p \in \mathcal{Q}_1(X) \}$$

(a*) $[p \in \mathcal{Q}_*(X) \land f : X \to \mathcal{Q}_*(X)] \to p f_o \in \mathcal{Q}_*(Y)$ is immediate from (a0) and Prop.5(c).

(b0) $\forall p \in \mathcal{Q}_0(X) \forall f : X \to \mathcal{Q}_0(Y) \exists p' \in \mathcal{Q}_0(N) \exists f' : N \to \mathcal{Q}_0(Y), p f_o = p' f'_o$.

As already stated the support $|p|$ is countable if $||p||$ exists and so there is an injection $i : |p| \to N$. Define a map $p' \in \mathcal{Q}(N)$ and a map $f' : N \to \mathcal{Q}(Y)$ by

$$p'_|[x] = \begin{cases} p|[x] & \text{if } \exists x \in |p|, n = xi \\ 0 & \text{otherwise} \end{cases}$$
and

\[ n f' = \begin{cases} \ x f & \text{if } \exists x \in [p] \ n = xi \\ 0 & \text{otherwise,} \end{cases} \]

respectively. Remark that \( n \in \{p'\} \) if and only if there exists \( x \in [p] \) such that \( n = xi \). Hence

\[ p f_\circ = \bigoplus_{x \in X} P_{[x]}(x f) \]
\[ = \bigoplus_{n \in \mathbb{N}} P_{[n]}(n f') \]
\[ = \ p' f'_\circ. \]

(b) In the case of \( \tau = * \):
Let \( p \in \mathcal{Q}_* (X) \) and \( f : X \to \mathcal{Q}_* (Y) \), and take the same \( p' \) and \( f' \) defined in (b_0). Then it is clear that \( p' \in \mathcal{Q}_* (\mathbb{N}) \) and \( f' : \mathbb{N} \to \mathcal{Q}_* (Y) \).

(b_1) In the case of \( \tau = 1 \):
Let \( p \in \mathcal{Q}_1 (X) \) and \( f : X \to \mathcal{Q}_1 (Y) \), and take the same \( p' \) defined in (b_0) and define \( f' : X \to \mathcal{Q}_1 (Y) \) by

\[ n f' = \begin{cases} \ x f & \text{if } \exists x \in [p] \ n = xi \\ \ y_0 & \text{otherwise,} \end{cases} \]

where \( y_0 \) is an arbitrary point of \( Y \). Then it is clear that \( p' \in \mathcal{Q}_1 (\mathbb{N}) \) and \( f' : \mathbb{N} \to \mathcal{Q}_1 (Y) \).

\[ \square \]

4. Convex composition

In the rest of the paper the subscript \( \tau \) is one of 0, 1 and *, unless otherwise stated. For a map \( f : X \to \mathcal{Q}_\tau (Y) \) the convex combination induces a map \( f_\circ : \mathcal{Q}_\tau (X) \to \mathcal{Q}_\tau (Y) \) by Proposition 6(a). We now list some basic properties of the induced maps.

**Proposition 7.** Let \( f : X \to \mathcal{Q}_\tau (Y) \), \( g : Y \to \mathcal{Q}_\tau (Z) \) and \( h : X \to \mathcal{Q}_\tau (X) \) be maps. Then the following hold:

(a) \( f_\circ g_\circ = (f g_\circ)_\circ \),
(b) \( e_X f_\circ = f \),
(c) \( (e_X)_\circ = \text{id}_{\mathcal{Q}_\tau (X)} \),
(d) \( h \subseteq f (\xi_X^\tau) \to h_\circ \subseteq f_\circ (\xi_X^\tau) \) for \( \tau \neq 1 \),
(e) \( (\xi_X^\tau)^2 f_\circ \subseteq f_\circ (\xi_X^\tau)^2 \) for \( \tau \neq 1 \),
(f) \( (\nabla X 10_Y)_\circ = \nabla_{\mathcal{Q}_\tau (X)10_Y} \) for \( \tau \neq 1 \).

\( \xi_X^\tau \) and \( h_\circ (\xi_X^\tau)^2 \) for \( \tau \neq 1 \).

Proof. (a) \( f_\circ g_\circ = (f g_\circ)_\circ : \)

\[ p(f g_\circ)_\circ = \bigoplus_{\lambda} p_{[\lambda]}(x f g_\circ) \]
\[ = \bigoplus_{\lambda} (p_{[\lambda]}(x f)) g_\circ \quad \{ \text{Prop.5(h)} \} \]
\[ = \bigoplus_{\lambda} \bigoplus_{\mu} (p_{[\lambda]}(x f))_{[\mu]} (y g) \]
\[ = \bigoplus_{\lambda} \bigoplus_{\mu} \bigoplus_{\nu} (p_{[\lambda]}(x f))_{[\mu]} (y g) \]
\[ = \bigoplus_{\lambda} \bigoplus_{\nu} (\sum_{\mu} p_{[\lambda]}(x f))_{[\nu]} (y g) \quad \{ \text{Prop.5(a)} \} \]
\[ = \bigoplus_{\lambda} (\sum_{\nu} p_{[\lambda]}(x f))_{[\nu]} (y g) \]
\[ = (p f_\circ)_{[\nu]} (y g) \]
\[ = p(f_\circ g_\circ)_{[\nu]} (y g) \]
\[ = p(f_\circ g_\circ). \]

(b) \( e_X f_\circ = f : \)

\[ xe_X f_\circ = \ x f_\circ \quad \{ xe_X = x \} \]
\[ = x f. \quad \{ \text{Prop.5(e)} \} \]
(e) \((e_X)_c = \text{id}_{Q_c(X)}\) is direct from Proposition 5 (f).

(d) \(h \subseteq f(\xi_X)^2 \rightarrow h_\circ \subseteq f_\circ (\xi_X)^2\):
For \(p \in Q_\tau(X)\) and \(x \in X\) we have

\[
(p_{h_\circ})_{[x]} = \sum_{x'} p_{[x']}(x'h)_{[x]} \leq \sum_{x'} p_{[x']}(x'f)_{[x]} \quad \{ x'h \leq x'f \leftarrow h \subseteq f(\xi_X)^2 \}
\]

which proves \(p_{h_\circ} \leq p_{f_\circ}\) and hence \(p_{h_\circ} \subseteq p_{f_\circ}(\xi_X)^2\).

(e) \((\xi_X)^2 f_\circ \subseteq f_\circ (\xi_X)^2\):

\[
\rightarrow \quad p \leq p' \rightarrow p_{f_\circ} \leq p'_{f_\circ} \\
\leftrightarrow \quad \xi_X \subseteq f_\circ \xi_X^2
\]

\[
\leftrightarrow \quad (\xi_X)^2 \subseteq f_\circ (\xi_X)^2
\]

\[
\leftrightarrow \quad (\xi_X)^2 h_\circ \subseteq f_\circ (\xi_X)^2
\quad \{ f_\circ : \text{tfn} \}
\]

(f) \((\nabla_X Y)_\circ = \nabla_{Q_\tau(X)} Y_\circ\):

\[
(\nabla_X Y)_\circ = \bigcup_{p \in Q_\tau(Y)} p(\nabla_X Y)_\circ \quad \{ \text{PA} \}
\]

\[
= \bigcup_{p \in Q_\tau(Y)} p^{||p||} 0_Y \quad \{ \text{Prop.5 (g)} \}
\]

\[
= \bigcup_{p \in Q_\tau(Y)} p^{||p||} 0_Y \quad \{ ||p|| = 0 \}
\]

\[
= \nabla_{Q_\tau(Y)} Y_\circ \quad \{ \text{PA} \}
\]

(g) \((\xi_X)^2 h_\circ(\xi_X)^2 = h_\circ(\xi_X)^4\):

\[
(\xi_X)^2 h_\circ(\xi_X)^2 \quad \subseteq \quad h_\circ(\xi_X)^2(\xi_X)^2 \quad \{ \text{e} \}
\]

\[
= \quad h_\circ(\xi_X)^2 \quad \{ (\xi_X)^2(\xi_X)^2 = (\xi_X)^2 \}
\]

\[
= \quad h_\circ(\xi_X)^2
\]

\[
\subseteq \quad (\xi_X)^2 h_\circ(\xi_X)^2.
\]

\[
\Box
\]

For a relation \(\alpha : X \rightarrow Q_\tau(Y)\) define a relation \(\alpha_\circ : Q_\tau(X) \rightarrow Q_\tau(Y)\) by

\[
\alpha_\circ = \bigcup_{f \subseteq \alpha} f_\circ,
\]

where \(f \subseteq \alpha\) means that \(f\) is a map \(f : X \rightarrow Q_\tau(Y)\) such that \(f \subseteq \alpha\). This notion allows convex composition to be treated in ordinary relational calculus.

**Remark.** By the relational axiom of choice (AC) there exists a map \(f \subseteq \alpha\) if \(\alpha\) is total. Such a map \(f\) is often called a choice function of \(\alpha\). Also \(\alpha_\circ\) is total if \(\alpha\) is total, and \(\alpha_\circ = \bigcup_{f \subseteq \alpha} f_\circ\) otherwise.

**Example 2.** Consider relations \(\gamma = g \cup h\) and \(\gamma' = h \cup e_X\) where \(g, h : X \rightarrow Q_\tau(X)\) appeared in Example 1. Since there is no maps included in \(\gamma = g \cup h\) other than \(g\) and \(h\), the identity \(\gamma_\circ = g_\circ \cup h_\circ\) holds.

For a relation \(\gamma'\), the identity \(\gamma'_\circ = h_\circ \cup e_{X_\circ}\) does not hold. Because \(\gamma'\) consists of four maps, that is \(h \cup e_X = f' \cup g \cup h \cup e_X\) where \(f', g : X \rightarrow Q_\tau(X)\) are maps defined by \(x f' = y f' = y, x g = x, y g = 0_X\). Therefore \(\gamma'_\circ = f'_\circ \cup g_\circ \cup h_\circ \cup e_{X_\circ}\) holds. \(\Box\)

**Proposition 8.** If \(\alpha : X \rightarrow Y\) is a total relation, then \(\alpha = \bigcup_{f \subseteq \alpha} f\).

Proof. Assume \(\alpha\) is total. By the axiom of choice (AC) there is a function \(f_0 : X \rightarrow Y\) such that \(f_0 \subseteq \alpha\). For each \((x_0, y_0) \in \alpha\) define a map \(f : X \rightarrow Y\) by

\[
\forall x \in X. \quad xf = \begin{cases} 
  y_0 & \text{if } x = x_0, \\
  f_0 & \text{otherwise}.
\end{cases}
\]

\[8\]
Then it is clear that \((x,q) \in f\) and so
\[
f = \bigcup_{x \in X} x^2xf \quad \{ \text{(PA)} \} \\
= x^4y_0 \bigcup (\bigcup_{x \neq x_0} x^2xf_0) \\
\subseteq \alpha \bigcup f_0 \quad \{ x^2x \subseteq \text{id}_X \} \\
= \alpha. \quad \{ f_0 \subseteq \alpha \}
\]
Hence
\[
(x_0, y_0) \in \alpha \rightarrow \exists f. (x_0, y_0) \in f \land (f \subseteq \alpha) \rightarrow (x_0, y_0) \in \bigcup_{\subseteq \alpha} f,
\]
which shows \(\alpha = \bigcup_{\subseteq \alpha} f\). \hfill \square

A relation \(\alpha : X \rightarrow Q_\tau(Y)\) is called down-closed if it satisfies \(\alpha(\xi^\tau_Y) = \alpha\). The next proposition indicates that a relation \(\alpha\) is total and down-closed if it is 0-included \([16]\), namely, \(0_Y \in x\alpha\) for each \(x \in X\).

**Proposition 9.** Let \(\tau \neq 1\). If \(\alpha : X \rightarrow Q_\tau(Y)\) is a total relation such that \(\alpha(\xi^\tau_Y) = \alpha\), then \(\nabla_X f 0_Y \subseteq \alpha\) (0-included).

Proof. Assume \(\alpha\) is total and \(\alpha(\xi^\tau_Y) = \alpha\). By the axiom of choice (AC) there is a function \(f_0 : X \rightarrow Q_\tau(Y)\) such that \(f_0 \subseteq \alpha\).

\[
\nabla_X f 0_Y = \bigcup_{x \in X} x^2 f 0_Y \quad \{ \text{(PA)} \} \\
\subseteq \bigcup_{x \in X} x^2 f_0 (\xi^\tau_Y)^2 \quad \forall q \in Q_\tau(Y). 0_Y \subseteq q(\xi^\tau_Y)^2 \} \\
= f_0 (\xi^\tau_Y)^2 \quad \{ \text{(PA)} \} \\
\subseteq \alpha (\xi^\tau_Y)^2 \quad \{ f_0 \subseteq \alpha \} \\
= \alpha. \quad \{ \alpha(\xi^\tau_Y)^2 = \alpha \}
\]

The diamond operator defined via convex combinations satisfies the following additional rules.

**Proposition 10.** Let \(\tau \neq 1\). For a map \(f : X \rightarrow Q_\tau(Y)\), a relation \(\alpha : X \rightarrow Q_\tau(Y)\) and \(t \in Q(Y)\) the following hold:

(a) \(f_t \cdot t = (ft)_o\),
(b) \(\alpha \cdot t_s \subseteq (\alpha t_s)_o\),
(c) \(\alpha (\xi^\tau_Y) \subseteq (\alpha(\xi^\tau_Y))_o\),
(d) \((f(\xi^\tau_X))_o = f_0(\xi^\tau_X)^t\).

Proof. (a) \(f_t \cdot t = (ft)_o\):
For \(p \in Q_\tau(X)\) we have
\[
p(f_t \cdot t) = (pf_t)_t \\
= (pf_t) \cdot t \\
= (\bigoplus_{x \in X} p[\cdot] \cdot (xf)) \cdot t \\
= \bigoplus_{x \in X} p[\cdot] \cdot (xf) \cdot t \\
= \bigoplus_{x \in X} p[\cdot] \cdot ((xf) \cdot t) \\
= \bigoplus_{x \in X} p[\cdot] \cdot (xf t_s) \\
= p(f_t)_o.
\]

(b) \(\alpha \cdot t_s \subseteq (\alpha t_s)_o\):
\[
\alpha \cdot t_s = (\bigcup_{\subseteq \alpha f_0} f_0) t_s \\
= \bigcup_{\subseteq \alpha f_0} f_0 t_s \\
= \bigcup_{\subseteq \alpha f_0} (ft)_o \quad \{ \text{(a)} \} \\
\subseteq \bigcup_{\subseteq \alpha f_0} (ft)_o \\
= (\alpha t_s)_o.
\]

9
(c) \( \alpha_\circ (\xi_Y^\sharp)^2 \subseteq (\alpha(\xi_Y^\sharp))^2 \) :

\[
\begin{align*}
\alpha_\circ (\xi_Y^\sharp)^2 &= \alpha_\circ (\cup_{t \in Q(Y)} t) \\
&= \cup_{t \in Q(Y)} \alpha_\circ t \\
&\subseteq \cup_{t \in Q(Y)} (\alpha_\circ t) \\
&\subseteq (\alpha(\xi_Y^\sharp)^2)_\circ. 
\end{align*}
\]

(d) \( (f(\xi_X^\sharp)^2)_\circ = f_\circ (\xi_X^\sharp)^2 : 

\[
(f(\xi_X^\sharp)^2)_\circ = \cup_{h \subseteq f(\xi_X^\sharp)} h_\circ \\
\subseteq f_\circ (\xi_X^\sharp)^2. \quad \{ \text{Prop.7 (d)} \}
\]

The opposite direction \( f_\circ (\xi_X^\sharp)^2 \subseteq (f(\xi_X^\sharp)^2)_\circ \) follows from (c). \hfill \Box

Now we will define a composition \([4, 7]\) for relations into algebras of probabilistic distributions.

**Definition 2.** Let \( : X \rightarrow Q_r(Y) \) and \( : Y \rightarrow Q_r(Z) \) be relations. The convex composite \( \circ : X \rightarrow Q_r(Z) \) of \( \circ \) followed by \( \circ \) is defined by

\[
\alpha \circ \beta = \alpha \beta_\circ.
\]

**Remark.** In some aspects, convex composition seems to be concrete examples of Kleisli composition of the powerset monads studied by Eklund and Gähler [2]. However, in our case the composition chooses a map from latter argument in nondeterministic way, whereas Kleisli composition chooses in deterministic way.

We show the basic properties on convex composition of relations.

**Proposition 11.** Let \( \alpha, \alpha' : X \rightarrow Q_r(Y), \beta, \beta' : Y \rightarrow Q_r(Z) \) and \( \gamma : Z \rightarrow Q_r(W) \) be relations. Then

\begin{enumerate}[(a)]
\item \( \beta \subseteq \beta' \rightarrow \beta_\circ \subseteq \beta'_\circ \),
\item \( \alpha \subseteq \alpha' \wedge \beta \subseteq \beta' \rightarrow \alpha \circ \beta \subseteq \alpha' \circ \beta' \),
\item \( \alpha_\circ \beta_\circ \subseteq (\alpha \circ \beta)_\circ \),
\item \( (\alpha \circ \beta) \circ \gamma \subseteq \alpha \circ (\beta \circ \gamma) \),
\item \( \alpha : \text{total} \rightarrow \alpha_\circ \),
\item \( \alpha : \text{total} \rightarrow \alpha_\circ \nabla Y_{I} 0_{Z} = \nabla X_{I} 0_{Z} \quad \text{for } \tau \neq 1 \),
\item \( \alpha : \text{total} \rightarrow 0_{X} \circ \alpha = 0_{Y} \quad \text{for } \tau \neq 1 \),
\item \( \alpha_\circ (\xi_X^\sharp)^2 \subseteq \alpha \circ (\xi_X^\sharp)^2 \quad \text{for } \tau \neq 1 \). \hfill \end{enumerate}

**Proof.** (a) \( \beta \subseteq \beta' \rightarrow \beta_\circ \subseteq \beta'_\circ \) :

Assume \( \beta \subseteq \beta' \). Then

\[
\beta_\circ = \cup_{g \subseteq \beta_\circ} g_\circ \\
\subseteq \cup_{g \subseteq \beta'_\circ} g_\circ \\
= \beta'_\circ.
\]

(b) \( \alpha \subseteq \alpha' \wedge \beta \subseteq \beta' \rightarrow \alpha \circ \beta \subseteq \alpha' \circ \beta' \) :

Assume \( \alpha \subseteq \alpha' \) and \( \beta \subseteq \beta' \). Then

\[
\alpha \circ \beta = \alpha \beta_\circ \\
\subseteq \alpha' \beta'_\circ \quad \{ \alpha \subseteq \alpha', \beta \subseteq \beta', (a) \}
\]

(c) \( \alpha_\circ \beta_\circ \subseteq (\alpha \circ \beta)_\circ \) :

Note that for maps \( f : X \rightarrow Q_r(Y) \) and \( g : Y \rightarrow Q_r(Z) \) such that \( f \subseteq \alpha \) and \( g \subseteq \beta \), we have

\[
(f_\circ g_\circ) \subseteq (f_\circ g_\circ) \subseteq (f_\circ g_\circ) \subseteq (f_\circ g_\circ). 
\]
Hence

\[ \alpha \circ \beta_\circ = (\sqcup_{f \subseteq \alpha} f_\circ) (\sqcup_{g \subseteq \beta} g_\circ) \]
\[ \sqsubseteq (\alpha \circ \beta)_\circ. \]

(d) \((\alpha \circ \beta) \circ \gamma \subseteq \alpha \circ (\beta \circ \gamma) :\)

\[ (\alpha \circ \beta) \circ \gamma = (\alpha \circ \beta_\circ) \gamma_\circ \]
\[ = \alpha (\beta_\circ \gamma_\circ) \]
\[ \sqsubseteq \alpha (\beta \circ \gamma)_\circ. \]

(e) \(\alpha : \text{total} \rightarrow e_X \alpha_\circ = \alpha :\)

\[ e_X \alpha_\circ = e_X (\sqcup_{f \subseteq \alpha} f_\circ) \]
\[ = \sqcup_{f \subseteq \alpha} e_X f_\circ \]
\[ = \sqcup_{f \subseteq \alpha} f. \] \(\text{Prop.7 (b)}\)
\[ = \alpha. \] \(\alpha : \text{total}\)

(f) \(\alpha : \text{total} \rightarrow \alpha \circ \nabla_Y 0_Z = \nabla_X 0_Z :\)

\[ \alpha \circ \nabla_Y 0_Z = \alpha (\nabla_Y 0_Z)_\circ \]
\[ = \alpha \nabla_{\nabla_\circ} (Y)_\circ \]
\[ = \nabla_X 0_Z. \] \(\alpha : \text{total}\)

(g) \(\alpha : \text{total} \rightarrow 0_X \circ \alpha = 0_Y :\)

\[ 0_X \circ \alpha = 0_X (\sqcup_{f \subseteq \alpha} f_\circ) \]
\[ = \sqcup_{f \subseteq \alpha} 0_X f_\circ \]
\[ = 0_Y. \] \(\text{Prop.5 (d)}\)

(h) \((\alpha \circ \beta)(\xi_Y)^2 \subseteq \alpha \circ \beta(\xi_Y)^2 :\)

\[ (\alpha \circ \beta)(\xi_Y)^2 = (\alpha \circ \beta_\circ)(\xi_Y)^2 \]
\[ = \alpha (\beta_\circ)(\xi_Y)^2 \]
\[ \sqsubseteq \alpha (\beta)(\xi_Y)^2. \] \(\text{Prop.10 (c)}\)
\[ = \alpha \circ (\beta)(\xi_Y)^2. \]

\(\square\)

By Proposition 7(c) and Proposition 11(e), if \(\alpha\) is total then \(e_X\) is neutral for convex composition, that is, \(\alpha \circ e_X = e_X \circ \alpha = \alpha\).

The following proposition shows that \(e_X\) is identity element for convex composition in the case of \(\tau \neq 1\) and \(\alpha(\xi_Y)^2 \subseteq \alpha \) (down-closed).

**Proposition 12.** Let \(\alpha : X \rightarrow Q_\tau (Y)\) be a total relation for \(\tau \neq 1\). Then the following holds:

(a) \(\alpha \subseteq e_X^c \circ \alpha \subseteq \alpha(\xi_Y)^2;\)

(b) \(\alpha \subseteq \alpha \circ e_Y^c \subseteq \alpha(\xi_Y)^2;\)

Proof. (a) \(\alpha \subseteq e_X^c \circ \alpha \subseteq \alpha(\xi_Y)^2 :\)

\[ \alpha = e_X \alpha_\circ \]
\[ \subseteq e_X (\xi_Y)^2 \alpha_\circ \]
\[ = e_X (\xi_Y)^2 (\sqcup_{f \subseteq \alpha} f_\circ) \]
\[ \subseteq e_X (\sqcup_{f \subseteq \alpha} f_\circ)(\xi_Y)^2 \]
\[ = e_X \alpha_\circ (\xi_Y)^2 \]
\[ = \alpha(\xi_Y)^2. \] \(\text{Prop.7 (c)}\)

(b) \(\alpha \subseteq \alpha \circ e_Y^c \subseteq \alpha(\xi_Y)^2 :\)

\[ \alpha = \alpha(e_Y)_\circ \]
\[ \subseteq \alpha(e_Y^c)_\circ \]
\[ \subseteq \alpha(\xi_Y^c)_\circ \]
\[ \subseteq \alpha(\xi_Y)^2. \] \(\text{Prop.7 (c)}\)

\(\square\)
5. Associative law

In this section we will introduce the convex relations and study the associative law of convex composition on their relations. For a relation $\gamma : Z \to Q_\tau(W)$ define a relation $\gamma^* : Z \to Q_\tau(W)$ by

$$\forall z \in Z. \ z\gamma^* = \nabla_{I\subset Q_\tau(n)} \circ \nabla_{N\parallel z}\gamma.$$  

Note that $\rho^* = \nabla_{I\subset Q_\tau(n)} \circ \nabla_{N\parallel \rho}$ for a relation $\rho : I \to Q_\tau(W)$.

**Remark.** The definition of $\gamma^*$ explicitly contains an element (or a variable) beyond preferable relational expressions.

The notion of $\gamma^*$ derives a property called convex for relations to satisfy the associativity of convex composition. A relation $\gamma : Z \to Q_\tau(W)$ is called convex if it satisfies $\gamma^* = \gamma$.

**Example 3.** Consider on the same expressions. Define a relation $\alpha : X \to Q_1(X)$ by $x\alpha = y\alpha = (\frac{1}{2})x$. Then $\alpha^* : X \to Q_1(X)$ satisfies $\alpha^* = \alpha$. However when we regard $\alpha$ as $\alpha : X \to Q_\tau(X)$, $\alpha^* : X \to Q_\tau(X)$ satisfies $x\alpha^* = y\alpha^* = (\frac{1}{2})X(\xi^\tau_X)^3$, that is $\alpha^* \neq \alpha$. For a relation $\gamma : X \to Q_\tau(X)$ which appeared in Example 2, we obtain that $\gamma^* \neq \gamma$ since $x\gamma^* = Q_\tau(X)$ though $x\gamma = \hat{x} \sqcup \hat{y}$.

**Proposition 13.** Let $\gamma : Z \to Q_\tau(W)$ be a relation.

(a) If $\gamma$ is total, then $\gamma \subseteq \gamma^*$.

(b) $\nabla_{I\subset Q_\tau(Y)} \circ \nabla_{Y\parallel z}\gamma \subseteq z\gamma^*$ for all sets $Y$.

(c) $\gamma^{**} \subseteq \gamma^*$.

(d) $\gamma^*(\xi^\tau_W)^\sharp \subseteq (\gamma(\xi^\tau_W)^\sharp)^\star$ ($\tau \neq 1$).

Proof. Set $\rho = z\gamma$. Then $\rho^* = z\gamma^*$, $\rho^{**} = z\gamma^*$ and $(\rho(\xi^\tau_W)^\sharp)^\star = z(\gamma(\xi^\tau_W)^\sharp)^\star$. Thus it suffices to show the following statements for $\rho$.

(a) $\gamma : \text{total} \to \rho \subseteq \rho^* :$

\[
\begin{align*}
\rho &= \nabla_{I\subset N\parallel N\parallel I\parallel \rho} \quad \{ N \neq \emptyset \} \\
&= \nabla_{I\subset N\parallel N\parallel I\parallel \rho} \quad \{ \text{Prop.11 (e), } \rho : \text{total} \} \\
&\subseteq \nabla_{I\subset Q_\tau(n)} \circ \nabla_{N\parallel I\parallel \rho} \\
&\subseteq \nabla_{I\subset Q_\tau(n)} \circ \nabla_{N\parallel I\parallel \rho}.
\end{align*}
\]

(b) $\nabla_{I\subset Q_\tau(Y)} \circ \nabla_{Y\parallel I\parallel \rho} \subseteq \nabla_{I\subset Q_\tau(n)} \circ \nabla_{N\parallel I\parallel \rho}$:

\[
\begin{align*}
\nabla_{I\subset Q_\tau(Y)} \circ \nabla_{Y\parallel I\parallel \rho} &= (\cup_{p\in Q_\tau(Y)} p(\cup_{f\subseteq \nabla_{Y\parallel I\parallel \rho}} f_\circ)) \\
&\subseteq (\cup_{p\in Q_\tau(Y)} p(\cup_{f\subseteq \nabla_{Y\parallel I\parallel \rho}} f_\circ)) \\
&\subseteq (\cup_{p\in Q_\tau(Y)} p(\cup_{f\subseteq \nabla_{Y\parallel I\parallel \rho}} f_\circ)) \\
&= \nabla_{I\subset Q_\tau(n)} \circ \nabla_{N\parallel I\parallel \rho}.
\end{align*}
\]

(c) Recall that

\[
\begin{align*}
q' \subseteq p^{**} &\implies \exists p' \in Q_\tau(N) \exists f' : N \to Q_\tau(X) \\
q' &= p'f_\circ' \land \forall n \in N. n f' \subseteq p^* \\
nf' \subseteq p^* &\implies \exists n_f \in Q_\tau(N) \exists f_n : N \to Q_\tau(X) \\
nf' &= p_n f_n \land \forall m \in N. m f_n \subseteq p,
\end{align*}
\]

and so

\[
\begin{align*}
q' &= p'f_\circ' \\
&= \bigoplus_{n \in N} p'_n(n f') \\
&= \bigoplus_{n \in N} p'_n(p_n f_n) \\
&= \bigoplus_{n \in N} p'_n[p_n f_n(m f_n)] \\
&= \bigoplus_{n \in N} \bigoplus_{m \in N} p'_n[p_n f_n(m f_n)].
\end{align*}
\]
Define \( \tilde{p} \in Q_{\tau}(N \times N) \) and \( \tilde{f} : N \times N \to Q_{\tau}(X) \) by \((n,m)\tilde{p} = p_{[n]}p_{[m]}\) and \((n,m)\tilde{f} = mf_{n} \), respectively. Then

\[
q' = \tilde{p}\tilde{f} \leq \nabla_{I Q_{\tau}(N \times N)}(\nabla_{N \times N} p) \leq \nabla_{I Q_{\tau}(N)}(\nabla_{N} p) \quad \{ \text{Prop. 10 (c)} \}
\]

This proves \( p^{\bullet \bullet} \leq p^{\bullet} \).

(d) \( \gamma^*(\xi_{W}^\sharp)^{\sharp} \leq (\gamma(\xi_{W}^\sharp)^{\sharp})^{\bullet} \): \( (\tau \neq 1) \)

\[
\begin{align*}
p^{\bullet}(\xi_{W}^\sharp)^{\sharp} & = \nabla_{I Q_{\tau}(N)}(\nabla_{N} p)(\xi_{W}^\sharp)^{\sharp} \subseteq \nabla_{I Q_{\tau}(N)}(\nabla_{N} p)(\xi_{W}^\sharp)^{\sharp} \quad \{ \text{Prop. 10 (c)} \} \\
& = (\rho(\xi_{W}^\sharp)^{\sharp})^{\bullet}.
\end{align*}
\]

Now we define two kinds of convex relations, named \( Q_{\tau} \)-convex relation and \( Q_{1} \)-convex relation.

\textbf{Definition 3.} A relation \( \alpha : X \to Q_{\tau}(Y) \) is called \( Q_{\tau} \)-convex if \( \text{id}_X \subseteq \alpha^\sharp \) (total), \( \alpha(\xi_{W}^\sharp)^{\sharp} = \alpha \) (down-closed) and \( \alpha^* = \alpha \) (convex). A relation \( \alpha : X \to Q_{1}(Y) \) is called \( Q_{1} \)-convex if \( \text{id}_X \subseteq \alpha^2 \) (total) and \( \alpha^* = \alpha \) (convex).

By Proposition 9, a \( Q_{\tau} \)-convex relation \( \alpha \) is 0-included, that is \( \alpha \) satisfies \( \nabla X \alpha \subseteq \alpha \).

We need the following lemma to derive the associative law of convex composition.

\textbf{Lemma 1.} Let \( f : Y \to Q_{\tau}(W) \) be a map, and \( \beta : Y \to Q_{\tau}(Z) \) and \( \gamma : Z \to Q_{\tau}(W) \) relations. If \( f \subseteq \beta \gamma_0 \), then \( f_0 \subseteq \beta_0(\gamma^\bullet)_0 \).

\textbf{Proof.} Let \( f \subseteq \beta \gamma_0 \) and \( p \in Q_{\tau}(Y) \). Then

1. \( \exists g \subseteq \beta. f \subseteq g \gamma_0 \):
   
   As \( f \subseteq \beta \gamma_0 \) it holds that
   \[
   \begin{align*}
   \text{id}_Y & = ff^2 \cap \text{id}_Y \quad \{ f : \text{tfn} \} \\
   & \subseteq \beta \gamma_0 f^2 \cap \text{id}_Y \quad \{ f \subseteq \beta \gamma_0 \} \\
   & \subseteq (\beta \cap f(\gamma_0)^2)(\beta^2 \cap \gamma_0 f^2). \quad \{ \text{DF}_{\gamma^\bullet} \}
   \end{align*}
   \]

Hence \( \beta \cap f(\gamma_0)^2 \) is total and by the axiom of choice (AC) there exists a tfn \( g : Y \to Q_{\tau}(Z) \) such that \( g \subseteq \beta \cap f(\gamma_0)^2 \), which is equivalent to \( g \subseteq \beta \) and \( f \subseteq g \gamma_0 \).

2. \( \forall y \in Y \exists h_y \subseteq \gamma. yf = yg(h_y)_0 \):
   
   Note that
   \[
   yf \subseteq yg \gamma_0 \quad \{ f \subseteq \gamma \} \\
   \quad = \sqcap h \subseteq ygh \gamma_0. \quad \{ \gamma_0 = \sqcap h \subseteq \gamma \}
   \]

Thus there exists \( h_y \subseteq \gamma \) such that \( yf = yg(h_y)_0 \).

3. Define a map \( r_z \in Q(Y) \) by

\[
r_{z}[y] = \begin{cases} 
p_{[y]}(yg)[z] & \text{if } (pg_0)[z] > 0, \\
p_{[y]} & \text{otherwise.}
\end{cases}
\]

4. \( (pg_0)[z]r_{z}[y] = p_{[y]}(yg)[z] \) and \( r_z \subseteq Q_{\tau}(Y) \), i.e., \( r_z \subseteq \nabla_{I Q_{\tau}(Y)} \):

If \( \tau = 1 \) then \( (pg_0)[z] = 0 \) implies \( (yg)[z] = 0 \) for each \( y \in Y \). Even if \( \tau \neq 1 \), \( (pg_0)[z] = 0 \) implies \( p = 0 \), or \( (yg)[z] = 0 \) for each \( y \in Y \). In each case it is clear that \( (pg_0)_{[z]}r_{z}[y] = p_{[y]}(yg)[z] \).
If \((pg_o)[z] = 0\) then \(r_z = p \in Q_r(Y)\). If \((pg_o)[z] \neq 0\) then

\[
||r_z|| = \sum_y p_{[y]}(yg)[z] / (pg_o)[z] \\
= \sum_y p_{[y]}(yg)[z] / (pg_o)[z] \\
= \frac{(pg_o)[z]}{(pg_o)[z]} \\
= 1,
\]

and \(|r_z| \subseteq |p|\), since \(p_{[y]} = 0\) implies \(r_z[y] = 0\). Hence \(r_z \in Q_r(Y)\).

(5) For all \(z \in Z\) define a map \(h_z : Y \rightarrow Q_r(W)\) by \(\forall y \in Y\), \(y h_z = z h_y\).

(6) \((h_z)_{o} \subseteq (\nabla_{Y z} y) o : \)

\[
\begin{align*}
\hat{h_z} &= \bigcup_{y \in Y} y^2 z h_y \\
&\subseteq \bigcup_{y \in Y} y^2 z \gamma \\
&= \nabla_{Y z} y, \\
&\subseteq \bigcup_{y \in Y} y^2 = \nabla_{Y I}
\end{align*}
\]

which implies \((\hat{h_z})_{o} \subseteq (\nabla_{Y z} y) o\).

(7) Define a map \(h : Z \rightarrow Q_r(W)\) by \(\forall z \in Z\), \(z h = r_z(\hat{h_z})_{o}\).

(8) \(h \subseteq \gamma^* : \)

\[
zh = r_z(\hat{h_z})_{o} \\
\subseteq \nabla_{Y Q_r(Y)}(\nabla_{Y z} y) o \\
\subseteq z \gamma^* .
\]

(9) \(p f_o = pg_o h_o : \)

\[
\begin{align*}
p f_o &= \bigoplus_y p_{[y]}(yf) \\
&= \bigoplus_y p_{[y]}(yg(h_y))_{o} \\
&= \bigoplus_y p_{[y]}(yg)(h_y)_{o} \\
&= \bigoplus_y (pg_o(z y)_{[z]})(zh_y) \\
&= \bigoplus_y ((pg_o[z] r z[y])_{[z]})(zh_y) \\
&= \bigoplus_y ((pg_o[z] r z[y])_{[z]})(y h_z) \\
&= \bigoplus_y ((pg_o[z] r z[y])_{[z]})(h_z)_{o} \\
&= \bigoplus_y ((pg_o[z] r z[h_z])_{o}) \\
&= \bigoplus_y (pg_o[z] r z[h_z])_{o} \\
&= \bigoplus_y (pg_o[z] r z[h_z])_{o} \subseteq (zh) \\
&= \bigoplus_y (pg_o[z] r z[h_z])_{o} \subseteq (zh) \\
&= (pg_o[h_o].
\end{align*}
\]

Note that \(h\) depends on \(p\) and so \(f_o = g_o h_o\) may not hold.

(10) \(f_o \subseteq \beta_o(\gamma^*)_{o} : \)

For each \(p \in Q_r(Y)\) we have

\[
\begin{align*}
p f_o &= pg_o h_o \\
&\subseteq p \beta_o(\gamma^*)_{o} , \quad (9) \subseteq (1) g \subseteq \beta, (8) h \subseteq \gamma^* \\
\end{align*}
\]

and hence \(f_o \subseteq \beta_o(\gamma^*)_{o}\). This completes the proof.

Corollary 2. Let \(\alpha : X \rightarrow Q_r(Y)\), \(\beta : Y \rightarrow Q_r(Z)\) and \(\gamma : Z \rightarrow Q_r(W)\) be relations. Then

(a) \(\alpha \circ (\beta \circ \gamma) \subseteq (\alpha \circ \beta) \circ \gamma^*\).

(b) If \(\gamma^* = \gamma\), then \(\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma\).
Proof. (a)

\[
\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta \circ \gamma) = \alpha (\beta \circ \gamma) = \alpha (\beta \circ \gamma) = (\alpha \beta) \circ \gamma
\]

(b)

\[
\alpha \circ (\beta \circ \gamma) \triangleq (\alpha \circ \beta) \circ \gamma^* \quad \{ \text{Prop. 11 (d)} \}
\]

We proved the associative law of convex composition for convex relations. However, the following example shows that the convex composition \( \circ \) need not be associative in general.

**Example 4.** Consider maps \( f, g, h : X \to Q_*(X) \) which appeared in Example 1. For all \( p \in Q_1(X) \) we have

\[
p_f = x, \quad p_g = p_{\{x\}} \cdot x, \quad \text{and} \quad p_h = p_{\{x\}} \cdot y.
\]

Thus \( f_g = f, \) \( xf_h = yf_h = y, \) and \( p(f_h) = y \) for \( p \in Q_1(X). \) Shown in Example 2, the identity \( \gamma_0 = g_0 \cup h_0 \) holds, and so \( p_{f_0\gamma_0} = \check{x} \sqcup y \) for all \( p \in Q_1(X). \) Note that \( \gamma^* \neq \gamma. \) On the other hand, except for two maps \( f_g \) and \( f_h \) there are just two maps \( k \) and \( k' \) included in \( f_\gamma, \) where \( xk = \check{x}, \) \( yk = \check{y}, \) \( xk' = \check{y}, \) \( yk' = \check{x}. \) Let \( p_0 = \left( \frac{1}{2} \right)_X, \) the middle point of \( \check{x} \) and \( \check{y}. \) Then we have

\[
p_0(f_\gamma_0) = p_0(f_g)_{\circ} \cup p_0(f_h)_{\circ} \cup p_0k_{\circ} \cup p_0k'_{\circ}
\]

\[
= \check{x} \sqcup \check{y} \sqcup p_0k_{\circ} \cup p_0k'_{\circ}
\]

\[
\neq \check{x} \sqcup \check{y}
\]

\[
= p_0f_\gamma_0,
\]

which proves that \( \alpha f_\gamma_0 \neq \alpha (f_\gamma_0) \) for a map \( \alpha : X \to Q_1(X) \) such that \( x\alpha = y\alpha = p_0. \) Therefore

\[
(\alpha \circ f) \circ \gamma = \alpha f_\gamma_0 \neq \alpha (f_\gamma_0) = \alpha \circ (f \circ \gamma).
\]

\[\square\]

6. Convex relations and distributivities

Now we discuss the convex relations and the distributive laws of convex composition over the joins.

**Proposition 14.** Let \( \alpha : X \to Q_*(Y) \) and \( \beta : Y \to Q_*(Z) \) be \( Q_*-\)convex relations. Then the following holds:

(a) \( \alpha \circ \beta \) is total,

(b) \( (\alpha \circ \beta)(\xi^*_Z) \sqsubseteq \alpha \circ \beta, \)

(c) \( (\alpha \circ \beta)^* = \alpha \circ \beta. \)

Proof. (a) \( \alpha \circ \beta \) is total:

Since \( \alpha \) and \( \beta \) are total, \( \beta_0 \) is total by the definition and so \( \alpha \circ \beta = \alpha \beta_0 \) is total.

(b) \( (\alpha \circ \beta)(\xi^*_Z) \sqsubseteq \alpha \circ \beta : \)

\[
(\alpha \circ \beta)(\xi^*_Z) = \alpha \beta_0(\xi^*_Z) \quad \{ \text{Def. 2} \}
\]

\[
\sqsubseteq \alpha(\beta(\xi^*_Z))_{\circ} \quad \{ \text{Prop. 10 (c)} \}
\]

\[
\sqsubseteq \alpha \beta_0. \quad \{ \beta(\xi^*_Z) = \beta \}
\]
Proposition 15. If $\alpha : X \rightarrow Q_1(Y)$ and $\beta : Y \rightarrow Q_1(Z)$ are $Q_1$-convex relations, then so is the convex composite $\alpha \circ \beta$.

Proof. The proof is the same as the proof (a) and (c) of Proposition 14. □

In the rest of paper, the subscript $\tau$ is one of 1 and $\ast$. For a set $\chi$ of $Q_\tau$-convex relations $\alpha : X \rightarrow Q_\tau(Y)$ define

$$\bigvee \chi = (\sqcup \chi)^\ast.$$ 
It is trivial that $\bigvee$ gives the join (the least upper bound) of $\chi$.

The following proposition shows the right distributivity over all joins.

Proposition 16. Let $\alpha : X \rightarrow Q_\tau(X)$ and $\beta : X \rightarrow Q_\tau(X)$ be relations.

(a) $\alpha^\ast \circ \beta \subseteq (\alpha \circ \beta)^\ast$,
(b) $(\bigvee \chi) \circ \beta = \bigvee (\chi \circ \beta)$.

Proof. (a) $\alpha^\ast \circ \beta \subseteq (\alpha \circ \beta)^\ast$:

$$\forall x \in X, x(\alpha^\ast \circ \beta) = x\alpha^\ast \beta_o$$
$$\subseteq \nabla_{Q_o(\beta)}(\nabla_{Q_o(\alpha)}(\alpha))$$
$$\subseteq \nabla_{Q_o(\beta)}(\nabla_{Q_o(\alpha)}(\alpha)) \beta_o \quad \{ \text{Prop.11 (c)} \}$$
$$= x(\alpha \beta_o)^\ast.$$ 

(b) $(\bigvee \chi) \circ \beta = \bigvee (\chi \circ \beta)$:

$$(\bigvee \chi) \circ \beta = (\sqcup \chi)^\ast \circ \beta$$
$$\subseteq ((\sqcup \chi)^\ast \circ \beta) \quad \{ (a) \}$$
$$= (\sqcup (\chi \circ \beta))^\ast$$
$$= \bigvee (\chi \circ \beta).$$

$\alpha \in \chi \rightarrow \alpha \circ \beta \subseteq (\bigvee \chi) \circ \beta \quad \{ \alpha \subseteq \bigvee \chi \}$$
$$\rightarrow \bigvee (\alpha \circ \beta) \subseteq (\bigvee \chi) \circ \beta.$$ □

The following example shows that the left distributivity $\alpha \circ \bigvee \chi = \bigvee (\alpha \circ \chi)$ needs not hold in general.

Example 5. Let $\alpha', \beta : X \rightarrow Q_\tau(X)$ be $Q_\tau$-convex relations such that $\alpha' = \alpha(\xi_X)^2$, $\beta = h(\xi_X)^2$ where $X$, $\alpha$ and $h$ are appeared in Example 4. Then $\alpha h_o \subseteq \alpha(\xi_X)^2 = \alpha'$ holds since $xah_o = yah_o = p_0h_o = \frac{1}{2} \gamma' \leq p_0 = x_0 = y$. Shown in Example 2, a relation $h \sqcup e_X$ consists of four maps, that is $h \sqcup e_X = f' \sqcup g \sqcup h \sqcup e_X$. Then we have

$$p_0f' \quad \subseteq \quad x(\alpha' \sqcup e_X) \quad \quad \{ \quad p_0 = xa \subseteq \alpha', \quad f' \subseteq (h \sqcup e_X) \}$$

$$= \quad x(\alpha' \sqcup h(\xi_X)^2)$$
$$x(\alpha' \sqcup e_X(\xi_X)^2)$$
$$= \quad x(\alpha' \circ (\beta \sqcup e_X))$$
$$= \quad x(\alpha' \circ (\beta \sqcup e_X)).$$
On the other hand, we have $p_0 f'_\chi = y \nsubseteq xa' = x (a' \circ \beta \lor a' \circ e'_{\chi})$ since

\[
\begin{align*}
a' \circ \beta \lor a' \circ e'_{\chi} &= a' \circ \beta \circ a' \circ (e'_{\chi})_o \\
&= a' (h (e'_{\chi})^\circ) \lor a' (e'_{\chi})^\circ \\
&= a' h_o (e'_{\chi})^\circ \lor a' (e'_{\chi})^\circ \\
&= ah_o (e'_{\chi})^\circ \lor a' \\
&= (Prop. 10 (d), 7 (c)) \\
&= \{ Prop. 7 (g), \quad a': convex \} \\
&= a'. \\
\end{align*}
\]

Therefore $a' \circ (\beta' \lor e'_{\chi}) \neq a' \circ \beta' \lor a' \circ e_{\chi}$.

Finally, we state several results about directed sets of $Q_\ast$-convex relations and their joins.

**Lemma 2.** If $\chi$ is a directed set of $Q_\ast$-convex relations $a : X \rightarrow Q_\ast (Y)$, then

\[
(\sqcup \chi)_o = \sqcup_{a \in \chi} a_o.
\]

**Proof.** The inclusion $\sqcup_{a \in \chi} a_o \subseteq (\sqcup \chi)_o$ is trivial. We will show that $p (\sqcup \chi)_o \subseteq \sqcup_{a \in \chi} p a_o$ for all $p \in Q_\ast (X)$.

For a map $f : X \rightarrow Q_\ast (Y)$ we have

\[
f \subseteq \sqcup \chi \quad \rightarrow \quad \forall x \in X. \quad xf \subseteq x (\sqcup \chi) \\
\rightarrow x f \subseteq \sqcup_{a \in \chi} x a_x \\
\rightarrow \exists a_x \in \chi. \quad xf \subseteq xa_x \\
\rightarrow \exists a_f \in \chi \forall x \in [p]. \quad xf \subseteq xa_f
\]

Note $\rightarrow$ follows from the assumption that $[p]$ is finite and $\chi$ is directed. Define a map $f' : X \rightarrow Q_\ast (Y)$ by

\[
\forall x \in X. \quad xf' = \begin{cases} \quad xf & \text{if } x \in [p], \\
\qquad 0_y & \text{otherwise.} \end{cases}
\]

It is clear that $pf_o = pf'_o$. Also $f' \subseteq a_f$ holds, because $\forall x \forall 0_y \subseteq a$ (as total and $a (e'_{\chi})^\circ = a$). Hence $pf_o = pf'_o \subseteq pa_o$ and so

\[
p (\sqcup \chi)_o = p (\sqcup_{f \subseteq \sqcup \chi} f_o) \\
= \sqcup_{f \subseteq \sqcup \chi} pf_o \\
\subseteq \sqcup_{a \in \chi} p a_o \\
\{ p f_o \subseteq p a_o \}
\]

\[
= p (\sqcup_{a \in \chi} a_o).
\]

The following proposition gives the directed join of $Q_\ast$-convex relations.

**Proposition 17.** Let $\chi$ be a directed set of $Q_\ast$-convex relations $a : X \rightarrow Q_\ast (X)$. Then

\[
\bigvee \chi = \bigcup \chi.
\]

**Proof.**

\[
\begin{align*}
x (\bigvee \chi) &= x (\sqcup \chi)^\bullet \\
&= \nabla_{I Q_\ast (\chi)} (\nabla_{N} x (\sqcup \chi))_o \\
&= \nabla_{I Q_\ast (\chi)} (\sqcup_{a \in \chi} \nabla_{N} xa)_o \\
&= \sqcup_{a \in \chi} \nabla_{I Q_\ast (\chi)} (\sqcup_{a \in \chi} \nabla_{N} xa)_o \\
&= \sqcup_{a \in \chi} \nabla_{I Q_\ast (\chi)} (\nabla_{N} xa)_o \\
&= \sqcup_{a \in \chi} \nabla_{I Q_\ast (\chi)} (\nabla_{N} xa). \\
\end{align*}
\]

The composition of $Q_\ast$-convex relations distributes all directed joins from the left-hand side.

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Proposition 18. Let $\alpha : X \to \mathcal{Q}_*(Y)$ be a $\mathcal{Q}_*$-convex relation and $\chi$ a directed set of $\mathcal{Q}_*$-convex relations $\beta : Y \to \mathcal{Q}_*(Z)$. Then

$$\alpha \circ \bigvee \chi = \bigvee (\alpha \circ \chi).$$

Proof.

\[
\begin{align*}
\alpha \circ \bigvee \chi &= \alpha (\bigcup \chi) \quad \{ \text{Prop.17} \} \\
&= \alpha (\bigcup_{\beta \in \chi} \beta) \quad \{ \text{Lemma. 2} \} \\
&= \bigcup_{\beta \in \chi} \alpha \beta \\
&= \bigcup_{\beta \in \chi} \alpha \circ \beta \\
&= \bigvee (\alpha \circ \chi) \\
&= \bigvee (\alpha \circ \chi). 
\end{align*}
\]

\[\square\]

7. Conclusion

In this paper we have studied the relations into algebras of probabilistic distributions using relational calculi, although McIver et al. [7] and Tsumagari [16] studied in set-theoretical way. We have shown the following.

- The set of $\mathcal{Q}_*$-convex relations forms a category with the convex composition, and the identity morphisms depending on $\tau \in \{*, 1\}$.

- For $\tau \in \{*, 1\}$ the convex composition of $\mathcal{Q}_*$-convex relations distributes over all non-empty joins from the right hand side.

- The convex composition of $\mathcal{Q}_*$-convex relations distributes over all non-empty directed joins even from the left hand side.

We have proved the associative law of convex composition for $\mathcal{Q}_*$-convex relations and $\mathcal{Q}_1$-convex relations in the same framework, though Tsumagari [16] had studied as their two convex-relations are different. Additionally we have given a counter example for the associative law of the convex composition in the absence of convexity.

The convex composition studied in this paper seems to be a generalization of reachability composition of multirelations. So we might be interested in the another composition of $\mathcal{Q}_*$-convex relations, corresponding to the composition of up-closed multirelations studied by Parikh [11, 12] and Rewitzky [6, 14].

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