Relations into Algebras of Probabilistic Distributions

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Abstract

The paper proposes two types of convex relations into algebras of probabilistic distributions as a relational algebraic foundation of semantic domains of probabilistic systems [4, 7, 8]. Following previous results by Tsumagari [16], we particularly focus on the associative law for the convex compositions defined via bounded combinations of probabilistic distributions, and prove that the convex compositions are associative for convex relations.

Keywords: algebras of probabilistic distributions, convex relations, associativity and distributivity of convex composition, relational calculus

1. Introduction

The concept of rings is basic in mathematics as a framework of numbers. Recently from a view point on algebraic study [1] of semantic domains for distributed algorithms, the importance of variants of rings, such as Kleene algebras [5] and idempotent semirings, has been increased. It is well-known that the set of all binary relations on a set forms a typical example of complete idempotent semirings.

When constructing a concrete model of semirings with preferable properties, we have to first focus our attention on the associativity of possible composition. For relations $\alpha : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown Z$ the ordinary composite $\alpha\beta : X \rightharpoondown Y$ is defined as

$$(x, z) \in \alpha\beta \;\leftrightarrow\; \exists y \in Y.\ (x, y) \in \alpha \wedge (y, z) \in \beta.$$

Of course the ordinary composition of relations is associative. A multirelation is a relation of a form $\alpha : X \rightharpoondown \wp(Y)$, where $\wp(X)$ denotes the power set of X. Depending on applications, two definitions of composition of multirelations are known. One of them is called the *reachability composition* studied by Peleg [13] and Goldblatt [3] for concurrent dynamic logic. The reachability composition $\alpha\cdot\beta$ of multirelations $\alpha : X \rightharpoondown \wp(Y)$ and $\beta : Y \rightharpoondown \wp(Z)$ is defined by

$$\begin{aligned}(x, T) \in \alpha \cdot \beta \quad \leftrightarrow \quad & \exists U \in \wp(Y).\ [\,(x, U) \in \alpha \wedge \exists \{T_y\}_{y\in U} \subseteq \wp(Z).\\ & \forall y \in U.\ (y, T_y) \in \beta \wedge T = \cup_{y\in U} T_y\,].\end{aligned}$$

Another composition of multirelations is given by Parikh and Rewitzky. Their composition $\alpha;\beta : X \rightharpoondown \wp(Z)$ of multirelations $\alpha : X \rightharpoondown \wp(Y)$ and $\beta : Y \rightharpoondown \wp(Z)$ is defined by

$$(x, T) \in \alpha;\beta \;\leftrightarrow\; \exists U \in \wp(Y).\ [(x, U) \in \alpha \wedge \forall y \in U.\ (y, T) \in \beta].$$

It is readily seen that the definition of $\alpha;\beta$ is making use of the membership relation and residual composition. For the associativity of this composition we need a condition called *up-closed*. Up-closed multirelations

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provide a model of Parikh's game logic [11, 12]. Rewitzky [6, 14] studied them as a semantic domain of predicate transformer semantics of nondeterministic programming language. Further Nishizawa, Tsumagari and Furusawa [10] demonstrated that the set of all up-closed multirelations forms a complete idempotent left semiring (complete IL-semiring) introduced by Möller [9].

On the other hand, McIver et al. [4, 7, 8] introduced a semantic domain of probabilistic programs and probabilistic Kleene algebra, and indicated that probabilistic Kleene algebras are useful to simplify a model of probabilistic distributed systems. Based on their works, Tsumagari [16] initially introduced two probabilistic (non-numerical) models of complete IL-semirings with the set of maps from a set into the unit interval $[0,1]$, and studied *probabilistic multirelations* and the point-wise convexity of them. The point-wise convexity plays an important rôle for both models to satisfy the associativity of composition.

The aim of the paper is to expand Tsumagari's work [16] and to give a relational foundation for relations into algebras of probabilistic distributions. Following his work we will reformulate probabilistic multirelations as certain convex relations, together with stepwise refinement. Then we will clarify how the convexity works in the associativity of composition of convex relations.

In section 2 we review the basic properties of algebras consist of maps from a set into the unit interval $[0,1]$ together with scalar products, multiplications and bounded sums. Section 3 studies convex combinations of probabilistic distributions. In section 4 we introduce convex composition of relations into algebras of probabilistic distributions by using convex combinations. In section 5 we show the associative law of convex composition. Section 6 introduces two types of convex relations, and study the distributive laws of convex composition over joins. Section 7 summarizes this work.

Notation. In the paper we will denote by I, a singleton set. A (binary) relation α from a set X to a set Y, written $\alpha : X \rightharpoondown Y$, is a subset $\alpha \subseteq X \times Y$. The empty relation $\emptyset_{XY} : X \rightharpoondown Y$ and the universal relation $\nabla_{XY} : X \rightharpoondown Y$ are defined by $\emptyset_{XY} = \emptyset$ and $\nabla_{XY} = X \times Y$ respectively. The converse of a relation $\alpha : X \rightharpoondown Y$ is denoted by $\alpha^\sharp$. The identity relation $\{(x,x) \mid x \in X\}$ over X is denoted by id_X. The ordinary composition of relations (which include functions) will be denoted by juxtaposition. For example, the composite of a relation $\alpha : X \rightharpoondown Y$ followed by $\beta : Y \rightharpoondown Z$ is denoted by $\alpha\beta$, and of course the composition of functions $f : X \to Y$ and $g : Y \to Z$ by fg. Also the traditional notation $f(x)$ will be written by xf as a composite of functions $x : I \to X$ and $f : X \to Y$. However, the evaluation of a map $p : X \to [0,1]$ at $x \in X$ will be expressed by $p_{[x]} \in [0,1]$. Note that the symbols of multiplication for reals and the ordinary composition of relations are omitted. Some proofs refer the point axiom (PA) and the Dedekind formula (DF$_*$), i.e.

$$\begin{array}{ll} \text{(PA)} & \sqcup_{x \in X} x = \nabla_{IX}, \\ (\text{DF}_*) & \alpha\beta \sqcap \gamma \sqsubseteq (\alpha \sqcap \gamma\beta^\sharp)(\beta \sqcap \alpha^\sharp\gamma), \end{array}$$

where $x \in X$ is identified as a function $x : I \to X$. Note that (PA) is equivalent to $\mathrm{id}_X = \sqcup_{x \in X} x^\sharp x$ and that so is (DF$_*$) to $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$. See [15] for more details on basic properties of relations.

2. Maps to the unit interval

We consider maps from a set X to the unit interval $[0,1]$. Such a map $p : X \to [0,1]$ is often called a fuzzy set. The support $\lfloor p \rfloor$ of a map p is the subset of X defined by $\lfloor p \rfloor = \{x \in X \mid p_{[x]} > 0\}$. The set of all maps from X to $[0,1]$ will be denoted by $\mathcal{Q}(X)$. As we will discuss later, maps in $\mathcal{Q}(X)$ will be restricted as to be probabilistic (sub-)distributions. The point-wise order $\leq$ on $\mathcal{Q}(X)$ is a binary relation such that

$$p \leq q \ \leftrightarrow\ \forall x \in X.\ p_{[x]} \leq q_{[x]}$$

for $p, q \in \mathcal{Q}(X)$. For a real $k \in [0,1]$ a map $k_X : X \to [0,1]$ such that $k_{X[x]} = k$ for all $x \in X$ is called the *constant map over* X with value k. For $a \in X$ define a map $\dot{a} : X \to [0,1]$ by

$$\dot{a}_{[x]} = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}$$

The constant maps 0_X and 1_X over X are the least and the greatest elements of $\mathcal{Q}(X)$, respectively.

We introduce the following operators to discuss algebras of probabilistic distributions. For a real $k \in [0,1]$ and maps $p,\ q \in \mathcal{Q}(X)$ we define maps $k \cdot p,\ p * q,\ p \oplus q \in \mathcal{Q}(X)$ by

$$(k \cdot p)_{[x]} = kp_{[x]}, \qquad (p * q)_{[x]} = p_{[x]}q_{[x]}$$

and

$$(p \oplus q)_{[x]} = \min\{p_{[x]} + q_{[x]},\ 1\}$$

for all $x \in X$, respectively. The set $\mathcal{Q}(X)$ forms an algebra called a prering.

Proposition 1. *Let $p, q \in \mathcal{Q}(X)$ and $k, k' \in [0,1]$. Then the following hold:*

(a) $(p * q) * r = p * (q * r), \quad p * q = q * p,$
(b) $(p \oplus q) \oplus r = p \oplus (q \oplus r), \quad p \oplus q = q \oplus p,$
(c) $k \cdot p = k_X * p, \quad 0 \cdot p = 0_X, \quad 1 \cdot p = p, \quad (kk') \cdot p = k \cdot (k' \cdot p),$
(d) $p \oplus 0_X = p, \quad p \oplus 1_X = 1_X,$
(e) *If $q_{[x]} + r_{[x]} \leq 1$ for all $x \in X$, then $p * (q \oplus r) = (p * q) \oplus (p * r)$.*
(f) *If $k + k' \leq 1$, then $(k + k') \cdot p = (k \cdot p) \oplus (k' \cdot p)$.*
(g) $p \leq p' \wedge q \leq q' \ \rightarrow\ p \oplus q \leq p' \oplus q' \ \wedge\ p * q \leq p' * q'$.

Proof is omitted. □

In general the distributive laws $p * (q \oplus r) = (p * q) \oplus (p * r)$ and $k \cdot (q \oplus r) = (k \cdot q) \oplus (k \cdot r)$ do not always hold.

The following proposition shows the basic properties about the support of maps in $\mathcal{Q}(X)$.

Proposition 2. *Let $p, q, r \in \mathcal{Q}(X)$ and $k \in [0,1]$. Then the following hold:*

(a) $\lfloor 0_X \rfloor = \emptyset$, *and* $\lfloor k_X \rfloor = X$ *if* $k > 0$,
(b) $\lfloor \dot{a} \rfloor = \{a\}$,
(c) $\lfloor p * q \rfloor = \lfloor p \rfloor \cap \lfloor q \rfloor$,
(d) $\lfloor p \oplus q \rfloor = \lfloor p \rfloor \cup \lfloor q \rfloor$.

Proof is omitted. □

Proposition 3. *Let $p,\ q \in \mathcal{Q}(X)$. Then*

$$q \leq p \ \leftrightarrow\ \exists t \in \mathcal{Q}(X).\ q = p * t.$$

Proof. $(\leftarrow)$ $q_{[x]} = p_{[x]}t_{[x]} \leq p_{[x]}$ since $t_{[x]}, p_{[x]} \in [0,1]$.
$(\rightarrow)$ Assume $q \leq p$. Define a map $t : X \to [0,1]$ by

$$\forall x \in X.\ t_{[x]} = \begin{cases} \dfrac{q_{[x]}}{p_{[x]}} & \text{if } p_{[x]} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is trivial that $q = p * t$. □

The *sum* of a map $p \in \mathcal{Q}(X)$ is the least upper bound of the set $\{\sum_{x \in F} p_{[x]} \mid F : \text{finite subset of } X\}$, which is denoted by $||p||$. It is well-known that the sum $||p||$ of p exists iff the above set is bounded. Also $\lfloor p \rfloor$ is a countable subset of X if $||p||$ exists. (For each positive integer n define a subset $\lfloor p \rfloor_n$ of X by $\lfloor p \rfloor_n = \{x \in X \mid 1/n < p_{[x]}\}$. Then each $\lfloor p \rfloor_n$ has finite (at most $(n-1) \cdot |n|$) members and so $\lfloor p \rfloor = \cup_{n>0} \lfloor p \rfloor_n$ is countable.)

We define three types of maps in $\mathcal{Q}(X)$ which are used in this paper.

Definition 1. Three subsets of $\mathcal{Q}(X)$ are defined as follows:

(a) $p \in \mathcal{Q}_0(X) \ \leftrightarrow\ p \in \mathcal{Q}(X) \ \wedge\ ||p|| \leq 1$,
(b) $p \in \mathcal{Q}_1(X) \ \leftrightarrow\ p \in \mathcal{Q}(X) \ \wedge\ ||p|| = 1$,
(c) $p \in \mathcal{Q}_*(X) \ \leftrightarrow\ p \in \mathcal{Q}_0(X) \ \wedge\ \lfloor p \rfloor$: finite subset of X. □

Note that $0_X \in \mathcal{Q}_*(X)$, $\dot{x} \in \mathcal{Q}_*(X) \cap \mathcal{Q}_1(X)$ and $p * q \in \mathcal{Q}_*(X)$ for all $x \in X$, p, $q \in \mathcal{Q}_*(X)$. Elements of $\mathcal{Q}_*(X)$ are probabilistic sub-distributions, and those of $\mathcal{Q}_*(X) \cap \mathcal{Q}_1(X)$ are probabilistic distributions. Essentially, McIver et al. [4, 7, 8] have studied either the case of $\mathcal{Q}_*(X)$ or the case of finite set X in order to develop models of probabilistic systems.

The restriction of the point-wise order $\leq: \mathcal{Q}(X) \rightharpoonup \mathcal{Q}(X)$ onto $\mathcal{Q}_\tau(X)$ is denoted by $\xi_X^\tau : \mathcal{Q}_\tau(X) \rightharpoonup \mathcal{Q}_\tau(X)$, that is,

$$\forall p, q \in \mathcal{Q}_\tau(X).\ (p, q) \in \xi_X^\tau \ \leftrightarrow\ p \leq q,$$

where the subscript/superscript τ is one of 0, 1, and $*$. Remark that the restricted order ξ_X^τ on $\mathcal{Q}_1(X)$ is discrete, that is, for $\xi_X^\tau = \mathrm{id}_{\mathcal{Q}_1(X)}$. Thus the order on $\mathcal{Q}_1(X)$ will not be used in the rest of the paper.

For $\tau \in \{0, *\}$ every map $t \in \mathcal{Q}(X)$ yields a map $t_* : \mathcal{Q}_\tau(X) \to \mathcal{Q}_\tau(X)$ by

$$\forall p \in \mathcal{Q}_\tau(X).\ p\, t_* = p * t.$$

Corollary 1. $(\xi_X^\tau)^\sharp = \sqcup_{t \in \mathcal{Q}(X)} t_*$ *for* $\tau \in \{0, *\}$.

Proof.

$$\begin{array}{lll} (p, q) \in (\xi_X^\tau)^\sharp & \leftrightarrow & q \leq p \\ & \leftrightarrow & \exists t \in \mathcal{Q}(X).\ q = p * t = p\, t_* \quad \{\ \text{Prop.3}\ \} \\ & \leftrightarrow & \exists t \in \mathcal{Q}(X).\ (p, q) \in t_* \\ & \leftrightarrow & (p, q) \in \sqcup_{t \in \mathcal{Q}(X)} t_*. \end{array}$$

□

A map $e_X : X \to \mathcal{Q}_\tau(X)$ is defined by $x e_X = \dot{x}$ for each $x \in X$, where $\tau \in \{0, 1, *\}$. Also, for $\tau \in \{0, *\}$ we define a relation $\varepsilon_X^\tau : X \rightharpoonup \mathcal{Q}_\tau(X)$ by $\varepsilon_X^\tau = e_X (\xi_X^\tau)^\sharp$. As discussed in detail later, e_X and ε_X^τ are the units of convex composition over certain convex relations, respectively.

3. Convex combinations

Extending finite bounded sums

$$\bigoplus_{j=1}^{n} q_j = q_1 \oplus q_2 \oplus \cdots \oplus q_n$$

of maps $q_1, q_2, \ldots, q_n \in \mathcal{Q}(X)$, we will define the bounded sum of an arbitrary set of maps in $\mathcal{Q}(X)$. For a set $\{q_j \mid j \in J\}$ of maps in $\mathcal{Q}(Y)$ define a map $\bigoplus_{j \in J} q_j$ in $\mathcal{Q}(Y)$ by

$$\forall y \in Y.\ \ (\bigoplus_{j \in J} q_j)_{[y]} = \begin{cases} \sum_{j \in J} (q_j)_{[y]} & \text{if } \sum_{j \in J} (q_j)_{[y]} \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Of course, we mean $(\bigoplus_{j \in J} q_j)_{[y]} = 1$ even if the sum $\sum_{j \in J} (q_j)_{[y]}$ diverges.

The support of bounded sums of a set of maps in $\mathcal{Q}(X)$ is given by the union of supports of their maps contained in the set.

Proposition 4. *For all subsets* $\{q_j \mid j \in J\} \subseteq \mathcal{Q}(X)$ *the following holds:*

$$\lfloor \bigoplus_{j \in J} q_j \rfloor = \cup_{j \in J} \lfloor q_j \rfloor.$$

Proof.

$$
\begin{array}{rcl}
y \notin \lfloor \bigoplus_{j\in J} q_j \rfloor & \leftrightarrow & (\bigoplus_{j\in J} q_j)_{[y]} = 0 \\
& \leftrightarrow & \sum_{j\in J} (q_j)_{[y]} = 0 \\
& \leftrightarrow & \forall j \in J.\ (q_j)_{[y]} = 0 \\
& \leftrightarrow & \forall j \in J.\ y \notin \lfloor q_j \rfloor \\
& \leftrightarrow & \neg[\exists j \in J.\ y \in \lfloor q_j \rfloor] \\
& \leftrightarrow & y \notin \cup_{j\in J} \lfloor q_j \rfloor.
\end{array}
$$

□

Let $f : X \to \mathcal{Q}(Y)$ be a map. Define a map $f_\diamond : \mathcal{Q}(X) \to \mathcal{Q}(Y)$ by

$$
pf_\diamond = \bigoplus_{x\in X} p_{[x]}\cdot(xf)
$$

where $p \in \mathcal{Q}(X)$. The map $pf_\diamond$ is called a *convex combination* of p and f. We need this notion to raise the composition of convex relations.

Example 1. Set $X = \{x, y\}$ and define maps f, f', g, $h : X \to \mathcal{Q}(X)$ by $xf = yf = \dot{x}$, $xf' = yf' = \dot{y}$, $xg = \dot{x}$, $yg = 0_X$, $xh = \dot{y}$ and $yh = 0_X$. We have $f_\diamond$, $f'_\diamond$, $g_\diamond$ and $h_\diamond$ such that

$$
pf_\diamond = p_{[x]}\cdot\dot{x} \oplus p_{[y]}\cdot\dot{x}, \quad pf'_\diamond = p_{[x]}\cdot\dot{y} \oplus p_{[y]}\cdot\dot{y}, \quad pg_\diamond = p_{[x]}\cdot\dot{x}, \quad ph_\diamond = p_{[x]}\cdot\dot{y},
$$

for all $p \in \mathcal{Q}(X)$. Especially for $p' \in \mathcal{Q}_1(X)$, $p'f_\diamond = \dot{x}$ and $p'f'_\diamond = \dot{y}$ hold. □

The basic properties of convex combinations are listed below.

Proposition 5. *Let $k \in [0, 1]$, $p \in \mathcal{Q}_0(X)$ and $f : X \to \mathcal{Q}(Y)$. Then the followings hold:*

(a) $(pf_\diamond)_{[y]} = \sum_{x\in X} p_{[x]}(xf)_{[y]}$,
(b) $||pf_\diamond|| = \sum_{x\in X} p_{[x]}||xf||$,
(c) $\lfloor pf_\diamond \rfloor = \cup_{x\in\lfloor p\rfloor} \lfloor xf \rfloor$,
(d) $0_X f_\diamond = 0_Y$,
(e) $\dot{x}f_\diamond = xf$,
(f) $p(e_X)_\diamond = p$,
(g) $p(\nabla_{XI} k_Y)_\diamond = (k||p||)_Y$,
(h) $k\cdot(pf_\diamond) = (k\cdot p)f_\diamond$.

Proof. (a) $(pf_\diamond)_{[y]} = \sum_{x\in X} p_{[x]}(xf)_{[y]}$:

$$
\begin{array}{rcll}
\sum_{x\in X} p_{[x]}(xf)_{[y]} & \leq & \sum_{x\in X} p_{[x]} & \{\ xf \in \mathcal{Q}(Y)\ \} \\
& \leq & 1. & \{\ p \in \mathcal{Q}_0(X)\ \}
\end{array}
$$

$$
\begin{array}{rcl}
(pf_\diamond)_{[y]} & = & \min\{\sum_{x\in X} p_{[x]}(xf)_{[y]}, 1\} \\
& = & \sum_{x\in X} p_{[x]}(xf)_{[y]}.
\end{array}
$$

(b) $||pf_\diamond|| = \sum_{x\in X} p_{[x]}||xf||$:

$$
\begin{array}{rcll}
||pf_\diamond|| & = & \sum_{y\in Y} (pf_\diamond)_{[y]} & \\
& = & \sum_{y\in Y} \sum_{x\in X} p_{[x]}(xf)_{[y]} & \{\ (a)\ \} \\
& = & \sum_{x\in X} p_{[x]} \sum_{y\in Y} (xf)_{[y]} & \\
& = & \sum_{x\in X} p_{[x]}||xf||. &
\end{array}
$$

(c) $\lfloor pf_\diamond \rfloor = \cup_{x\in\lfloor p\rfloor} \lfloor xf \rfloor$:

$$
\begin{array}{rcl}
\lfloor pf_\diamond \rfloor & = & \cup_{x\in X} \lfloor p_{[x]}\cdot(xf) \rfloor \\
& = & \cup_{x\in\lfloor p\rfloor} \lfloor xf \rfloor.
\end{array}
$$

(d) $0_X f_\diamond = 0_Y$:

$$\begin{array}{rcll} 0_X f_\diamond & = & \bigoplus_{x\in X}(0_X)_{[x]}\cdot(xf) & \\ & = & \bigoplus_{x\in X} 0\cdot(xf) & \\ & = & 0_Y. & \{\ 0\cdot q = 0_Y \text{ if } q\in\mathcal{Q}(Y)\ \} \end{array}$$

(e) $\dot{x}f_\diamond = xf$:

$$\begin{array}{rcl} \dot{x}f_\diamond & = & \bigoplus_{x'\in X}\dot{x}_{[x']}\cdot(x'f) \\ & = & xf. \end{array}$$

(f) $p(e_X)_\diamond = p$:

$$\begin{array}{rcll} (p(e_X)_\diamond)_{[x]} & = & \sum_{x'\in X} p_{[x']}(x'e_X)_{[x]} & \{\ \text{(a)}\ \} \\ & = & \sum_{x'\in X} p_{[x']}\dot{x}'_{[x]} & \\ & = & p_{[x]}. & \end{array}$$

(g) $p(\nabla_{XI}k_Y)_\diamond = (k||p||)_Y$:

$$\begin{array}{rcll} p(\nabla_{XI}k_Y)_\diamond & = & \bigoplus_{x\in X} p_{[x]}\cdot(x\nabla_{XI}k_Y) & \\ & = & \bigoplus_{x\in X} p_{[x]}\cdot k_Y & \{\ x\nabla_{XI} = \mathrm{id}_I\ \} \\ & = & (\sum_{x\in X} p_{[x]})\cdot k_Y & \{\ p\in\mathcal{Q}_0(X), \text{(a)}\ \} \\ & = & ||p||\cdot k_Y & \\ & = & (||p||k)_Y. & \{\ k'\cdot k_Y = (k'k)_Y\ \} \end{array}$$

(h) $k\cdot(pf_\diamond) = (k\cdot p)f_\diamond$:

$$\begin{array}{rcll} (k\cdot(pf_\diamond))_{[y]} & = & k(pf_\diamond)_{[y]} & \\ & = & k\sum_{x\in X} p_{[x]}(xf)_{[y]} & \{\ \text{(a)}\ \} \\ & = & \sum_{x\in X}(k\cdot p)_{[x]}(xf)_{[y]} & \\ & = & (k\cdot p)f_\diamond. & \end{array}$$

□

The convex combination also satisfies the following properties.

Proposition 6. *For $\tau\in\{0,1,*\}$ the following hold:*

(a) *If $p\in\mathcal{Q}_\tau(X)$ and $f: X\to\mathcal{Q}_\tau(Y)$, then $pf_\diamond\in\mathcal{Q}_\tau(Y)$.*
(b) *If $p\in\mathcal{Q}_\tau(X)$ and $f: X\to\mathcal{Q}_\tau(Y)$, then there exist $p'\in\mathcal{Q}_\tau(\mathbb{N})$ and $f':\mathbb{N}\to\mathcal{Q}_\tau(Y)$ such that $pf_\diamond = p'f'_\diamond$.*

Proof. (a_0) $[p\in\mathcal{Q}_0(X)\ \wedge\ f: X\to\mathcal{Q}_0(X)]\ \to\ pf_\diamond\in\mathcal{Q}_0(Y)$:

$$\begin{array}{rcll} ||pf_\diamond|| & = & \sum_{x\in X} p_{[x]}||xf|| & \{\ \text{Prop.5 (b)}\ \} \\ & \le & \sum_{x\in X} p_{[x]} & \{\ xf\in\mathcal{Q}_0(Y)\ \} \\ & = & ||p|| & \\ & \le & 1. & \{\ p\in\mathcal{Q}_0(X)\ \} \end{array}$$

(a_1) $[p\in\mathcal{Q}_1(X)\ \wedge\ f: X\to\mathcal{Q}_1(X)]\to pf_\diamond\in\mathcal{Q}_1(Y)$:

$$\begin{array}{rcll} ||pf_\diamond|| & = & \sum_{x\in\lfloor p\rfloor} p_{[x]}||xf|| & \{\ \text{Prop.5 (b)}\ \} \\ & = & \sum_{x\in\lfloor p\rfloor} p_{[x]} & \{\ xf\in\mathcal{Q}_1(Y)\ \} \\ & = & 1. & \{\ p\in\mathcal{Q}_1(Y)\ \} \end{array}$$

(a_*) $[p\in\mathcal{Q}_*(X)\wedge f: X\to\mathcal{Q}_*(X)]\to pf_\diamond\in\mathcal{Q}_*(Y)$ is immediate from (a_0) and Prop.5 (c).

(b_0) $\forall p\in\mathcal{Q}_0(X)\,\forall f: X\to\mathcal{Q}_0(Y)\,\exists p'\in\mathcal{Q}_0(\mathbb{N})\,\exists f':\mathbb{N}\to\mathcal{Q}_0(Y).\ pf_\diamond = p'f'_\diamond$:
As already stated the support $\lfloor p\rfloor$ is countable if $||p||$ exists and so there is an injection $i:\lfloor p\rfloor\to\mathbb{N}$. Define a map $p'\in\mathcal{Q}(\mathbb{N})$ and a map $f':\mathbb{N}\to\mathcal{Q}(Y)$ by

$$p'_{[n]} = \begin{cases} p_{[x]} & \text{if } \exists x\in\lfloor p\rfloor.\ n = xi \\ 0 & \text{otherwise} \end{cases}$$

and

$$nf' = \begin{cases} xf & \text{if } \exists x \in \lfloor p \rfloor.\ n = xi \\ 0_Y & \text{otherwise,} \end{cases}$$

respectively. Remark that $n \in \lfloor p' \rfloor$ if and only if there exists $x \in \lfloor p \rfloor$ such that $n = xi$. Hence

$$\begin{array}{rcl} pf_\diamond & = & \bigoplus_{x\in X} p_{[x]}\cdot(xf) \\ & = & \bigoplus_{n\in\mathbb{N}} p'_{[n]}\cdot(nf') \\ & = & p'f'_\diamond. \end{array}$$

(b_*) In the case of $\tau = *$:
Let $p \in \mathcal{Q}_*(X)$ and $f : X \to \mathcal{Q}_*(Y)$, and take the same p' and f' defined in (b_0). Then it is clear that $p' \in \mathcal{Q}_*(\mathbb{N})$ and $f' : \mathbb{N} \to \mathcal{Q}_*(Y)$.

(b_1) In the case of $\tau = 1$:
Let $p \in \mathcal{Q}_1(X)$ and $f : X \to \mathcal{Q}_1(Y)$, and take the same p' defined in (b_0) and define $f' : X \to \mathcal{Q}_1(Y)$ by

$$nf' = \begin{cases} xf & \text{if } \exists x \in \lfloor p \rfloor.\ n = xi \\ \dot{y}_0 & \text{otherwise,} \end{cases}$$

where y_0 is an arbitrary point of Y. Then it is clear that $p' \in \mathcal{Q}_1(\mathbb{N})$ and $f' : \mathbb{N} \to \mathcal{Q}_1(Y)$. □

4. Convex composition

In the rest of the paper the subscript τ is one of 0, 1 and $*$, unless otherwise stated. For a map $f : X \to \mathcal{Q}_\tau(Y)$ the convex combination induces a map $f_\diamond : \mathcal{Q}_\tau(X) \to \mathcal{Q}_\tau(Y)$ by Proposition 6 (a). We now list some basic properties of the induced maps.

Proposition 7. *Let $f : X \to \mathcal{Q}_\tau(Y)$, $g : Y \to \mathcal{Q}_\tau(Z)$ and $h : X \to \mathcal{Q}_\tau(X)$ be maps. Then the following hold:*

(a) $f_\diamond g_\diamond = (fg_\diamond)_\diamond$,
(b) $e_X f_\diamond = f$,
(c) $(e_X)_\diamond = \mathrm{id}_{\mathcal{Q}_\tau(X)}$,
(d) $h \sqsubseteq f(\xi^\tau_X)^\sharp \to h_\diamond \sqsubseteq f_\diamond(\xi^\tau_X)^\sharp$ *for* $\tau \neq 1$,
(e) $(\xi^\tau_X)^\sharp f_\diamond \sqsubseteq f_\diamond\,(\xi^\tau_Y)^\sharp$ *for* $\tau \neq 1$,
(f) $(\nabla_{XI}0_Y)_\diamond = \nabla_{\mathcal{Q}_\tau(X)I}0_Y$ *for* $\tau \neq 1$.
(g) $(\xi^\tau_X)^\sharp h_\diamond(\xi^\tau_X)^\sharp = h_\diamond(\xi^\tau_X)^\sharp$ *for* $\tau \neq 1$.

Proof. (a) $f_\diamond g_\diamond = (fg_\diamond)_\diamond$:

$$\begin{array}{rcll} p(fg_\diamond)_\diamond & = & \bigoplus_x p_{[x]}\cdot(xfg_\diamond) & \\ & = & \bigoplus_x (p_{[x]}\cdot(xf))g_\diamond & \{\ \text{Prop.5 (h)}\ \} \\ & = & \bigoplus_x \bigoplus_y (p_{[x]}(xf)_{[y]})\cdot(yg) & \\ & = & \bigoplus_y \bigoplus_x (p_{[x]}(xf)_{[y]})\cdot(yg) & \\ & = & \bigoplus_y (\sum_x p_{[x]}(xf)_{[y]})\cdot(yg) & \{\ \text{Prop.5 (a)}\ \} \\ & = & \bigoplus_y (pf_\diamond)_{[y]}\cdot(yg) & \\ & = & (pf_\diamond)g_\diamond & \\ & = & p(f_\diamond g_\diamond). & \end{array}$$

(b) $e_X f_\diamond = f$:

$$\begin{array}{rcll} xe_X f_\diamond & = & \dot{x}f_\diamond & \{\ xe_X = \dot{x}\ \} \\ & = & xf. & \{\ \text{Prop.5 (e)}\ \} \end{array}$$

(c) $(e_X)_\diamond = \mathrm{id}_{\mathcal{Q}_\tau(X)}$ is direct from Proposition 5 (f).
(d) $h \sqsubseteq f(\xi_X^\tau)^\sharp \ \rightarrow \ h_\diamond \sqsubseteq f_\diamond(\xi_X^\tau)^\sharp$:
For $p \in \mathcal{Q}_\tau(X)$ and $x \in X$ we have

$$\begin{array}{rcll} (ph_\diamond)_{[x]} & = & \sum_{x'} p_{[x']}(x'h)_{[x]} & \\ & \leq & \sum_{x'} p_{[x']}(x'f)_{[x]} & \{ \ x'h \leq x'f \leftarrow h \sqsubseteq f(\xi_X^\tau)^\sharp \ \} \\ & = & (pf_\diamond)_{[x]} & \end{array}$$

which proves $ph_\diamond \leq pf_\diamond$ and hence $ph_\diamond \sqsubseteq pf_\diamond(\xi_X^\tau)^\sharp$.
(e) $(\xi_X^\tau)^\sharp f_\diamond \sqsubseteq f_\diamond\,(\xi_Y^\tau)^\sharp$:

$$\begin{array}{cll} \rightarrow & p \leq p' \ \rightarrow \ pf_\diamond \leq p'f_\diamond & \\ \leftrightarrow & \xi_X^\tau \sqsubseteq f_\diamond\,\xi_Y^\tau f_\diamond^\sharp & \\ \leftrightarrow & (\xi_X^\tau)^\sharp \sqsubseteq f_\diamond\,(\xi_Y^\tau)^\sharp f_\diamond^\sharp & \\ \leftrightarrow & (\xi_X^\tau)^\sharp f_\diamond \sqsubseteq f_\diamond\,(\xi_Y^\tau)^\sharp. & \{ \ f_\diamond : \mathrm{tfn} \ \} \end{array}$$

(f) $(\nabla_{XI} 0_Y)_\diamond = \nabla_{\mathcal{Q}_\tau(X)\,I} 0_Y$:

$$\begin{array}{rcll} (\nabla_{XI} 0_Y)_\diamond & = & \sqcup_{p\in\mathcal{Q}_\tau(X)} p^\sharp p(\nabla_{XI} 0_Y)_\diamond & \{ \ (\mathrm{PA}) \ \} \\ & = & \sqcup_{p\in\mathcal{Q}_\tau(X)} p^\sharp(||p||\,0)_Y & \{ \ \mathrm{Prop.5\,(g)} \ \} \\ & = & \sqcup_{p\in\mathcal{Q}_\tau(X)} p^\sharp\,0_Y & \{ \ ||p||\,0 = 0 \ \} \\ & = & \nabla_{\mathcal{Q}_\tau(X)\,I} 0_Y. & \{ \ (\mathrm{PA}) \ \} \end{array}$$

(g) $(\xi_X^\tau)^\sharp h_\diamond (\xi_X^\tau)^\sharp = h_\diamond (\xi_X^\tau)^\sharp$:

$$\begin{array}{rcll} (\xi_X^\tau)^\sharp h_\diamond (\xi_X^\tau)^\sharp & \sqsubseteq & h_\diamond (\xi_X^\tau)^\sharp (\xi_X^\tau)^\sharp & \{ \ (\mathrm{e}) \ \} \\ & = & h_\diamond (\xi_X^\tau)^\sharp. & \{ \ (\xi_X^\tau)^\sharp (\xi_X^\tau)^\sharp = (\xi_X^\tau)^\sharp \ \} \end{array}$$

$$\begin{array}{rcl} h_\diamond (\xi_X^\tau)^\sharp & = & \mathrm{id}_{\mathcal{Q}_\tau(X)} h_\diamond (\xi_X^\tau)^\sharp \\ & \sqsubseteq & (\xi_X^\tau)^\sharp h_\diamond (\xi_X^\tau)^\sharp. \end{array}$$

□

For a relation $\alpha : X \rightharpoondown \mathcal{Q}_\tau(Y)$ define a relation $\alpha_\diamond : \mathcal{Q}_\tau(X) \rightharpoondown \mathcal{Q}_\tau(Y)$ by

$$\alpha_\diamond = \sqcup_{f \sqsubseteq \alpha} f_\diamond,$$

where $f \sqsubseteq \alpha$ means that f is a map $f : X \rightarrow \mathcal{Q}_\tau(Y)$ such that $f \sqsubseteq \alpha$. This notion allows convex composition to be treated in ordinary relational calculus.

Remark. By the relational axiom of choice (AC) there exists a map $f \sqsubseteq \alpha$ iff α is total. Such a map f is often called a choice function of α. Also $\alpha_\diamond$ is total if α is total, and $\alpha_\diamond = \emptyset_{\mathcal{Q}_\tau(X)\mathcal{Q}_\tau(Y)}$ otherwise.

Example 2. Consider relations $\gamma = g \sqcup h$ and $\gamma' = h \sqcup e_X$ where $g,\, h : X \rightarrow \mathcal{Q}_*(X)$ appeared in Example 1. Since there is no maps included in $\gamma = g \sqcup h$ other than g and h, the identity $\gamma_\diamond = g_\diamond \sqcup h_\diamond$ holds. For a relation γ', the identity $\gamma'_\diamond = h_\diamond \sqcup e_{X\diamond}$ does not hold. Because γ' consists of four maps, that is $h \sqcup e_X = f' \sqcup g \sqcup h \sqcup e_X$ where $f', g : X \rightarrow \mathcal{Q}_*(X)$ are maps defined by $xf' = yf' = \dot{y}$, $xg = \dot{x}$, and $yg = 0_X$. Therefore $\gamma'_\diamond = f'_\diamond \sqcup g_\diamond \sqcup h_\diamond \sqcup e_{X\diamond}$ holds. □

Proposition 8. *If $\alpha : X \rightharpoondown Y$ is a total relation, then $\alpha = \sqcup_{f \sqsubseteq \alpha} f$.*

Proof. Assume α is total. By the axiom of choice (AC) there is a function $f_0 : X \rightarrow Y$ such that $f_0 \sqsubseteq \alpha$. For each $(x_0, y_0) \in \alpha$ define a map $f : X \rightarrow Y$ by

$$\forall x \in X.\ xf = \begin{cases} y_0 & \text{if } x = x_0, \\ xf_0 & \text{otherwise.} \end{cases}$$

Then it is clear that $(x,q) \in f$ and so

$$\begin{array}{rcll} f & = & \sqcup_{x\in X} x^\sharp x f & \{ \text{(PA)} \ \} \\ & = & x_0^\sharp y_0 \sqcup (\sqcup_{x\neq x_0} x^\sharp x f_0) & \\ & \sqsubseteq & \alpha \sqcup f_0 & \{ \ x^\sharp x \sqsubseteq \mathrm{id}_X \ \} \\ & = & \alpha. & \{ \ f_0 \sqsubseteq \alpha \ \} \end{array}$$

Hence

$$\begin{array}{rcl} (x_0,y_0) \in \alpha & \to & \exists f.\ (x_0,y_0) \in f \wedge (f \sqsubseteq \alpha) \\ & \to & (x_0,y_0) \in \sqcup_{f\sqsubseteq\alpha} f, \end{array}$$

which shows $\alpha = \sqcup_{f\sqsubseteq\alpha} f$. □

A relation $\alpha : X \rightharpoondown \mathcal{Q}_\tau(Y)$ is called down-closed if it satisfies $\alpha(\xi_Y^\tau)^\sharp = \alpha$. The next proposition indicates that a relation α is total and down-closed iff it is 0-included [16], namely, $0_Y \in x\alpha$ for each $x \in X$.

Proposition 9. *Let $\tau \neq 1$. If $\alpha : X \rightharpoondown \mathcal{Q}_\tau(Y)$ is a total relation such that $\alpha(\xi_Y^\tau)^\sharp = \alpha$, then $\nabla_{XI} 0_Y \sqsubseteq \alpha$* (0-included).

Proof. Assume α is total and $\alpha(\xi_Y^\tau)^\sharp = \alpha$. By the axiom of choice (AC) there is a function $f_0 : X \to \mathcal{Q}_*(Y)$ such that $f_0 \sqsubseteq \alpha$.

$$\begin{array}{rcll} \nabla_{XI} 0_Y & = & \sqcup_{x\in X} x^\sharp 0_Y & \{ \text{(PA)} \ \} \\ & \sqsubseteq & \sqcup_{x\in X} x^\sharp x f_0 (\xi_Y^\tau)^\sharp & \{ \ \forall q \in \mathcal{Q}_\tau(Y).\ 0_Y \sqsubseteq q(\xi_Y^\tau)^\sharp \ \} \\ & = & f_0(\xi_Y^\tau)^\sharp & \{ \text{(PA)} \ \} \\ & \sqsubseteq & \alpha(\xi_Y^\tau)^\sharp & \{ \ f_0 \sqsubseteq \alpha \ \} \\ & = & \alpha. & \{ \ \alpha(\xi_Y^\tau)^\sharp = \alpha \ \} \end{array}$$

□

The diamond operator defined via convex combinations satisfies the following additional rules.

Proposition 10. *Let $\tau \neq 1$. For a map $f : X \to \mathcal{Q}_\tau(Y)$, a relation $\alpha : X \rightharpoondown \mathcal{Q}_\tau(Y)$ and $t \in \mathcal{Q}(Y)$ the following hold:*

(a) $f_\diamond t_* = (ft_*)_\diamond$,
(b) $\alpha_\diamond t_* \sqsubseteq (\alpha t_*)_\diamond$,
(c) $\alpha_\diamond (\xi_Y^\tau)^\sharp \sqsubseteq (\alpha(\xi_Y^\tau)^\sharp)_\diamond$,
(d) $(f(\xi_X^\tau)^\sharp)_\diamond = f_\diamond(\xi_X^\tau)^\sharp$.

Proof. (a) $f_\diamond t_* = (ft_*)_\diamond$:
For $p \in \mathcal{Q}_\tau(X)$ we have

$$\begin{array}{rcl} p(f_\diamond t_*) & = & (pf_\diamond)t_* \\ & = & (pf_\diamond)\cdot t \\ & = & (\bigoplus_{x\in X} p_{[x]}\cdot(xf))\cdot t \\ & = & \bigoplus_{x\in X} (p_{[x]}\cdot(xf))\cdot t \\ & = & \bigoplus_{x\in X} p_{[x]}\cdot((xf)\cdot t) \\ & = & \bigoplus_{x\in X} p_{[x]}\cdot(xft_*) \\ & = & p(ft_*)_\diamond. \end{array}$$

(b) $\alpha_\diamond t_* \sqsubseteq (\alpha t_*)_\diamond$:

$$\begin{array}{rcll} \alpha_\diamond t_* & = & (\sqcup_{f\sqsubseteq\alpha} f_\diamond)t_* & \\ & = & \sqcup_{f\sqsubseteq\alpha} f_\diamond t_* & \\ & = & \sqcup_{f\sqsubseteq\alpha} (ft_*)_\diamond & \{ \text{(a)} \ \} \\ & \sqsubseteq & \sqcup_{f'\sqsubseteq\alpha t_*} f'_\diamond & \{ \ ft_* \sqsubseteq \alpha t_* \ \} \\ & = & (\alpha t_*)_\diamond. & \end{array}$$

(c) $\alpha_\diamond(\xi_Y^\tau)^\sharp \sqsubseteq (\alpha(\xi_Y^\tau)^\sharp)_\diamond$:

$$\begin{array}{rcll} \alpha_\diamond(\xi_Y^\tau)^\sharp & = & \alpha_\diamond(\sqcup_{t\in\mathcal{Q}(Y)}t_*) & \{ (\xi_Y^\tau)^\sharp = \sqcup_{t\in\mathcal{Q}(Y)}t_* \} \\ & = & \sqcup_{t\in\mathcal{Q}(Y)}\alpha_\diamond t_* & \\ & \sqsubseteq & \sqcup_{t\in\mathcal{Q}(Y)}(\alpha t_*)_\diamond & \{ \text{(b)} \} \\ & \sqsubseteq & (\alpha(\xi_Y^\tau)^\sharp)_\diamond. & \{ t_* \sqsubseteq (\xi_Y^\tau)^\sharp \} \end{array}$$

(d) $(f(\xi_X^\tau)^\sharp)_\diamond = f_\diamond(\xi_X^\tau)^\sharp$:

$$\begin{array}{rcll} (f(\xi_X^\tau)^\sharp)_\diamond & = & \sqcup_{h\sqsubseteq f(\xi_X^\tau)^\sharp}h_\diamond & \\ & \sqsubseteq & f_\diamond(\xi_X^\tau)^\sharp. & \{ \text{Prop.7 (d)} \} \end{array}$$

The opposite direction $f_\diamond(\xi_X^\tau)^\sharp \sqsubseteq (f(\xi_X^\tau)^\sharp)_\diamond$ follows from (c). □

Now we will define a composition [4, 7] for relations into algebras of probabilistic distributions.

Definition 2. Let $\alpha : X \rightharpoondown \mathcal{Q}_\tau(Y)$ and $\beta : Y \rightharpoondown \mathcal{Q}_\tau(Z)$ be relations. The convex composite $\alpha \circ \beta : X \rightharpoondown \mathcal{Q}_\tau(Z)$ of α followed by β is defined by

$$\alpha \circ \beta = \alpha\beta_\diamond.$$

□

Remark. In some aspects, convex composition seems to be concrete examples of Kleisli composition of the powerset monads studied by Eklund and Gäehler [2]. However, in our case the composition chooses a map from latter argument in nondeterministic way, whereas Kleisli composition chooses in deterministic way.

We show the basic properties on convex composition of relations.

Proposition 11. *Let $\alpha, \alpha' : X \rightharpoondown \mathcal{Q}_\tau(Y)$, $\beta, \beta' : Y \rightharpoondown \mathcal{Q}_\tau(Z)$ and $\gamma : Z \rightharpoondown \mathcal{Q}_\tau(W)$ be relations. Then*

(a) $\beta \sqsubseteq \beta' \rightarrow \beta_\diamond \sqsubseteq \beta'_\diamond$,
(b) $\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \rightarrow \alpha \circ \beta \sqsubseteq \alpha' \circ \beta'$,
(c) $\alpha_\diamond\beta_\diamond \sqsubseteq (\alpha \circ \beta)_\diamond$,
(d) $(\alpha \circ \beta) \circ \gamma \sqsubseteq \alpha \circ (\beta \circ \gamma)$,
(e) $\alpha : \mathit{total} \rightarrow e_X\alpha_\diamond = \alpha$,
(f) $\alpha : \mathit{total} \rightarrow \alpha \circ \nabla_{YI}0_Z = \nabla_{XI}0_Z$ *for* $\tau \neq 1$,
(g) $\alpha : \mathit{total} \rightarrow 0_X \circ \alpha = 0_Y$ *for* $\tau \neq 1$,
(h) $(\alpha \circ \beta)(\xi_Z^\tau)^\sharp \sqsubseteq \alpha \circ \beta(\xi_Z^\tau)^\sharp$ *for* $\tau \neq 1$.

Proof. (a) $\beta \sqsubseteq \beta' \rightarrow \beta_\diamond \sqsubseteq \beta'_\diamond$:
Assume $\beta \sqsubseteq \beta'$. Then

$$\begin{array}{rcll} \beta_\diamond & = & \sqcup_{g\sqsubseteq\beta}g_\diamond & \\ & \sqsubseteq & \sqcup_{g\sqsubseteq\beta'}g_\diamond & \{ \beta \sqsubseteq \beta' \} \\ & = & \beta'_\diamond. & \end{array}$$

(b) $\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \rightarrow \alpha \circ \beta \sqsubseteq \alpha' \circ \beta'$:
Assume $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$. Then

$$\begin{array}{rcll} \alpha \circ \beta & = & \alpha\beta_\diamond & \\ & \sqsubseteq & \alpha'\beta'_\diamond & \{ \alpha \sqsubseteq \alpha', \beta \sqsubseteq \beta', \text{(a)} \} \\ & = & \alpha' \circ \beta'. & \end{array}$$

(c) $\alpha_\diamond\beta_\diamond \sqsubseteq (\alpha \circ \beta)_\diamond$:
Note that for maps $f : X \to \mathcal{Q}_\tau(Y)$ and $g : Y \to \mathcal{Q}_\tau(Z)$ such that $f \sqsubseteq \alpha$ and $g \sqsubseteq \beta$, we have

$$\begin{array}{rcll} f_\diamond g_\diamond & = & (fg_\diamond)_\diamond & \{ \text{Prop.7 (a)} \} \\ & \sqsubseteq & (\alpha\beta_\diamond)_\diamond & \{ \text{(a), (b)} \} \\ & = & (\alpha \circ \beta)_\diamond. & \end{array}$$

Hence

$$\begin{aligned}
\alpha_\diamond\beta_\diamond &= (\sqcup_{f\sqsubseteq\alpha} f_\diamond)(\sqcup_{g\sqsubseteq\beta} g_\diamond) \\
&= \sqcup_{f\sqsubseteq\alpha}\sqcup_{g\sqsubseteq\beta} f_\diamond g_\diamond \\
&\sqsubseteq (\alpha\circ\beta)_\diamond. && \{\ f_\diamond g_\diamond \sqsubseteq (\alpha\circ\beta)_\diamond\ \}
\end{aligned}$$

(d) $(\alpha\circ\beta)\circ\gamma \sqsubseteq \alpha\circ(\beta\circ\gamma)$:

$$\begin{aligned}
(\alpha\circ\beta)\circ\gamma &= (\alpha\beta_\diamond)\gamma_\diamond \\
&= \alpha(\beta_\diamond\gamma_\diamond) \\
&\sqsubseteq \alpha(\beta\circ\gamma)_\diamond && \{\ (c)\ \} \\
&= \alpha\circ(\beta\circ\gamma).
\end{aligned}$$

(e) α : total $\to$ $e_X\alpha_\diamond = \alpha$:

$$\begin{aligned}
e_X\alpha_\diamond &= e_X(\sqcup_{f\sqsubseteq\alpha} f_\diamond) \\
&= \sqcup_{f\sqsubseteq\alpha} e_X f_\diamond \\
&= \sqcup_{f\sqsubseteq\alpha} f && \{\ \text{Prop.7 (b)}\ \} \\
&= \alpha. && \{\ \alpha : \text{total}\ \}
\end{aligned}$$

(f) α : total $\to$ $\alpha\circ\nabla_{YI}0_Z = \nabla_{XI}0_Z$:

$$\begin{aligned}
\alpha\circ\nabla_{YI}0_Z &= \alpha(\nabla_{YI}0_Z)_\diamond \\
&= \alpha\nabla_{\mathcal{Q}_\tau(Y)I}0_Z && \{\ \text{Prop.7 (f)}\ \} \\
&= \nabla_{XI}0_Z. && \{\ \alpha : \text{total}\ \}
\end{aligned}$$

(g) α : total $\to$ $0_X\circ\alpha = 0_Y$:

$$\begin{aligned}
0_X\circ\alpha &= 0_X(\sqcup_{f\sqsubseteq\alpha} f_\diamond) \\
&= \sqcup_{f\sqsubseteq\alpha} 0_X f_\diamond \\
&= 0_Y. && \{\ \text{Prop.5 (d)}\ \}
\end{aligned}$$

(h) $(\alpha\circ\beta)(\xi^\tau_Z)^\sharp \sqsubseteq \alpha\circ\beta(\xi^\tau_Z)^\sharp$:

$$\begin{aligned}
(\alpha\circ\beta)(\xi^\tau_Z)^\sharp &= (\alpha\beta_\diamond)(\xi^\tau_Z)^\sharp \\
&= \alpha(\beta_\diamond(\xi^\tau_Z)^\sharp) \\
&\sqsubseteq \alpha(\beta(\xi^\tau_Z)^\sharp)_\diamond && \{\ \text{Prop.10 (c)}\ \} \\
&= \alpha\circ\beta(\xi^\tau_Z)^\sharp.
\end{aligned}$$

□

By Proposition 7(c) and Proposition 11(e), if α is total then e_X is neutral for convex composition, that is, $\alpha\circ e_X = e_X\circ\alpha = \alpha$.

The following proposition shows that ε^τ_X is identity element for convex composition in the case of $\tau\neq 1$ and $\alpha(\xi^\tau_Y)^\sharp \sqsubseteq \alpha$ (down-closed).

Proposition 12. *Let $\alpha : X \rightharpoonup \mathcal{Q}_\tau(Y)$ be a total relation for $\tau\neq 1$. Then the following holds:*

(a) $\alpha \sqsubseteq \varepsilon^\tau_X\circ\alpha \sqsubseteq \alpha(\xi^\tau_Y)^\sharp$,
(b) $\alpha \sqsubseteq \alpha\circ\varepsilon^\tau_Y \sqsubseteq \alpha(\xi^\tau_Y)^\sharp$.

Proof. (a) $\alpha \sqsubseteq \varepsilon^\tau_X\circ\alpha \sqsubseteq \alpha(\xi^\tau_Y)^\sharp$:

$$\begin{aligned}
\alpha &= e_X\alpha_\diamond && \{\ \alpha : \text{total, Prop.11 (e)}\ \} \\
&\sqsubseteq e_X(\xi^\tau_X)^\sharp\alpha_\diamond && \{\ e_X \sqsubseteq e_X(\xi^\tau_X)^\sharp = \varepsilon^\tau_X\ \} \\
&= e_X(\xi^\tau_X)^\sharp(\sqcup_{f\sqsubseteq\alpha} f_\diamond) \\
&\sqsubseteq e_X(\sqcup_{f\sqsubseteq\alpha} f_\diamond)(\xi^\tau_X)^\sharp && \{\ \text{Prop.7 (e)}\ \} \\
&= e_X\alpha_\diamond(\xi^\tau_X)^\sharp \\
&= \alpha(\xi^\tau_Y)^\sharp.
\end{aligned}$$

(b) $\alpha \sqsubseteq \alpha\circ\varepsilon^\tau_Y \sqsubseteq \alpha(\xi^\tau_Y)^\sharp$:

$$\begin{aligned}
\alpha &= \alpha(e_Y)_\diamond && \{\ \text{Prop.7 (c)}\ \} \\
&\sqsubseteq \alpha(\varepsilon^\tau_Y)_\diamond && \{\ e_Y \sqsubseteq \varepsilon^\tau_Y\ \} \\
&\sqsubseteq \alpha(\xi^\tau_Y)^\sharp. && \{\ (\varepsilon^\tau_Y)_\diamond \sqsubseteq (\xi^\tau_Y)^\sharp\ \}
\end{aligned}$$

□

5. Associative law

In this section we will introduce the convex relations and study the associative law of convex composition on their relations. For a relation $\gamma : Z \rightharpoonup \mathcal{Q}_\tau(W)$ define a relation $\gamma^\bullet : Z \rightharpoonup \mathcal{Q}_\tau(W)$ by

$$\forall z \in Z.\ z\gamma^\bullet = \nabla_{I\mathcal{Q}_\tau(\mathbb{N})} \circ \nabla_{\mathbb{N}I} z\gamma.$$

Note that $\rho^\bullet = \nabla_{I\mathcal{Q}_\tau(\mathbb{N})} \circ \nabla_{\mathbb{N}I}\rho$ for a relation $\rho : I \rightharpoonup \mathcal{Q}_\tau(W)$.

Remark. The definition of $\gamma^\bullet$ explicitly contains an element (or a variable) beyond preferable relational expressions.

The notion of $\gamma^\bullet$ derives a property called *convex* for relations to satisfy the associativity of convex composition. A relation $\gamma : Z \rightharpoonup \mathcal{Q}_\tau(W)$ is called convex if it satisfies $\gamma^\bullet = \gamma$.

Example 3. Consider on the same X as in previous examples. Define a relation $\alpha : X \rightharpoonup \mathcal{Q}_1(X)$ by $x\alpha = y\alpha = (\frac{1}{2})_X$. Then $\alpha^\bullet : X \rightharpoonup \mathcal{Q}_1(X)$ satisfies $\alpha^\bullet = \alpha$. However when we regard α as $\alpha : X \rightharpoonup \mathcal{Q}_*(X)$, $\alpha^\bullet : X \rightharpoonup \mathcal{Q}_*(X)$ satisfies $x\alpha^\bullet = y\alpha^\bullet = (\frac{1}{2})_X(\xi_X^*)^\sharp$, that is $\alpha^\bullet \neq \alpha$. For a relation $\gamma : X \rightharpoonup \mathcal{Q}_*(X)$ which appeared in Example 2, we obtain that $\gamma^\bullet \neq \gamma$ since $x\gamma^\bullet = \mathcal{Q}_*(X)$ though $x\gamma = \dot{x} \sqcup \dot{y}$. $\square$

Proposition 13. *Let $\gamma : Z \rightharpoonup \mathcal{Q}_\tau(W)$ be a relation.*

(a) *If γ is total, then $\gamma \sqsubseteq \gamma^\bullet$,*
(b) *$\nabla_{I\mathcal{Q}_\tau(Y)} \circ \nabla_{YI} z\gamma \sqsubseteq z\gamma^\bullet$ for all sets Y.*
(c) *$\gamma^{\bullet\bullet} \sqsubseteq \gamma^\bullet$.*
(d) *$\gamma^\bullet(\xi_W^\tau)^\sharp \sqsubseteq (\gamma(\xi_W^\tau)^\sharp)^\bullet \quad (\tau \neq 1)$.*

Proof. Set $\rho = z\gamma$. Then $\rho^\bullet = z\gamma^\bullet$, $\rho^{\bullet\bullet} = z\gamma^{\bullet\bullet}$ and $(\rho(\xi_W^\tau)^\sharp)^\bullet = z(\gamma(\xi_W^\tau)^\sharp)^\bullet$. Thus it suffices to show the following statements for ρ.
(a) γ : total $\to$ $\rho \sqsubseteq \rho^\bullet$:

$$\begin{array}{rcll} \rho & = & \nabla_{I\mathbb{N}}\nabla_{\mathbb{N}I}\rho & \{\ \mathbb{N} \neq \emptyset\ \} \\ & = & \nabla_{I\mathbb{N}} e_{\mathbb{N}}(\nabla_{\mathbb{N}I}\rho)_\diamond & \{\ \text{Prop.11 (e)},\ \rho : \text{total}\ \} \\ & \sqsubseteq & \nabla_{I\mathcal{Q}_\tau(\mathbb{N})}(\nabla_{\mathbb{N}I}\rho)_\diamond. & \end{array}$$

(b) $\nabla_{I\mathcal{Q}_\tau(Y)} \circ \nabla_{YI}\rho \sqsubseteq \nabla_{I\mathcal{Q}_\tau(\mathbb{N})} \circ \nabla_{\mathbb{N}I}\rho$:

$$\begin{array}{rcll} \nabla_{I\mathcal{Q}_\tau(Y)}(\nabla_{YI}\rho)_\diamond & = & (\sqcup_{p\in\mathcal{Q}_\tau(Y)}p)(\sqcup_{f\sqsubseteq\nabla_{YI}\rho}f_\diamond) & \\ & = & \sqcup_{p\in\mathcal{Q}_\tau(Y)} \sqcup_{f\sqsubseteq\nabla_{YI}\rho} pf_\diamond & \\ & \sqsubseteq & \sqcup_{p'\in\mathcal{Q}_\tau(\mathbb{N})} \sqcup_{f'\sqsubseteq\nabla_{\mathbb{N}I}\rho} (p'f'_\diamond) & \{\ \text{Prop.6 (b)}\ \} \\ & = & \nabla_{I\mathcal{Q}_\tau(\mathbb{N})}(\nabla_{\mathbb{N}I}\rho)_\diamond. & \end{array}$$

(c) Recall that

$$\begin{array}{rcl} q' \sqsubseteq \rho^{\bullet\bullet} & \to & \exists p' \in \mathcal{Q}_\tau(\mathbb{N})\,\exists f' : \mathbb{N} \to \mathcal{Q}_\tau(X) \\ & & q' = p'f'_\diamond \wedge \forall n \in \mathbb{N}.\ nf' \sqsubseteq \rho^\bullet \\ nf' \sqsubseteq \rho^\bullet & \to & \exists p_n \in \mathcal{Q}_\tau(\mathbb{N})\,\exists f_n : \mathbb{N} \to \mathcal{Q}_\tau(X) \\ & & nf' = p_n(f_n)_\diamond \wedge \forall m \in \mathbb{N}.\ mf_n \sqsubseteq \rho, \end{array}$$

and so

$$\begin{array}{rcl} q' & = & p'f'_\diamond \\ & = & \bigoplus_{n\in\mathbb{N}} p'_{[n]}(nf') \\ & = & \bigoplus_{n\in\mathbb{N}} p'_{[n]}(p_n(f_n)_\diamond) \\ & = & \bigoplus_{n\in\mathbb{N}} p'(n)(\bigoplus_{m\in\mathbb{N}} p_{n[m]}(mf_n)) \\ & = & \bigoplus_{n\in\mathbb{N}} \bigoplus_{m\in\mathbb{N}} p'_{[n]}p_{n[m]}(mf_n). \end{array}$$

Define $\hat{p} \in \mathcal{Q}_\tau(\mathbb{N}\times\mathbb{N})$ and $\hat{f} : \mathbb{N}\times\mathbb{N} \to \mathcal{Q}_\tau(X)$ by $(n,m)\hat{p} = p'_{[n]}p_{n[m]}$ and $(n,m)\hat{f} = mf_n$, respectively. Then

$$\begin{array}{rcll} q' & = & \hat{p}\hat{f}_\diamond & \\ & \sqsubseteq & \nabla_{I\mathcal{Q}_\tau(\mathbb{N}\times\mathbb{N})} \circ (\nabla_{\mathbb{N}\times\mathbb{N}I}\rho) & \\ & \sqsubseteq & \nabla_{I\mathcal{Q}_\tau(\mathbb{N})} \circ (\nabla_{\mathbb{N}I}\rho) & \{ \text{ (b) } \} \\ & = & \rho^\bullet. & \end{array}$$

This proves $\rho^{\bullet\bullet} \sqsubseteq \rho^\bullet$.
(d) $\gamma^\bullet(\xi^\tau_W)^\sharp \sqsubseteq (\gamma(\xi^\tau_W)^\sharp)^\bullet$: $\quad (\tau \neq 1)$

$$\begin{array}{rcll} \rho^\bullet(\xi^\tau_W)^\sharp & = & \nabla_{I\mathcal{Q}_*(\mathbb{N})}(\nabla_{\mathbb{N}I}\rho)_\diamond(\xi^\tau_W)^\sharp & \\ & \sqsubseteq & \nabla_{I\mathcal{Q}_*(\mathbb{N})}(\nabla_{\mathbb{N}I}\rho(\xi^\tau_W)^\sharp)_\diamond & \{ \text{ Prop. 10 (c) } \} \\ & = & (\rho(\xi^\tau_W)^\sharp)^\bullet. & \end{array}$$

□

Now we define two kinds of convex relations, named $\mathcal{Q}_*$-convex relation and $\mathcal{Q}_1$-convex relation.

Definition 3. A relation $\alpha : X \rightharpoonup \mathcal{Q}_*(Y)$ is called *$\mathcal{Q}_*$-convex* if $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$ (total), $\alpha(\xi^*_Y)^\sharp = \alpha$ (down-closed) and $\alpha^\bullet = \alpha$ (convex). A relation $\alpha : X \rightharpoonup \mathcal{Q}_1(Y)$ is called *$\mathcal{Q}_1$-convex* if $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$ (total) and $\alpha^\bullet = \alpha$ (convex). □

By Proposition 9, a $\mathcal{Q}_*$-convex relation α is 0-included, that is α satisfies $\nabla_{XI}0_Y \sqsubseteq \alpha$.
We need the following lemma to derive the associative law of convex composition.

Lemma 1. *Let $f : Y \to \mathcal{Q}_\tau(W)$ be a map, and $\beta : Y \rightharpoonup \mathcal{Q}_\tau(Z)$ and $\gamma : Z \rightharpoonup \mathcal{Q}_\tau(W)$ relations. If $f \sqsubseteq \beta\gamma_\diamond$, then $f_\diamond \sqsubseteq \beta_\diamond(\gamma^\bullet)_\diamond$.*

Proof. Let $f \sqsubseteq \beta\gamma_\diamond$ and $p \in \mathcal{Q}_\tau(Y)$. Then
(1) $\exists g \sqsubseteq \beta.\ f \sqsubseteq g\gamma_\diamond$:
As $f \sqsubseteq \beta\gamma_\diamond$ it holds that

$$\begin{array}{rcll} \mathrm{id}_Y & = & ff^\sharp \sqcap \mathrm{id}_Y & \{ f : \text{tfn} \} \\ & \sqsubseteq & \beta\gamma_\diamond f^\sharp \sqcap \mathrm{id}_Y & \{ f \sqsubseteq \beta\gamma_\diamond \} \\ & \sqsubseteq & (\beta \sqcap f(\gamma_\diamond)^\sharp)(\beta^\sharp \sqcap \gamma_\diamond f^\sharp). & \{ (\mathrm{DF}_*) \} \end{array}$$

Hence $\beta \sqcap f(\gamma_\diamond)^\sharp$ is total and by the axiom of choice (AC) there exists a tfn $g : Y \to \mathcal{Q}_\tau(Z)$ such that $g \sqsubseteq \beta \sqcap f(\gamma_\diamond)^\sharp$, which is equivalent to $g \sqsubseteq \beta$ and $f \sqsubseteq g\gamma_\diamond$.
(2) $\forall y \in Y\ \exists h_y \sqsubseteq \gamma.\ yf = yg(h_y)_\diamond$:
Note that

$$\begin{array}{rcll} yf & \sqsubseteq & yg\gamma_\diamond & \{ f \sqsubseteq g\gamma_\diamond \} \\ & = & \sqcup_{h \sqsubseteq \gamma} ygh_\diamond. & \{ \gamma_\diamond = \sqcup_{h\sqsubseteq\gamma} h_\diamond \} \end{array}$$

Thus there exists $h_y \sqsubseteq \gamma$ such that $yf = yg(h_y)_\diamond$.
(3) Define a map $r_z \in \mathcal{Q}(Y)$ by

$$r_{z[y]} = \begin{cases} \dfrac{p_{[y]}(yg)_{[z]}}{(pg_\diamond)_{[z]}} & \text{if } (pg_\diamond)_{[z]} > 0, \\ p_{[y]} & \text{otherwise.} \end{cases}$$

(4) $(pg_\diamond)_{[z]}r_{z[y]} = p_{[y]}(yg)_{[z]}$ and $r_z \in \mathcal{Q}_\tau(Y)$, i.e., $r_z \sqsubseteq \nabla_{I\mathcal{Q}_\tau(Y)}$:
If $\tau = 1$ then $(pg_\diamond)_{[z]} = 0$ implies $(yg)_{[z]} = 0$ for each $y \in Y$. Even if $\tau \neq 1$, $(pg_\diamond)_{[z]} = 0$ implies $p = 0_Y$ or $(yg)_{[z]} = 0$ for each $y \in Y$. In each case it is clear that $(pg_\diamond)_{[z]}r_{z[y]} = p_{[y]}(yg)_{[z]}$.

If $(pg_\diamond)_{[z]} = 0$ then $r_z = p \in \mathcal{Q}_\tau(Y)$. If $(pg_\diamond)_{[z]} \neq 0$ then

$$\begin{array}{rcl} ||r_z|| &=& \sum_y \dfrac{p_{[y]}(yg)_{[z]}}{(pg_\diamond)_{[z]}} \\ &=& \dfrac{\sum_y p_{[y]}(yg)_{[z]}}{(pg_\diamond)_{[z]}} \\ &=& \dfrac{(pg_\diamond)_{[z]}}{(pg_\diamond)_{[z]}} \\ &=& 1, \end{array}$$

and $\lfloor r_z \rfloor \subseteq \lfloor p \rfloor$, since $p_{[y]} = 0$ implies $r_{z[y]} = 0$. Hence $r_z \in \mathcal{Q}_\tau(Y)$.
(5) For all $z \in Z$ define a map $\hat{h}_z : Y \to \mathcal{Q}_\tau(W)$ by $\forall y \in Y.\ y\hat{h}_z = zh_y$.
(6) $(\hat{h}_z)_\diamond \sqsubseteq (\nabla_{YI} z\gamma)_\diamond$:

$$\begin{array}{rcll} \hat{h}_z &=& \sqcup_{y\in Y} y^\sharp z h_y & \\ &\sqsubseteq& \sqcup_{y\in Y} y^\sharp z\gamma & \{\ h_y \sqsubseteq \gamma\ \} \\ &=& \nabla_{YI} z\gamma, & \{\ \sqcup_{y\in Y}\ y^\sharp = \nabla_{YI}\ \} \end{array}$$

which implies $(\hat{h}_z)_\diamond \sqsubseteq (\nabla_{YI} z\gamma)_\diamond$.
(7) Define a map $h : Z \to \mathcal{Q}_\tau(W)$ by

$$\forall z \in Z.\ zh = r_z(\hat{h}_z)_\diamond.$$

(8) $h \sqsubseteq \gamma^\bullet$:

$$\begin{array}{rcll} zh &=& r_z(\hat{h}_z)_\diamond & \{\ (7)\ \} \\ &\sqsubseteq& \nabla_{I\mathcal{Q}_\tau(Y)}(\nabla_{YI} z\gamma)_\diamond & \{\ (4),\ (6)\ \} \\ &\sqsubseteq& z\gamma^\bullet. & \{\ \text{Prop.13 (b)}\ \} \end{array}$$

(9) $pf_\diamond = pg_\diamond h_\diamond$:

$$\begin{array}{rcll} pf_\diamond &=& \bigoplus_y p_{[y]}\cdot(yf) & \\ &=& \bigoplus_y p_{[y]}\cdot(yg(h_y)_\diamond) & \{\ (2)\ yf = yg(h_y)_\diamond\ \} \\ &=& \bigoplus_y (p_{[y]}\cdot(yg))(h_y)_\diamond & \{\ \text{Prop.5 (h)}\ \} \\ &=& \bigoplus_y \bigoplus_z (p_{[y]}(yg)_{[z]})\cdot(zh_y) & \\ &=& \bigoplus_z \bigoplus_y ((pg_\diamond)_{[z]} r_{z[y]})\cdot(zh_y) & \{\ (4)\ p_{[y]}(yg)_{[z]} = (pg_\diamond)_{[z]} r_{z[y]}\ \} \\ &=& \bigoplus_z \bigoplus_y ((pg_\diamond)_{[z]}\cdot r_z)_{[y]}\cdot(y\hat{h}_z) & \{\ (5)\ zh_y = y\hat{h}_z\ \} \\ &=& \bigoplus_z ((pg_\diamond)_{[z]}\cdot r_z)(\hat{h}_z)_\diamond & \\ &=& \bigoplus_z (pg_\diamond)_{[z]}\cdot(r_z(\hat{h}_z)_\diamond) & \{\ \text{Prop.5 (h)}\ \} \\ &=& \bigoplus_z (pg_\diamond)_{[z]}\cdot(zh) & \{\ (7)\ zh = r_z(\hat{h}_z)_\diamond\ \} \\ &=& (pg_\diamond)h_\diamond. & \end{array}$$

Note that h depends on p and so $f_\diamond = g_\diamond h_\diamond$ may not hold.
(10) $f_\diamond \sqsubseteq \beta_\diamond(\gamma^\bullet)_\diamond$:
For each $p \in \mathcal{Q}_*(Y)$ we have

$$\begin{array}{rcll} pf_\diamond &=& pg_\diamond h_\diamond & \{\ (9)\ \} \\ &\sqsubseteq& p\beta_\diamond(\gamma^\bullet)_\diamond, & \{\ (1)\ g \sqsubseteq \beta,\ (8)\ h \sqsubseteq \gamma^\bullet\ \} \end{array}$$

and hence $f_\diamond \sqsubseteq \beta_\diamond(\gamma^\bullet)_\diamond$. This completes the proof. □

Corollary 2. *Let $\alpha : X \rightharpoonup \mathcal{Q}_\tau(Y)$, $\beta : Y \rightharpoonup \mathcal{Q}_\tau(Z)$ and $\gamma : Z \rightharpoonup \mathcal{Q}_\tau(W)$ be relations. Then*

(a) $\alpha \circ (\beta \circ \gamma) \sqsubseteq (\alpha \circ \beta) \circ \gamma^\bullet$,
(b) *If $\gamma^\bullet = \gamma$, then $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$.*

Proof. (a)

$$\begin{array}{rcll} \alpha\circ(\beta\circ\gamma) & = & \alpha(\beta\circ\gamma)_\diamond & \\ & = & \alpha(\sqcup_{f\sqsubseteq\beta\circ\gamma} f_\diamond) & \\ & \sqsubseteq & \alpha(\beta_\diamond(\gamma^\bullet)_\diamond) & \{ \text{ Lem.1, } f_\diamond \sqsubseteq \beta_\diamond(\gamma^\bullet)_\diamond \ \} \\ & = & (\alpha\beta_\diamond)(\gamma^\bullet)_\diamond & \\ & = & (\alpha\circ\beta)\circ\gamma^\bullet. & \end{array}$$

(b)

$$\begin{array}{rcll} \alpha\circ(\beta\circ\gamma) & \sqsubseteq & (\alpha\circ\beta)\circ\gamma^\bullet & \{ \text{ (a) } \} \\ & = & (\alpha\circ\beta)\circ\gamma & \{ \ \gamma^\bullet = \gamma \ \} \\ & \sqsubseteq & \alpha\circ(\beta\circ\gamma). & \{ \text{ Prop.11 (d) } \} \end{array}$$

□

We proved the associative law of convex composition for convex relations. However, the following example shows that the convex composition $\circ$ need not be associative in general.

Example 4. Consider maps $f,\ g,\ h : X \to \mathcal{Q}_*(X)$ which appeared in Example 1. For all $p \in \mathcal{Q}_1(X)$ we have

$$pf_\diamond = \dot{x}, \quad pg_\diamond = p_{[x]}\cdot\dot{x}, \quad \text{and} \quad ph_\diamond = p_{[x]}\cdot\dot{y}.$$

Thus $fg_\diamond = f$, $xfh_\diamond = yfh_\diamond = \dot{y}$, and $p(fh_\diamond)_\diamond = \dot{y}$ for $p \in \mathcal{Q}_1(X)$. Shown in Example 2, the identity $\gamma_\diamond = g_\diamond \sqcup h_\diamond$ holds, and so $pf_\diamond\gamma_\diamond = \dot{x}\sqcup\dot{y}$ for all $p \in \mathcal{Q}_1(X)$. Note that $\gamma^\bullet \neq \gamma$. On the other hand, except for two maps $fg_\diamond$ and $fh_\diamond$ there are just two maps k and k' included in $f\gamma_\diamond$, where $xk = \dot{x}$, $yk = \dot{y}$, $xk' = \dot{y}$ and $yk' = \dot{x}$. Let $p_0 = (\frac{1}{2})_X$, the middle point of $\dot{x}$ and $\dot{y}$. Then we have

$$\begin{array}{rcl} p_0(f\gamma_\diamond)_\diamond & = & p_0(fg_\diamond)_\diamond \sqcup p_0(fh_\diamond)_\diamond \sqcup p_0k_\diamond \sqcup p_0k'_\diamond \\ & = & \dot{x}\sqcup\dot{y}\sqcup p_0k_\diamond \sqcup p_0k'_\diamond \\ & = & \dot{x}\sqcup\dot{y}\sqcup p_0 \\ & \neq & \dot{x}\sqcup\dot{y} \\ & = & p_0f_\diamond\gamma_\diamond, \end{array}$$

which proves that $\alpha f_\diamond\gamma_\diamond \neq \alpha(f\gamma_\diamond)_\diamond$ for a map $\alpha : X \to \mathcal{Q}_1(X)$ such that $x\alpha = y\alpha = p_0$. Therefore $(\alpha\circ f)\circ\gamma = \alpha f_\diamond\gamma_\diamond \neq \alpha(f\gamma_\diamond)_\diamond = \alpha\circ(f\circ\gamma)$. □

6. Convex relations and distributivities

Now we discuss the convex relations and the distributive laws of convex composition over the joins.

Proposition 14. *Let $\alpha : X \rightharpoonup \mathcal{Q}_*(Y)$ and $\beta : Y \rightharpoonup \mathcal{Q}_*(Z)$ be $\mathcal{Q}_*$-convex relations. Then the following holds:*

(a) *$\alpha\circ\beta$ is total,*
(b) *$(\alpha\circ\beta)(\xi_Z^*)^\sharp \sqsubseteq \alpha\circ\beta$,*
(c) *$(\alpha\circ\beta)^\bullet = \alpha\circ\beta$.*

Proof. (a) $\alpha\circ\beta$ is total :
Since α and β are total, $\beta_\diamond$ is total by the definition and so $\alpha\circ\beta = \alpha\beta_\diamond$ is total.
(b) $(\alpha\circ\beta)(\xi_Z^\tau)^\sharp \sqsubseteq \alpha\circ\beta$:

$$\begin{array}{rcll} (\alpha\circ\beta)\,(\xi_Z^\tau)^\sharp & = & \alpha\beta_\diamond\,(\xi_Z^\tau)^\sharp & \{ \text{ Def. 2 } \} \\ & \sqsubseteq & \alpha(\beta(\xi_Z^\tau)^\sharp)_\diamond & \{ \text{ Prop. 10 (c) } \} \\ & \sqsubseteq & \alpha\beta_\diamond. & \{ \ \beta(\xi_Z^\tau)^\sharp = \beta \ \} \end{array}$$

(c) $(\alpha \circ \beta)^\bullet = \alpha \circ \beta$:

$$\begin{array}{rcll} x(\alpha \circ \beta)^\bullet & = & \nabla_{I\mathcal{Q}_*(\mathbb{N})} \circ \nabla_{\mathbb{N}I} x(\alpha \circ \beta) & \\ & = & \nabla_{I\mathcal{Q}_*(\mathbb{N})} \circ (\nabla_{\mathbb{N}I} x\alpha \circ \beta) & \{ \alpha \circ \beta = \alpha\beta_\diamond \} \\ & = & (\nabla_{I\mathcal{Q}_*(\mathbb{N})} \circ \nabla_{\mathbb{N}I} x\alpha) \circ \beta & \{ \beta^\bullet = \beta, \text{ Associative law } \} \\ & = & x\alpha^\bullet \circ \beta & \\ & = & x\alpha \circ \beta & \{ \alpha^\bullet = \alpha \} \\ & = & x(\alpha \circ \beta). & \end{array}$$

□

Proposition 15. *If $\alpha : X \rightharpoondown \mathcal{Q}_1(Y)$ and $\beta : Y \rightharpoondown \mathcal{Q}_1(Z)$ are $\mathcal{Q}_1$-convex relations, then so is the convex composite $\alpha \circ \beta$.*

Proof. The proof is the same as the proof (a) and (c) of Proposition 14. □

In the rest of paper, the subscript τ is one of 1 and $*$. For a set χ of $\mathcal{Q}_\tau$-convex relations $\alpha : X \rightharpoondown \mathcal{Q}_\tau(Y)$ define

$$\bigvee \chi = (\sqcup \chi)^\bullet.$$

It is trivial that $\bigvee \chi$ gives the join (the least upper bound) of χ.

The following proposition shows the right distributivity over all joins.

Proposition 16. *Let $\alpha : X \rightharpoondown \mathcal{Q}_\tau(X)$ and $\beta : X \rightharpoondown \mathcal{Q}_\tau(X)$ be relations.*

(a) $\alpha^\bullet \circ \beta \sqsubseteq (\alpha \circ \beta)^\bullet$,
(b) $(\bigvee \chi) \circ \beta = \bigvee(\chi \circ \beta)$.

Proof. (a) $\alpha^\bullet \circ \beta \sqsubseteq (\alpha \circ \beta)^\bullet$:

$$\begin{array}{rcll} \forall x \in X.\ x(\alpha^\bullet \circ \beta) & = & x\alpha^\bullet \beta_\diamond & \\ & = & \nabla_{I\mathcal{Q}_\tau(\mathbb{N})} (\nabla_{\mathbb{N}\,I} x\alpha)_\diamond \beta_\diamond & \\ & \sqsubseteq & \nabla_{I\mathcal{Q}_\tau(\mathbb{N})} (\nabla_{\mathbb{N}\,I} x\alpha\beta_\diamond)_\diamond & \{ \text{Prop.11 (c)} \} \\ & = & x(\alpha\beta_\diamond)^\bullet. & \end{array}$$

(b) $(\bigvee \chi) \circ \beta = \bigvee(\chi \circ \beta)$:

$$\begin{array}{rcll} (\bigvee \chi) \circ \beta & = & (\sqcup \chi)^\bullet \circ \beta & \\ & \sqsubseteq & ((\sqcup \chi) \circ \beta)^\bullet & \{ \text{(a)} \} \\ & = & (\sqcup(\chi \circ \beta))^\bullet & \\ & = & \bigvee(\chi \circ \beta). & \end{array}$$

$$\begin{array}{rcll} \alpha \in \chi & \to & \alpha \circ \beta \sqsubseteq (\bigvee \chi) \circ \beta & \{ \alpha \sqsubseteq \bigvee \chi \} \\ & \to & \bigvee(\alpha \circ \beta) \sqsubseteq (\bigvee \chi) \circ \beta. & \end{array}$$

□

The following example shows that the left distributivity $\alpha \circ \bigvee \chi = \bigvee(\alpha \circ \chi)$ needs not hold in general.

Example 5. Let $\alpha', \beta : X \rightharpoondown \mathcal{Q}_*(X)$ be $\mathcal{Q}_*$-convex relations such that $\alpha' = \alpha(\xi_X^*)^\sharp$, $\beta = h(\xi_X^*)^\sharp$ where X, α and h are appeared in Example 4. Then $\alpha h_\diamond \sqsubseteq \alpha(\xi_X^*)^\sharp = \alpha'$ holds since $x\alpha h_\diamond = y\alpha h_\diamond = p_0 h_\diamond = \frac{1}{2} \cdot \dot{y} \leq p_0 = x\alpha = y\alpha$. Shown in Example 2, a relation $h \sqcup e_X$ consists of four maps, that is $h \sqcup e_X = f' \sqcup g \sqcup h \sqcup e_X$. Then we have

$$\begin{array}{rcll} p_0 f'_\diamond & \sqsubseteq & x\alpha'(h \sqcup e_X)_\diamond & \{ p_0 = x\alpha \sqsubseteq x\alpha',\ f'_\diamond \sqsubseteq (h \sqcup e_X)_\diamond \} \\ & \sqsubseteq & x\alpha'(h(\xi_X^*)^\sharp \sqcup e_X(\xi_X^*)^\sharp)_\diamond & \\ & = & x\alpha'(\beta \sqcup \varepsilon_X^*)_\diamond & \\ & = & x(\alpha' \circ (\beta \sqcup \varepsilon_X^*)) & \\ & \sqsubseteq & x(\alpha' \circ (\beta \vee \varepsilon_X^*)). & \end{array}$$

On the other hand, we have $p_0 f'_\diamond = \dot{y} \not\sqsubseteq x\alpha' = x(\alpha' \circ \beta \vee \alpha' \circ \varepsilon^*_X)$ since

$$\begin{array}{rcll}
\alpha' \circ \beta \sqcup \alpha' \circ \varepsilon^*_X & = & \alpha' \beta_\diamond \sqcup \alpha' (\varepsilon^*_X)_\diamond & \\
& = & \alpha' (h(\xi^*_X)^\sharp)_\diamond \sqcup \alpha' (e_X(\xi^*_X)^\sharp)_\diamond & \\
& = & \alpha' h_\diamond (\xi^*_X)^\sharp \sqcup \alpha' \mathrm{id}_{\mathcal{Q}_*(X)} (\xi^*_X)^\sharp & \{ \text{ Prop. 10 (d), 7 (c) } \} \\
& = & \alpha (\xi^*_X)^\sharp h_\diamond (\xi^*_X)^\sharp \sqcup \alpha' (\xi^*_X)^\sharp & \\
& = & \alpha h_\diamond (\xi^*_X)^\sharp \sqcup \alpha' & \{ \text{ Prop. 7 (g), } \alpha'\text{: convex } \} \\
& = & \alpha'. & \{ \alpha h_\diamond \sqsubseteq \alpha',\ \alpha'\text{: convex } \}
\end{array}$$

Therefore $\alpha' \circ (\beta' \vee \varepsilon^*_X) \neq \alpha' \circ \beta' \vee \alpha' \circ \varepsilon^*_X$. □

Finally, we state several results about directed sets of $\mathcal{Q}_*$-convex relations and their joins.

Lemma 2. *If χ is a directed set of $\mathcal{Q}_*$-convex relations $\alpha : X \rightharpoondown \mathcal{Q}_*(Y)$, then*

$$(\sqcup \chi)_\diamond = \sqcup_{\alpha \in \chi} \alpha_\diamond .$$

Proof. The inclusion $\sqcup_{\alpha\in\chi} \alpha_\diamond \sqsubseteq (\sqcup\chi)_\diamond$ is trivial. We will show that $p(\sqcup\chi)_\diamond \sqsubseteq \sqcup_{\alpha\in\chi} p\alpha_\diamond$ for all $p \in \mathcal{Q}_*(X)$. For a map $f : X \to \mathcal{Q}_*(Y)$ we have

$$\begin{array}{rcl}
f \sqsubseteq \sqcup\chi & \to & \forall x \in X.\ xf \sqsubseteq x(\sqcup\chi) \\
& \to & xf \sqsubseteq \sqcup_{\alpha\in\chi} x\alpha \\
& \to & \exists \alpha_x \in \chi.\ xf \sqsubseteq x\alpha_x \\
& \overset{*}{\to} & \exists \alpha_f \in \chi\, \forall x \in \lfloor p \rfloor.\ xf \sqsubseteq x\alpha_f
\end{array}$$

Note $\overset{*}{\to}$ follows from the assumption that $\lfloor p \rfloor$ is finite and χ is directed. Define a map $f' : X \to \mathcal{Q}_*(Y)$ by

$$\forall x \in X.\ xf' = \begin{cases} xf & \text{if } x \in \lfloor p \rfloor, \\ 0_Y & \text{otherwise.} \end{cases}$$

It is clear that $pf_\diamond = pf'_\diamond$. Also $f' \sqsubseteq \alpha_f$ holds, because $\nabla_{XI} 0_Y \sqsubseteq \alpha$ (α is total and $\alpha(\xi^\tau_Y)^\sharp = \alpha$). Hence $pf_\diamond = pf'_\diamond \sqsubseteq p\alpha_\diamond$ and so

$$\begin{array}{rcll}
p(\sqcup\chi)_\diamond & = & p(\sqcup_{f \sqsubseteq \sqcup\chi} f_\diamond) & \\
& = & \sqcup_{f \sqsubseteq \sqcup\chi} p f_\diamond & \\
& \sqsubseteq & \sqcup_{\alpha\in\chi} p\alpha_\diamond & \{ \ pf_\diamond \sqsubseteq p\alpha_\diamond \ \} \\
& = & p(\sqcup_{\alpha\in\chi} \alpha_\diamond). &
\end{array}$$

□

The following proposition gives the directed join of $\mathcal{Q}_*$-convex relations.

Proposition 17. *Let χ be a directed set of $\mathcal{Q}_*$-convex relations $\alpha : X \rightharpoondown \mathcal{Q}_*(X)$. Then*

$$\bigvee \chi = \bigsqcup \chi.$$

Proof.

$$\begin{array}{rcll}
x(\bigvee\chi) & = & x(\sqcup\chi)^\bullet & \\
& = & \nabla_{I\mathcal{Q}_*(\mathbb{N})} (\nabla_{\mathbb{N}I} x(\sqcup\chi))_\diamond & \\
& = & \nabla_{I\mathcal{Q}_*(\mathbb{N})} (\sqcup_{\alpha\in\chi} \nabla_{\mathbb{N}I} x\alpha)_\diamond & \\
& = & \nabla_{I\mathcal{Q}_*(\mathbb{N})} \sqcup_{\alpha\in\chi} (\nabla_{\mathbb{N}I} x\alpha)_\diamond & \{ \text{ Lemma 2 } \} \\
& = & \sqcup_{\alpha\in\chi} \nabla_{I\mathcal{Q}_*(\mathbb{N})} (\nabla_{\mathbb{N}I} x\alpha)_\diamond & \\
& = & \sqcup_{\alpha\in\chi} x\alpha^\bullet & \\
& = & \sqcup_{\alpha\in\chi} x\alpha & \\
& = & x(\sqcup\chi). &
\end{array}$$

□

The composition of $\mathcal{Q}_*$-convex relations distributes all directed joins from the left-hand side.

Proposition 18. *Let $\alpha : X \rightharpoondown \mathcal{Q}_*(Y)$ be a $\mathcal{Q}_*$-convex relation and χ a directed set of $\mathcal{Q}_*$-convex relations $\beta : Y \rightharpoondown \mathcal{Q}_*(Z)$. Then*

$$\alpha \circ \bigvee \chi = \bigvee(\alpha \circ \chi).$$

Proof.

$$\begin{array}{rcll} \alpha \circ \bigvee \chi & = & \alpha(\sqcup\chi)_\diamond & \{ \text{ Prop.17 } \} \\ & = & \alpha(\sqcup_{\beta\in\chi}\beta_\diamond) & \{ \text{ Lemma. 2 } \} \\ & = & \sqcup_{\beta\in\chi}\alpha\beta_\diamond & \\ & = & \sqcup_{\beta\in\chi}\alpha\circ\beta & \\ & = & \sqcup(\alpha\circ\chi) & \\ & = & \bigvee(\alpha\circ\chi). & \end{array}$$

□

7. Conclusion

In this paper we have studied the relations into algebras of probabilistic distributions using relational calculi, although McIver et al. [7] and Tsumagari [16] studied in set-theoretical way. We have shown the following.

- The set of $\mathcal{Q}_\tau$-convex relations forms a category with the convex composition, and the identity morphisms depending on $\tau \in \{*, 1\}$.
- For $\tau \in \{*, 1\}$ the convex composition of $\mathcal{Q}_\tau$-convex relations distributes over all non-empty joins from the right hand side.
- The convex composition of $\mathcal{Q}_*$-convex relations distributes over all non-empty directed joins even from the left hand side.

We have proved the associative law of convex composition for $\mathcal{Q}_*$-convex relations and $\mathcal{Q}_1$-convex relations in the same framework, though Tsumagari [16] had studied as their two convex-relations are different. Additionally we have given a counter example for the associative law of the convex composition in the absence of convexity.

The convex composition studied in this paper seems to be a generalization of reachability composition of multirelations. So we might be interested in the another composition of $\mathcal{Q}_\tau$-convex relations, corresponding to the composition of up-closed multirelations studied by Parikh [11, 12] and Rewitzky [6, 14].

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