ON THE RATIONAL K_2 OF A CURVE OF GL_2 TYPE OVER A GLOBAL FIELD OF POSITIVE CHARACTERISTIC

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Abstract. If $\mathscr{X}$ is an integral model of a smooth curve X over a global field k, there is a localization sequence comparing the K-theory of $\mathscr{X}$ and X. We show that $K_1(\mathscr{X})$ injects into $K_1(X)$ rationally, by showing that the previous boundary map in the localization sequence is rationally a surjection, for X of "GL_2 type" and k of positive characteristic not 2. Examples are given to show that the relative G_1 term can have large rank. Examples of such curves include non-isotrivial elliptic curves, Drinfeld modular curves, and the moduli of $\mathscr{D}$-elliptic sheaves of rank 2.

1. Introduction

In this secton, we state our main result (Theorem 1.1 and Corollary 1.3) and give an outline of the proof.

1.1. Let $\mathbb{F}_q$ be a finite field of q elements of characteristic p. Let C be a projective smooth (geometrically connected) curve over $\mathbb{F}_q$ and k be the function field of C. Consider the following cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, hook, "\eta"] & C \end{tikzcd}$$

where η is the generic point, f is proper flat, g is proper smooth, and $\mathscr{X}$ is regular. We remark here that for a given proper smooth curve X over k, there exists an $\mathscr{X}$ as above by [27, p.456, Section 10, Proposition 1.8].

Let C_0 denote the set of closed points of C. We regard an element in C_0 as a place of the global field k and vice versa. We write $(-)_{\mathbb{Q}} = (-) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $K_n(-)$ and $G_n(-)$ denote the higher algebraic K-theory and G-theory of Quillen [30]. (We write G for K' in [30].) The localization sequence of G-theory tensored by $\mathbb{Q}$ gives the exact sequence

$$\cdots \to K_2(\mathscr{X})_{\mathbb{Q}} \to K_2(X)_{\mathbb{Q}} \stackrel{\oplus\partial_\wp}{\to} \bigoplus_{\wp \in C_0} G_1(\mathscr{X}_{\kappa(\wp)})_{\mathbb{Q}} \to K_1(\mathscr{X})_{\mathbb{Q}} \to \cdots, \tag{1.1}$$

where $\kappa(\wp)$ is the residue field at $\wp$ and $\mathscr{X}_{\kappa(\wp)} = \mathscr{X} \times_C \operatorname{Spec} \kappa(\wp)$. In this article, we consider the following condition (GL_2) on a proper smooth k-scheme Z. Let us write $G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ for the absolute Galois group of a field L, and $Z_M = Z \times_{\operatorname{Spec} k} \operatorname{Spec} M$ for the base change, where M is a k-algebra. Let ℓ be a prime different from p. The condition is the following.

(GL_2) The G_k-representation $H^1_{\text{ét}}(Z_{k^{\mathrm{sep}}}, \overline{\mathbb{Q}}_\ell)$ is a direct sum of 2-dimensional irreducible representations.

Our main theorem is as follows.

Theorem 1.1. *Assume that the characteristic of k is greater than 2. Let X be a proper smooth curve over k. Suppose that X satisfies the condition (GL_2) (with $Z = X$). Let $\mathscr{X}$ be as above. Then the boundary map*

$$K_2(X)_{\mathbb{Q}} \stackrel{\oplus\partial_\wp}{\to} \bigoplus_{\wp \in C_0} G_1(\mathscr{X}_{\kappa(\wp)})_{\mathbb{Q}}$$

is surjective.

This result is motivated by Parshin's conjecture. Recall that Parshin's conjecture says that $K_i(Z) \otimes \mathbb{Q} = 0$ for all $i \geq 1$ when Z is a projective smooth scheme over $\mathbb{F}_q$. (We were not able to find a written account by Parshin on this conjecture, but see [10].) With our assumptions on $\mathscr{X}$, the validity of the conjecture then implies that the boundary map $\oplus_\wp \partial_\wp$ in Theorem 1.1 is an isomorphism in view of the exact sequence (1.1).

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Remark 1.2. The assumption that $p > 2$ is necessary for the use of the Tate conjecture proved by Zarhin [43]. For the case of $p = 2$, there is an unpublished work by Mori (see [29, pp.9–10 and pp.154–161]). Therefore if we use Mori's result, one can show that our main theorem holds also for $p = 2$.

We have the following corollary (see Corollary 6.1 for the precise statement).

Corollary 1.3. *Assume that the characteristic of k is greater than 2. The surjectivity statement of Theorem 1.1 holds when X is*

(1) an elliptic curve, which is not isotrivial,
(2) a Drinfeld modular curve,
(3) the moduli of $\mathscr{D}$-elliptic sheaves of rank 2, or
(4) one of the genus two curves constructed in Section 7.

In the four cases above, the curve X satisfies the condition (GL_2). Hence this corollary follows from Theorem 1.1. In the paper [22], the theorem for Case (1) has been proved.

1.2. Let us give a detailed outline of the proof of Theorem 1.1 below. From now on, we always assume that the characteristic of k is greater than 2. First, we translate the theorem into a statement about étale Chern class maps.

Proposition 1.4. *Let the notation be as in Theorem 1.1. Suppose that the condition (GL_2) holds. Then the composite map*

$$\begin{aligned} H^2_{\mathcal{M}}(X, \mathbb{Q}(2)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell &\to \textstyle\prod_{\wp \in C_0} H^2_{\mathcal{M}}(X_{k_\wp}, \mathbb{Q}(2)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \\ &\to \textstyle\prod_{\wp \in C_0} H^2_{\acute{e}t}(X_{k_\wp}, \mathbb{Q}_\ell(2)), \end{aligned}$$

where the first arrow is the product of pullback maps and the second arrow is the product of étale Chern class maps, is surjective.

To prove that Proposition 1.4 implies Theorem 1.1, we will use lemmas in the paper [22] (see Section 5.3 for details).

We remark that we can compute the dimension of target group explicitly (see Section 6) and see that the factor $H^2_{\text{ét}}(X_{k_\wp}, \mathbb{Q}_\ell(2))$ is zero for almost all $\wp$. It follows that the product of the étale cohomology groups is actually a (finite) direct sum.

1.3. To prove Proposition 1.4, we will show the following proposition.

Proposition 1.5. *Suppose $X = B$ is a simple abelian variety over k satisfying the condition (GL_2). Then the map*

$$\begin{aligned} H^{2,2}_{\mathcal{M}}(h^1(X)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell &\to \textstyle\prod_{\wp \in C_0} H^{2,2}_{\mathcal{M}}(h^1(X_{k_\wp})) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \\ &\to \textstyle\prod_{\wp \in C_0} H^{2,2}_{\acute{e}t}(h^1(X_{k_\wp})) \end{aligned}$$

is surjective. Here we refer to Section 4 for the notation concerning Chow motives such as $H^{2,2}_*$ and h^1. The first arrow is the pullback map and the second arrow is the étale Chern class map.

To see that Proposition 1.5 implies Proposition 1.4, we use the formalism of Chow motives. In the category of Chow motives (with rational equivalence and rational coefficients), we have an isomorphism

$$h(X) \cong \bigoplus_{i=0}^{2} h^i(X) \cong h^0(X) \oplus h^1(\mathrm{Jac}X) \oplus h^2(X) \cong h^0(X) \oplus \bigoplus_{i=1}^{m} h^1(B_i) \oplus h^2(X),$$

where each B_i is a k-simple abelian variety satisfying the condition (GL_2) such that $\mathrm{Jac}X$ is isogenous (over k) to $\prod_{i=1}^m B_i$ (see Section 5.2). Then as étale Chern classes respect the decomposition, and since the statement for the h^0 and h^2 is not an essential difficulty, Proposition 1.4 can be reduced to Proposition 1.5.

To prove Proposition 1.5, we proceed as follows. By the condition (GL_2), we have $H^1_{\text{ét}}(B_{k^{\mathrm{sep}}}, \overline{\mathbb{Q}}_\ell) \cong \oplus_{j=1}^n \rho_j$ where each ρ_j is an irreducible 2-dimensional representation of G_k with coefficients in $\overline{\mathbb{Q}}_\ell$. For a nonarchimedean local field L whose characteristic is prime to ℓ, define Sp to be the 2-dimensional representation of G_L with coefficients in $\mathbb{Q}_\ell$ which is the unique nontrivial extension of $\mathbb{Q}_\ell$ by $\mathbb{Q}_\ell(-1)$. By abuse of notation we also write Sp for the base change $\mathrm{Sp} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$. We consider two cases separately.

(1) There exists a pair $(j, \wp)$ of an integer $1 \le j \le n$ and a place $\wp$ of k such that $\rho_j|_{G_{k_\wp}}$ is isomorphic to Sp.
(2) The condition in (1) does not hold.

In Case (2), using the computation of Galois cohomology in Section 3, we show that the target of the Chern class map is zero (see Lemma 5.1). Hence the surjectivity becomes a trivial statement. In Case (1), we show that B is isogenous to a simple factor of the Jacobian of some Drinfeld modular curve such that $\wp$ is the fixed place at infinity. (We refer to Section 2.1 for the relevant definitions and properties of Drinfeld modular curves.) This is of course the function field analogue of Shimura's theory [35]. We note that the condition that there exists a place where the representation at the place $\wp$ is isomorphic to Sp is the function field analogue of (or rather the Drinfeld modular analogue of) modular forms of weight 2 (which is a condition at the place at infinity). As we did not find the construction in the literature, we give a detailed proof of this fact in Section 2. We note that we do not follow the proof for the case of the moduli of elliptic curves, but we use freely the Tate conjecture (a theorem of Zarhin) to make the exposition simple. With the Tate conjecture at our disposal, the only other necessary ingredient is some Atkin-Lehner theory over general global fields, but we use a theorem of Jacquet and Shalika for simplicity (see the proof of Lemma 2.3).

Thus we are reduced to proving Proposition 1.5 when B is a factor of the Jacobian of a Drinfeld modular curve. The idea, which does not apply to the case of a number field, and which already appeared in the paper [22], is used at this point. Note that B may appear as a factor of a Drinfeld modular curve for EVERY $\wp$ satisfying the condition in (1), so that we have as many "parametrizations" as such $\wp$. When B is an elliptic curve, this is really the modular parametrization as discussed in detail by Gekeler and Reversat [11]. We then invoke the result of [22]. The setup is that of Proposition 1.4, and X is a Drinfeld modular curve. The result in [22] says that there exists a certain subspace of $H^2_{\mathcal{M}}(X, \mathbb{Q}(2))$ which maps surjectively by the Chern classes at infinity (or $\wp$ in this case) and maps to zero at all the other places. By varying $\wp$, we obtain Proposition 1.5. This finishes the outline of the proof of Proposition 1.5 and hence of Theorem 1.1.

1.4. To see that the surjectivity statement is a nontrivial one, we will compute (Lemma 6.2) the dimension of the target group explicitly in terms of the G_k-representation that appears in the condition (GL_2). More precisely, if the representation is the direct sum $\oplus_{j=1}^n \rho_j$ of 2-dimensional representations ρ_j, then the dimension of the target equals the number of the pairs $(j, \wp)$ which appeared in the condition of (1).

In Corollary 1.3, we have four kinds of curves. For an elliptic curve, the condition on $\wp$ above is equivalent to that the elliptic curve has split multiplicative reduction at $\wp$. For a Drinfeld modular curve, the number of such pairs is bounded below by the genus since (j, ∞) always satisfy the condition of (1). Note that the genus grows as $\mathbb{K}$ becomes small. The genus 2 curve in (4) is constructed from Brumer-Hashimoto's family of genus 2 curves. We construct explicit examples within this family for which the number of pairs $(j, \wp)$ satisfying the condition of (1) is large.

1.5. We give a remark concerning our title and the condition (GL_2). Suppose a proper smooth curve X satisfies the condition (GL_2). Then the Jacobian of X is isogenous to a product of simple abelian varieties of GL_2-type if there exists a place $\wp_j$ such that $(j, \wp_j)$ satisfies the condition of (1) for each j. This can be seen using the procedure given in Section 5.1. When there does not exist such $\wp_j$'s, we do not know if this holds true.

1.6. The paper is organized as follows. In Section 2, given an automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_k)$ such that $\pi_\infty \cong \mathrm{St}$ at some place ∞ of k (see Section 2 for the notation), we construct a k-simple abelian variety A_π whose Tate module is related to π. Section 3 is devoted to the computation of Galois cohomology. We use the classification of representations of $\mathrm{GL}_2(F)$ (see Lemma 3.4) for a nonarchimedean local field F. The aim is to compute the local Galois cohomology of some 2-dimensional representations corresponding to the tempered ones via the local Langlands correspondence. In Section 4, we collect some facts concerning Chow motives, their cohomology theories, and the morphism between the cohomology theories. We are interested in the rational motivic cohomology and the absolute ℓ-adic cohomology. We refer to Jannsen's book [21] for cohomology theories and to Scholl's article [37] for generalities on Chow motives. In Section 5, we give a proof of Theorem 1.1 following the outline given in this introduction. Sections 5.1, 5.2, 5.3 are devoted to the proof of Propositions 1.5, 1.4 and Theorem 1.1 respectively. Section 7 is independent of the earlier sections aside from some general notations. The aim is to construct an explicit family of genus 2 curves satisfying the condition (GL_2) such that the dimension of the rational G_1 group is arbitrarily large.

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2. The construction of abelian varieties associated to automorphic representations of GL_2 in the function field case

The aim of this section is to prove Proposition 2.1. The abelian variety A_π will be used throughout this article. This proposition is the function field analogue of a theorem of Shimura [35, p.183, Thms 7.14, 7.15]. The proof is different in that we use the Tate conjecture (a theorem of Zarhin) in this paper. In doing so, we need not use the theory of modular forms (which does not seem to be available in the literature for the function field case) as much as Shimura did. Our construction has a shortcoming that the resulting abelian variety A_π is neither a sub nor a quotient of the Jacobian of a Drinfeld modular curve in a canonical way. Another output of this section is Corollary 2.5. This will be used in the proof of Lemma 5.2.

We fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. We use the results from the theory of automorphic forms with values in $\mathbb{C}$ as those with values in $\overline{\mathbb{Q}}_\ell$ via this fixed isomorphism.

2.1. We regard a place of k as a closed point of C and vice versa. Let us choose a closed point $\infty \in C_0$ and set $A = H^0(C \setminus \{\infty\}, \mathcal{O}_C)$. For a place $\wp$ of k, we let $k_\wp$ denote the completion of k at $\wp$. If $\wp \neq \infty$, we denote by the same symbol $\wp$ the prime ideal of A corresponding to $\wp$. Then the $\wp$-adic completion $A_\wp$ of A is equal to the ring of integers of $k_\wp$.

Let $I \subset A$ be a non-zero ideal. We let M_I^∞ denote the functor which classifies Drinfeld modules of rank 2 with a Drinfeld full level I structure on a k-scheme. When $\mathrm{Spec}\,(A/I)$ contains two or more closed points, it is representable by an affine curve over $\mathrm{Spec}\, k$ (see [8, p.576, Proposition 5.3] for construction of the moduli scheme over A; we consider its base change to k). We will denote the representing scheme by the same symbol. Let $\overline{M}_I^\infty$ denote the smooth compactification of M_I^∞. We set $\overline{M}^\infty = \varprojlim \overline{M}_I^\infty$ where the limit is taken over ideals of A such that $\mathrm{Spec}\,(A/I)$ contains two or more closed points.

Let $\mathbb{A} = \mathbb{A}_k$ denote the ring of adeles of k. We set $\widehat{A} = \varprojlim A/I$. Let $\mathbb{A}^\infty = \widehat{A} \otimes_A k$ denote the ring of finite adeles. We refrain from using the more common notation $\mathbb{A}_f$ to denote the ring of finite adeles because we will vary the place at infinity thus obtaining various rings of finite adeles. (For example, see the proof of Lemma 5.2.) The group $\mathrm{GL}_2(\mathbb{A}^\infty)$ acts on $\overline{M}^\infty$ (see [8, p.577]). Let $\mathbb{K} \subset \mathrm{GL}_2(\widehat{A}) \subset \mathrm{GL}_2(\mathbb{A}^\infty)$ be a compact open subgroup. We let $\overline{M}_\mathbb{K}^\infty = \mathbb{K}\backslash\overline{M}^\infty$ denote the quotient. For example ([8, p.578]), if $\mathbb{K} = \mathbb{K}_I = \mathrm{Ker}[\mathrm{GL}_2(\widehat{A}) \to \mathrm{GL}_2(\widehat{A}/I\widehat{A})]$, then we recover $\overline{M}_I^\infty = \mathbb{K}_I\backslash\overline{M}^\infty$ for I as in the previous paragraph. If L is a k-algebra, we write $\overline{M}_{\mathbb{K},L}^\infty = \overline{M}_\mathbb{K}^\infty \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$.

2.2. Let Π denote the set of (isomorphism classes of) cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$. Let St denote the Steinberg representation of GL_2 of a nonarchimedean local field. We have an isomorphism

$$\varinjlim_{\mathbb{K}\subseteq \mathrm{GL}_2(\mathbb{A}^\infty):\text{open compact subgroup}} H^1_{\text{ét}}(\overline{M}^\infty_{\mathbb{K},k^{\text{sep}}}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\pi\in\Pi,\pi_\infty\cong\mathrm{St}} (\otimes_{v\neq\infty}\pi_v)\otimes\rho(\pi) \tag{2.1}$$

of $\mathrm{GL}_2(\mathbb{A}^\infty) \times \mathrm{Gal}(k^{\text{sep}}/k)$-modules, where $\rho(\pi)$ is the 2-dimensional representation of $\mathrm{Gal}(k^{\text{sep}}/k)$ over $\overline{\mathbb{Q}}_\ell$ which is determined by π via the global Langlands correspondence and whose restriction to $\mathrm{Gal}(k_\infty^{\text{sep}}/k_\infty)$ is isomorphic to Sp. (See [6, Corollary 2.3].) Given a compact open subgroup $\mathbb{K} \subset \mathrm{GL}_2(\widehat{A})$, we may take the $\mathbb{K}$-invariant part of both sides above and obtain (see [11, Section 8] for the description even when $\mathbb{K}$ is large)

$$H^1_{\text{ét}}(\overline{M}^\infty_{\mathbb{K},k^{\text{sep}}}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\pi\in\Pi,\pi_\infty\cong\mathrm{St}} (\otimes_{v\neq\infty}\pi_v)^{\mathbb{K}}\otimes\rho(\pi). \tag{2.2}$$

For any place $\wp$ of k, the Galois representation $\rho(\pi)|_{G_{k_\wp}}$ (or the associated Weil-Deligne representation) corresponds to $\pi_\wp \otimes |\det|^{-\frac{1}{2}}$ via the local Langlands correspondence. (See [8, Theorem A]. See also [24, Corollaire VII.5].)

2.3. Let n be a non-zero ideal of A. Set

$$\mathbb{K}_{1,\wp}(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A_\wp) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod n \right\}$$

for a place $\wp \neq \infty$ and

$$\mathbb{K}_1(n) = \prod_{\wp\neq\infty} \mathbb{K}_{1,\wp}(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{A}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod n \right\}.$$

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that $\pi_\infty \cong \mathrm{St}$. By [4, p.302, Theorem 1.4], there exists a non-zero ideal $n_\pi \subset A$ such that $\dim_{\overline{\mathbb{Q}}_\ell} \otimes_{\wp\neq\infty} \pi_\wp^{\mathbb{K}_{1,\wp}(n_\pi)} = 1$. (This is the prime-to-$\infty$ part of the conductor of π. The conductor is then $n_\pi\infty$ since $\pi_\infty \cong \mathrm{St}$.) We consider the action of the Hecke operator $T_\wp$ on a nonzero element in $\otimes_{\wp\neq\infty}\pi_\wp^{\mathbb{K}_{1,\wp}(n_\pi)}$ for $\wp$ prime to $n_\pi\infty$. We denote the eigenvalue by $a_\wp(\pi)$. Let K_π denote the field generated by $a_\wp(\pi)$ over $\mathbb{Q}$ where $\wp$ runs over all primes prime to $n_\pi\infty$.

For an abelian variety B, we write $T_\ell B$ for the Tate module, set $V_\ell B = T_\ell B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $\overline{\mathrm{V}}_\ell B = V_\ell B \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$. For a $\overline{\mathbb{Q}}_\ell$-vector space V, we let $V^\vee = \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(V, \overline{\mathbb{Q}}_\ell)$ denote the dual.

Proposition 2.1. *(1) The field K_π is a number field. Let $I(\pi)$ denote the set of embeddings $K_\pi \hookrightarrow \overline{\mathbb{Q}}_\ell$.*

(2) For any $\tau \in I(\pi)$, there exists a cuspidal automorphic representation π^τ such that
 (a) $\pi^\tau_\infty \cong \mathrm{St}$,
 (b) $n_\pi = n_{\pi^\tau}$,
 (c) $a_v(\pi^\tau) = \tau(a_v(\pi))$ for all v prime to $n_\pi\infty$.

(3) There exists a k-simple abelian variety A_π such that $(\overline{\mathrm{V}}_\ell A_\pi)^\vee \cong \bigoplus_{\tau \in I(\pi)} \rho(\pi^\tau)$ and $\mathrm{End}_k(A_\pi) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_\pi$.

Note that by (c) and the strong multiplicity one theorem, π^τ is uniquely determined. Note also that by the Tate conjecture, the abelian variety A_π is uniquely determined up to isogeny.

2.4. We prove Proposition 2.1 inductively. Fix a nonzero ideal $n \subset A$. The inductive assumption is that

($*n$) There exists an abelian variety $A_{\pi'}$ satisfying the conditions in Proposition 2.1 for those π' such that $n_{\pi'}$ divides n and $n_{\pi'} \neq n$.

Note that ($*n$) for $n = (1)$ holds trivially. Assuming ($*n$), we prove the assertions for π such that $n_\pi = n$.

Set $d_{\pi,n} = \dim_{\overline{\mathbb{Q}}_\ell} \otimes_{\wp \neq \infty} \pi_\wp^{\mathbb{K}_{1,\wp}(n)}$. Let $\Pi_1^{\mathrm{new}}(n)$ denote the (finite) set of cuspidal automorphic representations π such that $\pi_\infty \cong \mathrm{St}$ and $d_{\pi,n} = 1$.

Lemma 2.2. *Assume that ($*n$) holds. There exist abelian varieties over k, denoted $J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}$ and $J^{\infty,\mathrm{old}}_{\mathbb{K}_1(n)}$, such that*

$$(\overline{\mathrm{V}}_\ell J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)})^\vee \cong \bigoplus_{\pi \in \Pi_1^{\mathrm{new}}(n)} \rho(\pi),$$
$$(\overline{\mathrm{V}}_\ell J^{\infty,\mathrm{old}}_{\mathbb{K}_1(n)})^\vee \cong \bigoplus_{d_{\pi,n} > 1} \rho(\pi)^{d_{\pi,n}}.$$

Moreover $J^{\infty,\mathrm{old}}_{\mathbb{K}_1(n)}$ is isogenous to an abelian variety of the form $\prod_\pi A_\pi$ (possibly with multiplicity) where $n_\pi | n$ and $n_\pi \neq n$ hold for each π appearing in the product.

Proof. The construction is done inductively. Suppose we are given an abelian variety B over k such that

$$(\overline{\mathrm{V}}_\ell B)^\vee \cong \left[\bigoplus_{d_{\pi,n}=1} \rho(\pi)\right] \oplus \left[\bigoplus_{d_{\pi,n}>1} \rho(\pi)^{b_\pi}\right]$$

with $b_\pi \geq 0$ for each π such that $d_{\pi,n} > 1$. Note that for π such that $d_{\pi,n} > 1$, we have $n_\pi \neq n$ (using [4, p.306, Corollary]) so that the inductive hypothesis applies, that is, there exists an abelian variety A_π for this π. We have

$$\begin{aligned}\mathrm{Hom}_k(B, A_\pi) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell &\cong \mathrm{Hom}_{\mathbb{Q}_\ell[G_k]}(V_\ell B, V_\ell A_\pi) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \\ &\cong \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[G_k]}(\overline{\mathrm{V}}_\ell B, \overline{\mathrm{V}}_\ell A_\pi)\end{aligned}$$

where the first isomorphism is the Tate conjecture (a theorem of Zarhin [43]), and the second is [5, Lemma 29.2, (29.5)]. If $b_\pi \geq 1$, then we can construct a nontrivial map $\overline{\mathrm{V}}_\ell B \to \overline{\mathrm{V}}_\ell A_\pi$ (namely, the projection to $\rho(\pi)$ followed by the canonical inclusion). Hence there exists a nontrivial morphism $B \to A_\pi$, which is surjective since A_π is simple. Let B^0 denote the connected component of the kernel of this morphism. Then $B' = B^0_{\mathrm{red}}$ is an abelian variety. (This fact is well known when the base field is perfect; we refer to [28, p.1, Theorem 1] for the case when the base field is not necessarily perfect.) Note that $\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[G_k]}(\rho(\pi), \rho(\pi')) = 0$ for π such that $d_{\pi,n} = 1$ and π' such that $d_{\pi,n} > 1$ since $\rho(\pi)$ and $\rho(\pi')$ then have different conductors and are not isomorphic. It follows that

$$(\overline{\mathrm{V}}_\ell B')^\vee \cong \left[\bigoplus_{d_{\pi,n}=1} \rho(\pi)\right] \oplus \left[\bigoplus_{d_{\pi,n}>1} \rho(\pi)^{b'_\pi}\right]$$

with $\sum_\pi b_\pi > \sum_\pi b'_\pi$, and B is isogenous to $B' \times A_\pi$.

Now we apply this procedure repeatedly. To prove the lemma, we start from the Jacobian $J^\infty_{\mathbb{K}_1(n)}$ of $\overline{M}^\infty_{\mathbb{K}_1(n)}$ and proceed until $\sum_\pi b'_\pi = 0$. Then we have $J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}$ as B' in the last step of the process. Also one has $J^{\infty,\mathrm{old}}_{\mathbb{K}_1(n)}$ as the product of all the A_π's that appeared in the process. It is clear that $J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}$ and $J^{\infty,\mathrm{old}}_{\mathbb{K}_1(n)}$ satisfy the desired properties. □

2.5. We now describe the decomposition of $J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}$ into the k-simple factors (that is, we find an abelian variety over k isogenous to $J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}$, which is a product of k-simple abelian varieties). Each simple factor will be the desired A_π for some π. We define a Hecke algebra $\mathbb{T}\subset \mathrm{End}_k(J^{\infty}_{\mathbb{K}_1(n)})$ as the subalgebra generated by the Hecke operators $T_\wp$ and $S_\wp$ for $\wp$ prime to n. We define $\mathbb{T}^{\mathrm{new}}\subset \mathrm{End}_k(J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)})\otimes_{\mathbb{Z}}\mathbb{Q}$ as follows. By Lemma 2.2 and the Tate conjecture, we obtain and fix an isogeny $\psi: J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}\times J^{\infty,\mathrm{old}}_{\mathbb{K}_1(n)}\to J^{\infty}_{\mathbb{K}_1(n)}$. If $T\in \mathrm{End}_k(J^{\infty}_{\mathbb{K}_1(n)})$, we obtain an element T' in $\mathrm{End}_k(J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)})$ as the composite

$$J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}\to J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}\times J^{\infty,\mathrm{old}}_{\mathbb{K}_1(n)}\xrightarrow{\psi} J^{\infty}_{\mathbb{K}_1(n)}\xrightarrow{T} J^{\infty}_{\mathbb{K}_1(n)}\xrightarrow{\psi^\vee} J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)}\times J^{\infty,\mathrm{old}}_{\mathbb{K}_1(n)}\to J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)},$$

where the first arrow is the canonical inclusion, $\psi^\vee$ is the dual isogeny, and the last arrow is the projection. We define $\mathbb{T}^{\mathrm{new}}\subset \mathrm{End}_k(J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)})\otimes_{\mathbb{Z}}\mathbb{Q}$ to be the subalgebra generated by the elements of the form $(\deg\psi)^{-1}T'$. Note that $\mathbb{T}\to\mathbb{T}^{\mathrm{new}}$ is an algebra homomorphism and that $\mathbb{T}^{\mathrm{new}}$ is a commutative algebra since $\mathbb{T}$ is commutative. By abuse of notation, we let $T_\wp\in\mathbb{T}^{\mathrm{new}}$ (resp. $S_\wp\in\mathbb{T}^{\mathrm{new}}$) denote the element corresponding to $T_\wp\in\mathbb{T}$ (resp. $S_\wp\in\mathbb{T}$).

From here, we proceed in a manner very similar to the proof of Proposition 4.2 in [39, p.228]. We have

$$\mathrm{End}_k(J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)})\otimes_{\mathbb{Z}}\overline{\mathbb{Q}}_\ell\cong \mathrm{End}_{\overline{\mathbb{Q}}_\ell[G_k]}(\overline{\mathrm{V}}_\ell J^{\infty,\mathrm{new}}_{\mathbb{K}_1(n)})\cong \mathrm{End}_{\overline{\mathbb{Q}}_\ell[G_k]}(\bigoplus_{\pi\in\Pi_1^{\mathrm{new}}(n)}\rho(\pi)^\vee)\cong\prod_{\pi\in\Pi_1^{\mathrm{new}}(n)}\overline{\mathbb{Q}}_\ell \tag{2.3}$$

where the first isomorphism is due to the Tate conjecture and the last isomorphism follows from the irreducibility of $\rho(\pi_i)$ and the fact that if $i\neq j$ then $\rho(\pi_i)$ and $\rho(\pi_j)$ are not isomorphic.

2.6. Let f^∞_π denote a nonzero element in $\otimes_{\wp\neq\infty}\pi_\wp^{\mathbb{K}_{1,\wp}(n)}$. We define a map

$$\phi:\bigoplus_{\pi\in\Pi_1^{\mathrm{new}}(n)}\otimes_{\wp\neq\infty}\pi_\wp^{\mathbb{K}_{1,\wp}(n)}\to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{T}^{\mathrm{new}},\overline{\mathbb{Q}}_\ell)$$

by $\phi(f^\infty_\pi)(T_\wp)=a_\wp(\pi)$ for $\wp$ prime to n (this map is independent of the choice of the f^∞_π).

Lemma 2.3. *The map ϕ is an isomorphism.*

Proof. We see from (2.3) that $\dim_{\mathbb{Q}}\mathbb{T}^{\mathrm{new}}$ is smaller than or equal to the dimension of the left hand side. It suffices to prove the injectivity of ϕ.

Let $f=\sum_{\pi\in\Pi_1^{\mathrm{new}}(n)}c_\pi f^\infty_\pi$ with $c_\pi\in\overline{\mathbb{Q}}_\ell$ be an element such that $\phi(f)(T)=0$ for all $T\in\mathbb{T}^{\mathrm{new}}$. It is enough to show $f=0$. For each π and each place $\wp$ such that $\wp\nmid n$, denote the Satake parameters at $\wp$ for π by $\alpha_\wp(\pi)$ and $\beta_\wp(\pi)$. For each $\wp$ with $\wp\nmid n$ and $r\geq 1$, it is easy to find an element $T_{\wp,r}\in\mathbb{T}^{\mathrm{new}}$ such that

$$\phi(f^\infty_\pi)(T_{\wp,r})=\alpha_\wp(\pi)^r+\beta_\wp(\pi)^r$$

for all $\pi\in\Pi_1^{\mathrm{new}}(n)$. Then we have

$$\sum_{\pi\in\Pi_1^{\mathrm{new}}(n)}c_\pi(\alpha_\wp(\pi)^r+\beta_\wp(\pi)^r)=0$$

for any $r\geq 1$. This implies that $c_\pi=0$ for all $\pi\in\Pi_1^{\mathrm{new}}(n)$ by [20, p.806, THEOREM]. □

2.7. Using Lemma 2.3, we obtain $\mathbb{T}^{\mathrm{new}}=\mathrm{End}_k(J^{\mathrm{new}}_{\mathbb{K}_1(n)})\otimes_{\mathbb{Z}}\mathbb{Q}$. Since $\mathbb{T}^{\mathrm{new}}$ is commutative and $\mathrm{End}_k(J^{\mathrm{new}}_{\mathbb{K}_1(n)})\otimes_{\mathbb{Z}}\mathbb{Q}$ is a semisimple $\mathbb{Q}$-algebra, we have $\mathbb{T}^{\mathrm{new}}\cong F_1\times\cdots\times F_s$ where each F_i is a number field by Wedderburn's theorem, for some $s\in\mathbb{Z}$. We set

$$J_i=J^{\mathrm{new}}_{\mathbb{K}_1(n)}/\mathrm{Ker}[\mathrm{End}_k(J^{\mathrm{new}}_{\mathbb{K}_1(n)})\to F_i]J^{\mathrm{new}}_{\mathbb{K}_1(n)}$$

for each i. Then $J^{\mathrm{new}}_{\mathbb{K}_1(n)}(n)$ is isogenous to $J_1\times\cdots\times J_s$.

We have

$$\overline{\mathrm{V}}_\ell J_i\cong\bigoplus_{\pi\in\Pi_i}(\otimes_{\wp\neq\infty}\pi_\wp^{\mathbb{K}_{1,\wp}(n)})\otimes\rho(\pi)$$

for some subset $\Pi_i\subset\Pi_1^{\mathrm{new}}(n)$. Take a $\pi\in\Pi_i$. We claim that J_i satisfies the conditions (1), (2) and (3) of Proposition 2.1 so that we may call this A_π.

We see from (2.3) that an element of $\mathbb{T}^{\mathrm{new}}$ acts as a scalar on each $(\otimes_{\wp\neq\infty}\pi_\wp^{\mathbb{K}_1(n)})\otimes\rho(\pi)$. Giving this action is equivalent to giving an algebra homomorphism

$$\mathbb{T}^{\mathrm{new}}\xrightarrow{\mathrm{pr}_i}F_i\xrightarrow{\varphi_\pi}\overline{\mathbb{Q}}_\ell$$

where pr_i is the projection to the i-th factor and φ_π is some embedding.

From Lemma 2.3, we have $\dim_{\overline{\mathbb{Q}}_\ell}\oplus_{\pi\in\Pi_i}\otimes_{\wp\neq\infty}\pi_\wp^{\mathbb{K}_{1,\wp}(n)}=\dim_{\mathbb{Q}}F_i$, hence if we fix an element $\pi\in\Pi_i$, then any other $\pi'\in\Pi_i$ corresponds to the algebra homomorphism

$$\mathbb{T}^{\mathrm{new}}\xrightarrow{\mathrm{pr}_i}F_i\xrightarrow{\varphi_{\pi'}}\overline{\mathbb{Q}}_\ell$$

for some embedding $\varphi_{\pi'}$.

Note that $K_\pi \subseteq F_i$. If $K_\pi \neq F_i$, then there exists some $\varphi_{\pi'}$ such that $\varphi_\pi|_{K_\pi} = \varphi_{\pi'}|_{K_\pi}$. This contradicts that π and π' are not isomorphic using the strong multiplicity one theorem, hence $K_\pi = F_i$. This completes the proof of Proposition 2.1. □

2.8. We will restate the result obtained in this section in the form to be used elsewhere. We will use Corollary 2.5 in the proof of Lemma 5.2.

We define an equivalence relation on the set Π^∞ of cuspidal automorphic representations such that $\pi_\infty \cong \mathrm{St}$ by $\pi \sim \pi'$ if and only if A_π is isogenous to $A_{\pi'}$.

Corollary 2.4. *The Jacobian $J^\infty_{\mathbb{K}_1(n)}$ of $\overline{M}^\infty_{\mathbb{K}_1(n)}$ is isogenous to $\prod_{\pi\in\Pi^\infty/\sim} A_\pi^{d_{\pi,n}}$.*

Proof. We see from Lemma 2.2 that $J^\infty_{\mathbb{K}_1(n)}$ is isogenous to some product of the form $\prod_\pi A_\pi$ (possibly with multiplicity). Then the description of $\overline{\mathrm{V}}_\ell A_\pi$ in Proposition 2.1(3) and (2.2) leads to the claim. □

In the following corollary, we are taking ∞ to be $\wp$.

Corollary 2.5. *Let π be a cuspidal automorphic representation. Let St_π denote the set of places v such that $\pi_\wp \cong \mathrm{St}$. Then, for each $\wp \in \mathrm{St}_\pi$, there exists a Drinfeld modular curve $M^\wp_{\mathbb{K}}$ such that A_π is a simple factor of $\mathrm{Jac}\overline{M}^\wp_{\mathbb{K}}$.*

Proof. Fix a prime $\wp \in \mathrm{St}_\pi$. As we had done in Section 2.3, we can find an ideal n of the coordinate ring $H^0(C \setminus \{v\}, \mathcal{O}_C)$ such that $\dim_{\overline{\mathbb{Q}}_\ell} \otimes_{\wp'\neq\wp} \pi_{\wp'}^{\mathbb{K}_{1,\wp'}(n)} \geq 1$. Then by Corollary 2.4, A_π is a simple factor of $\mathrm{Jac}\overline{M}^\wp_{\mathbb{K}}$. □

3. Galois cohomology of 2-dimensional representations

The aim of this section is to prove Proposition 3.1. The vanishing result will be used in the proof of Lemma 5.1. The nonzero result will be used in the proof of Lemma 6.2. For the application, σ below will be a local component of some 2-dimensional direct factor of the Jacobian of a curve over k.

3.1. Let F be a nonarchimedean local field of characteristic $p > 0$. We will not use them but the results of this section holds also for the case $\mathrm{char}(F) = 0$. Fix a uniformizer $\varpi \in F$ and denote the residue field of F by k_F. Let $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ denote the absolute Galois group. Let ℓ be a prime number prime to p. Let σ be a 2-dimensional ℓ-adic representation, that is, a continuous homomorphism $\sigma : G_F \to \mathrm{GL}(V_\sigma)$ where V_σ is a 2-dimensional E-vector space ($[E : \mathbb{Q}_\ell] < \infty$). Let $V_{\sigma,\overline{\mathbb{Q}}_\ell} := V_\sigma \otimes_E \overline{\mathbb{Q}}_\ell$. Let W_F denote the Weil group and $i : W_F \hookrightarrow G_F$ be the canonical inclusion. We also denote σ the representation of W_F on $V_{\sigma,\overline{\mathbb{Q}}_\ell}$ obtained by the restriction. We write $WD(\sigma)$ for the Weil-Deligne representation attached to σ (see Bushnell-Henniart [3, p.206, Theorem]) We let $\pi = \pi(\sigma)$ denote the irreducible admissible representation of $\mathrm{GL}_2(F)$ which corresponds to σ via the local Langlands correspondence (see Bushnell-Henniart [3, p.212, Langlands correspondence]). We fixed an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Let St denote the Steinberg representation of $\mathrm{GL}_2(F)$.

Proposition 3.1. *Let the notations be as above. Suppose that $\pi(\sigma) \otimes |\det|^{\frac{1}{2}}$ is tempered. Then*

$$\dim_E H^1(G_F, V_\sigma(n)) = \begin{cases} 1 & \text{if } \pi(\sigma) \otimes |\det|^{\frac{1}{2}} \cong \mathrm{St} \text{ and } n = 0 \text{ or } \pi(\sigma) \otimes |\det|^{\frac{1}{2}} \cong \mathrm{St} \text{ and } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

We define the L-factor for a G_F-module V by

$$L(s, V) = \det\left(1 - \mathrm{Frob}_\varpi \cdot q^{-s} \,|\, V^I\right)^{-1},$$

where $q = \# k_F$, Frob_ϖ is the geometric Frobenius element and I is the inertia subgroup of G_F.

Lemma 3.2. *We have*

$$\dim_E H^1(G_F, V_\sigma(n)) = -\mathrm{ord}_{s=0} L(s, V_\sigma(n)) - \mathrm{ord}_{s=0} L(s, V_\sigma(n)^*),$$

where $\mathrm{ord}_{s=0}$ means the order of zero at $s = 0$.

Proof. We have

$$\begin{aligned} \dim_E H^1(G_F, V_\sigma(n)) &= \dim_E H^0(G_F, V_\sigma(n)) + \dim_E H^2(G_F, V_\sigma(n)) \\ &= \dim_E H^0(G_F, V_\sigma(n)) + \dim_E H^0(G_F, (V_\sigma(n))^*) \\ &= -\mathrm{ord}_{s=0} L(s, V_\sigma(n)) - \mathrm{ord}_{s=0} L(s, V_\sigma(n)^*). \end{aligned}$$

The first equality uses that the Euler-Poincare characteristic is zero ([36, p.101, II.5.7. exercise 4]), the second equality uses the local Tate duality (see [34, Theorem 1.4.1]), and the third equality follows from the definition of the L-factor. □

Lemma 3.3. *Let $\chi : F^\times \to \mathbb{C}^\times$ be a unitary character. Let $n \in \mathbb{Z}$. Then we have*

$$\mathrm{ord}_{s=0} L(s + \frac{n}{2}, \chi) = \begin{cases} -1 & \chi = 1 \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The L-factor has a pole at $s = 0$ if and only if $\chi(\varpi^{-1})|\varpi^{-1}|^{\frac{n}{2}} = 1$ where ϖ is a uniformizer. If this holds then χ must be unramified. Since χ is unitary, $n = 0$ and $\chi(\varpi^{-1}) = 1$. □

For an irreducible admissible representation π of $\mathrm{GL}_2(F)$, we denote the L-factor for π by $L(s, \pi)$. For the definition of the L-factor, see Godement-Jacquet [13, Chapter I, Section 3] or Jacquet [18, Section 1]. We compute the L-factor in terms of $\pi(\sigma)$ as follows. We have

$$\begin{aligned} L(s, V_\sigma(n)) &= L(s, \pi(V_\sigma(n))) \\ &= L(s, \pi(\sigma) \otimes |\det|^{\frac{1}{2}} \otimes |\det|^{n-\frac{1}{2}}) \\ &= L(s + n - \tfrac{1}{2}, \pi(\sigma) \otimes |\det|^{\frac{1}{2}}) \end{aligned}$$

and

$$\begin{aligned} L(s, V_\sigma(n)^*) &= L(s, \pi(V_\sigma(n)^*)) \\ &= L(s, \pi(V_\sigma(n))^\vee(1)) \\ &= L(s, (\pi(\sigma) \otimes |\det|^{\frac{1}{2}})^\vee \otimes |\det|^{-n+\frac{3}{2}}) \\ &= L(s - n + \tfrac{3}{2}, (\pi(\sigma) \otimes |\det|^{\frac{1}{2}})^\vee). \end{aligned}$$

We use the following classification.

Lemma 3.4. *The tempered representations of $\mathrm{GL}_2(F)$ are (i) supercuspidals, (ii) unitary twists of the Steinberg representation, and (iii) unitary principal series representation $\mathrm{Ind}_B^{\mathrm{GL}_2}(\chi_1, \chi_2)$ with unitary characters χ_1, χ_2.*

Proof. See Jacquet [17]. See also Kudla [23, p.375, Example 2.5]. □

3.2. Proof of Proposition 3.1. It suffices to consider the three cases.
Case (i)[$\pi(\sigma) \otimes |\det|^{\frac{1}{2}}$ is supercuspidal]: Since the L-factor of a supercuspidal representation is 1, we have $L(s, V_\sigma(n)) = L(s, V_\sigma(n)^*) = 1$ in this case. Hence the proposition follows using Lemma 3.2.
Case (ii)[$\pi(\sigma) \otimes |\det|^{\frac{1}{2}}$ is a unitary twist $\mathrm{St} \otimes \chi$ of the Steinberg representation]: We have

$$\begin{aligned} L(s, V_\sigma(n)) &= L(s + n - \tfrac{1}{2}, \mathrm{St} \otimes \chi) \\ &= L(s + n - \tfrac{1}{2}, \chi), \\ L(s, V_\sigma(n)^*) &= L(s - n + \tfrac{3}{2}, \mathrm{St}^\vee \otimes \chi^{-1}) \\ &= L(s - n + \tfrac{3}{2}, \mathrm{St} \otimes \chi^{-1}) \\ &= L(s - n + \tfrac{3}{2}, \chi^{-1}), \end{aligned}$$

where we used that $\mathrm{St} \cong \mathrm{St}^\vee$ (See Bushnell-Henniart [3, p.68, (9.10.5)]) and formulas in [23, p.377, 3.1]. The claim follows then from Lemmas 3.2 and 3.3.
Case (iii)[$\pi(\sigma) \otimes |\det|^{\frac{1}{2}}$ is a unitary principal series representation $\mathrm{Ind}_B^{\mathrm{GL}_2}(\chi_1, \chi_2)$ with unitary characters χ_1, χ_2.]: The L-factors are

$$\begin{aligned} L(s, V_\sigma(n)) &= L(s + n - \tfrac{1}{2}, \mathrm{Ind}_B^{\mathrm{GL}_2}(\chi_1, \chi_2)) \\ &= L(s + n - \tfrac{1}{2}, \chi_1) L(s + n - \tfrac{1}{2}, \chi_2), \\ L(s, V_\sigma(n)^*) &= L(s - n + \tfrac{3}{2}, \mathrm{Ind}_B^{\mathrm{GL}_2}(\chi_1^{-1}, \chi_2^{-1})) \\ &= L(s - n + \tfrac{3}{2}, \chi_1^{-1}) L(s - n + \tfrac{3}{2}, \chi_2^{-1}). \end{aligned}$$

The claim follows then from Lemmas 3.2 and 3.3.

4. Motivic cohomology and ℓ-adic cohomology for Chow motives

We collect some definitions and properties concerning Chow motives and cohomology theories from Jannsen [21] and Scholl [37]. For the application, we take the base field L below to be either k or $k_\wp$. The Chern class or the map r below will be used only with the base field $k_\wp$.

4.1. We first recall some definitions and properties of twisted Poincaré duality theories ([1], see [21, Section 6]). For a quasi-projective scheme X over a field L and a closed subscheme $Z \subset X$, the (rational) motivic cohomology of X with support on Z and the (rational) motivic homology of X are defined to be

$$\begin{aligned} H^i_{\mathcal{M},Z}(X, \mathbb{Q}(j)) &= K_{2j-i}(X)^{(j)}, \\ H_a^{\mathcal{M}}(X, \mathbb{Q}(b)) &= K'_{a-2b}(X)^{(-b)}. \end{aligned}$$

We refer to [21, p.104, 6.12] for the precise definitions.

Let ℓ be a prime different from the characteristic of L. We will use the (absolute) ℓ-adic cohomology with support and the (absolute) ℓ-adic homology:

$$H^i_{\text{ét},Z}(X, \pi_X^*\mathbb{Q}_\ell(j)),$$
$$H_i^{\text{ét}}(X, \mathbb{Q}_\ell(j)) = H^{-i}(X, \pi_X^!\mathbb{Q}_\ell(-j))$$

where $\pi_X : X \to \operatorname{Spec} L$ is the structure morphism. (The reader may find convenient the torsion coefficient case given in [1, p.193, 2.1], and the geometric case with rational coefficient given in [21, p.87, 6.8].)

In [21], Jannsen considered the motivic theory and the geometric (as opposed to the absolute) ℓ-adic theory, and the morphism between the two twisted Poincaré duality theories. In a similar manner, one can define a morphism from the motivic theory to the absolute ℓ-adic theory above. We write

$$r : H^i_{\mathcal{M},Z}(X, \mathbb{Q}(j)) \to H^i_{\text{ét},Z}(X, \mathbb{Q}_\ell(j)),$$
$$r' : H_a^{\mathcal{M}}(X, \mathbb{Q}(b)) \to H_a^{\text{ét}}(X, \mathbb{Q}_\ell(b))$$

for the induced maps. We refer to [21, p.126, (8.4.1)] and the remarks preceding [21, p.125, 8.3 a)] for the details (on the geometric case).

4.2. We will use the setting for Chow motives given by Scholl [37, Section 1]. We recall some of the notations.

Let L be a field. We let $\mathcal{V}_L$ denote the category of projective smooth L-schemes. We let $\mathcal{M}_L$ denote the category of L-motives for rational equivalence ([37, p.165, 1.4]). Take $X, Y \in \mathcal{V}_L$. Suppose X is connected and of pure dimension d. We let

$$\operatorname{Corr}^r(X, Y) = \operatorname{CH}^{r+d}(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

denote the correspondences of degree r. For the definition of correspondences for general X and the composition $\circ$ of correspondences, we refer to [37, 1.3]. Recall that an object $(X, p, m) \in \mathcal{M}_L$ is a triple with $X \in \mathcal{V}_L$, an idempotent $p = p^2 \in \operatorname{Corr}^0(X, X)$, and an integer m. The hom group $\operatorname{Hom}_{\mathcal{M}_k}((X, p, m), (Y, q, n))$ is by definition $q \circ \operatorname{Corr}^{\dim X}(X, Y) \circ p$.

We extend the definition of the cohomology groups to objects in $\mathcal{M}_L$. Let $(X, p, m) \in \mathcal{M}_L$. Note that $p \in \operatorname{Corr}^0(X, X)$ defines a map $p : H^a_{\mathcal{M}}(X, \mathbb{Q}(n)) \to H^a_{\mathcal{M}}(X, \mathbb{Q}(n)))$ and a map $p_{\text{ét}} : H^a_{\text{ét}}(X, \mathbb{Q}_\ell(n)) \to H^a_{\text{ét}}(X, \mathbb{Q}_\ell(n))$. We refer to [38, Section 1] for the construction when the cohomology theory is the $\mathcal{K}$-cohomology theory. The procedure works for the two cohomology theories considered here. For integers a, b, we set

$$H^{a,b}_{\mathcal{M}}((X, p, m)) := \operatorname{Im}[p : H^a_{\mathcal{M}}(X, \mathbb{Q}(m+b)) \to H^a_{\mathcal{M}}(X, \mathbb{Q}(m+b))],$$
$$H^{a,b}_{\text{ét}}((X, p, m)) := \operatorname{Im}[p_{\text{ét}} : H^a_{\text{ét}}(X, \mathbb{Q}_\ell(m+b)) \to H^a_{\text{ét}}(X, \mathbb{Q}_\ell(m+b))].$$

Then the r and r' defined in the previous section induce a map

$$r : H^{a,b}_{\mathcal{M}}((X, p, m)) \to H^{a,b}_{\text{ét}}((X, p, m)).$$

5. Proof of Theorem 1.1

We give a proof of Theorem 1.1. We refer to Section 1 for the outline of proof.

5.1. We begin with the proof of Proposition 1.5.

By Assumption (GL_2), we have $H^1_{\text{ét}}(B_{k^{\text{sep}}}, \overline{\mathbb{Q}}_\ell) \cong \oplus_{j=1}^n \rho_j$ where each ρ_j is an irreducible 2-dimensional representation of G_k with coefficients in $\overline{\mathbb{Q}}_\ell$.

We consider the case when there does not exist a pair $(j, \wp)$ such that $\rho_j|_{G_{k_\wp}} \cong \mathrm{Sp}$.

Lemma 5.1. *In this case, we have $H^{2,2}_{\acute{e}t}(h^1(B_{k_\wp})) = 0$ for any $\wp$.*

Proof. We use the definition of h^1 in [37, p.176, Theorem 4.4]. It says in particular that there exists a curve $Z \hookrightarrow B$ such that the projector to h^1 factors as

$$H^{2,2}_{\text{ét}}(B_{k_\wp}) \to H^{2,2}_{\text{ét}}(Z_{k_\wp}) \to H^{2,2}_{\text{ét}}(h^1(B_{k_\wp}))$$

where the first map is the pullback. Using the Leray spectral sequences for $B_{k_\wp}$ and $Z_{k_\wp}$, we see that it suffices to prove that

$$\operatorname{Im}[H^p(G_{k_\wp}, H^q_{\text{ét}}(B_{k_\wp^{\text{sep}}}, \mathbb{Q}_\ell(2))) \to H^p(G_{k_\wp}, H^q_{\text{ét}}(Z_{k_\wp^{\text{sep}}}, \mathbb{Q}_\ell(2)))],$$

where the map is that induced by the pullback map, is zero for each pair (p, q) with $p + q = 2$. It is easy to see that the target is zero when $(p, q) = (2, 0)$. When $(p, q) = (0, 2)$, using the fact that Z is a curve, it follows from the weight argument that the target is zero. Suppose $(p, q) = (1, 1)$. Let π_j denote the cuspidal automorphic representation corresponding to ρ_j by the global Langlands correspondence [9, p.566, Theorem A]. By the Petersson-Ramanujan conjecture (see [9, p.566, Theorem B]), we have that each $\pi_{j,\wp} \otimes |\det|^{\frac{1}{2}}$ is tempered. Since we know that Sp corresponds to St via the local Langlands correspondence, we obtain $H^1(G_{k_\wp}, H^1_{\text{ét}}(B_{k_\wp^{\text{sep}}}, \mathbb{Q}_\ell(2))) = 0$ from Proposition 3.1. $\square$

Now suppose that there exists a pair $(j, \wp)$ for which $\rho_j|_{G_{k_\wp}} \cong \mathrm{Sp}$. Let π be the unitary cuspidal automorphic representation associated to ρ_j via the global Langlands correspondence. Since we have $\pi_\wp \cong \mathrm{St}$, we can apply Proposition 2.1 and find an abelian variety A_π. We use the argument in the proof of Lemma 2.2. We have

$$\mathrm{Hom}_k(B, A_\pi) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell \cong \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[G_k]}(\overline{\mathrm{V}}_\ell B, \overline{\mathrm{V}}_\ell A_\pi) \neq 0.$$

Therefore there is a nontrivial homomorphism $B \to A_\pi$ which is an isogeny since both B and A_π are simple. We may and will assume that $B = A_\pi$.

Lemma 5.2. *For any place $\wp$ of k there exists a finite dimensional $\mathbb{Q}$-vector space $V_{\pi,\wp} \subset H^{2,2}_{\mathcal{M}}(h^1(A_\pi))$ such that*

$$r_{\wp'}(V_{\pi,\wp}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \begin{cases} H^{2,2}_{\text{ét}}(h^1(A_{\pi,k_\wp})) & \text{if } \wp' = \wp, \\ 0 & \text{if } \wp' \neq \wp. \end{cases}$$

Proof. Define St_π to be the set of places $\wp$ of k such that $\pi_\wp \cong \mathrm{St}$.

(1) Case $\wp \notin \mathrm{St}_\pi$. We set $V_{\pi,\wp} = 0$. Then it suffices to show that $H^{2,2}_{\text{ét}}(h^1(A_{\pi,k_\wp})) = 0$ for $\wp \notin \mathrm{St}_\pi$. This can be proved in the same manner as in the proof of Lemma 5.1, hence is omitted.

(2) Case $\wp \in \mathrm{St}_\pi$. By Corollary 2.5, A_π is a k-simple factor of $\mathrm{Jac}\overline{M}^{\wp}_{\mathbb{K}}$ for some compact open subgroup $\mathbb{K} \subset \mathrm{GL}_2(\mathbb{A}^\wp)$. Here $\mathbb{A}^\wp$ is the finite adeles as defined in Section 2.1 with $\wp$ in place of ∞ there.

By [22, p.1090, Proposition 9.1], there exists a finite dimensional $\mathbb{Q}$-vector space

$$V^{\wp}_{\mathbb{K}} \subset H^{2,2}_{\mathcal{M}}(h^1(\overline{M}^{\wp}_{\mathbb{K}}))$$

such that

$$r_{\wp'}(V^{\wp}_{\mathbb{K}}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \begin{cases} H^{2,2}_{\text{ét}}(h^1(\overline{M}^{\wp}_{\mathbb{K},k_\wp})) & \text{if } \wp' = \wp, \\ 0 & \text{if } \wp' \neq \wp. \end{cases} \tag{5.1}$$

Since $r_{\wp'}$ are compatible with the decomposition of a Chow motive, if we set $V_{\pi,\wp}$ to be image of $V^{\wp}_{\mathbb{K}}$ by the projection to $H^{2,2}_{\mathcal{M}}(h^1(A_\pi))$, the desired conditions are satisfied. □

Remark 5.3. *In* [22]*, they use the étale Chern class map $c_{2,2}$ (introduced in* [12]*) instead of $r_\wp$. However, it is known that $r_\wp$ is two times $c_{2,2}$ (See* [12, p.233, Proof of Proposition 2.35]*). Therefore, we can apply the result of* [22] *to our situation.*

This completes the proof of Proposition 1.5. □

5.2. We prove Proposition 1.4. To prove this, we decompose the Chow motive $h(X)$ into smaller pieces as follows. Let us use the formalism of Chow motives as explained in [37].

By Poincaré reducibility theorem (see [28, p.1, Theorem 1] for the case where the base field is imperfect), the Jacobian $\mathrm{Jac}X$ of X is canonically (up to isogeny) isogenous to a product of k-simple abelian varieties: $\mathrm{Jac}X \to B_1 \times \cdots \times B_s$. Then we have

$$\begin{aligned} h(X) &\cong h^0(X) \oplus h^1(X) \oplus h^2(X) \\ &\cong h^0(X) \oplus h^1(\mathrm{Jac}X) \oplus h^2(X) \\ &\cong h^0(X) \oplus h^1(\textstyle\prod_{i=1}^s B_i) \oplus h^2(X) \\ &\cong h^0(X) \oplus \textstyle\bigoplus_{i=1}^s h^1(B_i) \oplus h^2(X), \end{aligned}$$

The first isomorphism is that in [37, p.172, 3.2]. The second isomorphism follows from [37, p.178, Proposition 4.5]. The third isomorphism follows from [37, p.182, Corollary 5.10]. For the fourth isomorphism, recall (see [7, p.216, Theorem 3.1]) that the projector corresponding to the h^i part of an abelian variety is characterized by the fact that the multiplication-by-n map acts as n^i on it. Now observe that $h(\prod_{i=1}^s B_i) \cong \otimes_{i=1}^s h(B_i) \cong \oplus_{i_1,\dots,i_s} h^{i_1}(B_1) \otimes \cdots \otimes h^{i_s}(B_s)$. Since $h^0(B_i) \cong h(\mathrm{Spec}\, k)$ for all i, the fourth isomorphism holds true.

Let us write $Y_L = Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$ for a k-scheme Y and a k-algebra L. To prove Proposition 1.4, it suffices to show the following three statements:

(1) The map $\oplus_\wp r_\wp : H^{2,2}_{\mathcal{M}}(h^0(X)) \to \oplus_\wp H^{2,2}_{\text{ét}}(h^0(X_{k_\wp}))$ induces a surjection when the source is tensored with $\mathbb{Q}_\ell$.
(2) The map $\oplus_\wp r_\wp : H^{2,2}_{\mathcal{M}}(h^1(B_i)) \to \oplus_\wp H^{2,2}_{\text{ét}}(h^1(B_{i,k_\wp}))$ induces a surjection when the source is tensored with $\mathbb{Q}_\ell$ for each $1 \leq i \leq s$.
(3) The map $\oplus_\wp r_\wp : H^{2,2}_{\mathcal{M}}(h^2(X)) \to \oplus_\wp H^{2,2}_{\text{ét}}(h^2(X_{k_\wp}))$ induces a surjection when the source is tensored with $\mathbb{Q}_\ell$.

Note that $h^0(X_{k_\wp}) \cong h(\operatorname{Spec} k_\wp)$ and $h^2(X_{k_\wp}) \cong (\operatorname{Spec} k_\wp, \mathrm{id}, -1) = \mathbb{L}$ (the Lefschetz motive). Hence the target group of the map in (1) and in (3) are zero, so the surjectivity follows trivially. The statement (2) follows from Proposition 1.5.

This finishes the proof of Proposition 1.4. □

5.3. Proof of Theorem 1.1. Let

$$\partial_{\wp,\mathbb{Q}_\ell} : K_2(X_{k_\wp})_{\mathbb{Q}_\ell} \to G_1(\mathscr{X}_{\kappa(\wp)})_{\mathbb{Q}_\ell}$$

be the boundary map at $\wp$ tensored with $\mathbb{Q}_\ell$ and

$$c_{2,2,\wp} : K_2(X_{k_\wp})_{\mathbb{Q}_\ell} \to H^2_{\text{ét}}(\mathscr{X}_{k_\wp}, \mathbb{Q}_\ell(2))$$

the étale Chern class map (defined in [12]) at $\wp$ tensored with $\mathbb{Q}_\ell$. As we remarked in Remark 5.3, the difference between $r_\wp$ and $c_{2,2,\wp}$ is a non-zero constant multiple.

Suppose

(1) $\partial_{\wp,\mathbb{Q}_\ell}(x) = 0$ if and only if $c_{2,2,\mathbb{Q}_\ell}(x) = 0$ for $x \in K_2(X_{k_\wp})_{\mathbb{Q}_\ell}$,
(2) $\partial_{\wp,\mathbb{Q}_\ell}$ is surjective if and only if $c_{2,2,\mathbb{Q}_\ell}$ is surjective.

Then it is easy to see that Theorem 1.1 follows from Proposition 1.4. Therefore it is enough to check (1) and (2). By [22, p.1059, Proposition 3.1], (1) holds true. Let $V \subset K_2(X_{k_\wp})_{\mathbb{Q}}$ be a finite dimensional vector space. Then, using (1), we have $\dim_{\mathbb{Q}_\ell} \partial_{\wp,\mathbb{Q}_\ell}(V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell} c_{2,2,\wp}(V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$. Since $\dim_{\mathbb{Q}} G_1(\mathscr{X}_{\kappa(\wp)})_{\mathbb{Q}} = \dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X_{k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell(2))$ by [22, p.1056, Proposition 2.1], (2) holds true. This completes the proof.

6. Proof of Corollary 1.3 and the target group

We give a precise form of Corollary 1.3 and its proof in this section. We also compute the dimension of the target group of the boundary map in Theorem 1.1.

6.1.

Corollary 6.1. *Theorem 1.1 holds in all of the following cases.*

(1) X is an elliptic curve, which is not isotrivial.
(2) X is a (compactified) Drinfeld modular curve. That is, when $X = \overline{M}^\infty_{\mathbb{K}}$ for some ∞ and $\mathbb{K}$ as in Section 2.1.
(3) $X = M^\infty_{B,\mathbb{K}}$ is the moduli scheme of $\mathscr{D}$-elliptic sheaves of rank 2 with level $\mathbb{K}$-structure as defined in [26]. *Here, B is an indefinite quaternion algebra over k, $\mathscr{D}$ is a maximal order of B and $\mathbb{K}$ is an open compact subgroup of $(B \otimes_k \mathbb{A}_k)^\times$.*
(4) X is the curve X_s of Section 7 for $s \in k$ such that $D(s) \neq 0$.

Proof. We check that the condition (GL_2) holds in each case. It holds trivially in Case (1). It follows from (2.2) in Case (2). For Case (3), let Π_B denote the set of cuspidal automorphic representations of $(B \otimes_k \mathbb{A}_k)^\times$. Then we have

$$H^1_{\text{ét}}(M^\infty_{B,\mathbb{K},k^{\mathrm{sep}}}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\pi \in \Pi_B, \pi_\infty \cong \mathrm{St}} (\otimes_{v \neq \infty} \pi_v)^{\mathbb{K}} \otimes \rho(\pi)$$

where $\rho(\pi)$ is an irreducible 2-dimensional representation of G_k. To see this, one proceeds as in the proof of [8, p.590, Proposition 10.3]. (See also [11, Theorem 4.13.1].) The key ingredient is the uniformization of the moduli space, which, in the $\mathscr{D}$-elliptic sheaf case, is provided by [2, p.171, Theorem 4.4.11]. The irreducibility of $\rho(\pi)$ follows from the L-factor computation given in [26, Theorem 14.9(ii)]. For Case (4), it follows from Corollary 7.2. □

6.2. Let X be as in Theorem 1.1. Write $H^1_{\text{ét}}(X_{k^{\mathrm{sep}}}, \overline{\mathbb{Q}}_\ell) \cong \oplus_{j=1}^n \rho_j$ where each ρ_j is a 2-dimensional irreducible representation of G_k. Let Sp_X denote the number of pairs $(j, \wp)$ such that $\rho_j|_{G_{k_\wp}} \cong \mathrm{Sp}$.

Lemma 6.2. *We have* $\dim_{\mathbb{Q}_\ell} \oplus_{\wp \in C_0} H^{2,2}_{\acute{e}t}(h^1(X_{k_\wp})) = \mathrm{Sp}_X$.

Proof. It suffices to prove this for $X = B$ a simple abelian variety. If $\mathrm{Sp}_B = 0$, this holds by Lemma 5.1. When $\mathrm{Sp}_B \neq 0$, as we have seen in the discussion after Lemma 5.1 that $B = A_\pi$ for some π. In this case, the claim follows from Proposition 3.1. □

Corollary 6.3. *Let X be as in Theorem 1.1. Then*

- $\dim_{\mathbb{Q}} K_2(X)_{\mathbb{Q}} \geq \mathrm{Sp}_X$,
- $\dim_{\mathbb{Q}} H^2_{\mathcal{M}}(X, \mathbb{Q}(2)) \geq \mathrm{Sp}_X$.

Proof. This follows from Lemma 6.2, using Theorem 1.1 and Proposition 1.4 respectively. □

Consider Case (4) in Corollary 6.1. We show (Proposition 7.1) that, given an integer n, there exists an element $s \in k$ such that $D(s) \neq 0$ and $\mathrm{Sp}_{X_s} \geq 2n$. Hence we have

Corollary 6.4. *Suppose the characteristic of k is greater than 5. For any $n \in \mathbb{Z}$, there exists a genus two curve X such that $K_2(X)_{\mathbb{Q}} \geq n$.*

Proof. This follows from Corollary 6.3 and Proposition 7.1 immediately. $\square$

Note that in [22, p.1054, Corollary 1.3], a similar statement where genus two is replaced by genus one is proved.

7. Abelian varieties of GL_2-type

We keep the notations in Section 2. In this section we will be concerned with some classes of abelian varieties over k which admit many endomorphisms. Such abelian varieties are originally considered by Ribet [33] in which the ground fields are $\mathbb{Q}$ and they are called abelian varieties of GL_2-type. We will define an analogue of such objects in the function fields setting and construct examples of 2-dimensional abelian varieties over k of GL_2-type which has multiplicative reduction at given finite places.

Let B be an abelian variety over k of dimension g. We say that B is of GL_2-type if there exist a number field F of degree g over $\mathbb{Q}$ and an embedding $\iota : F \hookrightarrow \mathrm{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}$-algebras. The abelian varieties A_{π_i} of Proposition 2.1 (3) are examples of abelian varieties over k of GL_2-type.

Let B be an abelian variety over k of GL_2-type endowed with an embedding $\iota : F \hookrightarrow \mathrm{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let ℓ be a prime number which is different from the characteristic of k. Via ι and the natural embedding $\mathrm{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \mathrm{End}_{\mathbb{Q}_\ell[G_k]}(V_\ell B)$, the field F acts on $V_\ell B$ and its action is non-trivial and faithful because F is a field and it contains 1. Since $[F : \mathbb{Q}] = g$ and $\mathrm{rank}_{\mathbb{Q}_\ell} V_\ell B = 2g$, we can view $V_\ell B$ as a free $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank 2 by [39, Lemma 2.1-(i)]. Then we have the 2-dimensional Galois representation attached to B:

$$\rho : G_k \longrightarrow \mathrm{Aut}_{F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell}(V_\ell B) \simeq \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = \prod_{v|\ell} \mathrm{GL}_2(F_v)$$

where F_v is the completion of F at a finite place v of F dividing ℓ. We denote by ρ_v the composition of ρ and the projection $\prod_{v|\ell} \mathrm{GL}_2(F_v) \xrightarrow{\mathrm{pr}_v} \mathrm{GL}_2(F_v)$. It is easy to see that $L(s, V_\ell B) = \prod_{v|\ell} L(s, \rho_v)$.

We next construct examples of 2-dimensional abelian varieties over k of GL_2-type by using Brumer-Hashimoto's family.

Let $p > 5$ be a prime number and q be a power of p. Let k and A be as in Section 1.2. Let $H(X; a, b, c)$ be the polynomial defined by [16, p.479, (10)]. For any $s \in k$, we define the following polynomial by using $H(X; a, b, c)$:

$$(7.1)\quad H(x; -\frac{7}{4}, s+1, -1) := x^6 - (2s+3)x^5 + (s^2 + 4s + \frac{13}{4})x^4 + (2s^2 - 2s - \frac{3}{2})x^3 + (s^2 + 2s + \frac{1}{4})x^2 - 2sx$$

with the discriminant

$$D(s) := s^6(-9 + 1160s + 1512s^2 + 992s^3 + 48s^4)^2 \neq 0$$

with respect to x. For $s \in k$ so that $D(s) \neq 0$, gluing the two finite surjective morphism $f_1 : \mathrm{Spec}\, k[x, y]/(y^2 - H(x; -\frac{7}{4}, s+1, -1)) \longrightarrow \mathbb{A}^1_{k,x}$, $(x, y) \mapsto x$ and $f_2 : \mathrm{Spec}\, k[x_1, y_1]/(y_1^2 - x_1^6 H(\frac{1}{x_1}; -\frac{7}{4}, s+1, -1)) \longrightarrow \mathbb{A}^1_{k,x_1}$, $(x_1, y_1) \mapsto x_1$ by the isomorphism $(x, y) \longleftrightarrow (x_1, y_1) = (\frac{1}{x}, \frac{y}{x^3})$, we have a finite surjective morphism $X_s \longrightarrow \mathbb{P}^1$. The resulting variety X_s is a projective smooth curve of genus 2 over k. Put $J_s = \mathrm{Jac}X_s$. Then by [16] we have an embedding $\iota : F = \mathbb{Q}(\sqrt{5}) \hookrightarrow \mathrm{End}_k(J_s) \otimes_{\mathbb{Z}} \mathbb{Q}$. Actually the results of [16] stated for which ground fields is the rational function fields over $\mathbb{Q}$, but it is easy to check that the algebraic operations performed there work over any $\mathbb{Z}[\frac{1}{2\cdot 3\cdot 5}]$-algebra. Hence we can apply the results in [16].

Let $\wp_1, \ldots, \wp_n$ be the non-zero prime ideals of A. Take a non-zero element $s \in \cap_{i=1}^n \wp_i$. Then it is easy to see $D(s) \neq 0$. For such s and each finite place $v|\ell$ of $F = \mathbb{Q}(\sqrt{5})$ we attach the 2-dimensional Galois representation $\rho_{s,v} : G_k \longrightarrow \mathrm{GL}_2(F_v)$ from J_s as above. Then the local behaviour of $\rho_{s,v}$ at each $\wp_i$ is as follows.

Proposition 7.1. *For each $i \in \{1, \ldots, n\}$, $\rho_{s,v}|_{G_{k_{\wp_i}}} \simeq \mathrm{Sp}$.*

Proof. For simplicity put $\wp = \wp_i$. Let $A_\wp$ (resp. $k_\wp$) be the completion of A (resp. k) at $\wp$. Fix a uniformizer ϖ of $A_\wp$, put $\kappa := A_\wp/\varpi$ and $q_\varpi = \sharp\kappa$. Let $W_{k_\wp}$ be the Weil group of $k_\wp$ (see [41] for basic properties). Fix a lift $\mathrm{Frob}_\varpi \in W_{k_\wp}$ of the geometric Frobenius element of G_κ. Then each element $\sigma \in W_{k_\wp}$ can be written uniquely as $\sigma = \tau \cdot \mathrm{Frob}_\varpi^n$, $\tau \in I_{k_\wp}$, $n \in \mathbb{Z}$. Put $n(\sigma) = n$ for such $\sigma \in W_{k_\wp}$. Since $W_{k_\wp}$ is dense in $G_{k_\wp}$, the restriction map from the category of continuous $G_{k_\wp}$-representations to the category of $W_{k_\wp}$-representations is injective. So it suffices to prove $\mathrm{WD}(\rho_{s,v}|_{G_{k_{\wp_i}}}) \simeq \mathrm{WD}(\mathrm{Sp})$.

Gluing the two finite surjective morphism

$$\widetilde{f_1} : \mathfrak{X}_{s,1} := \mathrm{Spec}\, A_\wp[x, y]/(y^2 - H(x; -\frac{7}{4}, s+1, -1)) \longrightarrow \mathbb{A}^1_{A_\wp, x},\ (x, y) \mapsto x$$

and

$$\widetilde{f_2} : \mathfrak{X}_{s,2} := \operatorname{Spec} A_\wp[x_1, y_1]/(y_1^2 - x_1^6 H(\frac{1}{x_1}; -\frac{7}{4}, s+1, -1)) \longrightarrow \mathbb{A}^1_{A_\wp, x_1},\ (x_1, y_1) \mapsto x_1$$

by the isomorphism $(x, y) \longleftrightarrow (x_1, y_1) = (\frac{1}{x}, \frac{y}{x^3})$, we have a finite surjective morphism $\mathfrak{X}_s \longrightarrow \mathbb{P}^1_{A_\wp}$. The resulting variety $\mathfrak{X}_s$ is a projective flat scheme over $A_\wp$ with relative dimension one. The projectivity follows from the ampleness of the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^1_{A_\wp}$ and [15, III, Exercise 5.8-(d)]. Since the special fiber of $\mathfrak{X}_s$ is obtained by gluing the following affine models:

$$\mathfrak{X}_{s,1} \otimes_{A_\wp} \kappa = \operatorname{Spec} \kappa[x, y]/(y^2 - x^2(x-1)^2(x - \frac{1}{2})^2),\ \mathfrak{X}_{s,2} \otimes_{A_\wp} \kappa = \operatorname{Spec} \kappa[x_1, y_1]/(y_1^2 - (1-x_1)^2(1 - \frac{1}{2}x_1)^2),$$

the possible singularities are at the points $P_i \in \mathfrak{X}_{s,1}$ corresponding to the ideals $(x - i, y, \varpi)$ for $i \in \{0, 1, \frac{1}{2}\}$. It is easy to see that $\mathfrak{X}_s$ is regular at $P_{\frac{1}{2}}$. We denote by $\widetilde{\mathfrak{X}}_s$ the blowing up of $\mathfrak{X}_s$ along to the ideals (x, y, ϖ) and $(x-1, y, \varpi)$. Then $\widetilde{\mathfrak{X}}_s$ is a proper strictly semistable model of X_s over $A_\wp$ and its special fiber is obtained by gluing two copies of $\mathbb{P}^1_\kappa$ corresponding to two irreducible components of $\mathfrak{X}_s \otimes \kappa$ at three points P_0, P_1, and $P_{\frac{1}{2}}$ and then replacing P_i by a chain of $i+1$ projective lines for $i = 0, 1$. Hence $\mathfrak{X}_s \otimes \kappa$ is the union of five divisors $D_1, \ldots, D_5$ which are isomorphic to $\mathbb{P}^1_\kappa$ over κ. Here D_1 and D_2 correspond to irreducible components of $\mathfrak{X}_s \otimes \kappa$ and D_3 (resp. D_i, $i = 4, 5$) is one chain (resp. two chains) between D_1 and D_2. These components intersect at κ-rational points. Then by [14, Section 12], we have the claim by Picard-Lefschetz formula which is well-known for experts. However we give a proof here in details for the reader's convenience and further use the weight spectral sequence of Rapoport and Zink instead of Picard-Lefschetz formula.

Henceforth H^* means the étale cohomology. To compute the action of Weil group $W_{k_\wp}$ of $k_\wp$ on $V_\ell J_s$ we use the isomorphism $V_l J_s \simeq H^1(X_{s,k^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ and the weight spectral sequence of Rapoport and Zink for the semistable curve X_s. Put $D^{(0)} = \coprod_{i=1}^5 D_i$ and $D^{(1)} = \coprod_{i<j} D_i \cap D_j$. Note that $D^{(1)}$ consists of six κ-rational points. Then by [32] we have the following spectral sequence:

$$E_1^{-r,w+r} = \bigoplus_{i \geq \max\{0,-r\}} H^{w-r-2i}(D_{\overline{\kappa}}^{(-r+2i)}, \mathbb{Q}_\ell(-i)) \Longrightarrow H^w(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell).$$

This gives the weight filtration Fil_i^W on $H^w(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$. The essential E_1-terms are as follows:

$$\begin{array}{ccc} E_1^{-1,2} & E_1^{0,2} & \\ & & \\ & E_1^{0,1} & \\ & & \\ & E_1^{0,0} & E_1^{0,1}. \end{array}$$

We can compute the E_1-terms as follows:

$$E_1^{-1,2} = H^0(D_{\overline{\kappa}}^{(1)}, \mathbb{Q}_\ell(-1)) \simeq \mathbb{Q}_\ell(-1)^{\oplus 6},\ E_1^{0,2} = H^2(D_{\overline{\kappa}}^{(0)}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-1)^{\oplus 5},$$

$$E_1^{0,1} = H^1(D_{\overline{\kappa}}^{(0)}, \mathbb{Q}_\ell) = 0,\ E_1^{0,0} = H^0(D_{\overline{\kappa}}^{(0)}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^{\oplus 5},\ E_1^{1,0} = H^0(D_{\overline{\kappa}}^{(1)}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^{\oplus 6}.$$

For each $\sigma \in W_{k_\wp}$, since

$$\sum_w (-1)^w \mathrm{tr}(\sigma | H^w(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)) = \sum_{r,w} (-1)^w \mathrm{tr}(\sigma | E_1^{-r,w+r}) = 5 - 6 + 5q_\varpi^{n(\sigma)} - 6q_\varpi^{n(\sigma)} = -1 - q_\varpi^{n(\sigma)},$$

$\mathrm{tr}(\sigma | H^0(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)) = 1$, and $\mathrm{tr}(\sigma | H^2(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)) = q_\varpi^{n(\sigma)}$, we have that

$$\mathrm{tr}(\sigma | H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)) = 2 + 2q_\varpi^{n(\sigma)}.$$

Since $H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ is a free $F_\ell := F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank 2 and $\det(\sigma | H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)) = q_\varpi^{2n(\sigma)}$, the eigenvalues of σ on F_ℓ-module $H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ are $1, q_\varpi^{n(\sigma)}$. Therefore as $F_\ell[W_{k_\wp}]$-modules we have that

$$H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)^{\mathrm{ss}} \simeq F_\ell \oplus F_\ell(-1).$$

We next compute the monodromy operator N on $H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ which is defined by Grothendieck's monodromy theorem. By weight monodromy theorem for curves, we have that

(1) $N(\mathrm{Fil}_i^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)) \subset \mathrm{Fil}_{i-2}^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$,

(2) $N : \mathrm{gr}_2^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-1)^{\oplus 2} \xrightarrow{\sim} \mathrm{gr}_0^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^{\oplus 2}$.

Recall that F acts on $H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell))$ via the embeddings $F \overset{\iota}{\hookrightarrow} \mathrm{End}_k(J_s) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \mathrm{End}_{\mathbb{Q}_\ell[G_k]}(V_\ell J_s)$ and $V_\ell J_s \simeq H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell))$ as $\mathbb{Q}_\ell[G_k]$-modules. Therefore the action of $F = \mathbb{Q}(\sqrt{5})$ on $H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell))$ commutes with the Galois action, hence F acts also on $\mathrm{gr}_0^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell) = \mathrm{Fil}_0^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell) = \mathrm{Ker}(N)$

and $\mathrm{gr}_2^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$. Note that $\mathrm{gr}_i^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ is a free F_ℓ-module of rank one for $i = 0, 2$. So take a basis e_2 of $\mathrm{gr}_0^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell) = \mathrm{Fil}_0^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ over F_ℓ. Then by (1) as above, $N(e_2) = 0$ since $\mathrm{Fil}_{-2}^W H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell) = 0$. Take a lift $e_1 \in H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ of the element corresponding to e_2 via the isomorphism of (2). Then we have $N(e_1) = e_2$. Clearly $\{e_1, e_2\}$ forms a basis of $H^1(X_{s,k_\wp^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ over F_ℓ. Therefore the matrix representation of N with respect to this basis is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Hence we have $WD(\rho_v|_{G_{k_\wp}}) \simeq WD(\mathrm{Sp})$. □

Corollary 7.2. *The notations are same as above. The curve X_s satisfies the condition* (GL_2).

Proof. Recall that we have assumed that $p > 5$ for the construction of X_s. Then by Tate conjecture [43], $V_\ell J_s \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell = \bigoplus_{v|\ell} \rho_v \otimes_{F_v} \overline{\mathbb{Q}}_\ell$ is a semi-simple $\overline{\mathbb{Q}}_\ell[G_k]$-module. If $\rho_v \otimes_{F_v} \overline{\mathbb{Q}}_\ell$ is reducible for some $v|\ell$, then the rank of the monodromy operator N on $V_\ell J_s$ is 0 or 1. However by Proposition 7.1 the rank of N has to be 2 which gives a contradiction. Hence $\rho_v \otimes_{F_v} \overline{\mathbb{Q}}_\ell$ is irreducible for any $v|\ell$. □

Remark 7.3. *It seems to be meaningful to discuss about k-simpleness of J_s because if J_s is not k-simple, then J_s is isogenous to the product of two elliptic curves over k which have split multiplicative reduction at each $\wp_i$. In this case, Corollary 6.4 comes down to* [22, Corollary 1.3] *and therefore does not create essentially a new example.*

In general, it seems to be difficult to check the k-simpleness of J_s for any k. However for a given k we can check the k-simpleness of J_s as follows. Assume that X_s has good reduction at some non-zero $\wp \in \mathrm{Spec}\, A$. We have the maps

$$\mathrm{End}_k(J_s) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\alpha} \mathrm{End}_{k_\wp}(J_{s,k_\wp}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\mathrm{sp}} \mathrm{End}_\kappa(J_{s,\kappa}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The homomorphism α is a natural embedding and the second map sp *is the specialization map which is known to be injective (cf.* [25, p.45, Theorem 3.2]*). Hence the composition of α and* sp *is injective. If $L(s, V_\ell J_{s,\kappa})^{-1}$ is an irreducible polynomial in $\mathbb{Q}[q^{-s}]$, then $J_{s,\kappa}$ is κ-simple by* [40]*, hence $\mathrm{End}_\kappa(J_{s,\kappa}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division field. Hence J_s is also k-simple because of the injectivity of* $\mathrm{sp} \circ \alpha$ *as above. From this we can easily check the k-simpleness of J_s when $k = \mathbb{F}_p(s)$ and $5 < p < 1000$. Here the calculation is done by using Mathematica version 8.0.*

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