General relativistic symmetry of electron spin vorticity

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ABSTRACT: The general relativistic symmetry of QED (quantum electrodynamics) predicts that the spin vorticity of electron contributes to the kinetic momentum of electron. The canonical quantization of QED is performed by using new b-photon, f-electron, and f^c-positron algebras. These algebras work for interacting particles and are useful for nonperturbationally solving the dual Cauchy problems of QED.

Key Words: spin vorticity; general relativity; QED; b-photon; f-electron; f^c-positron

1. Introduction

Let us ask a simple but "odd" question: what is momentum of electron spin? "How odd this question is" should be obvious since electron is considered a point particle and spin is its internal degree of freedom and then spin is considered to have nothing to do with momentum. On the contrary to this obvious common sense, we shall prove that the spin vorticity of electron contributes to the kinetic momentum of electron.

The key idea is the general relativistic symmetry of QED (quantum electrodynamics) [1,2]. QED is reformulated in a way that is covariant under general coordinate transformation [3-5]. The consequence gives the right answer to the odd question raised above.

In QED, every dynamical variable is given by the quantized q-number field operator defined on the background Minkowski spacetime $x^\mu = (ct, \vec{x})$ with symbol on it, e.g. $\hat{F}(x)$, distinct from countable c-number, e.g. $F(x)$ [1]. *Surprisingly, it is not the velocity* $\partial\hat{\vec{s}}(x)/\partial t$ *but the vorticity* $\mathrm{rot}\hat{\vec{s}}(x)$ *of the spin vector field operator* $\hat{\vec{s}}(x)$ *that contributes to the kinetic momentum operator* $\hat{\vec{\Pi}}(x)$ *under the covariant symmetry of the general coordinate transformation* (with factor 1/2; see Eq. (2.21) in Section 2.3).

In Section 2, we shall first quickly review our preceding publications on general relativistic symmetry of QED [6-9] in such a way that it warrants the derivation of the general relativistic symmetry of electron spin vorticity. In Section 3, the canonical commutation relationship of the electromagnetic field of photon and the canonical anticommutation relationship of the Dirac field of electron and positron is studied nonpertubationally using new devices called the b-photon, f-electron, and f^c-positron algebras. In Section 4, we construct the ket vector with wave function for the dual Cauchy problems of QED and conclude the concrete space-time resolved simulation of the c-number $\langle \hat{F}(x) \rangle$ for the q-number $\hat{F}(x)$. Mathematical details are summarized in Appendix A for the Minkowski spacetime and Appendix B for general relativity.

2. Spin vorticity

2.1. Covariant derivative on the background Minkowski spacetime

On the background Minkowski spacetime, the Dirac equation of the Dirac spinor operator $\hat{\psi}$ with the covariant derivative $\hat{D}_\mu$ of QED is given as [1]

$$\left(i\hbar\gamma^{\mu}\hat{D}_{\mu} - mc\right)\hat{\psi} = 0 \tag{2.1}$$

$$\hat{D}_{\mu} = \partial_{\mu} + i\frac{q}{\hbar c}\hat{A}_{\mu} \tag{2.2}$$

where m is the mass of electron, c is the speed of light in vacuum, $q = -e$ is the charge of electron, and $\hat{A}^{\mu}$ the Abelian gauge potential of photon in the Coulomb gauge. The charge current

$$\hat{j}^{\mu} = cq\hat{\bar{\psi}}\gamma^{\mu}\hat{\psi}, \quad \hat{\bar{\psi}} = \hat{\psi}^{\dagger}\gamma^{0} \tag{2.3}$$

satisfies the conservation law

$$\partial_{\mu}\hat{j}^{\mu} = 0 \tag{2.4}$$

and the kinetic momentum

$$\hat{\vec{\Pi}} = \frac{1}{2}\left(\hat{\psi}^{\dagger}\left(i\hbar\hat{\vec{D}}\right)\hat{\psi} + h.c.\right) \tag{2.5}$$

satisfies the equation of motion [6]

$$\frac{\partial}{\partial t}\hat{\vec{\Pi}} = \hat{\vec{L}} + \hat{\vec{\tau}}^{\Pi} \tag{2.6}$$

In the right hand side, the force is composed of the Lorentz force $\hat{\vec{L}}$ and the tension $\hat{\vec{\tau}}^{\Pi}$

$$\hat{\vec{\tau}}^{\Pi} = \mathrm{div}\,\hat{\vec{\vec{\tau}}}^{\Pi}, \quad \hat{\tau}^{\Pi k} = \partial_{\ell}\hat{\tau}^{\Pi k\ell} \tag{2.7}$$

which is the divergence of the stress tensor $\hat{\vec{\vec{\tau}}}^{\Pi}$

$$\hat{\tau}^{\Pi}_{\mu\nu} = \frac{c}{2}\left(\hat{\bar{\psi}}\gamma_{\nu}\left(-i\hbar\hat{D}_{\mu}\right)\hat{\psi} + h.c.\right) \tag{2.8}$$

The stress tensor itself is not defined uniquely since mathematically any tensor whose divergence is zero can be added to.

2.2. Covariant derivative of general relativity

To seek for the variation principle of the equation of motion on the background curved spacetime, the semiclassical Einstein-Hilbert action integral has been used under the symmetry of the general coordinate transformation of gravity [1]

$$\delta\hat{I}=0,\quad \hat{I}=\frac{c}{2\kappa}\int R\sqrt{-g}\,d^4x+\frac{1}{c}\int\hat{L}\sqrt{-g}\,d^4x,\ \ \kappa=\frac{8\pi G}{c^2} \tag{2.9}$$

where R is the Ricci scalar, G is the universal gravitational constant, and $\hat{L}$ is the Lagrangian density of QED including the interaction with gravity.

The gravitational covariant derivative $\hat{D}_\mu(g)$ is then given as [3-5]

$$\begin{aligned}\hat{D}_\mu(g)&=\partial_\mu+i\frac{q}{\hbar c}\hat{A}_\mu++i\frac{1}{2\hbar}\gamma_{ab\mu}J^{ab}\\&=\hat{D}_\mu+i\frac{1}{2\hbar}\gamma_{ab\mu}J^{ab}\end{aligned} \tag{2.10}$$

with the spin angular momentum

$$J^{ab}=\frac{i\hbar}{4}\left[\gamma^a,\gamma^b\right] \tag{2.11}$$

and spin connection

$$\gamma_a{}^b{}_\mu=e_{a\nu;\mu}\eta^{bc}e_c{}^\nu \tag{2.12}$$

Using the gravitational covariant derivative, the stress tensor of electron $\hat{\tau}^{\Pi}_{\mu\nu}(\mathrm{g})$ becomes [6,7]

$$\hat{\tau}^{\Pi}_{\mu\nu}(\mathrm{g})=\frac{c}{2}\left(\hat{\bar{\psi}}\gamma_\nu\left(-i\hbar\hat{D}_\mu(g)\right)\hat{\psi}+h.c.\right) \tag{2.13}$$

2.3. Spin vorticity

In this variation principle, *due to the presence of the spin connection $\gamma_{ab\mu}$, a new symmetry-polarized geometrical tensor $\hat{\varepsilon}^{\Pi}_{\mu\nu}$ appears and whose antisymmetric component cancels* *with that of* $\hat{\tau}^{\Pi}_{\mu\nu}(g)$

$$\hat{\varepsilon}^{A\mu\nu} + \hat{\tau}^{A\mu\nu}(g) = 0 \qquad (2.14)$$

where

$$\hat{\varepsilon}^{A\mu\nu} = \frac{1}{2}\left(\hat{\varepsilon}^{\Pi\mu\nu} - \hat{\varepsilon}^{\Pi\nu\mu}\right) \qquad (2.15)$$

$$\hat{\tau}^{A\mu\nu}(g) = \frac{1}{2}\left(\hat{\tau}^{\Pi\mu\nu}(g) - \hat{\tau}^{\Pi\nu\mu}(g)\right) \qquad (2.16)$$

This cancellation is originated from the fact that in order to satisfy the symmetry under the general coordinate transformation the energy-momentum tensor $\hat{T}_{\mu\nu}$ should be symmetric

$$\hat{T}_{\mu\nu} = \hat{T}_{\nu\mu} \qquad (2.17)$$

It follows that the electronic part of the energy-momentum tensor $\hat{T}_{e\mu\nu}$ of $\hat{T}_{\mu\nu}$ should be symmetric

$$\hat{T}_{e\mu\nu} = -\hat{\varepsilon}^{\Pi}_{\mu\nu} - \hat{\tau}^{\Pi}_{\mu\nu}(g) = \hat{T}_{e\nu\mu} \qquad (2.18)$$

Consequently, the cancelling is mandatory.

What is the physical meaning of Eq. (2.14)? The answer is two-fold as is found if we take the limit to the Minkowski spacetime. First, for the time sector with $\mu = 0$, $\nu = 1,2,3$ we obtain

$$\mathrm{rot}\hat{\vec{s}} + \hat{\vec{\Pi}} - \frac{1}{2}\left(\hat{\bar{\psi}}\vec{\gamma}\left(i\hbar\hat{D}_0\right)\hat{\psi} + h.c.\right) = 0 \qquad (2.19)$$

Second, for the space sector with $\mu, \nu = 1,2,3$ we obtain

$$\frac{\partial}{\partial t}\hat{\vec{s}} - \hat{\vec{t}} - \hat{\vec{\zeta}} = 0 \qquad (2.20)$$

with torque $\hat{\vec{t}}$ and zeta force $\hat{\vec{\zeta}}$. *Furthermore, similarly taking the limit of Eq. (2.18) to the*

Minkowski spacetime, it is found that half the vorticity, $\frac{1}{2}\mathrm{rot}\hat{\vec{s}}$, *appears as the component of the momentum added to the kinetic momentum*

$$\hat{P}_{\mathrm{e}} = \hat{\vec{\Pi}} + \frac{1}{2}\mathrm{rot}\hat{\vec{s}} \tag{2.21}$$

(see, AppendixB, Eq. (B.22)).

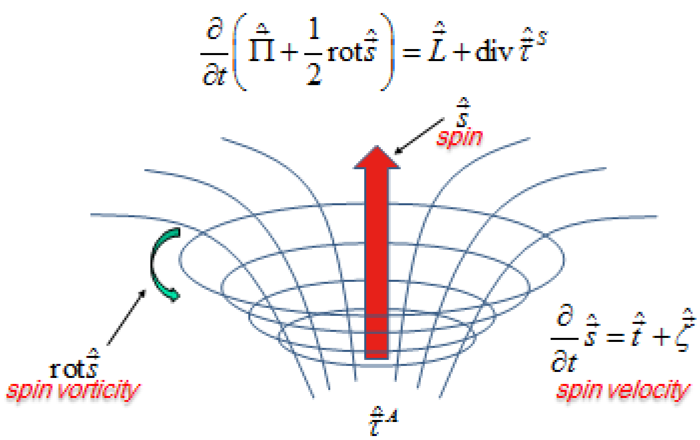

Figure 1. Symmetry of the stress tensor of the Dirac field of electron and positron. Antisymmetric stress tensor drives spin torque and zeta force through vorticity.

Consequently, the left hand side of Eq. (2.6) should change from $\frac{\partial}{\partial t}\hat{\vec{\Pi}}$ to $\frac{\partial}{\partial t}\left(\hat{\vec{\Pi}} + \frac{1}{2}\mathrm{rot}\hat{\vec{s}}\right)$; for this purpose, we need to use Eq. (2.20), and after some manipulations, we finally arrive at

$$\frac{\partial}{\partial t}\hat{P}_{\mathrm{e}} = \hat{\vec{L}} + \hat{\vec{\tau}}^{S} \tag{2.22}$$

$$\hat{\vec{\tau}}^{S} = \mathrm{div}\,\hat{\vec{\tau}}^{S}, \quad \hat{\vec{\tau}}^{Sk} = \partial_{\ell}\hat{\tau}^{Sk\ell} \tag{2.23}$$

$$\hat{\tau}^{S\mu\nu} = \frac{1}{2}\left(\hat{\tau}^{\Pi\mu\nu} + \hat{\tau}^{\Pi\nu\mu}\right) \tag{2.24}$$

This assures the equation of motion using solely the symmetric part of the tensor $\hat{\tau}^{S}_{k\ell}$ *in the right hand side.* This is schematically shown in Figure 1.

2.4. The Cauchy problem

In QED, the dynamics of $\hat{\vec{s}}\left(x\right)$ is mediated by the electromagnetic field, and the associated charge current Eq. (2.3) is conventionally represented as

$$\hat{j}^{\mu}\left(x\right) = \left(c\hat{\rho}\left(x\right), \hat{\vec{j}}\left(x\right)\right) \tag{2.25}$$

The Cauchy problem of the QED operator dynamics in the Heisenberg representation has been elaborated elegantly by Nakanishi using ghost field in the Landau gauge [2]. Here in this paper we use the Coulomb gauge for the vector potential $\hat{\vec{A}}(x)$ as

$$\mathrm{div}\hat{\vec{A}}(x) = 0 \tag{2.26}$$

with the conjugate transversal electric field

$$\hat{\vec{E}}_{T}(x) = -\frac{1}{c}\frac{\partial}{\partial t}\hat{\vec{A}}(x) \tag{2.27}$$

and we do not invoke the additional ghost field.

To solve for the Cauchy problem of QED, clocks at different space points are synchronized at $t = t_0$, when canonical quantization is performed with the definition of the vacuum ket vector $\left|0\right\rangle$.The $\hat{j}^{\mu}\left(x\right)$ develops forward $t > t_0$ with the retarded interactions mediated by photon. The vacuum and field operators are not defined backward $t < t_0$ (see, Figure 2).

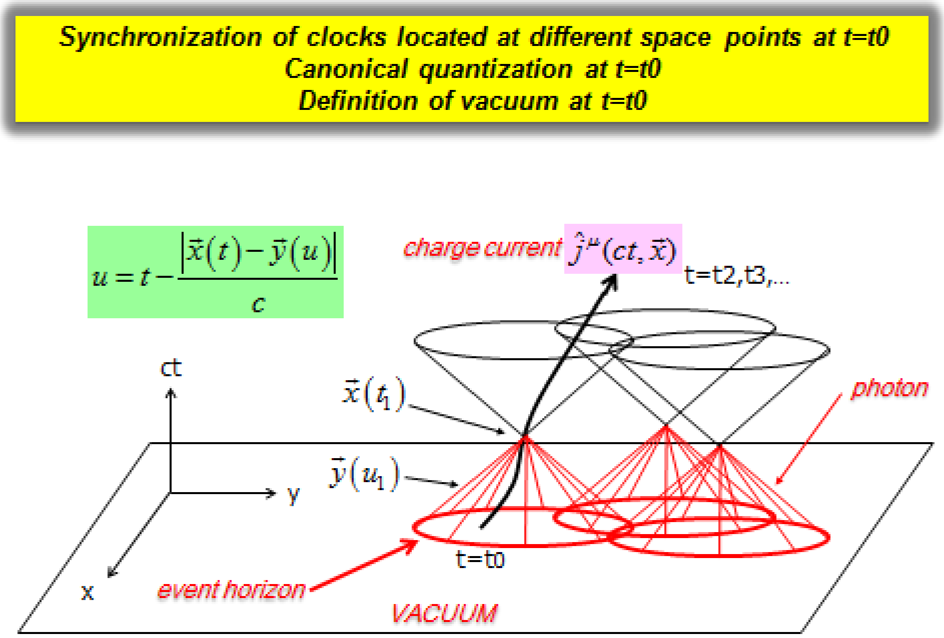

Figure 2. Synchronization of clocks. The charge current develops forward $t > t_0$ with the retarded interactions mediated by photon. The vacuum and field operators are not defined backward $t < t_0$.

The equal-time canonical quantization of the electromagnetic field leads to the equal-time commutation relationships

$$\left[\hat{A}^i(x), \hat{A}^j(y) \right]_{x^0=y^0} = 0 \tag{2.28}$$

$$\left[\hat{E}_T{}^i(x), \hat{E}_T{}^j(y) \right]_{x^0=y^0} = 0 \tag{2.29}$$

$$\frac{1}{4\pi c}\left[\hat{A}^i(x), \hat{E}_T{}^j(y) \right]_{x^0=y^0} = i\hbar\left(\eta^{ij}\delta^3(\vec{x}-\vec{y}) + \partial^i\partial^j\left(-\frac{1}{4\pi}\cdot\frac{1}{|\vec{x}-\vec{y}|} \right) \right) \tag{2.30}$$

Second, the equal-time canonical quantization of the Dirac field leads to the equal-time anti-commutation relationships

$$\left\{\hat{\psi}_{\ell}\left(x\right),\hat{\psi}_{\ell'}\left(y\right)\right\}_{x^0=y^0}=\left\{\hat{\psi}_{\ell}^{\dagger}\left(x\right),\hat{\psi}_{\ell'}^{\dagger}\left(y\right)\right\}_{x^0=y^0}=0 \tag{2.31}$$

$$\left\{\hat{\psi}_{\ell}\left(x\right),\hat{\psi}_{\ell'}^{\dagger}\left(y\right)\right\}_{x^0=y^0}=\delta_{\ell\ell'}\delta^3\left(\vec{x}-\vec{y}\right) \tag{2.32}$$

The $\hat{\psi}\left(x\right)$ commutes with $\hat{\vec{A}}(x)$

$$\left[\hat{\psi}\left(x\right),\hat{\vec{A}}(x)\right]=0 \tag{2.33}$$

These fields should of course be renormalized in a step-by-step way, reflecting the time-dependent minimal coupling.

The time-development of $\hat{\vec{s}}\left(x\right)$, or any field operator $\hat{F}(x)$ obeys the Heisenberg equation of motion

$$i\hbar\frac{\partial}{\partial t}\hat{F}(x)=\left[\hat{F}(x),\hat{H}_{\text{QED}}\right] \tag{2.34}$$

with the QED Hamiltonian $\hat{H}_{\text{QED}}$. Note that $\hat{H}_{\text{QED}}$ is made to be independent of time

$$\frac{\partial}{\partial t}\hat{H}_{\text{QED}}=0 \tag{2.35}$$

The $\hat{H}_{\text{QED}}$ is given in the Coulomb gauge using the normal order denoted as : : modulo c-number albeit infinity if any

$$\hat{H}_{\text{QED}}=\int d^3\vec{x}:\hat{H}_{\text{QED}}(x): \tag{2.36}$$

$$\begin{aligned}\hat{H}_{\text{QED}}(x)&=\frac{1}{8\pi}\left(\left(\hat{\vec{E}}_T(x)\right)^2+\left(\text{rot}\hat{\vec{A}}(x)\right)^2\right)\\&-\frac{1}{c}\hat{\vec{j}}(x)\bullet\hat{\vec{A}}(x)+\frac{1}{2c}\hat{j}_0(x)\hat{A}_0(x)+\hat{\bar{\psi}}(x)\left(-i\hbar\gamma^k\partial_k+mc\right)\hat{\psi}(x)\times c\end{aligned} \tag{2.37}$$

$$\hat{A}_0(x) = \int_{-\infty}^{\infty} d^3\vec{y}\, \frac{\hat{\rho}(y)\big|_{y^0=x^0}}{|\vec{x}-\vec{y}|} \tag{2.38}$$

In due course, for application to realistic situation in experiments of spin dynamics, we need to set up wave function in order to discriminate numbers of electrons, positrons and photons, and calculate

$$\left\langle \hat{F}(x) \right\rangle = \frac{{}_H\langle \Psi | \hat{F}(x) | \Psi \rangle_H}{{}_H\langle \Psi | \Psi \rangle_H} \tag{2.39}$$

where $|\Psi\rangle_H$ denotes the time-independent ket vector in the Heisenberg representation. This is another Cauchy problem in QED (see Section 4).

3. New algebras

3.1. Causality and initial condition

To obtain $\hat{F}(x)$ with $x^\mu = (ct, \vec{x})$ at position $\vec{x}$ with time t in the Minkowski spacetime, we may collect information of $\hat{j}^\mu(y)$ with $y^\mu = (cu, \vec{y})$ at distant $\vec{y}$ with the retarded time $u = t - \frac{|\vec{x}-\vec{y}|}{c}$ satisfying causality

$$\hat{j}^\mu(cu, \vec{y}) = 0, \quad u > t \tag{3.1}$$

and initial condition (see Figure 3)

$$\hat{j}^\mu(cu, \vec{y}) = 0, \quad u < t_0 \tag{3.2}$$

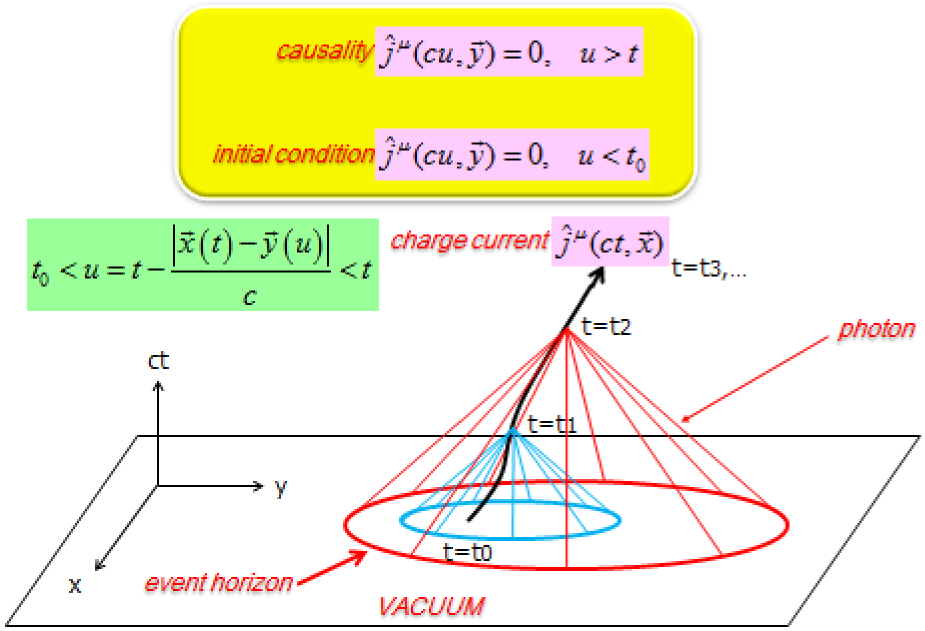

Figure 3. Causality and initial condition.

For this purpose, in the following discussions we may use that any function $F(u)$ satisfying

$$F(u)=0,\quad u<t_0,\quad u>t \tag{3.3}$$

may be obtained at u with $t_0<u=t-\frac{|\vec{x}-\vec{y}|}{c}<t$ as [8]

$$\begin{aligned} F(u)\Big|_{u=t-\frac{|\vec{x}-\vec{y}|}{c}} &= \int_{-\infty}^{\infty} du' F(u')\delta\left(u'-\left(t-\frac{|\vec{x}-\vec{y}|}{c}\right)\right) \\ &= \frac{|\vec{x}-\vec{y}|}{c\pi}\int_{t_0}^{t} du' \int_{-\infty}^{\infty} d\alpha F(u') e^{i\alpha\left((u'-t)^2-\frac{(\vec{x}-\vec{y})^2}{c^2}\right)} \end{aligned} \tag{3.4}$$

where we have used the delta function

$$\delta((u'-t)^2-a^2)=\frac{1}{2a}\left(\delta((u'-t)-a)+\delta((u'-t)+a)\right),\quad a>0 \tag{3.5}$$

with

$$\delta((u'-t)^2 - \frac{(\vec{x}-\vec{y})^2}{c^2}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha e^{i\alpha\left((u'-t)^2 - \frac{(\vec{x}-\vec{y})^2}{c^2}\right)} \tag{3.6}$$

3.2. Electromagnetic field

The vector potential $\hat{\vec{A}}(x)$ should satisfy the Maxwell equation

$$\Box\hat{\vec{A}}(x) = \frac{4\pi}{c}\hat{\vec{j}}_T(x) \tag{3.7}$$

with the transversal charge current

$$\hat{\vec{j}}_T(x) = \hat{\vec{j}}(x) - \frac{1}{4\pi}\,\mathrm{grad}\frac{\partial}{\partial t}\hat{A}_0(x) \tag{3.8}$$

Using the standard Green function, we have [1]

$$\hat{\vec{A}}(x) = \hat{\vec{A}}_{\mathrm{radiation}}(x) + \hat{\vec{A}}_A(x) \tag{3.9}$$

$$\begin{aligned}\hat{\vec{A}}_A(ct,\vec{x}) &= \frac{1}{c}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d^3\vec{y}\,\frac{\hat{\vec{j}}_{\mathrm{T}}(cu,\vec{y})}{|\vec{x}-\vec{y}|}\,\delta\left(u - \left(t - \frac{|\vec{x}-\vec{y}|}{c}\right)\right) \\ &= \frac{1}{c^2\pi}\int_{t_0}^{t} du \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d^3\vec{y}\,\hat{\vec{j}}_{\mathrm{T}}(cu,\vec{y})\,e^{i\alpha\left((t-u)^2 - \frac{(\vec{x}-\vec{y})^2}{c^2}\right)}\end{aligned} \tag{3.10}$$

where $\hat{\vec{A}}_{\mathrm{radiation}}(x)$ denotes the radiation vector potential. It should be noted that we have used Eq. (3.4) using the causality and initial condition and then obtained the retarded potential $\hat{\vec{A}}_A(x)$ with separation of space-time variables (see Figure 4).

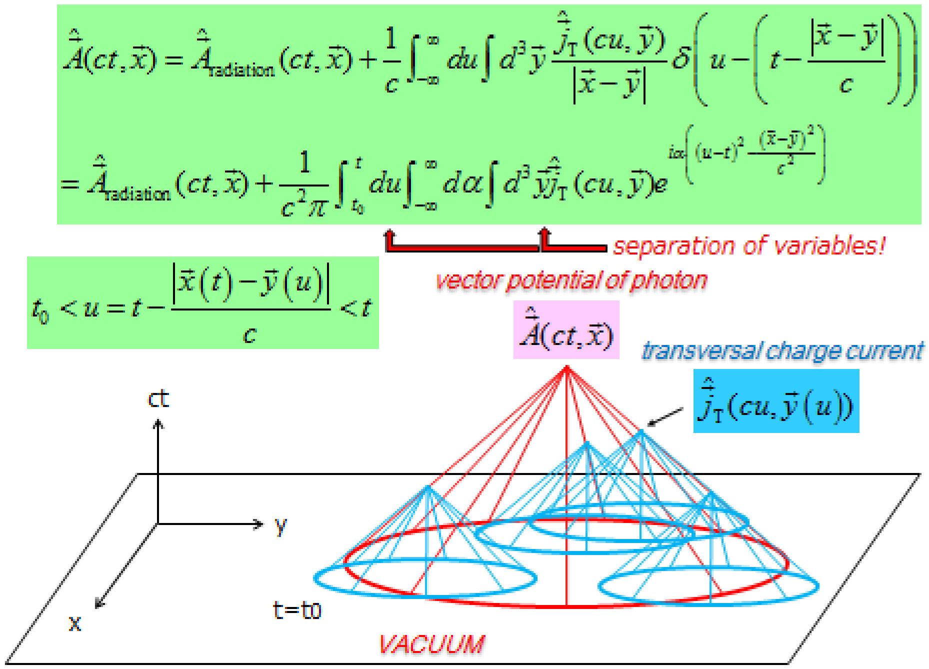

Figure 4. Separation of variables for real-time simulation.

The $\hat{\vec{A}}_{\text{radiation}}(x)$ is given by the $a_{\text{radiation}}$ -photon field

$$\hat{\vec{A}}_{\text{radiation}}(x) = \hat{\vec{a}}_{\text{radiation}}(x) + \hat{\vec{a}}^{\dagger}_{\text{radiation}}(x) \tag{3.11}$$

$$\hat{\vec{a}}_{\text{radiation}}(x) = \frac{\sqrt{4\pi\hbar^2 c}}{\sqrt{(2\pi\hbar)^3}} \sum_{\sigma=\pm1} \int_{-\infty}^{\infty} \frac{d^3\vec{p}}{\sqrt{2p_{\text{radiation}}{}^0}} \hat{a}_{\text{radiation}}(\vec{p},\sigma) e^{-ix_\mu p_{\text{radiation}}{}^{\mu}/\hbar} \vec{e}(\vec{p},\sigma) \tag{3.12}$$

with the usual dispersion relationship of spectrum

$$p_{\text{radiation}}{}^{\mu} = \left(p_{\text{radiation}}{}^0, \vec{p}\right), \quad p_{\text{radiation}}{}^0 = \frac{h\nu_{\text{radiation}}}{c} = |\vec{p}| \tag{3.13}$$

and the polarization vector $\vec{e}(\vec{p},\sigma)$

$$\vec{p} \bullet \vec{e}(\vec{p},\sigma) = 0 \tag{3.14}$$

$$\sum_{\sigma=\pm1} e^i\left(\vec{p},\sigma\right) e^{j*}\left(\vec{p},\sigma\right) = -\eta^{ij} + \frac{p^i p^j}{-\left|\vec{p}\right|^2} \tag{3.15}$$

$$\sum_{i=1}^{3} e^i\left(\vec{p},\sigma\right) e^{i*}\left(\vec{p},\sigma'\right) = \delta_{\sigma\sigma'} \tag{3.16}$$

Note the usual commutation algebra of the $a_{\text{radiation}}$ -photon field

$$\left[\hat{a}_{\text{radiation}}\left(\vec{p},\sigma\right), \hat{a}_{\text{radiation}}\left(\vec{q},\sigma'\right)\right] = \left[\hat{a}^{\dagger}_{\text{radiation}}\left(\vec{p},\sigma\right), \hat{a}^{\dagger}_{\text{radiation}}\left(\vec{q},\sigma'\right)\right] = 0 \tag{3.17}$$

$$\left[\hat{a}_{\text{radiation}}\left(\vec{p},\sigma\right), \hat{a}^{\dagger}_{\text{radiation}}\left(\vec{q},\sigma'\right)\right] = \delta_{\sigma\sigma'}\delta^3\left(\vec{p}-\vec{q}\right) \tag{3.18}$$

The generic solution may be given by using the b -photon field defined as follows

$$\hat{\vec{A}}(x) = \hat{\vec{b}}(x) + \hat{\vec{b}}^{\dagger}(x) \tag{3.19}$$

$$\hat{\vec{b}}(x) = \frac{\sqrt{4\pi\hbar^2 c}}{\sqrt{\left(2\pi\hbar\right)^3}} \sum_{\sigma=\pm1} \int_0^{\infty} d\nu \int_{-\infty}^{\infty} \frac{d^3\vec{p}}{\sqrt{2p^0\left(\nu,\left|\vec{p}\right|\right)}} \hat{b}\left(\nu,\vec{p},\sigma\right) e^{-i2\pi\nu t} \vec{e}(\vec{p},\sigma) e^{i\vec{x}\bullet\vec{p}/\hbar} \tag{3.20}$$

By using the integral form of the current

$$\hat{\vec{j}}_T(x) = \frac{1}{\sqrt{\left(2\pi\hbar\right)^3}} \int_0^{\infty} d\nu \int_{-\infty}^{\infty} d^3\vec{p} \left(\hat{\vec{j}}_T\left(\nu,\vec{p}\right) e^{-i2\pi\nu t} e^{i\vec{x}\bullet\vec{p}/\hbar} + \hat{\vec{j}}_T^{\dagger}\left(\nu,\vec{p}\right) e^{+i2\pi\nu t} e^{-i\vec{x}\bullet\vec{p}/\hbar}\right) \tag{3.21}$$

the b -photon field may be represented as

$$\frac{\sqrt{4\pi\hbar^2 c}}{\sqrt{2p^0\left(\nu,\left|\vec{p}\right|\right)}} \left(-\left(\frac{2\pi\nu}{c}\right)^2 + \frac{\left|\vec{p}\right|^2}{\hbar^2}\right) \sum_{\sigma=\pm1} \hat{b}\left(\nu,\vec{p},\sigma\right) \vec{e}(\vec{p},\sigma) = \frac{4\pi}{c} \hat{\vec{j}}_T\left(\nu,\vec{p}\right) \tag{3.22}$$

Comparing Eq. (3.22) with Eqs. (3.9), (3.11), and (3.19), we may observe that the $a_{\text{radiation}}$ -photon fields are sticking to the b -photon field through $\hat{\vec{j}}_T(x)$. This sticking process may be called "thermalization" of the $a_{\text{radiation}}$ -photon fields to the b -photon field. Note that the real positive number $p^0\left(\nu,\left|\vec{p}\right|\right)$ in Eq. (3.20) is the counterpart of $p_{\text{radiation}}{}^0$ in Eqs. (3.12) and (3.13). The

$p^0\left(\nu,|\vec{p}|\right)$ is a function of ν and $|\vec{p}|$ serving as the thermalized solution of Eq. (3.22).

The field algebra in Eqs. (2.28)-(2.30) are recovered if we assume the b-photon algebra

$$\left[\hat{b}(\nu,\vec{p},\sigma),\hat{b}(\nu',\vec{q},\sigma')\right]=\left[\hat{b}^{\dagger}(\nu,\vec{p},\sigma),\hat{b}^{\dagger}(\nu',\vec{q},\sigma')\right]=0 \tag{3.23}$$

$$\left[\hat{b}(\nu,\vec{p},\sigma),\hat{b}^{\dagger}(\nu',\vec{q},\sigma')\right]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{q})\delta\left(\nu-\nu(|\vec{p}|)_b\right)\delta\left(\nu'-\nu(|\vec{q}|)_b\right) \tag{3.24}$$

where $\nu(|\vec{p}|)_b$ denotes real positive frequency that depends on $|\vec{p}|$. The b-photon field apparently includes the $a_{\text{radiation}}$-photon field in a delta-function form

$$\hat{b}(\nu,\vec{p},\sigma)\ \supset\ \hat{a}_{\text{radiation}}(\vec{p},\sigma)\delta(\nu-\nu_{\text{radiation}}) \tag{3.25}$$

Then, the electromagnetic part of $\hat{H}_{\text{QED}}$ (modulo c-number vacuum energy) in Eqs. (2.36) and (2.37) is given as

$$\begin{aligned}\hat{H}_{\text{QED}}\ &\supset\ \int d^3\vec{x}:\frac{1}{8\pi}\left(\left(\hat{\vec{E}}_T(x)\right)^2+\left(\text{rot}\hat{\vec{A}}(x)\right)^2\right):\\ &=\hbar^2 c\sum_{\sigma=\pm1}\int_0^{\infty}d\nu\int_0^{\infty}d\nu'\int_{-\infty}^{\infty}\frac{d^3\vec{p}}{\sqrt{2p^0\left(\nu,|\vec{p}|\right)}\sqrt{2p^0\left(\nu',|\vec{p}|\right)}}\\ &\times\left(\left(\frac{2\pi\nu}{c}\right)\left(\frac{2\pi\nu'}{c}\right)+\frac{|\vec{p}|^2}{\hbar^2}\right)\hat{b}^{\dagger}(\nu,\vec{p},\sigma)\hat{b}(\nu',\vec{p},\sigma)e^{i2\pi(\nu-\nu')t}\\ &\quad(\text{modulo c-number})\end{aligned} \tag{3.26}$$

which part may depend on t and t_0 although $\hat{H}_{\text{QED}}$ is independent of t. Moreover, Eq. (3.26) includes the radiation part (modulo time-independent c-number vacuum energy) given as

$$\begin{aligned}&\int d^3\vec{x}:\frac{1}{8\pi}\left(\left(\hat{\vec{E}}_T(x)\right)^2+\left(\text{rot}\hat{\vec{A}}(x)\right)^2\right):\\ &\supset\ \int d^3\vec{x}:\frac{1}{8\pi}\left(\left(\hat{\vec{E}}_{T_{\text{radiation}}}(x)\right)^2+\left(\text{rot}\hat{\vec{A}}_{\text{radiation}}(x)\right)^2\right):\\ &=\sum_{\sigma=\pm1}\int_{-\infty}^{\infty}d^3\vec{p}\,c\,p_{\text{radiation}}{}^0\hat{a}_{\text{radiation}}{}^{\dagger}(\vec{p},\sigma)\hat{a}_{\text{radiation}}(\vec{p},\sigma)\\ &(\text{modulo time-independent c-number})\end{aligned} \tag{3.27}$$

which is manifestly independent of t as well as t_0.

3.3. The Dirac field

The $\hat{\psi}(x)$ may be given by using the spinor Green function $K(x,y)$ as [1]

$$\hat{\psi}(x)=\hat{\psi}_{\text{free}}(x)+\frac{1}{i\hbar}\int d^4yK(x,y)\left(-\frac{q}{c}\hat{A}(y)\right)\hat{\psi}(y) \tag{3.28}$$

$$(-i\hbar\partial\!\!\!/+mc)K(x,y)=i\hbar\delta^4(x-y) \tag{3.29}$$

where $\hat{\psi}_{\text{free}}(x)$ denotes the free field. The $\hat{\psi}_{\text{free}}(x)$ is given by the free e_{free}-electron and $e_{\text{free}}{}^{c}$-positron fields

$$\hat{\psi}_{\text{free}}(x)=\hat{e}_{\text{free}}(x)+\hat{e}_{\text{free}}{}^{c\dagger}(x) \tag{3.30}$$

$$\hat{e}_{\text{free}_\ell}(x)=\frac{1}{\sqrt{(2\pi\hbar)^3}}\sum_{\sigma=\pm\frac{1}{2}}\int_{-\infty}^{\infty}d^3\vec{p}\hat{e}_{\text{free}}(\vec{p},\sigma)e^{-ix_\mu p_{\text{free}}{}^{\mu}/\hbar}u_\ell(\vec{p},\sigma) \tag{3.31}$$

$$\hat{e}_{\text{free}_\ell}{}^{c\dagger}(x)=\frac{1}{\sqrt{(2\pi\hbar)^3}}\sum_{\sigma=\pm\frac{1}{2}}\int_{-\infty}^{\infty}d^3\vec{p}\hat{e}_{\text{free}}{}^{c\dagger}(\vec{p},\sigma)e^{+ix_\mu p_{\text{free}}{}^{\mu}/\hbar}v_\ell(\vec{p},\sigma) \tag{3.32}$$

with the usual dispersion relationship of spectrum

$$p_{\text{free}}{}^{\mu}=\left(p_{\text{free}}{}^{0},\vec{p}\right),\quad p_{\text{free}}{}^{0}=\frac{h\nu_{\text{free}}}{c}=\sqrt{(mc)^2+|\vec{p}|^2} \tag{3.33}$$

and the anti-commutation algebra

$$\begin{aligned}&\left\{\hat{e}_{\text{free}}(\vec{p},\sigma),\hat{e}_{\text{free}}(\vec{q},\sigma')\right\}=\left\{\hat{e}_{\text{free}}{}^{c}(\vec{p},\sigma),\hat{e}_{\text{free}}{}^{c}(\vec{q},\sigma')\right\}\\&=\left\{\hat{e}_{\text{free}}{}^{\dagger}(\vec{p},\sigma),\hat{e}_{\text{free}}{}^{\dagger}(\vec{q},\sigma')\right\}=\left\{\hat{e}_{\text{free}}{}^{c\dagger}(\vec{p},\sigma),\hat{e}_{\text{free}}{}^{c\dagger}(\vec{q},\sigma')\right\}=0\end{aligned} \tag{3.34}$$

$$\left\{\hat{e}_{\text{free}}(\vec{p},\sigma),\hat{e}_{\text{free}}{}^{\dagger}(\vec{q},\sigma')\right\}=\left\{\hat{e}_{\text{free}}{}^{c}(\vec{p},\sigma),\hat{e}_{\text{free}}{}^{c\dagger}(\vec{q},\sigma')\right\}=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{q}) \tag{3.35}$$

The Dirac spinors $u(\vec{p},\sigma)$ for electron and $v(\vec{p},\sigma)$ for positron satisfy

$$\left(p_{\text{free}}{}^{\mu}\gamma_{\mu} - mc \right) u(\vec{p},\sigma) = 0 \tag{3.36}$$

$$\left(p_{\text{free}}{}^{\mu}\gamma_{\mu} + mc \right) v(\vec{p},\sigma) = 0 \tag{3.37}$$

$$\sum_{\sigma=\pm\frac{1}{2}} u(\vec{p},\sigma)\overline{u}(\vec{p},\sigma) = \frac{1}{2\,p_{\text{free}}{}^{0}} \left(p_{\text{free}}{}^{\mu}\gamma_{\mu} + mc \right) \tag{3.38}$$

$$\sum_{\sigma=\pm\frac{1}{2}} v(\vec{p},\sigma)\overline{v}(\vec{p},\sigma) = \frac{1}{2\,p_{\text{free}}{}^{0}} \left(p_{\text{free}}{}^{\mu}\gamma_{\mu} - mc \right) \tag{3.39}$$

$$\overline{u}(\vec{p},\sigma)\gamma^{\mu}u(\vec{p},\sigma') = \overline{v}(\vec{p},\sigma)\gamma^{\mu}v(\vec{p},\sigma') = \left(p_{\text{free}}{}^{\mu} / p_{\text{free}}{}^{0} \right)\delta_{\sigma\sigma'} \tag{3.40}$$

$$\overline{u}(\vec{p},\sigma)\gamma^{0}v(-\vec{p},\sigma') = \overline{v}(\vec{p},\sigma)\gamma^{0}u(-\vec{p},\sigma') = 0 \tag{3.41}$$

The generic solution may be given by using the f -electron and f^{c} -positron fields defined as follows

$$\hat{\psi}(x) = \hat{f}(x) + \hat{f}^{c\dagger}(x) \tag{3.42}$$

$$\hat{f}_{\ell}(x) = \frac{1}{\sqrt{(2\pi\hbar)^{3}}} \sum_{\sigma=\pm\frac{1}{2}} \int_{0}^{\infty} d\nu \int_{-\infty}^{\infty} d^{3}\vec{p}\, \hat{f}\left(\nu,\vec{p},\sigma\right) e^{-i2\pi\nu t} u_{\ell}(\vec{p},\sigma) e^{i\vec{x}\bullet\vec{p}/\hbar} \tag{3.43}$$

$$\hat{f}_{\ell}^{c\dagger}(x) = \frac{1}{\sqrt{(2\pi\hbar)^{3}}} \sum_{\sigma=\pm\frac{1}{2}} \int_{0}^{\infty} d\nu \int_{-\infty}^{\infty} d^{3}\vec{p}\, \hat{f}^{c\dagger}\left(\nu,\vec{p},\sigma\right) e^{+i2\pi\nu t} v_{\ell}(\vec{p},\sigma) e^{-i\vec{x}\bullet\vec{p}/\hbar} \tag{3.44}$$

Applying the first thermalization of the b -photon field Eq. (3.22) to Eq. (2.1), we obtain the second thermalization of the f -electron field

$$\frac{q}{c}\gamma^0\int_0^\infty d\nu'\int_{-\infty}^\infty d^3\vec{q}\hat{A}_0\left(\nu-\nu',\vec{p}-\vec{q}\right)\sum_{\sigma=\pm\frac{1}{2}}\hat{f}\left(\nu',\vec{q},\sigma\right)u(\vec{q},\sigma)$$

$$=\frac{4\pi}{c}\int_0^\infty d\nu'\int_{-\infty}^\infty \frac{d^3\vec{q}}{\left(-\left(\frac{2\pi\nu'}{c}\right)^2+\frac{|\vec{q}|^2}{\hbar^2}\right)}\left(\gamma_k\hat{j}_T{}^k\left(\nu',\vec{q}\right)\sum_{\sigma=\pm\frac{1}{2}}\hat{f}\left(\nu-\nu',\vec{p}-\vec{q},\sigma\right)u(\vec{p}-\vec{q},\sigma)\right. \tag{3.45}$$

$$\left.+\gamma_k\hat{j}_T{}^{\dagger k}\left(\nu',\vec{q}\right)\sum_{\sigma=\pm\frac{1}{2}}\hat{f}\left(\nu+\nu',\vec{p}+\vec{q},\sigma\right)u(\vec{p}+\vec{q},\sigma)\right)$$

with

$$\hat{A}_0\left(\nu,\vec{p}\right)=\frac{q}{\left(2\pi\hbar\right)^3}\sum_{\sigma=\pm\frac{1}{2}}\sum_{\sigma'=\pm\frac{1}{2}}\int_0^\infty d\nu'\int_{-\infty}^\infty d^3\vec{q}$$

$$\times\left(\hat{f}^\dagger\left(\nu',\vec{q},\sigma\right)\hat{f}\left(\nu+\nu',\vec{p}+\vec{q},\sigma'\right)u^\dagger(\vec{q},\sigma)u(\vec{p}+\vec{q},\sigma')\right.$$

$$+\hat{f}^\dagger\left(\nu',\vec{q},\sigma\right)\hat{f}^{c\dagger}\left(-\nu-\nu',-\vec{p}-\vec{q},\sigma'\right)u^\dagger(\vec{q},\sigma)v(-\vec{p}-\vec{q},\sigma') \tag{3.46}$$

$$+\hat{f}^c\left(\nu',\vec{q},\sigma\right)\hat{f}\left(\nu-\nu',\vec{p}-\vec{q},\sigma'\right)v^\dagger(\vec{q},\sigma)u(\vec{p}-\vec{q},\sigma')$$

$$\left.+\hat{f}^c\left(\nu',\vec{q},\sigma\right)\hat{f}^{c\dagger}\left(-\nu+\nu',-\vec{p}+\vec{q},\sigma'\right)v^\dagger(\vec{q},\sigma)v(-\vec{p}+\vec{q},\sigma')\right)$$

and the third thermalization of the f^c-positron field

$$\frac{q}{c}\gamma^0\int_0^\infty d\nu'\int_{-\infty}^\infty d^3\vec{q}\hat{A}_0^\dagger\left(\nu-\nu',\vec{p}-\vec{q}\right)\sum_{\sigma=\pm\frac{1}{2}}\hat{f}^{c\dagger}\left(\nu',\vec{q},\sigma\right)v(\vec{q},\sigma)$$

$$=\frac{4\pi}{c}\int_0^\infty d\nu'\int_{-\infty}^\infty \frac{d^3\vec{q}}{\left(-\left(\frac{2\pi\nu'}{c}\right)^2+\frac{|\vec{q}|^2}{\hbar^2}\right)}\left(\gamma_k\hat{j}_T{}^k\left(\nu',\vec{q}\right)\sum_{\sigma=\pm\frac{1}{2}}\hat{f}^{c\dagger}\left(\nu+\nu',\vec{p}+\vec{q},\sigma\right)v(\vec{p}+\vec{q},\sigma)\right. \tag{3.47}$$

$$\left.+\gamma_k\hat{j}_T{}^{\dagger k}\left(\nu',\vec{q}\right)\sum_{\sigma=\pm\frac{1}{2}}\hat{f}^{c\dagger}\left(\nu-\nu',\vec{p}-\vec{q},\sigma\right)v(\vec{p}-\vec{q},\sigma)\right)$$

The field algebra in Eqs. (2.31) and (2.32) are recovered if we assume the f-electron and f^c-positron algebras

$$\left\{ \hat{f}\left(\nu,\vec{p},\sigma\right),\hat{f}\left(\nu',\vec{q},\sigma'\right)\right\} = \left\{ \hat{f}^{c}\left(\nu,\vec{p},\sigma\right),\hat{f}^{c}\left(\nu',\vec{q},\sigma'\right)\right\}$$
$$= \left\{ \hat{f}^{\dagger}\left(\nu,\vec{p},\sigma\right),\hat{f}^{\dagger}\left(\nu',\vec{q},\sigma'\right)\right\} = \left\{ \hat{f}^{c\dagger}\left(\nu,\vec{p},\sigma\right),\hat{f}^{c\dagger}\left(\nu',\vec{q},\sigma'\right)\right\} = 0 \quad (3.48)$$

$$\left\{ \hat{f}\left(\nu,\vec{p},\sigma\right),\hat{f}^{\dagger}\left(\nu',\vec{q},\sigma'\right)\right\} = \left\{ \hat{f}^{c}\left(\nu,\vec{p},\sigma\right),\hat{f}^{c\dagger}\left(\nu',\vec{q},\sigma'\right)\right\}$$
$$= \delta_{\sigma\sigma'}\delta^{3}\left(\vec{p}-\vec{q}\right)\delta\left(\nu-\nu\left(\left|\vec{p}\right|\right)_{f}\right)\delta\left(\nu'-\nu\left(\left|\vec{q}\right|\right)_{f}\right) \quad (3.49)$$

where $\nu\left(\left|\vec{p}\right|\right)_{f}$ denotes real positive frequency that depends on $\left|\vec{p}\right|$. Also, Eq. (2.33) is recovered if we assume

$$\left[\hat{f}\left(\nu,\vec{p},\sigma\right),\hat{b}\left(\nu',\vec{q},\sigma'\right)\right] = \left[\hat{f}^{c}\left(\nu,\vec{p},\sigma\right),\hat{b}\left(\nu',\vec{q},\sigma'\right)\right]$$
$$= \left[\hat{f}^{\dagger}\left(\nu,\vec{p},\sigma\right),\hat{b}\left(\nu',\vec{q},\sigma'\right)\right] = \left[\hat{f}^{c\dagger}\left(\nu,\vec{p},\sigma\right),\hat{b}\left(\nu',\vec{q},\sigma'\right)\right] = 0 \quad (3.50)$$

The f -electron and f^{c} -positron fields apparently include the e_{free}-electron and $e_{\mathrm{free}}{}^{c}$ -positron fields respectively in the delta-function forms

$$\hat{f}\left(\nu,\vec{p},\sigma\right) \supset \hat{e}_{\mathrm{free}}\left(\vec{p},\sigma\right)\delta\left(\nu-\nu_{\mathrm{free}}\right) \quad (3.51)$$

$$\hat{f}^{c}\left(\nu,\vec{p},\sigma\right) \supset \hat{e}_{\mathrm{free}}{}^{c}\left(\vec{p},\sigma\right)\delta\left(\nu-\nu_{\mathrm{free}}\right) \quad (3.52)$$

Then, the Dirac part of $\hat{H}_{\mathrm{QED}}$ (modulo c-number vacuum energy) in Eqs. (2.36) and (2.37) is given as

$$\hat{H}_{\mathrm{QED}} \supset \int d^{3}\vec{x} : \hat{\bar{\psi}}(x)\left(-i\hbar\gamma^{k}\partial_{k}+mc\right)\hat{\psi}(x)\times c :$$
$$= \sum_{\sigma=\pm1}\int_{0}^{\infty}d\nu\int_{0}^{\infty}d\nu'\int_{-\infty}^{\infty}d^{3}\vec{p}\,c p_{\mathrm{free}}{}^{0}$$
$$\times\left(\hat{f}^{\dagger}\left(\nu,\vec{p},\sigma\right)\hat{f}\left(\nu',\vec{p},\sigma\right)e^{+i2\pi(\nu-\nu')t}+\hat{f}^{c\dagger}\left(\nu',\vec{p},\sigma\right)\hat{f}^{c}\left(\nu,\vec{p},\sigma\right)e^{-i2\pi(\nu-\nu')t}\right)$$
$$\left(\text{modulo c-number}\right) \quad (3.53)$$

which part may depend on t and t_0 although $\hat{H}_{\mathrm{QED}}$ is independent of t. Moreover, Eq. (3.53) includes the free part (modulo time-independent c-number vacuum energy) given as

$$\int d^3\vec{x} : \hat{\bar{\psi}}(x)\left(-i\hbar\gamma^k\partial_k + mc\right)\hat{\psi}(x)\times c :$$
$$\supset \int d^3\vec{x} : \hat{\bar{\psi}}_{\text{free}}(x)\left(-i\hbar\gamma^k\partial_k + mc\right)\hat{\psi}_{\text{free}}(x)\times c :$$
$$= \sum_{\sigma=\pm\frac{1}{2}}\int_{-\infty}^{\infty} d^3\vec{p}cp_{\text{free}}{}^0\left(\hat{e}_{\text{free}}{}^{\dagger}(\vec{p},\sigma)\hat{e}_{\text{free}}(\vec{p},\sigma)+\hat{e}_{\text{free}}{}^{c\dagger}(\vec{p},\sigma)\hat{e}_{\text{free}}{}^{c}(\vec{p},\sigma)\right) \tag{3.54}$$
$$\left(\text{modulo time-independent c-number}\right)$$

which is manifestly independent of t as well as t_0.

4. Conclusion

The wave function $\Phi_N(\omega_1,\cdots,\omega_N)$ in the Hilbert space of QED is equipped with the ket vector

$$\left|\Psi\right\rangle_{H\text{ or }S} = \sum_{N=0}^{\infty}\int d\omega_1\cdots d\omega_N \left|\omega_1,\cdots,\omega_N\right\rangle_{H\text{ or }S}\Phi_N(\omega_1,\cdots,\omega_N) \tag{4.1}$$

in term of the Heisenberg (H) or Schrődinger (S) representation satisfying the Heisenberg equation

$$i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle_H = 0 \tag{4.2}$$

or the Schrődinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle_S = \hat{H}_{\text{QED}}\left|\Psi\right\rangle_S \quad , \quad \left|\Psi\right\rangle_S = e^{\frac{1}{i\hbar}\hat{H}_{\text{QED}}(t-t_0)}\left|\Psi\right\rangle_H \tag{4.3}$$

The ω denotes the collected set of variables for expansion of the wave function using the basis ket vectors; a primitive choice may be given with the obvious notation

$$\left|\omega_1,\cdots,\omega_N\right\rangle_{H\text{ at }t=t_0} = \frac{1}{\sqrt{N_b!}}\hat{b}^{\dagger}\left(\omega_{1_b}\right)\cdots\hat{b}^{\dagger}\left(\omega_{N_b}\right)$$
$$\times\frac{1}{\sqrt{N_f!}}\hat{f}^{\dagger}\left(\omega_{1_f}\right)\cdots\hat{f}^{\dagger}\left(\omega_{N_f}\right) \tag{4.4}$$
$$\times\frac{1}{\sqrt{N_{f^c}!}}\hat{f}^{c\dagger}\left(\omega_{1_{f^c}}\right)\cdots\hat{f}^{c\dagger}\left(\omega_{N_{f^c}}\right)\left|0\right\rangle$$

$$\Phi_N(\omega_1,\cdots,\omega_N)=\Phi_N(\omega_{1_b},\cdots,\omega_{N_b},\omega_{1_f},\cdots,\omega_{N_f},\omega_{1_{f^c}},\cdots,\omega_{N_{f^c}}) \tag{4.5}$$

$$N=N_b\oplus N_f\oplus N_{f^c} \tag{4.6}$$

$$\omega=\omega_b\otimes\omega_f\otimes\omega_{f^c} \tag{4.7}$$

$$\omega_b,\omega_f,\omega_{f^c}=\{\nu,\vec{p},\sigma\} \tag{4.8}$$

For permutation P of variables

$$\Phi_N(\omega_{P1},\cdots,\omega_{PN})=\Phi_N(\omega_{P_b1_b},\cdots,\omega_{P_bN_b},\omega_{P_f1_f},\cdots,\omega_{P_fN_f},\omega_{P_{f^c}1_{f^c}},\cdots,\omega_{P_{f^c}N_{f^c}}) \tag{4.9}$$

$$P=P_b\otimes P_f\otimes P_{f^c} \tag{4.10}$$

the wave function changes the antisymmetric $(-)$ sign

$$\operatorname{sgn}(P)\Phi_N(\omega_{P1},\cdots,\omega_{PN})=\Phi_N(\omega_1,\cdots,\omega_N) \tag{4.11}$$

$$\operatorname{sgn}(P)=(-)^{P_f}(-)^{P_{f^c}} \tag{4.12}$$

Using the primitive choice described above, the basis vectors are orthonormal

$$\begin{aligned}
&{}_H\langle\omega_1,\cdots,\omega_N\,|\,\omega'_1,\cdots,\omega'_M\rangle_H\\
&=\delta_{N_bM_b}\frac{1}{N_b!}\sum_{P_b}\delta_b\left(\omega_{1_b}-\omega'_{P_b1_b}\right)\cdots\delta_b\left(\omega_{N_b}-\omega'_{P_bN_b}\right)\\
&\times\delta_{N_fM_f}\frac{1}{N_f!}\sum_{P_f}(-)^{P_f}\delta_f\left(\omega_{1_f}-\omega'_{P_f1_f}\right)\cdots\delta_f\left(\omega_{N_f}-\omega'_{P_fN_f}\right)\\
&\times\delta_{N_{f^c}M_{f^c}}\frac{1}{N_{f^c}!}\sum_{P_{f^c}}(-)^{P_{f^c}}\delta_{f^c}\left(\omega_{1_{f^c}}-\omega'_{P_{f^c}1_{f^c}}\right)\cdots\delta_{f^c}\left(\omega_{N_{f^c}}-\omega'_{P_{f^c}N_{f^c}}\right)
\end{aligned} \tag{4.13}$$

with

$$\delta_b\left(\omega_b-\omega'_b\right)=\delta_{\sigma\sigma'}\delta^3\left(\vec{p}-\vec{p}\,'\right)\delta\left(\nu-\nu\left(|\vec{p}|\right)_b\right)\delta\left(\nu'-\nu\left(|\vec{p}\,'|\right)_b\right) \tag{4.14}$$

$$\delta_f\left(\omega_f-\omega'_f\right)=\delta_{f^c}\left(\omega_{f^c}-\omega'_{f^c}\right)=\delta_{\sigma\sigma'}\delta^3\left(\vec{p}-\vec{p}\,'\right)\delta\left(\nu-\nu\left(|\vec{p}|\right)_f\right)\delta\left(\nu'-\nu\left(|\vec{p}\,'|\right)_f\right) \tag{4.15}$$

This demonstrates another Cauchy problem in QED. Namely, for an event α_i starting at t_i

with $t_0 < t_i \;\; ; \;\; i = 1, 2, 3, \cdots$, we set up the initial ket vector $\left|\Psi\left(\alpha_i, t_i\right)\right\rangle_H$ for Eq. (4.2) but need to obtain the wave function $\Phi_N(\alpha_i, t_i; \omega_1, \cdots, \omega_N, t)$ developing from t_i to t with $t_i < t$ onward obeying

$$\begin{aligned} & i\hbar \frac{\partial}{\partial t} \Phi_N(\alpha_i, t_i; \omega_1, \cdots, \omega_N, t) \\ & = \sum_{M=0}^{\infty} \int d\omega'_1 \cdots d\omega'_M \; H_{NM}(\omega_1, \cdots, \omega_N, \omega'_1, \cdots, \omega'_M) \Phi_M(\alpha_i, t_i; \omega'_1, \cdots, \omega'_M, t) \end{aligned} \tag{4.16}$$

using the time-independent function

$$\begin{aligned} & H_{NM}(\omega_1, \cdots, \omega_N, \omega'_1, \cdots, \omega'_M) \\ & = {}_S\left\langle \omega_1, \cdots, \omega_N \left| \hat{H}_{\mathrm{QED}} \right| \omega'_1, \cdots, \omega'_M \right\rangle_S = {}_H\left\langle \omega_1, \cdots, \omega_N \left| \hat{H}_{\mathrm{QED}} \right| \omega'_1, \cdots, \omega'_M \right\rangle_H \end{aligned} \tag{4.17}$$

$$\frac{\partial}{\partial t} H_{NM}(\omega_1, \cdots, \omega_N, \omega'_1, \cdots, \omega'_M) = 0 \tag{4.18}$$

Finally, substituting this time-dependent $\Phi_N(\alpha_i, t_i; \omega_1, \cdots, \omega_N, t)$ into Eq. (4.1), we calculate

$\left\langle \hat{F}(x) \right\rangle_{\alpha_i, t_i} = \dfrac{{}_H\left\langle \Psi\left(\alpha_i, t_i\right) \right| \hat{F}(x) \left| \Psi\left(\alpha_i, t_i\right) \right\rangle_H}{{}_H\left\langle \Psi\left(\alpha_i, t_i\right) | \Psi\left(\alpha_i, t_i\right) \right\rangle_H}$ for each event α_i starting at t_i with

$t_0 < t_i < t \;\; ; \;\; i = 1, 2, 3, \cdots$ developing onward with $x^\mu = (ct, \vec{x})$ at position $\vec{x}$ with time t in using Eq. (2.39).

This concludes the way for solving the dual Cauchy problems in QED using the new b-photon, f-electron, and f^c-positron algebras. These new algebras work for interacting particles through the first thermalization Eq. (3.22), the second Eq. (3.45), and the third Eq. (3.47). As compared with the conventional Gell-Mann-Low relationship using covariant perturbational approach [1], this present approach paves the way for realizing nonperturbationally space-time resolved simulation of QED.

Appendix A

In this Appendix, we may first quickly review basic mathematics. The coordinate x with the contravariant components x^μ and the covariant components x_μ and the metric tensor $\eta_{\mu\nu} = \eta^{\mu\nu}$ of the Minkowski spacetime, together with the inner product of two 4-vectors A and B written as $A \cdot B$ as well as the inner product of the Dirac gamma matrices γ^μ and a 4-vector A written as the Dirac slash $\not{A}$ are defined as

$$x^\mu = \left(x^0, x^k\right) = \left(x^0, x^1, x^2, x^3\right) = \left(ct, x, y, z\right) = \left(ct, \vec{r}\right) = \left(ct, \vec{x}\right) \qquad \text{(A.1)}$$

$$x_\mu = \eta_{\mu\nu} x^\nu = \left(x_0, x_k\right) = \left(x_0, x_1, x_2, x_3\right) = \left(ct, -x, -y, -z\right) = \left(ct, -\vec{r}\right) = \left(ct, -\vec{x}\right) \qquad \text{(A.2)}$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu}, \quad \eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu{}_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases} \qquad \text{(A.3)}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \text{grad}\right) \qquad \text{(A.4)}$$

$$\partial^\mu = \eta^{\mu\nu}\frac{\partial}{\partial x^\nu} = \left(\frac{\partial}{\partial x^0}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\text{grad}\right) \qquad \text{(A.5)}$$

$$A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A} \bullet \vec{B}, \quad \vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z \qquad \text{(A.6)}$$

$$\not{A} = \eta_{\mu\nu} \gamma^\mu A^\nu = \gamma^0 A^0 - \vec{\gamma} \bullet \vec{A}, \quad \vec{\gamma} \bullet \vec{A} = \gamma^1 A_x + \gamma^2 A_y + \gamma^3 A_z \qquad \text{(A.7)}$$

$$\Box = \partial^2 = \left(\frac{1}{c}\frac{\partial}{\partial t}\right)^2 - \Delta, \quad \Delta = \left(\vec{\nabla}\right)^2 \qquad \text{(A.8)}$$

$$\left\{A, B\right\} = AB + BA = \left[A, B\right]_+; \quad \left[A, B\right] = AB - BA = \left[A, B\right]_- \qquad \text{(A.9)}$$

where the Einstein summation convention is used.

The spinor $\psi(x)$ in the chiral representation $\psi_{\text{chiral}}(x)$ is constructed by the undotted spinor $\psi_R(x)=\xi^A(x)$ with right-handed chirality and the dotted spinor $\psi_L(x)=\eta_{\dot{U}}(x)$ with left-handed chirality as

$$\psi=\psi_{\text{chiral}}=\begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix}=\begin{pmatrix}\xi^A\\ \eta_{\dot{U}}\end{pmatrix} \tag{A.10}$$

$$\xi^A=\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix},\quad \eta_{\dot{U}}=\begin{pmatrix}\eta_{\dot{1}}\\ \eta_{\dot{2}}\end{pmatrix} \tag{A.11}$$

The undotted and dotted capital Latin letters run from 1 to 2 and change position by using the antisymmetric matrix ε as

$$\xi_A=\xi^B\varepsilon_{BA},\quad \eta^{\dot{U}}=\varepsilon^{\dot{U}\dot{V}}\eta_{\dot{V}} \tag{A.12}$$

$$\xi^A=\varepsilon^{AB}\xi_B,\quad \eta_{\dot{U}}=\eta^{\dot{V}}\varepsilon_{\dot{V}\dot{U}} \tag{A.13}$$

$$\varepsilon_{AB}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\varepsilon^{AB},\quad \varepsilon^{\dot{U}\dot{V}}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\varepsilon_{\dot{U}\dot{V}} \tag{A.14}$$

where the Einstein summation convention is used.

The Pauli matrix σ with the contravariant components σ^μ and the covariant components σ_μ

$$\sigma^\mu=\left(\sigma^0,\sigma^k\right)=\left(\sigma^0,\sigma^1,\sigma^2,\sigma^3\right)=\left(1,\sigma_x,\sigma_y,\sigma_z\right)=\left(1,\vec{\sigma}\right) \tag{A.15}$$

$$\sigma_\mu=\eta_{\mu\nu}\sigma^\nu=\left(\sigma_0,\sigma_k\right)=\left(\sigma_0,\sigma_1,\sigma_2,\sigma_3\right)=\left(1,-\sigma_x,-\sigma_y,-\sigma_z\right)=\left(1,-\vec{\sigma}\right) \tag{A.16}$$

(note the use of 1 as the unit matrix) are cast into the MTW (Misner-Thorne-Wheeler) representation [10]

$$
\begin{aligned}
\left(\sigma_0\right)^{A\dot{U}} &= \left(\sigma^0\right)_{\dot{V}B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^0 \\
\left(\sigma_1\right)^{A\dot{U}} &= \left(\sigma^1\right)_{\dot{V}B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \\
\left(\sigma_2\right)^{A\dot{U}} &= \left(\sigma^2\right)_{\dot{V}B} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y \\
\left(\sigma_3\right)^{A\dot{U}} &= \left(\sigma^3\right)_{\dot{V}B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z
\end{aligned}
\tag{A.17}
$$

Also, the Dirac gamma matrices γ^μ and the chiral matrix γ_5

$$
\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.18}
$$

are given in the chiral representation using the MTW representation of the Pauli matrices as

$$
\begin{aligned}
\gamma^0 &= \begin{pmatrix} 0 & \left(\sigma_0\right)^{A\dot{U}} \\ \left(\sigma^0\right)_{\dot{V}B} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\gamma^k &= \begin{pmatrix} 0 & -\left(\sigma_k\right)^{A\dot{U}} \\ \left(\sigma^k\right)_{\dot{V}B} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix} \\
\gamma_5 &= \begin{pmatrix} \left(\sigma^0\right)^{A}{}_{B} & 0 \\ 0 & -\left(\sigma^0\right)_{\dot{U}}{}^{\dot{V}} \end{pmatrix} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\gamma^5
\end{aligned}
\tag{A.19}
$$

where the following MTW representation is found for the diagonal block

$$
\begin{aligned}
\left(\sigma^0\right)^{A}{}_{B} &= \left(\sigma^0\right)_{\dot{U}}{}^{\dot{V}} = \sigma^0 \\
\left(\sigma^1\right)^{A}{}_{B} &= \left(\sigma^1\right)_{\dot{U}}{}^{\dot{V}} = \sigma_x \\
\left(\sigma^2\right)^{A}{}_{B} &= \left(\sigma^2\right)_{\dot{U}}{}^{\dot{V}} = \sigma_y \\
\left(\sigma^3\right)^{A}{}_{B} &= \left(\sigma^3\right)_{\dot{U}}{}^{\dot{V}} = \sigma_z
\end{aligned}
\tag{A.20}
$$

The Clifford algebra of the Dirac gamma matrices should be

$$\left\{\gamma^{\mu},\gamma^{\nu}\right\}=2\eta^{\mu\nu}\begin{pmatrix}\left(\sigma^{0}\right)^{A}{}_{B} & 0\\ 0 & \left(\sigma^{0}\right)_{\dot{U}}{}^{\dot{V}}\end{pmatrix}=2\eta^{\mu\nu}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=2\eta^{\mu\nu} \tag{A.21}$$

Appendix B

In this Appendix B, we quickly review variation principle of QED with gravitation.

B.1. Semiclassical Einstein-Hilbert gravitational action for QED

The semiclassical Einstein-Hilbert gravitational action I_G is added to the system action I_S and made stationary

$$\delta I=0,\quad I=I_G+I_S \tag{B.1}$$

First, for the variation $\delta g^{\mu\nu}$ of the symmetric metric tensor $g^{\mu\nu}=g^{\nu\mu}$, the Einstein equation is derived as

$$G_{\mu\nu}=Y_{\mu\nu} \tag{B.2}$$

$$G_{\mu\nu}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\frac{2\kappa}{c}I_G=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=G_{\nu\mu} \tag{B.3}$$

$$Y_{\mu\nu}=-\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\frac{2\kappa}{c}I_S=-\frac{\kappa}{c^2}T_{\mu\nu}=Y_{\nu\mu} \tag{B.4}$$

In QED system, the variation principle leads to the Dirac equation of electron

$$\left(i\hbar\gamma^{a}e_{a}{}^{\mu}D_{\mu}\left(g\right)-mc\right)\psi=0 \tag{B.5}$$

and the Maxwell equation of photon

$$\nabla_{\nu}F^{\mu\nu}=\frac{4\pi}{c}j^{\mu} \tag{B.6}$$

with the continuity equation of current

$$\partial_\mu j^\mu = 0 \tag{B.7}$$

In terms of the vector potential, we have the field equation

$$\nabla^\nu \nabla_\nu A^\mu + R^\mu{}_\nu A^\nu - \nabla^\mu \nabla_\nu A^\nu = \frac{4\pi}{c} j^\mu \tag{B.8}$$

Let the Coulomb gauge be given as

$$\nabla_i A^i = 0 \tag{B.9}$$

Then, we get the Laplace equation

$$\nabla^i \nabla_i A^0 + R^0{}_\nu A^\nu = \frac{4\pi}{c} j^0 \tag{B.10}$$

and the d'Alembert equation

$$\nabla^\mu \nabla_\mu A^i + R^i{}_\nu A^\nu - \nabla^i \nabla_0 A^0 = \frac{4\pi}{c} j^i \tag{B.11}$$

We may further introduce the longitudinal and the transversal currents as

$$j^i = j_T{}^i + j_L{}^i \tag{B.12}$$

in such a way that Eq. (B.11) is reduced to a separable form

$$-\nabla^i \nabla_0 A^0 = \frac{4\pi}{c} j_L{}^i \tag{B.13}$$

$$\nabla^\mu \nabla_\mu A^i + R^i{}_\nu A^\nu = \frac{4\pi}{c} j_T{}^i \tag{B.14}$$

The symmetric energy-momentum tensor

$$T_{\mu\nu} = -\varepsilon^\Pi{}_{\mu\nu} - \tau^\Pi{}_{\mu\nu}(g) - \frac{1}{4\pi} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - g_{\mu\nu}\left(L_{\mathrm{EM}} + L_{\mathrm{e}}\right) = T_{\nu\mu} \tag{B.15}$$

$$T_{\mu\nu} = T_{\mathrm{EM}\mu\nu} + T_{\mathrm{e}\mu\nu} \tag{B.16}$$

$$T_{\mathrm{EM}\mu\nu} = -\frac{1}{4\pi} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - g_{\mu\nu} L_{\mathrm{EM}} = T_{\mathrm{EM}\mu\nu} \tag{B.17}$$

$$T_{\mathrm{e}\mu\nu} = -\varepsilon^{\Pi}{}_{\mu\nu} - \tau^{\Pi}{}_{\mu\nu}(g) - g_{\mu\nu} L_{\mathrm{e}} = T_{\mathrm{e}\mu\nu} \tag{B.18}$$

satisfies the conservation law

$$\nabla_{\lambda} T^{\lambda}{}_{\mu} = 0 \tag{B.19}$$

Also the antisymmetric angular momentum tensor

$$M^{\lambda\mu\nu} = x^{\mu} T^{\lambda\nu} - x^{\nu} T^{\lambda\mu} = -M^{\lambda\nu\mu} \tag{B.20}$$

satisfies the conservation law

$$\partial_{\lambda} M^{\lambda k \ell} = 0 \tag{B.21}$$

B.2. Energy-momentum tensor and spin vorticity

In the limit to non-gravitation field, the energy-momentum tensor $T_{\mathrm{e}\mu\nu}$ is reduced to

$$T_{\mathrm{e}}{}^{\mu\nu} \to \begin{pmatrix} \frac{1}{2}(M + h.c.) & c\left(\vec{\Pi} + \frac{1}{2}\mathrm{rot}\vec{s}\right)_x & c\left(\vec{\Pi} + \frac{1}{2}\mathrm{rot}\vec{s}\right)_y & c\left(\vec{\Pi} + \frac{1}{2}\mathrm{rot}\vec{s}\right)_z \\ c\left(\vec{\Pi} + \frac{1}{2}\mathrm{rot}\vec{s}\right)_x & -\tau^{S}{}_{xx} + L_{\mathrm{e}} & -\tau^{S}{}_{xy} & -\tau^{S}{}_{xz} \\ c\left(\vec{\Pi} + \frac{1}{2}\mathrm{rot}\vec{s}\right)_y & -\tau^{S}{}_{yx} & -\tau^{S}{}_{yy} + L_{\mathrm{e}} & -\tau^{S}{}_{yz} \\ c\left(\vec{\Pi} + \frac{1}{2}\mathrm{rot}\vec{s}\right)_z & -\tau^{S}{}_{zx} & -\tau^{S}{}_{zy} & -\tau^{S}{}_{zz} + L_{\mathrm{e}} \end{pmatrix} \tag{B.22}$$

with the mass term M. The energy-momentum tensor $T_{\mathrm{EM}\mu\nu}$ is then reduced to

$$T_{\mathrm{EM}}{}^{\mu\nu} \to \begin{pmatrix} H_{\gamma} & cG_x & cG_y & cG_z \\ cG_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cG_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cG_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{B.23}$$

with the Poynting vector $\vec{G}$ and the Maxwell stress tensor $\vec{\vec{\sigma}}$. The conservation law Eq. (B.19) of energy and momentum is then reduced to

$$\nabla_\nu T^{\nu 0} = 0 \rightarrow \frac{\partial}{\partial t} cP^0 + c^2 \mathrm{div}\vec{P} = 0 \tag{B.24}$$

$$\nabla_\nu T^{\nu k} = 0 \rightarrow \frac{\partial}{\partial t} \vec{P} + \mathrm{div}\left(\vec{\sigma} - \vec{\tau}^S\right) = 0 \tag{B.25}$$

$$P^\mu = \left(\frac{\frac{1}{2}\left(M + h.c.\right) + H_\gamma}{c},\ \vec{\Pi} + \frac{1}{2}\mathrm{rot}\vec{s} + \vec{G} \right) \tag{B.26}$$

The conservation law Eq. (B.21) of angular momentum is then reduced to

$$\partial_\lambda M^{\lambda k \ell} = 0 \quad \rightarrow \quad \frac{\partial}{\partial t} \vec{J} + \mathrm{div}\left(\vec{r} \times \left(\vec{\sigma} - \vec{\tau}^S\right)\right) = 0 \tag{B.27}$$

$$\frac{1}{c} M^{0 k \ell} \quad \rightarrow \quad \vec{J} = \vec{r} \times \vec{\Pi} + \vec{r} \times \frac{1}{2}\mathrm{rot}\vec{s} + \vec{r} \times \vec{G} \tag{B.28}$$

Now that the vorticity plays an important role as momentum, and it is associated with antisymmetric electronic stress tensor $\vec{\tau}^A$, we may further prove that symmetric electronic stress tensor $\vec{\tau}^S$ plays an important role as tension $\vec{\tau}^S = \mathrm{div}\vec{\tau}^S$ compensating the Lorentz force $\vec{L}$ as

$$\frac{\partial}{\partial t}\left(\vec{\Pi} + \frac{1}{2}\mathrm{rot}\vec{s} \right) = \vec{L} + \vec{\tau}^S \tag{B.29}$$

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