

# **Studies on Matrix Eigenvalue Problems in Terms of Discrete Integrable Systems**

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# Chapter 1

## Introduction

The theme of the thesis is to analyze the convergence and develop some algorithms for matrix eigenvalue problems from the viewpoint of discrete integrable systems. Matrix eigenvalue problems are fundamental and important problems in numerical linear algebra. In the long history of numerical linear algebra, many numerical algorithms with linear and nonlinear recursion formulae have been developed to solve matrix eigenvalue problems. On the other hand, integrable systems are unique in the study of nonlinear dynamical systems in physics since the solutions to integrable systems are concretely written down. A skillful time-discretization of some integrable system is called a discrete integrable system. In recent studies, surprisingly, it has turned out that matrix eigenvalue problems and discrete integrable systems are in a close connection with each other, though the two subjects have different backgrounds.

In this chapter, firstly, historical backgrounds of numerical algorithms for matrix eigenvalue problems and discrete integrable systems are explained by showing typical examples, respectively. Secondly, the relationship between matrix eigenvalue problems and discrete integrable systems is described. The purposes and the outline of the thesis are given in the latter part of this chapter.

For an  $m$ -by- $m$  matrix  $A \in \mathbb{C}^{m \times m}$ , a complex constant  $\lambda \in \mathbb{C}$  and a nonzero vector  $\boldsymbol{x} \in \mathbb{C}^m$  which satisfy

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \tag{1.1}$$

are called an eigenvalue and an eigenvector of  $A$ , respectively. A pair of an eigenvalue  $\lambda$  and an eigenvector  $\boldsymbol{x}$  is an eigenpair. The standard eigenvalue

problem of  $A$  is the problem of finding the whole or a part of eigenpairs of  $A$ .

The Bernoulli method is developed for computing the solutions of the greatest absolute value to a given univariate algebraic equation by Bernoulli [4] in 1730s. The Bernoulli method is applicable to compute the eigenvalue of the greatest absolute value of a matrix if the characteristic polynomial of a given matrix is known. Let  $\lambda_1, \lambda_2, \dots, \lambda_m$  be the eigenvalues of a given matrix  $A$  such that  $|\lambda_1| > |\lambda_2| > \dots > |\lambda_m| > 0$ . In the Bernoulli method, a sequence  $\{f_n\}_{n=0,1,\dots}$  is generated by the linear recurrence equation

$$f_{n+m} + a_1 f_{n+m-1} + a_2 f_{n+m-2} + \dots + a_{m-1} f_{n+1} + a_m f_n = 0, \quad (1.2)$$

where  $a_1, a_2, \dots, a_m$  are the coefficients of the characteristic polynomial of  $A$ . Then, it holds that

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \lambda_1. \quad (1.3)$$

The Bernoulli method is also applicable to compute the limit of a sequence  $\{f_n\}_{n=0,1,\dots}$

An extension of the Bernoulli method is proposed by Aitken [1] in 1920s for computing all solutions to a univariate algebraic equation by using determinants whose entries are given by  $\{f_n\}_{n=0,1,\dots}$ .

The power method is used for computing matrix eigenvalues and eigenvectors by using a sequence  $\{f_n\}_{n=0,1,\dots}$  given by

$$f_n = \mathbf{x}^\top A^n \mathbf{x}, \quad (1.4)$$

where  $A$  is a given matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$  and  $\mathbf{x}$  is an initial vector. The inner product  $f_n$  of the vectors  $\mathbf{x}$  and  $A^n \mathbf{x}$  is called a moment. Then, it holds that (1.3). In 1950s, the Lanczos algorithm [41] is developed for reducing matrices to tridiagonal matrices. The characteristic of the Lanczos algorithm is to use the moments similar to the power method.

The qd algorithm

$$q_k^{(n+1)} + e_{k-1}^{(n+1)} = q_k^{(n)} + e_k^{(n)}, \quad q_k^{(n+1)} e_k^{(n+1)} = q_{k+1}^{(n)} e_k^{(n)}, \quad (1.5)$$

$$e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, \quad k = 1, 2, \dots, \quad n = 0, 1, \dots, \quad (1.6)$$

is proposed for finding poles of rational functions by Rutishauser [51] in 1954. In the following years, it is shown that the qd algorithm is applicable to

compute all the eigenvalues of tridiagonal matrices called Jacobi matrices by Rutishauser [52]. The variables of the qd algorithm are expressed as follows,

$$q_k^{(n)} = \frac{H_k^{(n+1)} H_{k-1}^{(n)}}{H_{k-1}^{(n+1)} H_k^{(n)}}, \quad e_k^{(n)} = \frac{H_{k-1}^{(n+1)} H_{k+1}^{(n)}}{H_k^{(n+1)} H_k^{(n)}}, \quad (1.7)$$

where  $H_k^{(n)}$  are Hankel determinants defined by

$$H_{-1}^{(n)} \equiv 0, \quad H_0^{(n)} \equiv 1, \quad H_k^{(n)} := \det(f_{n+i+j-2})_{1 \leq i, j \leq k}. \quad (1.8)$$

Then, it follows that

$$\lim_{n \rightarrow \infty} q_k^{(n)} = \lambda_k, \quad \lim_{n \rightarrow \infty} e_k^{(n)} = 0. \quad (1.9)$$

The  $LR$  algorithm based on a matrix decomposition is developed by Rutishauser [53]. The qd algorithm is equivalent to the  $LR$  algorithm for Jacobi matrices [28]. In 1890s, the determinant identities and the expansion expressions of Hankel matrices are shown by Hadamard [27], and Henrici [28, 30] describes the qd algorithm through Hadamard's techniques. The qd algorithm is then regarded as an extension of the Bernoulli method. The power method is also regarded as a specialization of the qd algorithm for which a different sequence  $\{f_n\}_{n=0,1,\dots}$  is given. The Lanczos method is equivalent to the qd algorithm for which a different sequence  $\{f_n\}_{n=0,1,\dots}$  is given.

The term “integrable systems” is used for nonlinear systems whose solutions can be concretely written down.

One of the most typical integrable systems is the Toda lattice equation [58]

$$\begin{aligned} \frac{d}{dt} V_{k+1} &= V_{k+1}(J_{k+1} - J_k), \quad \frac{d}{dt} J_k = V_{k+1} - V_k, \\ k &= 1, 2, \dots, m, \quad t \geq 0, \end{aligned} \quad (1.10)$$

which describes physical phenomena such as chains of particles with nonlinear interaction and ladder-type LC circuits [60]. The infinite Toda equation, on the infinite lattice  $n \in \mathbb{Z}$ , possesses an  $N$ -soliton solution. The Lax pair of the infinite Toda equation is given by Flaschka [15] as follows,

$$\Psi_{k+1} + J_k \Psi_k + V_k \Psi_{k-1} = \lambda \Psi_k, \quad (1.11)$$

$$\frac{d\Psi_k}{dt} = -V_k \Psi_{k-1}. \quad (1.12)$$

The “discrete integrable system” is a skillful time-discretization of an integrable system. A remarkable property of discrete integrable systems is that discrete integrable systems have the same structure of solutions as continuous-time integrable systems. The discrete Toda equation [32, 33, 34] is given as follows,

$$q_k^{(n+1)} + e_{k-1}^{(n+1)} = q_k^{(n)} + e_k^{(n)}, \quad q_k^{(n+1)} e_k^{(n+1)} = q_{k+1}^{(n)} e_k^{(n)}. \quad (1.13)$$

Equation (1.13) with boundary conditions  $e_0^{(n)} \equiv 0$  and  $e_m^{(n)} \equiv 0$  for  $n = 0, 1, \dots$  is called the discrete finite Toda equation. Hereinafter, the discrete Toda equation means the discrete finite Toda equation.

In 1993, a surprising fact in the studies of integrable systems and numerical linear algebra is shown that the discrete Toda equation (1.13) is equivalent to the recursion formulae of the qd algorithm (1.5) [34]. This fact suggests that eigenvalue problems and discrete integrable systems are in a close relation with each other.

Some numerical algorithms for computing matrix eigenvalues are designed based on the similarity transformation of matrices associated with discrete integrable systems. It is known that such algorithms, called the integrable algorithms, have high relative accuracy. Such good properties stem from discrete integrable systems. In 2001, the dLV algorithm for computing singular values of bidiagonal matrices is designed by using the discrete Lotka-Volterra (dLV) system, which stands for the prey-predator model in mathematical biology, by Tsujimoto et al. [63]. By Fukuda et al. [16], the dhLV algorithm for computing a kind of band matrices with complex eigenvalues is developed based on the discrete hungry Lotka-Volterra (dhLV) system, which is a generalization of the dLV system. The dhToda algorithm for computing eigenvalues of totally nonnegative (TN) matrices of Hessenberg form is proposed based on the discrete hungry (dh) Toda equation, which is an extension of the discrete Toda equation by Fukuda et al. [18]. TN matrices are entry-wise nonnegative matrices whose minors are all nonnegative.

The main purpose of the thesis is to analyze and solve matrix eigenvalue problems through a sequence  $\{f_n\}_{n=0,1,\dots}$  appearing in discrete integrable systems.

As is mentioned above, in the history of numerical algorithms, the linear sequence  $\{f_n\}_{n=0,1,\dots}$  plays a key role in developing numerical algorithms related with the qd algorithm. Under certain conditions, the recurrence equations (1.2) hold for the linear sequence  $\{f_n\}_{n=0,1,\dots}$  appearing in discrete

integrable systems. In fact, the determinant expressions (1.7) and the Hankel determinants (1.8) indicate that the variables  $q_k^{(n)}$  and  $e_k^{(n)}$  of the qd algorithm are expressed by using the linear sequence  $\{f_n\}_{n=0,1,\dots}$  in (1.2). Thus, reconsidering the linear sequence  $\{f_n\}_{n=0,1,\dots}$  appearing in discrete integrable systems is very important in order to investigate discrete integrable systems and the integrable algorithms deeply.

According to the fact that the Bernoulli methods is used for computing the limit of a given sequence of numbers, it is expected that the convergence of a sequence  $\{f_n\}_{n=0,1,\dots}$  could be investigated by means of the discrete integrable systems. Thus, the asymptotic behavior of a special solution to a discrete integrable system given by the Fibonacci sequence, which is one of the most famous and interesting sequences in combinatorics, is discussed in the thesis.

The convergence of the integrable algorithms has been proved using their recursion formulae or the explicit expression of determinant solutions only in the case where a given matrix has distinct and positive eigenvalues [52, 30, 16, 18]. With the developments of computational environment, the analysis of the convergence of numerical algorithms becomes significant even in the case of multiple eigenvalues in order to show the robustness of numerical algorithms for numerical error. However, the proof of the convergence in the case of multiple eigenvalues has not be completely finished even for the qd algorithm. The second purpose of the thesis is to analyze the asymptotic behavior of the qd algorithm for a tridiagonal matrix with multiple eigenvalues.

In matrix eigenvalue problems, inverse problems which are called inverse eigenvalue problems are important subjects. One of interesting topics in inverse eigenvalue problems is to construct a matrix with prescribed eigenvalues. There are several papers on such inverse eigenvalue problems for a tridiagonal matrix [8, 5, 26]. However, effective approaches have not been yet found for inverse eigenvalue problems for TN matrices because of the hard restriction of TN matrices. According to (1.7), (1.8) and (1.9), the variables of the qd algorithm are expressed by using the eigenvalues of a given tridiagonal matrix. The facts suggest that the solutions to discrete integrable systems might be expressed by using the eigenvalues of the associated matrices. The third purpose of the thesis is to approach inverse eigenvalue problems by making use of expressions of the linear sequence  $\{f_n\}_{n=0,1,\dots}$  appearing in determinant solutions to discrete integrable systems.

The thesis is organized as follows.

In Chapter 2 [A1], the asymptotic behavior of a special solution to the discrete Lotka-Volterra system given by some Fibonacci sequences is shown from the viewpoint that the qd algorithm is an extension of the Bernoulli method for computing the limit of a sequence of numbers. It is proved that a special solution to the discrete Lotka-Volterra system given by the Fibonacci sequence converges to a special constant such as the golden ratio.

In Chapter 3 [A4], the asymptotic behavior of the qd algorithm is discussed in the case where a tridiagonal matrix with multiple eigenvalues is given. The convergence rate of the variables of the qd algorithm is estimated.

In Chapter 4 [A2], an inverse eigenvalue problem for tridiagonal matrices with multiple eigenvalues is considered. A method for constructing a tridiagonal matrix with the same eigenvalues, except for multiplicities, as a given nonsymmetric dense matrix is proposed by using the properties of the qd algorithm. Some numerical examples are given.

In Chapter 5 [A3], an inverse eigenvalue problem for Hessenberg-type TN matrices is taken up. A finite-step procedure for constructing Hessenberg-type TN matrices with prescribed eigenvalues is designed based on the discrete hungry Toda equation. As examples, some TN matrices are constructed by using the proposed procedure.

In Chapter 6 [A5], an extension of the procedure in Chapter 5 [A3] for dense TN matrices is proposed. The recursion formula for constructing dense matrices with prescribed eigenvalues is derived by use of Hankel-like determinants and Hadamard-like polynomials. The initial conditions for the constructed matrices to be TN are found. The relationships between the proposed procedure and the numerical algorithms based on the Toda-type discrete integrable systems are also mentioned.

Chapter 7 is devoted to concluding remarks of the thesis.

## Chapter 2

# Some Fibonacci sequences in the discrete Lotka-Volterra system

In Chapter 1, we give an explanation of the relationship among the Bernoulli method, the qd algorithm and the discrete Toda equation. The Bernoulli method is applicable to compute the limit of a given sequence of numbers. Thus, it is interesting to verify that discrete integrable systems could compute the limit of a sequence. In this chapter, a special solution to the discrete Lotka-Volterra system, which is one of discrete integrable systems, given by the famous Fibonacci sequence is discussed.

### 2.1 The discrete Lotka-Volterra system

One of the prey-predator dynamics of  $(2m - 1)$  species in mathematical biology is described by the following integrable Lotka-Volterra (LV) system [66],

$$\begin{aligned} \frac{dU_k(t)}{dt} &= U_k(t)(U_{k+1}(t) - U_{k-1}(t)), \quad k = 1, 2, \dots, 2m - 1, \\ U_0(t) &\equiv 0, \quad U_{2m}(t) \equiv 0, \quad t \geq 0, \end{aligned} \quad (2.1)$$

where  $U_k(t)$  denotes the number of the  $k$ th species at the continuous time  $t$ . The LV system (2.1) describes that the  $k$ th species falls prey to the  $(k - 1)$ th

and preys on the  $(k+1)$ th. In [9], the LV system (2.1) is called the Kac-Van Moerbeke lattice, and its solution is represented as

$$U_{2k-1}(t) = \frac{\hat{\tau}_k(t)\tau_{k-1}(t)}{\tau_k(t)\hat{\tau}_{k-1}(t)}, \quad U_{2k}(t) = \frac{\tau_{k+1}(t)\hat{\tau}_{k-1}(t)}{\hat{\tau}_k(t)\tau_k(t)},$$

with two kinds of the Hankel determinants

$$\tau_k(t) := \begin{vmatrix} \omega_0(t) & \omega_1(t) & \cdots & \omega_{k-1}(t) \\ \omega_1(t) & \omega_2(t) & \cdots & \omega_k(t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k-1}(t) & \omega_k(t) & \cdots & \omega_{2k-2}(t) \end{vmatrix},$$

$$\hat{\tau}_k(t) := \begin{vmatrix} \omega_1(t) & \omega_2(t) & \cdots & \omega_k(t) \\ \omega_2(t) & \omega_3(t) & \cdots & \omega_{k+1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_k(t) & \omega_{k+1}(t) & \cdots & \omega_{2k-1}(t) \end{vmatrix}.$$

The index  $k$  of  $\tau_k(t)$  and  $\hat{\tau}_k(t)$  denotes the dimension of them. The Hankel determinant  $\hat{\tau}_k(t)$  differs from  $\tau_k(t)$  in that the (1,1) entry is  $\omega_1(t)$  not  $\omega_0(t)$ . The every other entry of  $\hat{\tau}_k(t)$  is also distinct to that of  $\tau_k(t)$ . A skillful time-discretization of (2.1) yields

$$u_k^{(n+1)}(1 + u_{k-1}^{(n+1)}) = u_k^{(n)}(1 + u_{k+1}^{(n)}), \quad k = 1, 2, \dots, 2m-1, \quad (2.2)$$

$$u_0^{(n)} \equiv 0, \quad u_{2m}^{(n)} \equiv 0, \quad n = 0, 1, \dots,$$

where  $u_k^{(n)}$  is the number of the  $k$ th species at the discrete time  $n$  [35, 63]. Here (2.2) is called the discrete Lotka-Volterra (dLV) system. A determinant solution to (2.2) is shown in [46, 55, 63] as

$$u_{2k-1}^{(n)} = \frac{\hat{H}_k^{(n)} H_{k-1}^{(n+1)}}{H_k^{(n)} \hat{H}_{k-1}^{(n+1)}}, \quad u_{2k}^{(n)} = \frac{H_{k+1}^{(n)} \hat{H}_{k-1}^{(n+1)}}{\hat{H}_k^{(n)} H_k^{(n+1)}}, \quad (2.3)$$

$$H_k^{(n)} := \begin{vmatrix} f_0^{(n)} & f_1^{(n)} & \cdots & f_{k-1}^{(n)} \\ f_1^{(n)} & f_2^{(n)} & \cdots & f_k^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k-1}^{(n)} & f_k^{(n)} & \cdots & f_{2k-2}^{(n)} \end{vmatrix}, \quad \hat{H}_k^{(n)} := \begin{vmatrix} f_1^{(n)} & f_2^{(n)} & \cdots & f_k^{(n)} \\ f_2^{(n)} & f_3^{(n)} & \cdots & f_{k+1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ f_k^{(n)} & f_{k+1}^{(n)} & \cdots & f_{2k-1}^{(n)} \end{vmatrix}, \quad (2.4)$$

$$H_0^{(n)} \equiv 1, \quad H_{m+1}^{(n)} \equiv 0, \quad \hat{H}_0^{(n)} \equiv 1. \quad (2.5)$$

The Hankel determinants  $H_k^{(n)}$  and  $\hat{H}_k^{(n)}$  have a structure similar to  $\tau_k(t)$  and  $\hat{\tau}_k(t)$  on their dimensions and entries. The Hankel determinants  $H_k^{(n)}$  and  $\hat{H}_k^{(n)}$  satisfy

$$\hat{H}_k^{(n)} H_{k-1}^{(n+1)} = \hat{H}_{k-1}^{(n)} H_k^{(n+1)} - H_k^{(n)} \hat{H}_{k-1}^{(n+1)}, \quad (2.6)$$

$$H_{k+1}^{(n)} \hat{H}_{k-1}^{(n+1)} = H_k^{(n)} \hat{H}_k^{(n+1)} - \hat{H}_k^{(n)} H_k^{(n+1)}, \quad (2.7)$$

which is called Hirota's bilinear form [35, 38, 46, 62, 63]. This is useful for proving the determinant solution (2.3). The proof is also derived from the study of orthogonal polynomials [56]. With respect to the entry  $f_j^{(n)}$  in  $H_k^{(n)}$  and  $\hat{H}_k^{(n)}$ , the time evolution from  $n$  to  $n+1$  is given by the recursion formula

$$f_j^{(n+1)} = f_j^{(n)} + f_{j+1}^{(n)}, \quad j = 0, 1, \dots, 2m-2. \quad (2.8)$$

It is emphasized here that the time evolution from  $n$  to  $n+1$  of  $f_{2m-1}^{(n)}$  is not given by (2.8). Hirota's bilinear form plays a key role for getting the formula concerning the time evolution of  $f_{2m-1}^{(n)}$ , which is not included in (2.8).

An algorithm for matrix singular values is designed based on the dLV system (2.2) [38, 39, 63]. However, in [38, 39, 63], the entry  $f_j^{(n)}$  is not mainly discussed. In this chapter, by focusing on  $f_j^{(n)}$ , we first consider the determinant solution (2.3) given by the  $m$ -step Fibonacci sequences [43] which cover the well-known Fibonacci, Tribonacci sequences [64] and so on. This study is mainly twofold. One is to derive a relationship between the Hankel determinants  $\hat{H}_m^{(n)}$  and  $\hat{H}_m^{(n+1)}$  by considering Hirota's bilinear form. The second is to clarify a time evolution from  $n$  to  $n+1$  of  $f_{2m-1}^{(n)}$  through the obtained relationship between  $\hat{H}_m^{(n)}$  and  $\hat{H}_m^{(n+1)}$ , and is to prove that if  $\{f_j^{(n)}\}_{j=0,1,\dots}$ , for fixed  $n$ , is an  $m$ -step Fibonacci sequence, then  $\{f_j^{(n+1)}\}_{j=0,1,\dots}$  is also so. We next show that, as  $n \rightarrow \infty$ , the dLV variable  $u_1^{(n)}$  has an interesting relationship with the ratio of two successive  $m$ -step Fibonacci numbers. We also demonstrate some examples in order to confirm theoretical results numerically.

## 2.2 $m$ -step Fibonacci sequence in determinant solution

The  $m$ -step Fibonacci sequence  $\{F_j\}_{j=0,1,\dots}$  is given by the recursion formula

$$F_{j+m} = F_j + F_{j+1} + \dots + F_{j+m-1}, \quad j = 0, 1, \dots, \quad (2.9)$$

for some integer  $F_0, F_1, \dots, F_{m-1}$  [43]. The 2-step Fibonacci sequence is well-known as the basic Fibonacci sequence such that the ratio  $F_{j+1}/F_j$  converges to the golden ratio  $\tau_2 = (\sqrt{5} + 1)/2$  as  $j \rightarrow \infty$  [64]. The 3-step Fibonacci sequence is also called the Tribonacci sequence. In this section, we show that  $f_j^{(n)}$  is associated with the  $m$ -step Fibonacci sequence.

As is shown in (2.4), the Hankel determinants  $H_1^{(n)}, H_2^{(n)}, \dots, H_m^{(n)}$  and  $\hat{H}_1^{(n)}, \hat{H}_2^{(n)}, \dots, \hat{H}_m^{(n)}$  are composed of  $f_0^{(n)}, f_1^{(n)}, \dots, f_{2m-1}^{(n)}$ . The all entries of  $H_1^{(n+1)}, H_2^{(n+1)}, \dots, H_m^{(n+1)}$  and  $\hat{H}_1^{(n+1)}, \hat{H}_2^{(n+1)}, \dots, \hat{H}_m^{(n+1)}$  are also represented in terms of  $f_0^{(n)}, f_1^{(n)}, \dots, f_{2m-1}^{(n)}$  through (2.8). In other words, the determinant solution (2.3) is given by  $f_0^{(n)}, f_1^{(n)}, \dots, f_{2m-1}^{(n)}$ . Let us assume here that  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$ , for fixed  $n$ , is an  $m$ -step Fibonacci sequence, namely,

$$f_{j+m}^{(n)} = f_j^{(n)} + f_{j+1}^{(n)} + \dots + f_{j+m-1}^{(n)}, \quad j = 0, 1, \dots, m-1, \quad (2.10)$$

for some integer  $f_0^{(n)}, f_1^{(n)}, \dots, f_{m-1}^{(n)}$ . Then we give a lemma concerning the evolution from  $n$  to  $n+1$  of the Hankel determinant  $H_m^{(n)}$  with the determinant dimension  $m$ .

**Lemma 2.2.1.** *Let  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$ , for fixed  $n$ , be an  $m$ -step Fibonacci sequence as in (2.10). If  $m$  is even, then it holds that*

$$H_m^{(n+1)} = H_m^{(n)}. \quad (2.11)$$

Otherwise,

$$H_m^{(n+1)} = 2H_m^{(n)}. \quad (2.12)$$

*Proof.* First, let us discuss the case where  $m$  is even. Let  $m = 2\ell$ . Let us begin with considering the Hankel determinant

$$H_{2\ell}^{(n+1)} = \begin{vmatrix} f_0^{(n+1)} & f_1^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} \\ f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} & \cdots & f_{4\ell-2}^{(n+1)} \end{vmatrix}, \quad (2.13)$$

which appears in the left hand side of (2.11). By using (2.8) and (2.10), we may rewrite the  $(2\ell, 1)$  entry of  $H_{2\ell}^{(n+1)}$  as

$$\begin{aligned} f_{2\ell-1}^{(n+1)} &= f_{2\ell-1}^{(n)} + f_{2\ell}^{(n)} \\ &= f_{2\ell-1}^{(n)} + (f_0^{(n)} + f_1^{(n)} + \cdots + f_{2\ell-1}^{(n)}) \\ &= f_{2\ell-1}^{(n)} + [(f_0^{(n)} + f_1^{(n)}) + (f_2^{(n)} + f_3^{(n)}) + \cdots + (f_{2\ell-2}^{(n)} + f_{2\ell-1}^{(n)})] \\ &= f_{2\ell-1}^{(n)} + (f_0^{(n+1)} + f_2^{(n+1)} + \cdots + f_{2\ell-2}^{(n+1)}). \end{aligned}$$

Similarly, for the  $(2\ell, 2)$ , the  $(2\ell, 3)$ ,  $\dots$ , the  $(2\ell, 2\ell)$  entries of  $H_{2\ell}^{(n+1)}$ , it holds that

$$f_{2\ell+k-2}^{(n+1)} = f_{2\ell+k-2}^{(n)} + (f_{k-1}^{(n+1)} + f_{k+1}^{(n+1)} + \cdots + f_{2\ell+k-3}^{(n+1)}), \quad k = 2, 3, \dots, 2\ell.$$

Thus,  $H_{2\ell}^{(n+1)}$  is represented as

$$\begin{aligned} H_{2\ell}^{(n+1)} &= \sum_{j=0}^{\ell-1} H_{2\ell, 2j}^{(n+1)} + \begin{vmatrix} f_0^{(n+1)} & f_1^{(n+1)} & \cdots & f_{2\ell-2}^{(n+1)} & f_{2\ell-1}^{(n+1)} \\ f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-2}^{(n+1)} & f_{2\ell-1}^{(n+1)} & \cdots & f_{4\ell-4}^{(n+1)} & f_{4\ell-3}^{(n+1)} \\ f_{2\ell-1}^{(n)} & f_{2\ell}^{(n)} & \cdots & f_{4\ell-3}^{(n)} & f_{4\ell-2}^{(n)} \end{vmatrix}, \\ H_{2\ell, 2j}^{(n+1)} &:= \begin{vmatrix} f_0^{(n+1)} & f_1^{(n+1)} & \cdots & f_{2\ell-2}^{(n+1)} & f_{2\ell-1}^{(n+1)} \\ f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-2}^{(n+1)} & f_{2\ell-1}^{(n+1)} & \cdots & f_{4\ell-4}^{(n+1)} & f_{4\ell-3}^{(n+1)} \\ f_{2j}^{(n+1)} & f_{2j+1}^{(n+1)} & \cdots & f_{2j+2\ell-1}^{(n+1)} & f_{2j+2\ell-2}^{(n+1)} \end{vmatrix}. \end{aligned}$$

For every  $j$ , the  $(2j+1)$ th row of  $H_{2\ell,2j}^{(n+1)}$  is just equal to the  $2\ell$ th one. So, it is obvious that  $H_{2\ell,0}^{(n+1)} = 0$ ,  $H_{2\ell,2}^{(n+1)} = 0$ ,  $\dots$ ,  $H_{2\ell,2\ell-2}^{(n+1)} = 0$ , and then, by using (2.8), we derive

$$\begin{aligned}
H_{2\ell}^{(n+1)} &= \begin{vmatrix} f_0^{(n+1)} & f_1^{(n+1)} & \cdots & f_{2\ell-2}^{(n+1)} & f_{2\ell-1}^{(n+1)} \\ f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-2}^{(n+1)} & f_{2\ell-1}^{(n+1)} & \cdots & f_{4\ell-4}^{(n+1)} & f_{4\ell-3}^{(n+1)} \\ f_{2\ell-1}^{(n)} & f_{2\ell}^{(n)} & \cdots & f_{4\ell-3}^{(n)} & f_{4\ell-2}^{(n)} \end{vmatrix} \\
&= \begin{vmatrix} f_0^{(n)} + f_1^{(n)} & f_1^{(n)} + f_2^{(n)} & \cdots & f_{2\ell-2}^{(n)} + f_{2\ell-1}^{(n)} & f_{2\ell-1}^{(n)} + f_{2\ell}^{(n)} \\ f_1^{(n)} + f_2^{(n)} & f_2^{(n)} + f_3^{(n)} & \cdots & f_{2\ell-1}^{(n)} + f_{2\ell}^{(n)} & f_{2\ell}^{(n)} + f_{2\ell+1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-2}^{(n)} + f_{2\ell-1}^{(n)} & f_{2\ell-1}^{(n)} + f_{2\ell}^{(n)} & \cdots & f_{4\ell-4}^{(n)} + f_{4\ell-3}^{(n)} & f_{4\ell-3}^{(n)} + f_{4\ell-2}^{(n)} \\ f_{2\ell-1}^{(n)} & f_{2\ell}^{(n)} & \cdots & f_{4\ell-3}^{(n)} & f_{4\ell-2}^{(n)} \end{vmatrix} \\
&= \begin{vmatrix} f_0^{(n)} & f_1^{(n)} & \cdots & f_{2\ell-2}^{(n)} & f_{2\ell-1}^{(n)} \\ f_1^{(n)} & f_2^{(n)} & \cdots & f_{2\ell-1}^{(n)} & f_{2\ell}^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-2}^{(n)} & f_{2\ell-1}^{(n)} & \cdots & f_{4\ell-4}^{(n)} & f_{4\ell-3}^{(n)} \\ f_{2\ell-1}^{(n)} & f_{2\ell}^{(n)} & \cdots & f_{4\ell-3}^{(n)} & f_{4\ell-2}^{(n)} \end{vmatrix} \\
&= H_{2\ell}^{(n)}.
\end{aligned}$$

Therefore, we have (2.11) if  $m$  is even.

Next, let us discuss the case  $m = 2\ell + 1$  in a way similar to the case  $m = 2\ell$ . Note that, from (2.8) and (2.10),

$$f_{2\ell+k-1}^{(n+1)} = 2f_{2\ell+k-1}^{(n)} + (f_{k-1}^{(n+1)} + f_{k+1}^{(n+1)} + \cdots + f_{2\ell+k-3}^{(n+1)}), \quad k = 1, 2, \dots, 2\ell + 1.$$

Thus,  $H_{2\ell+1}^{(n+1)}$  is rewritten as

$$H_{2\ell+1}^{(n+1)} = \sum_{j=0}^{\ell-1} H_{2\ell+1,2j}^{(n+1)} + 2 \begin{vmatrix} f_0^{(n+1)} & f_1^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} \\ f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} & \cdots & f_{4\ell-2}^{(n+1)} & f_{4\ell-1}^{(n+1)} \\ f_{2\ell}^{(n)} & f_{2\ell+1}^{(n)} & \cdots & f_{4\ell-1}^{(n)} & f_{4\ell}^{(n)} \end{vmatrix},$$

$$\hat{H}_{2\ell+1,2j}^{(n+1)} := \begin{vmatrix} f_0^{(n+1)} & f_1^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} \\ f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} & \cdots & f_{4\ell-2}^{(n+1)} & f_{4\ell-1}^{(n+1)} \\ f_{2j}^{(n+1)} & f_{2j+1}^{(n+1)} & \cdots & f_{2j+2\ell-1}^{(n+1)} & f_{2j+2\ell}^{(n+1)} \end{vmatrix}.$$

By taking account that  $H_{2\ell+1,0}^{(n+1)} = 0$ ,  $H_{2\ell+1,2}^{(n+1)} = 0$ ,  $\dots$ ,  $H_{2\ell+1,2\ell-2}^{(n+1)} = 0$ , we derive

$$H_{2\ell+1}^{(n+1)} = 2 \begin{vmatrix} f_0^{(n+1)} & f_1^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} \\ f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} & \cdots & f_{4\ell-2}^{(n+1)} & f_{4\ell-1}^{(n+1)} \\ f_{2\ell}^{(n)} & f_{2\ell+1}^{(n)} & \cdots & f_{4\ell-1}^{(n)} & f_{4\ell}^{(n)} \end{vmatrix}. \quad (2.14)$$

Obviously, from (2.8), the right hand side of (2.14) becomes  $2H_{2\ell+1}^{(n)}$ . Therefore, we have (2.12) if  $m$  is odd.  $\square$

Lemma 2.2.1 leads to a lemma concerning the evolution from  $n$  to  $n+1$  of  $\hat{H}_m^{(n)}$  with the determinant dimension  $m$ .

**Lemma 2.2.2.** *Let  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$ , for fixed  $n$ , be an  $m$ -step Fibonacci sequence as in (2.10). Moreover, let  $H_m^{(n)} \neq 0$ . If  $m$  is even, then it holds that*

$$\hat{H}_m^{(n+1)} = \hat{H}_m^{(n)}. \quad (2.15)$$

Otherwise,

$$\hat{H}_m^{(n+1)} = 2\hat{H}_m^{(n)}. \quad (2.16)$$

*Proof.* Eq. (2.7) with  $k = m$  becomes

$$H_m^{(n)} \hat{H}_m^{(n+1)} - \hat{H}_m^{(n)} H_m^{(n+1)} = 0,$$

since it holds that  $H_{m+1}^{(n)} = 0$ . By combining it with Lemma 2.2.1, we derive

$$H_m^{(n)} \hat{H}_m^{(n+1)} = H_m^{(n)} \hat{H}_m^{(n)},$$

if  $m$  is even. Otherwise,

$$H_m^{(n)} \hat{H}_m^{(n+1)} = 2H_m^{(n)} \hat{H}_m^{(n)}.$$

Thus, by taking account that  $H_m^{(n)} \neq 0$ , we have (2.15) and (2.16).  $\square$

Lemma 2.2.2 is useful for observing the time evolution from  $n$  to  $n + 1$  of  $a_{2m-1}^{(n)}$ . So, with the help of Lemma 2.2.2, we have a proposition for the sequence  $\{f_j^{(n+1)}\}_{j=0,1,\dots,2m-1}$ .

**Proposition 2.2.3** (Akaiwa-Iwasaki [A1]). *Let  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$ , for fixed  $n$ , be an  $m$ -step Fibonacci sequence as in (2.10). Moreover, let  $H_m^{(n)} \neq 0$ . Then  $\{f_j^{(n+1)}\}_{j=0,1,\dots,2m-1}$  also becomes an  $m$ -step Fibonacci sequence, namely,*

$$f_{j+m}^{(n+1)} = f_j^{(n+1)} + f_{j+1}^{(n+1)} + \cdots + f_{j+m-1}^{(n+1)}, \quad j = 0, 1, \dots, m-1. \quad (2.17)$$

*Proof.* It is easy to prove (2.17), except for the case where  $j = m-1$ . From (2.8) and (2.10), we derive, for  $j = 0, 1, \dots, m-2$ ,

$$\begin{aligned} f_{j+m}^{(n+1)} &= f_{j+m}^{(n)} + f_{j+m+1}^{(n)} \\ &= (f_j^{(n)} + f_{j+1}^{(n)} + \cdots + f_{j+m-1}^{(n)}) + (f_{j+1}^{(n)} + f_{j+2}^{(n)} + \cdots + f_{j+m}^{(n)}) \\ &= (f_j^{(n)} + f_{j+1}^{(n)}) + (f_{j+1}^{(n)} + f_{j+2}^{(n)}) + \cdots + (f_{j+m-1}^{(n)} + f_{j+m}^{(n)}) \\ &= f_j^{(n+1)} + f_{j+1}^{(n+1)} + \cdots + f_{j+m-1}^{(n+1)}. \end{aligned}$$

This implies that  $\{f_j^{(n+1)}\}_{j=0,1,\dots,2m-2}$  is an  $m$ -step Fibonacci sequence. Hereinafter, we show that  $\{f_j^{(n+1)}\}_{j=0,1,\dots,2m-1}$ , given by adding  $f_{2m-1}^{(n+1)}$  to

$\{f_j^{(n+1)}\}_{j=1,2,\dots,2m-2}$ , is also an  $m$ -step Fibonacci sequence, namely,  $f_{2m-1}^{(n+1)} = f_{m-1}^{(n+1)} + f_m^{(n+1)} + \dots + f_{2m-2}^{(n+1)}$ .

First, let us discuss the case where  $m = 2\ell$  in Lemma 2.2.2. The Hankel determinant  $\hat{H}_{2\ell}^{(n)}$  in (2.15) is transformed by (2.8) into

$$\begin{aligned} \hat{H}_{2\ell}^{(n)} &= \begin{vmatrix} f_1^{(n)} + f_2^{(n)} & f_2^{(n)} + f_3^{(n)} & \cdots & f_{2\ell-1}^{(n)} + f_{2\ell}^{(n)} & f_{2\ell}^{(n)} + f_{2\ell+1}^{(n)} \\ f_2^{(n)} + f_3^{(n)} & f_3^{(n)} + f_4^{(n)} & \cdots & f_{2\ell}^{(n)} + f_{2\ell+1}^{(n)} & f_{2\ell+1}^{(n)} + f_{2\ell+2}^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-1}^{(n)} + f_{2\ell}^{(n)} & f_{2\ell}^{(n)} + f_{2\ell+1}^{(n)} & \cdots & f_{4\ell-3}^{(n)} + f_{4\ell-2}^{(n)} & f_{4\ell-2}^{(n)} + f_{4\ell-1}^{(n)} \\ f_{2\ell}^{(n)} & f_{2\ell+1}^{(n)} & \cdots & f_{4\ell-2}^{(n)} & f_{4\ell-1}^{(n)} \end{vmatrix} \\ &= \begin{vmatrix} f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} \\ f_2^{(n+1)} & f_3^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} & \cdots & f_{4\ell-3}^{(n+1)} & f_{4\ell-2}^{(n+1)} \\ f_{2\ell}^{(n)} & f_{2\ell+1}^{(n)} & \cdots & f_{4\ell-2}^{(n)} & f_{4\ell-1}^{(n)} \end{vmatrix}. \end{aligned} \quad (2.18)$$

Let us add the 1st, the 3rd,  $\dots$ , the  $(2\ell - 1)$ th one. Then, by using (2.8) and (2.10), we may rewrite the  $(2\ell, 1)$ , the  $(2\ell, 2)$ ,  $\dots$ , the  $(2\ell, 2\ell - 1)$  entries in the  $2\ell$ th row of (2.18) as

$$\begin{aligned} & f_{2\ell+j}^{(n)} + (f_{j+1}^{(n+1)} + f_{j+3}^{(n+1)} + \dots + f_{2\ell+j-1}^{(n+1)}) \\ &= f_{2\ell+j}^{(n)} + [(f_{j+1}^{(n)} + f_{j+2}^{(n)}) + (f_{j+3}^{(n)} + f_{j+4}^{(n)}) + \dots + (f_{2\ell+j-1}^{(n)} + f_{2\ell+j}^{(n)})] \\ &= f_{2\ell+j}^{(n)} + f_{2\ell+j+1}^{(n)} \\ &= f_{2\ell+j}^{(n+1)}, \quad j = 0, 1, \dots, 2\ell - 2. \end{aligned}$$

Simultaneously, the  $(2\ell, 2\ell)$  entry becomes

$$\begin{aligned} & f_{4\ell-1}^{(n)} + f_{2\ell}^{(n+1)} + f_{2\ell+2}^{(n+1)} + \dots + f_{4\ell-2}^{(n+1)} \\ &= [(f_{2\ell-1}^{(n)} + f_{2\ell}^{(n)}) + (f_{2\ell+1}^{(n)} + f_{2\ell+2}^{(n)}) + \dots + (f_{4\ell-3}^{(n)} + f_{4\ell-2}^{(n)})] \\ &\quad + (f_{2\ell}^{(n+1)} + f_{2\ell+2}^{(n+1)} + \dots + f_{4\ell-2}^{(n+1)}) \\ &= (f_{2\ell-1}^{(n+1)} + f_{2\ell+1}^{(n+1)} + \dots + f_{4\ell-3}^{(n+1)}) + (f_{2\ell}^{(n+1)} + f_{2\ell+2}^{(n+1)} + \dots + f_{4\ell-2}^{(n+1)}) \\ &= f_{2\ell-1}^{(n+1)} + f_{2\ell}^{(n+1)} + \dots + f_{4\ell-2}^{(n+1)}. \end{aligned}$$

Thus, it follows that

$$\hat{H}_{2\ell}^{(n)} = \begin{vmatrix} f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} \\ f_2^{(n+1)} & f_3^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell-1}^{(n+1)} & f_{2\ell}^{(n+1)} & \cdots & f_{4\ell-3}^{(n+1)} & f_{4\ell-2}^{(n+1)} \\ f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} & \cdots & f_{4\ell-2}^{(n+1)} & f_{2\ell-1}^{(n+1)} + f_{2\ell}^{(n+1)} + \cdots + f_{4\ell-2}^{(n+1)} \end{vmatrix}. \quad (2.19)$$

Let us recall here that  $\hat{H}_{2\ell}^{(n+1)} = \hat{H}_{2\ell}^{(n)}$ . Therefore, by comparing (2.19) with  $\hat{H}_{2\ell}^{(n+1)}$ , we have  $f_{4\ell-1}^{(n+1)} = f_{2\ell-1}^{(n+1)} + f_{2\ell}^{(n+1)} + \cdots + f_{4\ell-2}^{(n+1)}$ , namely, (2.17) with  $j = m - 1$  if  $m$  is even.

Next, let us discuss the case where  $m = 2\ell + 1$ . Similarly as the case where  $m = 2\ell$ , by using (2.8) and (2.10), we derive

$$\begin{aligned} \hat{H}_{2\ell+1}^{(n)} = & \begin{vmatrix} f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} \\ f_2^{(n+1)} & f_3^{(n+1)} & \cdots & f_{2\ell+1}^{(n+1)} & f_{2\ell+2}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} & \cdots & f_{4\ell-1}^{(n+1)} & f_{4\ell}^{(n+1)} \\ f_{2\ell+1}^{(n+1)} & f_{2\ell+2}^{(n+1)} & \cdots & f_{4\ell}^{(n+1)} & f_{2\ell}^{(n+1)} + f_{2\ell+1}^{(n+1)} + \cdots + f_{4\ell}^{(n+1)} \end{vmatrix} \\ & + \begin{vmatrix} f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} \\ f_2^{(n+1)} & f_3^{(n+1)} & \cdots & f_{2\ell+1}^{(n+1)} & f_{2\ell+2}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} & \cdots & f_{4\ell-1}^{(n+1)} & f_{4\ell}^{(n+1)} \\ f_{2\ell}^{(n)} & f_{2\ell+1}^{(n)} & \cdots & f_{4\ell-1}^{(n)} & f_{4\ell}^{(n)} \end{vmatrix}. \end{aligned} \quad (2.20)$$

The 2nd term in the right hand side of (2.20) is transformed by (2.8) into  $-\hat{H}_{2\ell+1}^{(n)}$ . Thus, it follows that

$$2\hat{H}_{2\ell+1}^{(n)} = \begin{vmatrix} f_1^{(n+1)} & f_2^{(n+1)} & \cdots & f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} \\ f_2^{(n+1)} & f_3^{(n+1)} & \cdots & f_{2\ell+1}^{(n+1)} & f_{2\ell+2}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2\ell}^{(n+1)} & f_{2\ell+1}^{(n+1)} & \cdots & f_{4\ell-1}^{(n+1)} & f_{4\ell}^{(n+1)} \\ f_{2\ell+1}^{(n+1)} & f_{2\ell+2}^{(n+1)} & \cdots & f_{4\ell}^{(n+1)} & f_{2\ell}^{(n+1)} + f_{2\ell+1}^{(n+1)} + \cdots + f_{4\ell}^{(n+1)} \end{vmatrix}. \quad (2.21)$$

Therefore, by combining (2.21) with Lemma 2.2.2, we have  $f_{4\ell+1}^{(n+1)} = f_{2\ell}^{(n+1)} + f_{2\ell+1}^{(n+1)} + \cdots + f_{4\ell}^{(n+1)}$ , namely, (2.17) with  $m = 2\ell + 1$  if  $m$  is odd.  $\square$

For  $n = 0, 1, \dots$ , we successively have a main theorem in this section by induction of Proposition 2.2.3.

**Theorem 2.2.4** (Akaiwa-Iwasaki [A1]). *Let  $\{f_j^{(0)}\}_{j=0,1,\dots,2m-1}$  be an  $m$ -step Fibonacci sequence such that  $H_m^{(0)} \neq 0$ . Then it holds that, for every  $n$ ,  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$  is also an  $m$ -step Fibonacci sequence.*

In particular, if  $\{f_j^{(0)}\}_{j=0,1,2,3}$  is a 2-step Fibonacci sequence in the case where  $m = 2$ , then, by taking account that  $f_2^{(n)} = f_0^{(n)} + f_1^{(n)} = f_0^{(n+1)}$  and  $f_3^{(n)} = f_1^{(n)} + f_2^{(n)} = f_1^{(n+1)}$ , we give the following corollary.

**Corollary 2.2.5.** *Let  $\{f_j^{(0)}\}_{j=0,1,2,3}$  be a 2-step Fibonacci sequence such that  $H_2^{(0)} \neq 0$ . Then  $\{f_0^{(0)}, f_1^{(0)}, f_0^{(1)}, f_1^{(1)}, \dots, f_0^{(n)}, f_1^{(n)}, \dots\}$  is also a 2-step Fibonacci sequence.*

## 2.3 Convergence to the ratio of two successive $m$ -step Fibonacci numbers

It is well-known that, in the case where  $\{F_j\}_{j=0,1,\dots}$  is a 2-step Fibonacci sequence,  $F_{k+1}/F_k$ , which is the ratio of two successive 2-step Fibonacci numbers, converges to  $\tau_2 = (\sqrt{5} + 1)/2$  called the golden ratio, as  $k \rightarrow \infty$ . The ratio of two successive and sufficiently large 3-step Fibonacci numbers approaches to the irrational number  $1.839286755214161\dots$ . According to [43], in an  $m$ -step Fibonacci sequence, the ratio  $F_{k+1}/F_k$  converges to the constant  $\tau_m$  as  $k \rightarrow \infty$ , and  $\tau_m$  is equal to one of the real solutions to the  $m$ -degree algebraic equation

$$x^m - x^{m-1} - \cdots - x - 1 = 0. \quad (2.22)$$

Obviously,  $1 < \tau_m < 2$ . This is because  $1 < 1 + (F_{k-m+1} + F_{k-m+2} + \cdots + F_{k-1})/F_k = F_{k+1}/F_k = 2 - F_{k-m}/F_k < 2$  for every  $k$  in the case where  $\{F_j\}_{j=0,1,\dots}$  is a positive or negative  $m$ -step Fibonacci sequence. Moreover, it is shown in [6] that the continued fraction

$$1 + \frac{b|}{|1} + \frac{b|}{|1} + \cdots \quad (2.23)$$

becomes the largest solution to the quadratic equation  $x^2 - x - b = 0$ , through considering an extended 2-step Fibonacci sequence  $\{\mathcal{F}_j\}_{j=0,1,\dots}$  given by the recursion formula  $\mathcal{F}_{j+2} = \mathcal{F}_{j+1} + b\mathcal{F}_j$ . It is proved in [64] that the continued fraction (2.23) with  $b = 1$  is just equal to the golden ratio  $\tau_2$ . So, the discussion for the case where  $b = 1$  in [6] properly coincides with that for the case where  $m = 2$  in [43].

In this section, with the help of Theorem 2.2.4, we show a relationship of  $u_1^{(n)}$  with the constant  $\tau_m = \lim_{k \rightarrow \infty} F_{k+1}/F_k$ . Let us recall here that  $u_1^{(n)} = f_1^{(n)}/f_0^{(n)}$ . We first derive a proposition for the behavior of  $f_0^{(n)}$  and  $f_1^{(n)}$  as  $n$  increases.

**Proposition 2.3.1** (Akaiwa-Iwasaki [A1]). *Let  $\{f_j^{(0)}\}_{j=0,1,\dots,2m-1}$  be a positive or negative  $m$ -step Fibonacci sequence. Moreover, let  $H_m^{(n)} \neq 0$ . Then both  $f_0^{(n)}$  and  $f_1^{(n)}$  have the same sign, and their absolute values become larger monotonically as  $n$  grows larger.*

*Proof.* Let us assume that  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$  is a positive  $m$ -step Fibonacci sequence. Then it is obvious from (2.8) that  $f_0^{(n+1)}, f_1^{(n+1)}, \dots, f_{2m-2}^{(n+1)}$  are positive. As is shown in Proposition 2.2.3,  $\{f_j^{(n+1)}\}_{j=0,1,\dots,2m-1}$  is an  $m$ -step Fibonacci sequence. Thus,  $f_{2m-1}^{(n+1)} = f_{m-1}^{(n+1)} + f_m^{(n+1)} + \dots + f_{2m-2}^{(n+1)}$  is positive, and then  $\{f_j^{(n+1)}\}_{j=0,1,\dots,2m-1}$  is also a positive sequence. By induction for  $n = 0, 1, \dots$ , it is proved that if  $\{f_j^{(0)}\}_{j=0,1,\dots,2m-1}$  is a positive sequence, then  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$ , for each  $n$ , is also so. Moreover, by using (2.8), we derive  $f_j^{(0)} < f_j^{(0)} + f_{j+1}^{(0)} = f_j^{(1)} < f_j^{(1)} + f_{j+1}^{(1)} = f_j^{(2)} < \dots$  for  $j = 0, 1$ . Similarly, if  $\{f_j^{(0)}\}_{j=0,1,\dots,2m-1}$  is negative, then  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$  is also so and  $f_j^{(0)} > f_j^{(0)} + f_{j+1}^{(0)} = f_j^{(1)} > f_j^{(1)} + f_{j+1}^{(1)} = f_j^{(2)} > \dots$  for  $j = 0, 1$ .  $\square$

Under the assumption that  $H_k^{(0)}$  and  $\hat{H}_k^{(0)}$  are nonzero constants, we next give a proposition concerning the nonzero boundedness of  $H_k^{(n)}$  and  $\hat{H}_k^{(n)}$ .

**Proposition 2.3.2** (Akaiwa-Iwasaki [A1]). *Let  $\{f_j^{(0)}\}_{j=0,1,\dots,2m-1}$  be an  $m$ -step Fibonacci sequence. Moreover, let  $H_k^{(0)}$  and  $\hat{H}_k^{(0)}$  be nonzero constants. Then, for each  $n$ ,  $H_k^{(n)}$  and  $\hat{H}_k^{(n)}$  are also nonzero constants.*

*Proof.* Let us assume that  $H_k^{(n)}$  and  $\hat{H}_k^{(n)}$  are nonzero constant. Let us in-

introduce new dependent variables

$$w_{2k-1}^{(n)} := \frac{\hat{H}_k^{(n)} H_{k-1}^{(n)}}{H_k^{(n)} \hat{H}_{k-1}^{(n)}}, \quad w_{2k}^{(n)} := \frac{H_{k+1}^{(n)} \hat{H}_{k-1}^{(n)}}{\hat{H}_k^{(n)} H_k^{(n)}}.$$

In [39], it is shown that

$$\sum_{k=1}^{2m-1} w_k^{(n+1)} = \sum_{k=1}^{2m-1} w_k^{(n)}. \quad (2.24)$$

Under the assumption of the proposition, it holds that  $\sum_{k=1}^{2m-1} w_k^{(n)}$  becomes some constant  $C$ . By combining it with (2.24), we derive

$$\sum_{k=1}^{2m-1} w_k^{(n+1)} = C. \quad (2.25)$$

It is remarkable here that  $H_1^{(n+1)}, \hat{H}_1^{(n+1)}, H_2^{(n+1)}, \hat{H}_2^{(n+1)}, \dots, H_{m-1}^{(n+1)}, \hat{H}_{m-1}^{(n+1)}$  may be regarded as the polynomials of  $f_1^{(n)}, f_2^{(n)}, \dots, f_{2m-1}^{(n)}$ . So, let  $H_k^{(n+1)} = M_k$  and  $\hat{H}_k^{(n+1)} = \hat{M}_k$  for  $k = 1, 2, \dots, m-1$ , where  $M_k$  and  $\hat{M}_k$  are some constant. From Lemmas 2.2.1 and 2.2.2, there exist some nonzero constant  $M_m$  and  $\hat{M}_m$  such that  $H_m^{(n+1)} = M_m$  and  $\hat{H}_m^{(n+1)} = \hat{M}_m$ . Thus,  $\sum_{k=1}^{2m-1} w_k^{(n+1)}$  is represented, in terms of  $M_1, M_2, \dots, M_m$  and  $\hat{M}_1, \hat{M}_2, \dots, \hat{M}_m$ , as

$$\sum_{k=1}^{2m-1} w_k^{(n+1)} = \sum_{k=1}^{m-1} \left( \frac{\hat{M}_k M_{k-1}}{M_k \hat{M}_{k-1}} + \frac{M_{k+1} \hat{M}_{k-1}}{\hat{M}_k M_k} \right) + \frac{\hat{M}_m M_{m-1}}{M_m \hat{M}_{m-1}}. \quad (2.26)$$

Some denominators and numerators become 0 in the right hand side of (2.26), if, at least, one of  $M_1, M_2, \dots, M_{m-1}$  and  $\hat{M}_1, \hat{M}_2, \dots, \hat{M}_{m-1}$  is 0. This contradicts (2.25). Therefore it is concluded that  $M_k \neq 0$  and  $\hat{M}_k \neq 0$  for  $k = 1, 2, \dots, m$ . The proof is completed by induction for  $n = 0, 1, \dots$ .  $\square$

Proposition 2.3.2 implies that the sequence  $\{u_1^{(n)}\}_{n=1,2,\dots}$  is theoretically computable without overflow through the evolution from  $n$  to  $n+1$  by the dLV system (2.2).

By combining Propositions 2.3.1 and 2.3.2 with Theorem 2.2.4, we finally have a theorem for the asymptotic convergence of  $u_1^{(n)}$  as  $n \rightarrow \infty$ .

**Theorem 2.3.3** (Akaiwa-Iwasaki [A1]). *Let  $\{f_j^{(0)}\}_{j=0,1,\dots,2m-1}$  be a positive or negative  $m$ -step Fibonacci sequence. Moreover, let  $H_k^{(0)}$  and  $\hat{H}_k^{(0)}$  be nonzero constant. Then the dLV variable  $u_1^{(n)} = f_1^{(n)}/f_0^{(n)}$  converges to  $\tau_m = \lim_{k \rightarrow \infty} F_{k+1}/F_k$  as  $n \rightarrow \infty$ .*

Hereinafter, for  $m = 2, 3, 4, 5$ , we numerically confirm Theorem 2.3.3, namely, the convergence of  $u_1^{(n)}$  to  $\tau_m$  as  $n \rightarrow \infty$ . Numerical examples have been carried out with a computer with OS: Mac OS 10.6.2, CPU: Intel Core 2 Duo 3.06GHz, RAM: 4GB. We also use Wolfram Mathematica 6.0 with 16-digits precision arithmetic.

Let  $\{f_j^{(0)}\}_{j=0,1,\dots,2m-1}$  be set as the positive  $m$ -step Fibonacci sequence such that  $H_k^{(n)} \neq 0$  and  $\hat{H}_k^{(n)} \neq 0$  for  $k = 1, 2, \dots, m$ . For example, in the case where  $m = 2$ , let  $f_0^{(0)} = 1$ ,  $f_1^{(0)} = 2$ ,  $f_2^{(0)} = 3$  and  $f_3^{(0)} = 5$ . Then  $H_1^{(0)} = 1$ ,  $\hat{H}_1^{(0)} = 2$ ,  $H_2^{(0)} = -1$ ,  $\hat{H}_2^{(0)} = 1$  and  $u_1^{(0)} = 2$ ,  $u_2^{(0)} = -1/6$ ,  $u_3^{(0)} = -3/5$ . For  $n = 0, 1, \dots, 50$ , the values of  $u_1^{(n)}$  are plotted by the mark  $\circ$  in Figure 2.1. As is shown in Proposition 2.3.2, it is guaranteed that  $H_1^{(n)} \neq 0$ ,  $\hat{H}_1^{(n)} \neq 0$ ,  $H_2^{(n)} \neq 0$  and  $\hat{H}_2^{(n)} \neq 0$  for  $n = 0, 1, \dots, 50$ , provided that  $H_1^{(0)} \neq 0$ ,  $\hat{H}_1^{(0)} \neq 0$ ,  $H_2^{(0)} \neq 0$  and  $\hat{H}_2^{(0)} \neq 0$ . So, by using the dLV system (2.2),  $u_1^{(1)}$ ,  $u_1^{(2)}$ ,  $\dots$ ,  $u_1^{(50)}$  are numerically computable without overflow. Figure 2.1 agrees with this fact. Figure 2.1 also describes that, as is shown in Theorem 2.3.3,  $u_1^{(n)}$  converges to some constant as  $n$  becomes larger. It is remarkable here that  $u_1^{(n)}$  converges even if all of  $u_k^{(n)}$  are not always positive. This asymptotic behavior differs from that shown in [38, 39, 63] for the case where all of  $u_k^{(n)}$  are positive. The 1st row of Table 2.1 shows the values of  $\tau_2$ , derived from solving (2.22), and  $u_1^{(50)}$ . Table 2.1 tells us that  $u_1^{(50)}$  coincides with the golden ratio  $\tau_2$  in 16-digits precision arithmetic.

Similarly, in the cases where  $m = 3, 4, 5$ , the asymptotic convergence of  $u_1^{(n)}$  to  $\tau_3 = 1.83928675\ 5214161$ ,  $\tau_4 = 1.927561975482925$ ,  $\tau_5 = 1.9659482366\ 45485$  are shown in Figure 2.1 and Table 2.1. It is numerically confirmed from Figure 2.1 and Table 2.1 that Theorem 2.3.3 also holds for  $m = 3, 4, 5$ . Additionally, it is observed that  $u_1^{(50)}$  tends to 2, as  $m$  becomes larger.

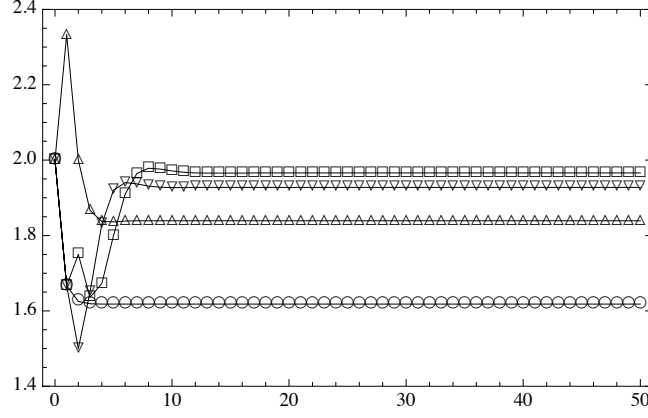


Figure 2.1: A graph of the variable  $n$  ( $x$ -axis) and the values of  $u_1^{(n)}$  ( $y$ -axis) in the cases where  $m = 2, 3, 4, 5$ . The symbols  $\bigcirc$ ,  $\triangle$ ,  $\nabla$  and  $\square$  indicate the cases  $m = 2, 3, 4$  and  $5$ , respectively.

Table 2.1: The real solutions to (2.22) and the values of  $u_1^{(50)}$  in the case where  $m = 2, 3, 4, 5$ .

$m$	The real solutions to (2.22)		The values of $u_1^{(50)}$
2	$\tau_2 = 1.618033988749895$	$-0.6180339887498948$	1.618033988749895
3	$\tau_3 = 1.839286755214161$		1.839286755214161
4	$\tau_4 = 1.927561975482925$	$-0.7748041132154339$	1.927561975482924
5	$\tau_5 = 1.965948236645485$		1.965948236645486

## Chapter 3

# Convergence of the qd algorithm for tridiagonal matrices with multiple eigenvalues

It is well-known that the quotient-difference (qd) algorithm is applicable to compute all the eigenvalues of a tridiagonal matrix. In most papers on the convergence of the qd algorithm, it is assumed that a tridiagonal matrix has distinct eigenvalues. Generally, the convergence of numerical algorithms is proved by using their recursion formula. Thus, it is not easy to analyze sensitively the convergence of numerical algorithms with respect to the multiplicity of eigenvalues.

In this chapter, for tridiagonal matrices with multiple eigenvalues, the convergence and the convergence rate of the qd algorithm are shown by making use of good properties of the qd algorithm.

### 3.1 The quotient-difference algorithm

The quotient-difference (qd) algorithm [51], which is one of the most important algorithms in numerical linear algebra, is originally proposed by Rutishauser for the purpose of computing eigenvalues of tridiagonal matrices and zeros of polynomials. The qd algorithm also has several applications to computing singular values of bidiagonal matrices [11, 47] and the Laplace

transformations of rational functions [45].

Symmetric dense matrices are transformed into symmetric tridiagonal matrices by a sequence of Householder transformations without changing eigenvalues [25]. Eigenvalues of symmetric dense matrices can be thus computed by combining the qd algorithm with Householder transformations. In finite arithmetic, a variant of the qd algorithm, which is called the differential qd with shift (dqds) algorithm, generates eigenvalues of tridiagonal matrices with high relative accuracy [47]. The dqds algorithm is also useful for computing singular values of bidiagonal matrices, which is equivalent to computing eigenvalues of symmetric tridiagonal matrices [11]. The well-known Linear Algebra PACKage (LAPACK) [42] thus adopts basic concept of the qd algorithm in its eigenvalue and singular value solvers.

Similarity transformations for symmetric tridiagonal matrices give tridiagonal matrices of the form

$$T = \begin{pmatrix} u_1 & 1 & & & \\ v_1 & u_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & v_{m-1} & u_m \end{pmatrix}. \quad (3.1)$$

Though symmetric tridiagonal matrices are always nonsymmetrized, the tridiagonal matrices  $T$  are not always symmetrized. In fact,  $T$  cannot be symmetrized if they have multiple eigenvalues. In other words,  $T$  essentially contain symmetric tridiagonal matrices. So, in this chapter we focus on the qd algorithm for the tridiagonal matrices  $T$ .

Almost all the articles concerning the qd algorithm deal with the case where the eigenvalues of  $T$  are real and distinct. It is shown in Rutishauser [52, 54] and Henrici [30] that the variables  $q_k^{(n)}$  and  $e_k^{(n)}$  appearing in the qd algorithm are expressed by using the Hankel determinants, and the asymptotic behavior  $q_k^{(n)} \rightarrow \lambda_k$  for  $k = 1, 2, \dots, m$  and  $e_k^{(n)} \rightarrow 0$  for  $k = 1, 2, \dots, m-1$  as  $n \rightarrow \infty$  is shown where  $\lambda_1, \lambda_2, \dots, \lambda_m$  denote the eigenvalues of  $T$ . The Hankel determinants associated with rational functions whose poles are of order two or higher is analyzed by Golomb [24], but the asymptotic behavior of the qd variables is substantially considered only in the case where  $T$  has real and distinct eigenvalues. In the case where all the eigenvalues of  $T$  are the same, by Ferreira and Parlett [12], the convergence of the  $LR$  algorithm, which can be regarded as a generalization of the qd algorithm, is examined.

In this chapter, we clarify the asymptotic behavior of the qd variables in the case where  $T$  has any kinds of multiple eigenvalues.

The remainder of this chapter is organized as follows. In Section 3.2, we describe basic properties of the qd algorithm for the tridiagonal matrices. In Section 3.3, we derive expressions of the qd variables in terms of the Hankel determinants in the case where  $T$  has multiple eigenvalues, and then in Section 3.4, we present the asymptotic expansion of the Hankel determinants. In Section 3.5, we thus show the asymptotic behavior of the qd variables.

## 3.2 The qd algorithm for the tridiagonal matrix

In this section, we briefly explain basic properties of the qd algorithm for computing eigenvalues of the tridiagonal matrix  $T$ .

The qd algorithm for  $T$  employs the recursion formula

$$\begin{cases} q_k^{(n+1)} + e_{k-1}^{(n+1)} = q_k^{(n)} + e_k^{(n)}, & k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \\ q_k^{(n+1)} e_k^{(n+1)} = q_{k+1}^{(n)} e_k^{(n)}, & k = 1, 2, \dots, m-1, \quad n = 0, 1, \dots, \end{cases} \quad (3.2)$$

under the boundary conditions

$$e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, \quad n = 0, 1, \dots, \quad (3.3)$$

and the initial conditions

$$\begin{cases} q_k^{(0)} + e_{k-1}^{(0)} = u_k, & k = 1, 2, \dots, m, \\ q_k^{(0)} e_k^{(0)} = v_k, & k = 1, 2, \dots, m-1, \end{cases} \quad (3.4)$$

where the superscript  $n$  denotes the iteration number of the qd recursion formula (3.2) and the subscript  $k$  corresponds to the position of an entry of  $T$ . The qd recursion formula (3.2) leads to the matrix identity

$$L^{(n+1)} R^{(n+1)} = R^{(n)} L^{(n)}, \quad n = 0, 1, \dots, \quad (3.5)$$

where  $L^{(n)}$  and  $R^{(n)}$  are the lower and the upper bidiagonal matrices with the qd variables  $e_1^{(n)}, e_2^{(n)}, \dots, e_{m-1}^{(n)}$  and  $q_1^{(n)}, q_2^{(n)}, \dots, q_m^{(n)}$ , respectively, given

as

$$L^{(n)} := \begin{pmatrix} 1 & & & \\ e_1^{(n)} & 1 & & \\ & \ddots & \ddots & \\ & & e_{m-1}^{(n)} & 1 \end{pmatrix}, \quad R^{(n)} := \begin{pmatrix} q_1^{(n)} & 1 & & \\ & q_2^{(n)} & \ddots & \\ & & \ddots & 1 \\ & & & q_m^{(n)} \end{pmatrix}. \quad (3.6)$$

Let us introduce a tridiagonal matrix  $T^{(n)}$  with the same form as the tridiagonal matrix  $T$  as the product  $L^{(n)}R^{(n)}$ , namely,

$$\begin{aligned} T^{(n)} &= L^{(n)}R^{(n)} \\ &= \begin{pmatrix} q_1^{(n)} & 1 & & \\ q_1^{(n)}e_1^{(n)} & q_2^{(n)} + e_1^{(n)} & \ddots & \\ & \ddots & \ddots & 1 \\ & & q_{m-1}^{(n)}e_{m-1}^{(n)} & q_m^{(n)} + e_{m-1}^{(n)} \end{pmatrix}. \end{aligned} \quad (3.7)$$

With respect to the evolution from  $T^{(n)}$  to  $T^{(n+1)}$ , we then derive

$$T^{(n+1)} = R^{(n)}J^{(n)}(R^{(n)})^{-1}, \quad n = 0, 1, \dots, \quad (3.8)$$

which implies that eigenvalues of  $T^{(n)}$  and  $T^{(n+1)}$  are equal to each other. In other words, the similarity transformation from  $T^{(n)}$  to  $T^{(n+1)}$  can be given by the qd recursion formula (3.2). The matrices  $T^{(n)}$  for  $n = 1, 2, \dots$  become similar to the tridiagonal matrix  $T$  if  $T^{(0)} = T$ .

Let us introduce a formal power series with respect to  $z \in \mathbb{C}$ ,

$$F(z) := f_0 + f_1z + f_2z^2 + f_3z^3 + \dots. \quad (3.9)$$

Let  $H_k^{(n)}$  be the Hankel determinants associated with  $F(z)$  as  $H_{-1}^{(n)} \equiv 0$ ,  $H_0^{(n)} \equiv 1$  and

$$H_k^{(n)} := \begin{vmatrix} f_n & f_{n+1} & \cdots & f_{n+k-1} \\ f_{n+1} & f_{n+2} & \cdots & f_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+k-1} & f_{n+k} & \cdots & f_{n+2k-2} \end{vmatrix}, \quad k = 1, 2, \dots, \quad n = 0, 1, \dots. \quad (3.10)$$

Henrici's book [30, pp. 594–640] provides several theorems for the qd algorithm from the viewpoint of the formal power series  $F(z)$  under the settings  $q_1^{(n)} = f_{n+1}/f_n$  for  $n = 0, 1, \dots$ .

Here, let us consider the case where  $F(z)$  is a rational function whose denominator coincides with the characteristic polynomial of  $T$ , namely,  $\det(zI_m - T)$  where  $I_m$  denotes the  $m$ -dimensional identity matrix. Moreover, let  $T_k$  be the  $k$ -by- $k$  principal submatrices of  $T$  defined by

$$T_k = \begin{pmatrix} u_1 & 1 & & \\ v_1 & u_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & v_{k-1} & u_k \end{pmatrix}, \quad k = 1, 2, \dots, m \quad (3.11)$$

whose characteristic polynomials are expanded as

$$\det(zI_k - T_k) = z^k + a_1^{(k)}z^{k-1} + a_2^{(k)}z^{k-2} + \dots + a_k^{(k)}, \quad k = 1, 2, \dots, m, \quad (3.12)$$

where  $a_1^{(k)}, a_2^{(k)}, \dots, a_k^{(k)}$  are the constant coefficients. Then it is obvious that all the coefficients  $a_1^{(k)}, a_2^{(k)}, \dots, a_k^{(k)}$  are uniquely determined.

Let  $\lambda_1, \lambda_2, \dots, \lambda_m$  denote eigenvalues of  $J$  such that  $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_m|$ . Moreover, let  $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_N$  be distinct eigenvalues of  $J$  with multiplicities  $m_1, m_2, \dots, m_N$ , respectively. Of course,  $m_1 + m_2 + \dots + m_N = m$  and  $|\hat{\lambda}_1| \geq |\hat{\lambda}_2| \geq \dots \geq |\hat{\lambda}_N|$ . Then the characteristic polynomial of  $J$  is factorized as

$$\det(zI_m - T) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m), \quad (3.13)$$

and

$$\det(zI_m - T) = (z - \hat{\lambda}_1)^{m_1} (z - \hat{\lambda}_2)^{m_2} \cdots (z - \hat{\lambda}_N)^{m_N}. \quad (3.14)$$

The following theorem gives important properties concerning the qd algorithm for the tridiagonal matrix  $T$ .

**Theorem 3.2.1** (Henrici [30, pp.596–613]). *For any  $T$ , let us assume that the sequence  $\{f_n\}_{n=0,1,\dots}$  satisfies a system of linear equations*

$$f_{n+m} + a_1^{(m)}f_{n+m-1} + \dots + a_{m-1}^{(m)}f_{n+1} + a_m^{(m)}f_n = 0, \quad n = 0, 1, \dots, \quad (3.15)$$

where  $f_0, f_1, \dots, f_{m-1}$  are given by

$$f_0 = 1, \quad f_k = -a_1^{(k)} f_{k-1} - a_2^{(k)} f_{k-2} - \dots - a_k^{(k)} f_0, \quad k = 1, 2, \dots, m-1. \quad (3.16)$$

If  $H_k^{(n)} \neq 0$  for  $k = 1, 2, \dots, m$  and  $n = 0, 1, \dots$ , then the  $q$ d variables  $q_k^{(n)}$  and  $e_k^{(n)}$  in (3.2) with (3.3) and (3.4) are expressed as

$$q_k^{(n)} = \frac{H_k^{(n+1)} H_{k-1}^{(n)}}{H_k^{(n)} H_{k-1}^{(n+1)}}, \quad k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \quad (3.17)$$

$$e_k^{(n)} = \frac{H_{k+1}^{(n)} H_{k-1}^{(n+1)}}{H_k^{(n)} H_k^{(n+1)}}, \quad k = 0, 1, \dots, m-1, \quad n = 0, 1, \dots \quad (3.18)$$

For each  $k$  such that  $|\lambda_k| > |\lambda_{k+1}|$  where  $\lambda_{m+1} = 0$  there exists a constant  $K_k \neq 0$  independently of  $n$  such that, for any  $\rho$  satisfying  $|\lambda_k| > \rho > |\lambda_{k+1}|$ ,

$$H_k^{(n)} = K_k (\lambda_1 \lambda_2 \dots \lambda_k)^n \left( 1 + O \left( \left( \frac{\rho}{|\lambda_k|} \right)^n \right) \right), \quad (3.19)$$

as  $n \rightarrow \infty$ . Moreover, it holds that, for each  $k$  such that  $|\lambda_{k-1}| > |\lambda_k| > |\lambda_{k+1}|$  where  $\lambda_0 = \infty$  and  $\lambda_{m+1} = 0$ ,

$$\lim_{n \rightarrow \infty} q_k^{(n)} = \lambda_k, \quad (3.20)$$

and, for each  $k$  such that  $|\lambda_k| > |\lambda_{k+1}|$ ,

$$\lim_{n \rightarrow \infty} e_k^{(n)} = 0. \quad (3.21)$$

As an example for the statements of Theorem 3.2.1, let us consider a 3-by-3 tridiagonal matrix  $T$  where  $\lambda_1 = \lambda_2 \neq \lambda_3$  and  $|\lambda_1| = |\lambda_2| > |\lambda_3|$ . Though it turns out from Theorem 3.2.1 that  $\lim_{n \rightarrow \infty} q_3^{(n)} = \lambda_3$  and  $\lim_{n \rightarrow \infty} e_2^{(n)} = 0$ , it is not clear whether three limits  $\lim_{n \rightarrow \infty} q_1^{(n)}$ ,  $\lim_{n \rightarrow \infty} q_2^{(n)}$  and  $\lim_{n \rightarrow \infty} e_1^{(n)}$  exist. Even if  $\lim_{n \rightarrow \infty} q_1^{(n)} = q_1^*$ ,  $\lim_{n \rightarrow \infty} q_2^{(n)} = q_2^*$  and  $\lim_{n \rightarrow \infty} e_1^{(n)} = e_1^*$ , then it is not shown that  $T^{(n)}$  converges to a bidiagonal matrix as  $n \rightarrow \infty$ . This is because

$$\lim_{n \rightarrow \infty} T^{(n)} = \lim_{n \rightarrow \infty} \begin{pmatrix} q_1^{(n)} & 1 & 0 \\ q_1^{(n)} e_1^{(n)} & q_2^{(n)} + e_1^{(n)} & 1 \\ 0 & q_2^{(n)} e_2^{(n)} & q_3^{(n)} + e_2^{(n)} \end{pmatrix} = \left( \begin{array}{cc|c} q_1^* & 1 & 0 \\ q_1^* e_1^* & q_2^* + e_1^* & 1 \\ 0 & 0 & \lambda_3 \end{array} \right). \quad (3.22)$$

Golomb [24] showed the asymptotic expansions of the Hankel determinant  $H_m^{(n)}$  associated with the formal power series  $F(z)$  whose poles are of order two or higher. However, the Hankel determinants  $H_1^{(n)}, H_2^{(n)}, \dots, H_{m-1}^{(n)}$  themselves and their asymptotic expansion have not been investigated yet.

### 3.3 Entries in the Hankel determinants

In this section, we give expressions of entries of the Hankel determinants  $H_k^{(n)}$ , appearing in the determinant expressions of the qd variables  $q_k^{(n)}$  and  $e_k^{(n)}$ , in terms of eigenvalues of the tridiagonal matrix  $T$ .

The subscript  $\ell$  of  $\lambda_\ell$  can be uniquely expressed by using two integers  $i \in \{1, 2, \dots, N\}$  and  $j \in \{0, 1, \dots, m_i - 1\}$  as

$$\ell = m_1 + m_2 + \dots + m_{i-1} + j + 1. \quad (3.23)$$

For simplicity, let  $\delta_\ell$  be the differential operator that

$$\delta_\ell := \frac{1}{j!} \left( \frac{\partial}{\partial \lambda_\ell} \right)^j. \quad (3.24)$$

Then, it follows that

$$\delta_\ell \lambda_\ell^n = \binom{n}{j} \hat{\lambda}_i^{n-j}. \quad (3.25)$$

Let us introduce  $k$ -by- $m$  matrices

$$\Lambda_k^{(n)} = \begin{pmatrix} \delta_1 \lambda_1^n & \delta_2 \lambda_2^n & \dots & \delta_m \lambda_m^n \\ \delta_1 \lambda_1^{n+1} & \delta_2 \lambda_2^{n+1} & \dots & \delta_m \lambda_m^{n+1} \\ \delta_1 \lambda_1^{n+2} & \delta_2 \lambda_2^{n+2} & \dots & \delta_m \lambda_m^{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1 \lambda_1^{n+k-1} & \delta_2 \lambda_2^{n+k-1} & \dots & \delta_m \lambda_m^{n+k-1} \end{pmatrix}, \quad (3.26)$$

$$k = 1, 2, \dots, m, \quad n = 0, 1, \dots$$

The  $m$ -by- $m$  matrix  $\Lambda_m^{(0)}$  is an extended Vandermonde matrix. The following lemma describes the expansion of  $\det(\Lambda_m^{(0)})$  in terms of  $\hat{\lambda}_k$ .

**Lemma 3.3.1** (Golomb [24]). *The extended Vandermonde determinant is expanded as*

$$\det(\Lambda_m^{(0)}) = \prod_{1 \leq i < j \leq N} (\hat{\lambda}_j - \hat{\lambda}_i)^{m_i m_j} > 0. \quad (3.27)$$

With the help of Lemma 3.3.1, we derive expressions of  $f_n$  in terms of  $\lambda_1, \lambda_2, \dots, \lambda_m$  or  $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_N$ .

**Theorem 3.3.2** (Akaiwa et al. [A4]). *For any  $T$ , let us assume that  $\{f_n\}_{n=0,1,\dots}$  satisfies (3.15) with (3.16). Then  $f_n$  are expressed as*

$$f_n = \sum_{\ell=1}^m c_\ell \delta_\ell \lambda_\ell^n, \quad n = 0, 1, \dots, \quad (3.28)$$

or

$$f_n = \sum_{i=1}^N \sum_{j=0}^{m_i-1} \hat{c}_j^{(i)} \binom{n}{j} \hat{\lambda}_i^{n-j}, \quad n = 0, 1, \dots, \quad (3.29)$$

where  $c_1, c_2, \dots, c_m$  and  $\hat{c}_0^{(1)}, \hat{c}_1^{(1)}, \dots, \hat{c}_{m_N-1}^{(N)}$  are given by

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = (\Lambda_m^{(0)})^{-1} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{m-1} \end{pmatrix}, \quad (3.30)$$

$$\hat{c}_j^{(i)} = c_\ell. \quad (3.31)$$

*Proof.* By substituting  $f_n$  in (3.28) into (3.15) and by interchanging the summations, we can rewrite the left hand side of (3.15) as

$$\sum_{\ell=1}^m c_\ell \delta_\ell \left[ \lambda_\ell^n \left( \sum_{i=0}^m a_i^{(m)} \lambda_\ell^{m-i} \right) \right]. \quad (3.32)$$

Since it is obvious from (3.12) and (3.13) that  $\sum_{i=0}^m a_i^{(m)} \lambda_\ell^{m-i} = 0$ , (3.32) becomes 0. Thus,  $f_n$  in (3.28) satisfy (3.15) with (3.16). By combining (3.25) with (3.28), we immediately have (3.29).

Equation (3.28) leads to  $\Lambda_m^{(0)}(c_1, c_2, \dots, c_m)^\top = (f_0, f_1, \dots, f_{m-1})^\top$ . Since it turns out from Lemma 3.3.1 that  $(\Lambda_m^{(0)})^{-1}$  exists,  $c_1, c_2, \dots, c_m$  are given by (3.30). From (3.25) and (3.28), we derive (3.31).  $\square$

### 3.4 Expansions of the Hankel determinants

In this section, we give asymptotic expansions of the Hankel determinants  $H_k^{(n)}$  by examining dominant terms of the Hankel determinants  $H_k^{(n)}$ .

Let us begin with recognizing the Hankel determinants  $H_k^{(n)}$  as determinants of products of matrices. Let us introduce an  $m$ -by- $m$  matrix of the block diagonal form

$$\hat{C} = \text{diag}(\hat{C}_1, \hat{C}_2, \dots, \hat{C}_N), \quad (3.33)$$

where  $\hat{C}_1, \hat{C}_2, \dots, \hat{C}_N$  are  $m_i$ -by- $m_i$  upper anti-triangular matrices whose entries are  $\hat{c}_j^{(i)}$  in Theorem 3.3.2,

$$\hat{C}_i = \begin{pmatrix} \hat{c}_0^{(i)} & \hat{c}_1^{(i)} & \cdots & \hat{c}_{m_i-1}^{(i)} \\ \hat{c}_1^{(i)} & & \ddots & \\ \vdots & & & \\ \hat{c}_{m_i-1}^{(i)} & & & 0 \end{pmatrix}, \quad i = 1, 2, \dots, N. \quad (3.34)$$

Also, for convenience in the following lemmas, let us prepare two integers  $\mu_k$  and  $\nu_k$  which satisfy  $m_1 + m_2 + \cdots + m_{\mu_k} + \nu_k + 1 = k$  for  $k = 1, 2, \dots, \ell$ . Of course, if  $k = \ell$  then  $\mu_\ell$  and  $\nu_\ell$  coincide with  $i$  and  $j$  in (3.23), respectively. Then, by using Theorem 3.3.2, we obtain expressions of the Hankel determinants  $H_k^{(n)}$  in terms of  $\Lambda_k^{(n)}$  in (3.26) and  $\hat{C}$  in (3.33).

**Lemma 3.4.1.** *Let us assume that  $\{f_n\}_{n=0,1,\dots}$  satisfies (3.29). Then,  $H_k^{(n)}$  are expressed as*

$$H_k^{(n)} = \det(\Lambda_k^{(n)} \hat{C} (\Lambda_k^{(0)})^\top), \quad k = 1, 2, \dots, m, \quad n = 0, 1, \dots \quad (3.35)$$

*Proof.* From Theorem 3.3.2, we can express the  $(s, t)$  entry of  $H_k^{(n)}$  as

$$f_{n+s+t-2} = \sum_{i=1}^N \sum_{j=0}^{m_i-1} \hat{c}_j^{(i)} \binom{n+s+t-2}{j} \hat{\lambda}_i^{n+s+t-2-j}. \quad (3.36)$$

Applying a formula for binomial coefficients

$$\binom{n+s+t-2}{j} = \sum_{r=0}^j \binom{n+s-1}{j-r} \binom{t-1}{r}$$

to (3.36), we derive

$$f_{n+s+t-2} = \sum_{i=1}^N \sum_{j=0}^{m_i-1} \sum_{r=0}^j \hat{c}_j^{(i)} \binom{n+s-1}{j-r} \hat{\lambda}_i^{n+s-1-(j-r)} \binom{t-1}{r} \hat{\lambda}_i^{t-1-r}. \quad (3.37)$$

The replacements  $j-r = j'$  and  $r = r'$  in (3.37) lead to

$$f_{n+s+t-2} = \sum_{i=1}^N \sum_{j'=0}^{m_i-1} \binom{n+s-1}{j'} \hat{\lambda}_i^{n+s-1-j'} \left( \sum_{r'=0}^{m_i-1-j'} \hat{c}_{j'+r'}^{(i)} \binom{t-1}{r'} \hat{\lambda}_i^{t-1-r'} \right).$$

Since it holds that  $i = \mu_\ell$ ,  $j' = \nu_\ell$  and  $\hat{\lambda}_i = \lambda_\ell$  in the 1st term, and  $i = \mu_{\ell+r'}$ ,  $r' = \nu_{\ell+r'}$  and  $\hat{\lambda}_i = \lambda_{\ell+r'}$  in the 2nd term, we obtain

$$f_{n+s+t-2} = \sum_{\ell=1}^m \delta_\ell \lambda_\ell^{n+s-1} \left( \sum_{r'=0}^{m_i-1-\nu_\ell} \hat{c}_{\nu_\ell+r'}^{(\nu_\ell)} \delta_{\ell+r'} \lambda_{\ell+r'}^{t-1} \right).$$

Since  $\delta_\ell \lambda_\ell^{n+s-1}$  is the  $(s, \ell)$  entry of  $\Lambda_k^{(n)}$  and  $\sum_{r'=0}^{m_i-1-\nu_\ell} \hat{c}_{\nu_\ell+r'}^{(\mu_\ell)} \delta_{\ell+r'} \lambda_{\ell+r'}^{t-1}$  is the  $(\ell, t)$  entry of  $\hat{C}(\Lambda_k^{(0)})^\top$ , we thus have (3.35).  $\square$

Let  $W_k := \{(i_1, i_2, \dots, i_k) \mid 1 \leq i_1 < i_2 < \dots < i_k \leq m\}$ . For  $a = (i_1, i_2, \dots, i_k) \in W_k$  and  $b = (j_1, j_2, \dots, j_k) \in W_k$ , let  $(X)_b^a$  denote the submatrix obtained from  $X$  by deleting rows and columns except for the  $i_1$ th,  $i_2$ th,  $\dots$ ,  $i_k$ th rows and the  $j_1$ th,  $j_2$ th,  $\dots$ ,  $j_k$ th columns. An generalization of Cauchy-Binet's formula [50] enables us to divide  $H_k^{(n)} = \det(\Lambda_k^{(n)} \hat{C}(\Lambda_k^{(0)})^\top)$  into sums of determinants.

**Lemma 3.4.2.** *Let us assume that  $\{f_n\}_{n=0,1,\dots}$  satisfies (3.29). Then  $H_k^{(n)}$  are expressed as*

$$H_k^{(n)} = \sum_{\sigma \in W_k} \det \left( (\Lambda_k^{(n)})_{\sigma}^{\alpha_k} \right) \kappa_{\sigma}, \quad k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \quad (3.38)$$

$$\kappa_{\sigma} := \sum_{\omega \in W_k} \det \left( (\hat{C})_{\omega}^{\sigma} \right) \det \left( (\Lambda_k^{(0)})_{\omega}^{\alpha_k} \right), \quad (3.39)$$

where  $\alpha_k := (1, 2, \dots, k) \in W_k$ .

From the definition of  $\Lambda_k^{(n)}$  in (3.26), it follows that

$$\det((\Lambda_k^{(n)})_{\sigma}^{\alpha_k}) = \delta_{\sigma(1)} \delta_{\sigma(2)} \cdots \delta_{\sigma(k)} (\lambda_{\sigma(1)}^n \lambda_{\sigma(2)}^n \cdots \lambda_{\sigma(k)}^n V_{\sigma}), \quad (3.40)$$

where  $\sigma(i)$  denotes the  $i$ th element of  $\sigma$  and  $V_\sigma$  is defined as

$$V_\sigma := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{\sigma(1)} & \lambda_{\sigma(2)} & \cdots & \lambda_{\sigma(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{\sigma(1)}^{k-1} & \lambda_{\sigma(2)}^{k-1} & \cdots & \lambda_{\sigma(k)}^{k-1} \end{vmatrix}. \quad (3.41)$$

Let  $\delta_\ell^j$  be the differential operator defined by

$$\delta_\ell^p := \frac{1}{p!} \left( \frac{\partial}{\partial \lambda_\ell} \right)^p, \quad p = 0, 1, \dots, \nu_\ell. \quad (3.42)$$

The differential operator  $\delta_\ell$  in (3.24) is equal to the case where  $p = \nu_\ell$  in (3.42). Applying Leibniz rule to (3.40), we obtain the following lemma for expressions of  $(\Lambda_k^{(n)})_\sigma^{\alpha_k}$ .

**Lemma 3.4.3.** *Let us assume that  $\{f_n\}_{n=0,1,\dots}$  satisfies (3.29). Then  $\det((\Lambda_k^{(n)})_\sigma^{\alpha_k})$  are expressed as*

$$\begin{aligned} & \det((\Lambda_k^{(n)})_\sigma^{\alpha_k}) \\ &= \sum_{r_1=0}^{\nu_{\sigma(1)}} \sum_{r_2=0}^{\nu_{\sigma(2)}} \cdots \sum_{r_k=0}^{\nu_{\sigma(k)}} \left( \prod_{s=1}^k \delta_{\sigma(s)}^{\nu_{\sigma(s)} - r_s} \lambda_{\sigma(s)}^n \right) \left( \delta_{\sigma(1)}^{r_1} \delta_{\sigma(2)}^{r_2} \cdots \delta_{\sigma(k)}^{r_k} V_\sigma \right), \quad (3.43) \\ & \quad k = 1, 2, \dots, m, \quad n = 0, 1, \dots \end{aligned}$$

It is here worth noting that the subscript  $k$  can be rewritten as

$$k = M_{\xi-1} + \eta,$$

where  $\xi := \mu_k + 1$ ,  $\eta := \nu_k + 1$  and

$$M_0 = 0, \quad M_{\xi-1} := m_1 + m_2 + \cdots + m_{\xi-1}. \quad (3.44)$$

Lemmas 3.4.2 and 3.4.3 yield the asymptotic expansions of  $H_k^{(n)}$  as  $n \rightarrow \infty$ .

**Lemma 3.4.4.** *The asymptotic expansions as  $n \rightarrow \infty$  of  $H_k^{(n)}$  are given as*

$$H_k^{(n)} = \kappa_{\beta_k} \left( \prod_{i=1}^{M_{\xi-1}} \lambda_i^n \right) \left( \delta_i^{m_\xi - \eta} \lambda_k^n \right)^\eta \det \left( (\Lambda_k^{(0)})_{\alpha_k}^{\alpha_k} \right) (1 + O(n^{-1})). \quad (3.45)$$

*Proof.* From Lemmas 3.4.2 and 3.4.3, we can regard the Hankel determinants  $H_k^{(n)}$  as sums of  $(\nu_{\sigma(1)} + \nu_{\sigma(2)} + \cdots + \nu_{\sigma(k)})$  terms. Each term is divided into the part involving  $n$  and the other. The part involving  $n$  in each term is expressed by using  $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(k))$  and  $r = (r_1, r_2, \dots, r_k)$  as

$$\prod_{s=1}^k \delta_{\sigma(s)}^{\nu_{\sigma(s)} - r_s} \lambda_{\sigma(s)}^n = \prod_{s=1}^k \binom{n}{\nu_{\sigma(s)} - r_s} \lambda_{\sigma(s)}^{n - \sigma(s)}. \quad (3.46)$$

The key point for the asymptotic expansions of  $H_k^{(n)}$  as  $n \rightarrow \infty$  is thus to find  $\sigma$  and  $r$  such that (3.46) becomes the most dominant as  $n \rightarrow \infty$ . By taking into account that  $\binom{n}{k}$  is the  $k$ th order polynomial with respect to  $n$ , we observe that (3.46) becomes the most dominant as  $n \rightarrow \infty$  if  $\sigma$  is given as

$$\begin{cases} \sigma(s) = s, & s = 1, 2, \dots, M_{\xi-1}, \\ \sigma(s) \in \{M_{\xi-1} + 1, M_{\xi-1} + 2, \dots, M_{\xi} - 1, M_{\xi}\}, & s = M_{\xi-1} + 1, M_{\xi-1} + 2, \dots, k, \end{cases} \quad (3.47)$$

and  $r$  enables us to maximize  $\nu_{\sigma(s)} - r_s$  for every  $s = 1, 2, \dots, k$ . From the definition of  $V_{\sigma}$  in (3.41), we see that the minimum of  $r$  satisfying  $\delta_{\sigma(1)}^{r_1} \delta_{\sigma(2)}^{r_2} \cdots \delta_{\sigma(k)}^{r_k} V_{\sigma} \neq 0$  are given as, for  $i = 1, 2, \dots, k$ ,

$$r_i = \begin{cases} r_{i-1} + 1, & \lambda_{\sigma(i)} = \lambda_{\sigma(i-1)}, \\ 0, & \lambda_{\sigma(i)} \neq \lambda_{\sigma(i-1)}, \end{cases}$$

where  $\lambda_{\sigma(0)} \equiv 0$ . The choice  $\sigma = \beta_k$  also maximizes  $\nu_{\sigma(s)}$  with keeping (3.47), where

$$\beta_k := (\overbrace{1, 2, \dots, M_{\xi-1}}^{M_{\xi-1}}, \overbrace{M_{\xi} - \eta + 1, \dots, M_{\xi} - 1, M_{\xi}}^{\eta}).$$

Thus,  $\nu_{\sigma(s)} - r_s$  becomes maximal if  $r$  is given by

$$\begin{cases} r_i = \nu_i, & i = 1, 2, \dots, M_{\xi-1}, \\ r_{M_{\xi-1}+j+1} = j, & j = 0, 1, \dots, \eta - 1. \end{cases}$$

Putting the above discussions together, we can express the dominant term of  $H_k^{(n)}$  as  $n \rightarrow \infty$  as

$$\left( \prod_{s=1}^{M_\xi-1} \delta_s^0 \lambda_s^n \right) \left( \prod_{s=0}^{\eta-1} \delta_{\gamma+s}^{\nu_{\gamma+s}-s} \lambda_{\gamma+s}^n \right) \delta_1^{\nu_1} \delta_2^{\nu_2} \cdots \delta_{M_\xi-1}^{\nu_{M_\xi-1}} \delta_\gamma^0 \delta_{\gamma+1}^1 \cdots \delta_{\gamma+\eta-1}^{\eta-1} V_{\beta_k},$$

where  $\gamma := M_\xi - \eta + 1$ . Since it holds that  $\nu_{\gamma+s} = (m_\xi - 1) - (\eta - 1) + s$  and  $\delta_{\gamma+s}^{m_\xi-\eta} \lambda_{\gamma+s} = \delta_k^{m_\xi-\eta} \lambda_k$ , it follows that

$$\prod_{s=0}^{\eta-1} \delta_{\gamma+s}^{\nu_{\gamma+s}-s} \lambda_{\gamma+s}^n = \prod_{s=0}^{\eta-1} \delta_k^{m_\xi-\eta} \lambda_k^n = (\delta_k^{m_\xi-\eta} \lambda_k^n)^\eta.$$

From  $\delta_s^{\nu_s} = \delta_s$  for  $s = 1, 2, \dots, M_\xi$  and  $\delta_{\gamma+s}^s = \delta_{M_\xi-1+s+1}$  for  $s = 0, 1, \dots, \eta-1$ , we also derive

$$\delta_1^{\nu_1} \delta_2^{\nu_2} \cdots \delta_{M_\xi-1}^{\nu_{M_\xi-1}} \delta_\gamma^0 \delta_{\gamma+1}^1 \cdots \delta_{\gamma+\eta-1}^{\eta-1} = \delta_0 \delta_1 \cdots \delta_k V_{\alpha_k} = \det \left( (\Lambda_k^{(0)})_{\alpha_k}^{\alpha_k} \right).$$

Of course, the  $(\nu_{\sigma(1)} + \nu_{\sigma(2)} + \cdots + \nu_{\sigma(k)})$  terms in the sums in Lemma 3.4.3 contain some nondominant terms. Therefore, by taking into account that the degrees with respect to  $n$  of them have smaller than the dominant terms, we have (3.45).  $\square$

Lemma 3.3.1 immediately gives the expansions of the determinants of  $(\Lambda_k^{(0)})_{\alpha_k}^{\alpha_k}$ .

**Lemma 3.4.5.** *The determinants  $\det(\Lambda_k^{(0)})_{\alpha_k}^{\alpha_k}$  are expanded as*

$$\det \left( (\Lambda_k^{(0)})_{\alpha_k}^{\alpha_k} \right) = \left[ \prod_{1 \leq i < j \leq \xi-1} (\hat{\lambda}_j - \hat{\lambda}_i)^{m_i m_j} \right] \prod_{1 \leq r \leq \xi-1} (\hat{\lambda}_\xi - \hat{\lambda}_r)^{m_r \eta}. \quad (3.48)$$

From (3.33) and (3.34), we also derive the expansions of determinants of  $(\hat{C})_{\alpha_k}^{\beta_k}$ .

**Lemma 3.4.6.** *The determinants  $\det((\hat{C})_{\alpha_k}^{\beta_k})$  are expressed as*

$$\det \left( (\hat{C})_{\alpha_k}^{\beta_k} \right) = (-1)^{k-\xi} \left( \prod_{i=1}^{\xi-1} (\hat{c}_{m_i-1}^{(i)})^{m_i} \right) (\hat{c}_{m_\xi-1}^{(\xi)})^\eta. \quad (3.49)$$

*Proof.* From (3.34), it immediately follows that

$$\det(\hat{C}_i) = (-1)^{m_i-1}(\hat{c}_{m_i-1}^{(i)})^{m_i}, \quad i = 1, 2, \dots, N. \quad (3.50)$$

By combining (3.50) with (3.33) and by taking into account that  $k = m_1 + m_2 + \dots + m_{\xi-1} + \eta$ , we obtain

$$\begin{aligned} \det((\hat{C})_{\alpha_k}^{\beta_k}) &= \left( \prod_{i=1}^{t-1} \det(\hat{C}_i) \right) \begin{vmatrix} \hat{c}_{m_\xi-\eta}^{(\xi)} & \hat{c}_{m_\xi-\eta+1}^{(\xi)} & \cdots & \hat{c}_{m_\xi-1}^{(\xi)} \\ \hat{c}_{m_\xi-\eta+1}^{(\xi)} & & \ddots & \\ \vdots & & \ddots & \\ \hat{c}_{m_\xi-1}^{(\xi)} & & & 0 \end{vmatrix} \\ &= (-1)^{M_{\xi-1}-(\xi-1)+\eta-1} \left( \prod_{i=1}^{\xi-1} (\hat{c}_{m_i-1}^{(i)})^{m_i} \right) (\hat{c}_{m_\xi-1}^{(\xi)})^\eta. \quad \square \end{aligned}$$

Consequently, by using Lemmas 3.4.2–3.4.6, we complete the asymptotic expansions of the Hankel determinants  $H_k^{(n)}$  in terms of the eigenvalues  $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_N$ .

**Theorem 3.4.7** (Akaiwa et al. [A4]). *Let us assume that  $H_k^{(n)} \neq 0$  for  $k = 1, 2, \dots, m$  and  $n = 0, 1, \dots$ . As  $n \rightarrow \infty$ , the Hankel determinants  $H_k^{(n)}$  are expanded as*

$$\begin{aligned} H_k^{(n)} &= \hat{K}_k n^{\eta(m_\xi-\eta)} (\hat{\lambda}_1 \hat{\lambda}_2 \cdots \hat{\lambda}_{\xi-1})^n \hat{\lambda}_\xi^{\eta(n-m_\xi+\eta)} (1 + O(n^{-1})), \quad (3.51) \\ k &= M_{\xi-1} + 1, M_{\xi-1} + 2, \dots, M_{\xi-1} + m_\xi - 1. \end{aligned}$$

If  $k = M_\xi$ , then, for  $\hat{\rho}$  satisfying  $|\hat{\lambda}_{\xi+1}| < \hat{\rho} < |\hat{\lambda}_\xi|$ , it holds that, as  $n \rightarrow \infty$ ,

$$H_k^{(n)} = \hat{K}_k (\hat{\lambda}_1^{m_1} \hat{\lambda}_2^{m_2} \cdots \hat{\lambda}_\xi^{m_\xi})^n \left( 1 + O \left( \left( \frac{\hat{\rho}}{|\hat{\lambda}_\xi|} \right)^n \right) \right), \quad (3.52)$$

where  $\hat{K}_k$  is a nonzero constant independently of  $n$  given as

$$\begin{aligned} \hat{K}_k &:= \frac{1}{((m_\xi - \eta)!)^\eta} \times (-1)^{k-\xi} \left( \prod_{i=1}^{\xi-1} (\hat{c}_{m_i-1}^{(i)})^{m_i} \right) (\hat{c}_{m_\xi-1}^{(\xi)})^\eta \\ &\quad \times \prod_{1 \leq i < j \leq \xi-1} (\hat{\lambda}_j - \hat{\lambda}_i)^{4m_i m_j} \prod_{i=1}^{\xi-1} (\hat{\lambda}_\xi - \hat{\lambda}_i)^{4m_i \eta}. \quad (3.53) \end{aligned}$$

*Proof.* Since it holds that  $\hat{\lambda}_\xi = \lambda_i$  for  $i = M_\xi - \eta + 1, M_\xi - \eta + 2, \dots, M_\xi$ , the terms with respect to  $\delta_i^{m_\xi - \eta}$  in (3.45) can be simplified as

$$\left(\delta_i^{m_\xi - \eta} \lambda_k^n\right)^\eta = \left(\binom{n}{m_\xi - \eta} \lambda_k^{n - m_\xi + \eta}\right)^\eta = \frac{n^{\eta(m_\xi - \eta)} \hat{\lambda}_\xi^{\eta(n - m_\xi + \eta)}}{((m_\xi - \eta)!)^p} (1 + O(n^{-1})).$$

By combining it with Lemmas 3.4.2–3.4.4, we obtain (3.51) and (3.52). From Lemma 3.4.2, it is obvious that  $\kappa_{\beta_k} = \det((\hat{C})_{\alpha_k}^{\beta_k}) \det((\Lambda_k^{(0)})_{\alpha_k}^{\alpha_k})$ . Thus, from Lemma 3.4.5 and 3.4.6, we see that  $\hat{K}_k = \kappa_{\beta_k} / ((m_\xi - \eta)!)^p$  becomes (3.53). Moreover, it follows that  $\hat{K}_k \neq 0$  since  $H_m^{(n)} \neq 0$  leads to  $\hat{c}_{m_i - 1}^{(i)} \neq 0$ .  $\square$

### 3.5 Convergence of the qd variables

In this section, by combining expressions of the qd variables  $q_k^{(n)}$  and  $e_k^{(n)}$  using the Hankel determinants  $H_k^{(n)}$  with expansions of the Hankel determinants  $H_k^{(n)}$ , we clarify the asymptotic behavior of the qd algorithm as  $n \rightarrow \infty$  in the case where the tridiagonal matrix  $T$  has multiple eigenvalues.

Theorems 3.2.1 and 3.4.7 give the most important theorem in this chapter, which shows the convergence of the qd variables  $q_k^{(n)}$  and  $e_k^{(n)}$  as  $n \rightarrow \infty$ .

**Theorem 3.5.1** (Akaiwa et al. [A4]). *Let us assume that  $H_k^{(n)} \neq 0$  for  $k = 1, 2, \dots, m$  and  $n = 0, 1, \dots$ . Moreover, let  $|\hat{\lambda}_1| > |\hat{\lambda}_2| > \dots > |\hat{\lambda}_N|$ . Then, for  $k = M_{\xi-1} + \eta$ , it holds that*

$$\lim_{n \rightarrow \infty} q_k^{(n)} = \hat{\lambda}_\xi, \quad (3.54)$$

$$\lim_{n \rightarrow \infty} e_k^{(n)} = 0. \quad (3.55)$$

*Proof.* Let  $k = M_{\xi-1} + 1$  in Theorem 3.4.7. Then, the convergence as  $n \rightarrow \infty$  of  $q_k^{(n)} = H_k^{(n+1)} H_{k-1}^{(n)} / (H_k^{(n)} H_{k-1}^{(n+1)})$ , is given as

$$\begin{aligned} \lim_{n \rightarrow \infty} q_k^{(n)} &= \lim_{n \rightarrow \infty} \frac{\left( \kappa_k n^{(m_\xi - 1)} \prod_{i=1}^{\xi-1} \hat{\lambda}_i^{m_i(n+1)} \hat{\lambda}_\xi^{n+1-(m_\xi-1)} \right) \left( \kappa_{k-1} \prod_{i=1}^{\xi-1} \hat{\lambda}_i^{m_i n} \right)}{\left( \kappa_k n^{(m_\xi - 1)} \prod_{i=1}^{\xi-1} \hat{\lambda}_i^{m_i n} \hat{\lambda}_\xi^{n-(m_\xi-1)} \right) \left( \kappa_{k-1} \prod_{i=1}^{\xi-1} \hat{\lambda}_i^{m_i(n+1)} \right)} \\ &= \lim_{n \rightarrow \infty} \hat{\lambda}_\xi^{n+1-(m_\xi-1)-n+(m_\xi-1)} = \hat{\lambda}_\xi. \end{aligned}$$

We also obtain  $\lim_{n \rightarrow \infty} q_k^{(n)} = \hat{\lambda}_\xi$  for  $k = M_{\xi-1} + 2, M_{\xi-1} + 3, \dots, M_\xi$ .

In a way similar to the case of  $q_k^{(n)}$ , we derive

$$\lim_{n \rightarrow \infty} e_k^{(n)} = \lim_{n \rightarrow \infty} \tilde{K}_k n^{-2} \hat{\lambda}_1 = 0, \quad k = 1, 2, \dots, m_1 - 1,$$

where  $\tilde{K}_k := \hat{K}_{k-1} \hat{K}_{k+1} / (\hat{K}_k)^2$ . For  $k = m_1$ , it follows that

$$\lim_{n \rightarrow \infty} e_k^{(n)} = \lim_{n \rightarrow \infty} \tilde{K}_k n^{m_1+m_2-1} \hat{\lambda}_1^{-2m_1} \hat{\lambda}_2^{-(m_2-1)} \left( \frac{\hat{\lambda}_2}{\hat{\lambda}_1} \right)^n = 0.$$

The cases where  $k = M_{\xi-1} + 1, M_{\xi-1} + 2, \dots, M_{\xi-1} + m_\xi - 1$  and  $k = M_2, M_3, \dots, M_{N-1}$  lead to, respectively,

$$\begin{aligned} \lim_{n \rightarrow \infty} e_k^{(n)} &= \lim_{n \rightarrow \infty} \tilde{K}_k n^{-2} \hat{\lambda}_\xi^{b_1} = 0, \\ \lim_{n \rightarrow \infty} e_k^{(n)} &= \lim_{n \rightarrow \infty} \tilde{K}_k n^{b_2} \hat{\lambda}_\xi^{b_3} \hat{\lambda}_{\xi+1}^{b_4} \left( \frac{\hat{\lambda}_{\xi+1}}{\hat{\lambda}_\xi} \right)^n = 0, \end{aligned}$$

where  $b_1, b_2, b_3$  and  $b_4$  are some constants independent of  $n$ . □

It is therefore concluded that the qd variables  $q_k^{(n)}$  and  $e_k^{(n)}$  converge to eigenvalues of the tridiagonal matrix and 0 as  $n \rightarrow \infty$ , respectively, in not only the case of real distinct eigenvalues but also the case of multiple eigenvalues.

## Chapter 4

# Construction of tridiagonal matrices with multiple eigenvalues based on the qd formula

Inverse eigenvalue problems (IEPs) for tridiagonal matrices have been discussed in several papers. However, there do not exist many papers on IEPs for tridiagonal matrices with multiple eigenvalues.

In this chapter, as a beachhead for solving IEPs by using discrete integrable systems and the associated integrable algorithms, a procedure for constructing tridiagonal matrices with multiple eigenvalues is proposed based on the qd recursion formula.

### 4.1 Inverse eigenvalue problems for tridiagonal matrices

One of the important problems in linear algebra is to construct matrices with prescribed eigenvalues. This is an inverse eigenvalue problem which is classified in Structured Inverse Eigenvalue Problem (SIEP) in [8]. The main purpose of this chapter is to design a procedure for solving an SIEP in the case where the constructed matrix has tridiagonal form with multiple eigenvalues, through reconsidering the quotient-difference (qd) formula. It is known that the qd formula has the applications to computing a continued

fraction expansion of power series [31], zeros of polynomial [29], eigenvalues of a tridiagonal matrix called a Jacobi matrix [54] and so on. Though Rutishauser's book [54] refers to an aspect similar to that in the following sections, it gives only an anticipated comment without proof in the case of multiple eigenvalues. There is no observation about numerical examples for verifying it. The key point for the purpose is to investigate the Hankel determinants appearing in the determinant solution to the qd formula with the help of the Jordan canonical form. In this chapter, we focus on the unsettled case in order to design a procedure for constructing a tridiagonal matrix with prescribed multiple eigenvalues, based on the qd formula. The reason why the case with multiple eigenvalues has not been sufficiently discussed is expected that multiple-precision arithmetic and symbolic computing around the published year of Rutishauser's works for the qd formula have not been sufficiently developed. The qd formula, strictly speaking the differential form of it, for computing tridiagonal eigenvalues acts with high relative accuracy in single-precision or double-precision arithmetic [47], while, actually, the qd formula for constructing a tridiagonal matrix gives rise to not small errors. Thus, the qd formula serving for constructing a tridiagonal matrix is not so worth in single-precision or double-precision arithmetic. In recent computers, it is not difficult to employ not only single or double precision arithmetic but also arbitrary-precision arithmetic or symbolic computing. In fact, symbolic computation can be performed on scientific computing software such as Wolfram Mathematica and Maple. Numerical errors frequently occur in finite-precision arithmetic, so that constructed tridiagonal matrices may not have prescribed multiple eigenvalues. The procedure developed in this chapter is assumed to be carried out with symbolic computation.

This chapter is organized as follows. In Section 4.2, we first give a short explanation of some already obtained properties concerning the qd formula. In Section 4.3, we observe a tridiagonal matrix whose characteristic polynomial is associated with the minimal polynomial of a general matrix through reconsidering the qd formula. The tridiagonal matrix essentially differs from the Jacobi matrix in that it is not always symmetrized. We also discuss the characteristic and the minimal polynomials of a tridiagonal matrix in Section 4.4. In Section 4.5, we design a procedure for constructing a tridiagonal matrix with prescribed multiple eigenvalues, and then demonstrate four tridiagonal matrices as examples of the procedure.

## 4.2 Some properties for the qd recursion formula

In this section, we briefly review two theorems in [30] concerning the qd formula from the viewpoint of a generating function, the Hankel determinant and tridiagonal matrices.

Let us introduce Hankel determinants  $H_1^{(n)}, H_2^{(n)}, \dots$  given in terms of a complex sequence  $\{f_n\}_0^\infty$  given by

$$H_s^{(n)} := \begin{vmatrix} f_n & f_{n+1} & \cdots & f_{n+s-1} \\ f_{n+1} & f_{n+2} & \cdots & f_{n+s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+s-1} & f_{n+s} & \cdots & f_{n+2s-2} \end{vmatrix},$$

$$s = 1, 2, \dots, \quad n = 0, 1, \dots, \quad (4.1)$$

where  $H_{-1}^{(n)} \equiv 0$  and  $H_0^{(n)} \equiv 1$  for  $n = 0, 1, \dots$ . Moreover, let  $F(z)$  be a generating function associated with  $\{f_n\}_0^\infty$  as

$$F(z) = \sum_{n=0}^{\infty} f_n z^n = f_0 + f_1 z + f_2 z^2 + \cdots. \quad (4.2)$$

Let us assume that  $F(z)$  is a rational function with respect to  $z$  with a pole of order  $l_0 \geq 0$  at infinity and finite poles  $z_k \neq 0$  of order  $l_k \leq 1$  for  $k = 1, 2, \dots, L$ . The sum of the orders of the finite poles is denoted by  $l = l_1 + l_2 + \cdots + l_L$ , and  $F(z)$  is assumed to satisfy

$$F(z) = G_0(z) + \frac{G(z)}{(z - z_1)^{l_1} (z - z_2)^{l_2} \cdots (z - z_L)^{l_L}}, \quad (4.3)$$

where  $G(z)$  is a polynomial of degree at most  $l$ , and  $G_0(z)$  is a polynomial of degree  $l_0$  if  $l_0 > 0$ , or  $G_0(z) = 0$  if  $l_0 = 0$ . The following theorem gives the determinant solution to the qd recursion formula

$$\begin{cases} q_s^{(n+1)} + e_{s-1}^{(n+1)} = q_s^{(n)} + e_s^{(n)}, \\ q_s^{(n+1)} e_s^{(n+1)} = q_{s+1}^{(n)} e_s^{(n)}, \end{cases} \quad s = 1, 2, \dots, \quad n = 0, 1, \dots. \quad (4.4)$$

**Theorem 4.2.1** (Henrici [30, pp. 596, 603, 610]). *Let  $F(z)$  be expressed as in (4.3). Then it holds that*

$$H_s^{(n)} = 0, \quad s = l+1, l+2, \dots, \quad n = l_0+1, l_0+2, \dots \quad (4.5)$$

*Let us assume that*

$$H_s^{(n)} \neq 0, \quad s = 1, 2, \dots, l, \quad n = 0, 1, \dots \quad (4.6)$$

*Then the qd formula (4.4) with the initial condition*

$$e_0^{(n)} = 0, \quad q_1^{(n)} = \frac{f_{n+1}}{f_n}, \quad n = 0, 1, \dots, \quad (4.7)$$

*admits the determinant solution*

$$q_s^{(n)} = \frac{H_s^{(n+1)} H_{s-1}^{(n)}}{H_s^{(n)} H_{s-1}^{(n+1)}}, \quad s = 1, 2, \dots, l, \quad n = 0, 1, \dots, \quad (4.8)$$

$$e_s^{(n)} = \frac{H_{s+1}^{(n)} H_{s-1}^{(n+1)}}{H_s^{(n)} H_s^{(n+1)}}, \quad s = 0, 1, \dots, l, \quad n = 0, 1, \dots \quad (4.9)$$

From (4.9) with (4.5), it follows that  $e_l^{(n)} = 0$  for  $n = 0, 1, \dots$ . Moreover, it turns out that  $q_s^{(n)}$  and  $e_s^{(n)}$  for  $s = l+1, l+2, \dots$  and  $n = 0, 1, \dots$  are not given in the same form as (4.8) and (4.9).

Let us introduce  $s$ -by- $s$  tridiagonal matrices,

$$T_s^{(n)} = \begin{pmatrix} q_1^{(n)} & q_1^{(n)} e_1^{(n)} & & \\ 1 & q_2^{(n)} + e_1^{(n)} & \ddots & \\ & \ddots & \ddots & q_{s-1}^{(n)} e_{s-1}^{(n)} \\ & & 1 & q_s^{(n)} + e_{s-1}^{(n)} \end{pmatrix}, \quad s = 1, 2, \dots, l, \quad n = 0, 1, \dots, \quad (4.10)$$

of the qd variables  $q_s^{(n)}$  and  $e_s^{(n)}$ . Let  $I_s$  be the  $s$ -by- $s$  identity matrix. Then we obtain a theorem for the characteristic polynomial of  $T_l^{(n)}$ .

**Theorem 4.2.2.** ([30, pp. 626, 635]) *Let  $F(z)$  be expressed as in (4.3). Let us assume that  $H_s^{(n)}$  satisfies (4.6). For  $n = 0, 1, \dots$ , it holds that*

$$\det(zI_l - T_l^{(n)}) = (z - z_1^{-1})^{l_1} (z - z_2^{-1})^{l_2} \dots (z - z_L^{-1})^{l_L}. \quad (4.11)$$

### 4.3 Tridiagonal matrix associated with general matrix

In this section, from the viewpoint of the characteristic and the minimal polynomials, we associate a general  $M$ -by- $M$  complex matrix  $A$  with the tridiagonal matrix  $T_l^{(n)}$ .

Let  $\lambda_1, \lambda_2, \dots, \lambda_N$  be the distinct eigenvalues of  $A$ , which are indexed as  $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_N|$ . It is noted that some of  $|\lambda_1|, |\lambda_2|, \dots, |\lambda_N|$  may have the same values in the case where some of  $\lambda_1, \lambda_2, \dots, \lambda_N$  are negative eigenvalues or complex eigenvalues. Let  $M_k$  be the algebraic multiplicity of  $\lambda_k$ , where  $M = M_1 + M_2 + \dots + M_N$ . For the identity matrix  $I_M \in \mathbb{R}^{M \times M}$ , let  $\phi_A(z) = \det(zI_M - A)$  be the characteristic polynomial of  $A$ , namely,

$$\phi_A(z) = (z - \lambda_1)^{M_1} (z - \lambda_2)^{M_2} \dots (z - \lambda_N)^{M_N}. \quad (4.12)$$

Let us prepare the sequence  $\{f_n\}_0^\infty$  given by

$$f_n = \mathbf{w}^H A^n \mathbf{u}, \quad n = 0, 1, \dots, \quad (4.13)$$

for some nonzero  $M$ -dimensional complex vectors  $\mathbf{u}$  and  $\mathbf{w}$ , where the superscript  $H$  denotes the Hermitian transpose. Originally,  $f_0, f_1, \dots$  have been called the Schwarz constants, but today they are usually called the moments or the Markov parameters [10]. Since the matrix power series  $\sum_{n=0}^\infty (zA)^n$  is a Neumann series (cf. [44]),  $F(z) = \sum_{n=0}^\infty \mathbf{w}^H (zA)^n \mathbf{u}$  converges absolutely in the disk  $D : |z| < |\lambda_1|^{-1}$ . Moreover, we derive  $F(z) = \mathbf{w}^H (I_M - zA)^{-1} \mathbf{u}$  which implies that  $F(z)$  is a rational function with the denominator  $\det(I_M - zA) = z^M \phi_A(z^{-1})$  as follows,

$$F(z) = \frac{\tilde{G}(z)}{(1 - \lambda_1 z)^{M_1} (1 - \lambda_2 z)^{M_2} \dots (1 - \lambda_N z)^{M_N}}, \quad (4.14)$$

where  $\tilde{G}(z)$  is some polynomial with respect to  $z$ . It is remarkable that the numerator  $\tilde{G}(z)$  may have the same factors as the denominator  $(1 - \lambda_1 z)^{M_1} (1 - \lambda_2 z)^{M_2} \dots (1 - \lambda_N z)^{M_N}$ . In other words,  $F(z)$  has the poles  $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_N^{-1}$  whose orders are less than or equal to  $M_1, M_2, \dots, M_N$ , respectively.

Let us introduce the Jordan canonical form of  $A$  in order to investigate the poles of  $F(z)$  with (4.13) even in the case where  $A$  has multiple eigenvalues. Let  $\mathcal{M}_k$  be the geometric multiplicity of  $\lambda_k$  which indicates the dimension of eigenspace  $\text{Ker}(A - \lambda_k I_M)$ . It is noted that  $\mathcal{M}_k$  is less than or equal to the

algebraic multiplicity  $M_k$ . The matrix  $A$  has  $\mathcal{M}_k$  eigenvectors corresponding to  $\lambda_k$ , and then the eigenvectors, denoted by  $\mathbf{v}_{k,1}, \mathbf{v}_{k,2}, \dots, \mathbf{v}_{k,\mathcal{M}_k}$ , satisfy

$$A\mathbf{v}_{k,j} = \lambda_k \mathbf{v}_{k,j}, \quad j = 1, 2, \dots, \mathcal{M}_k. \quad (4.15)$$

Hereinafter, for  $j = 1, 2, \dots, \mathcal{M}_k$ , let  $\mathbf{v}_{k,j}(1) = \mathbf{v}_{k,j}$ . Moreover, for  $j = 1, 2, \dots, \mathcal{M}_k$ , let  $\mathbf{v}_{k,j}(2), \mathbf{v}_{k,j}(3), \dots, \mathbf{v}_{k,j}(m_{k,j})$  be generalized eigenvectors associated with the eigenvectors  $\mathbf{v}_{k,j}(1)$ , where  $m_{k,j}$  is the maximal integer such that  $\mathbf{v}_{k,j}(1), \mathbf{v}_{k,j}(2), \dots, \mathbf{v}_{k,j}(m_{k,j})$  are linearly independent. Of course,  $m_{k,1} + m_{k,2} + \dots + m_{k,\mathcal{M}_k} = M_k$ . The generalized eigenvectors  $\mathbf{v}_{k,j}(2), \mathbf{v}_{k,j}(3), \dots, \mathbf{v}_{k,j}(m_{k,j})$  are indexed so that

$$\begin{aligned} A\mathbf{v}_{k,j}(i) &= \lambda_k \mathbf{v}_{k,j}(i) + \mathbf{v}_{k,j}(i-1), \\ i &= 2, 3, \dots, m_{k,j}, \quad j = 1, 2, \dots, \mathcal{M}_k. \end{aligned} \quad (4.16)$$

From (4.15) and (4.16), we derive the Jordan canonical form of  $A$  as

$$V^{-1}AV = J \quad (4.17)$$

with the nonsingular matrix

$$V = (V_1 \ V_2 \ \dots \ V_N) \in \mathbb{C}^{M \times M}, \quad (4.18)$$

and the block diagonal matrix

$$J = \text{diag}(J_1, J_2, \dots, J_N) \in \mathbb{C}^{M \times M}, \quad (4.19)$$

where

$$V_k = (V_{k,1} \ V_{k,2} \ \dots \ V_{k,\mathcal{M}_k}) \in \mathbb{C}^{M \times M_k}, \quad (4.20)$$

$$V_{k,j} = (\mathbf{v}_{k,j}(1) \ \mathbf{v}_{k,j}(2) \ \dots \ \mathbf{v}_{k,j}(m_{k,j})) \in \mathbb{C}^{M \times m_{k,j}}, \quad (4.21)$$

$$J_k = \text{diag}(J_{k,1}, J_{k,2}, \dots, J_{k,\mathcal{M}_k}) \in \mathbb{C}^{M_k \times M_k}, \quad (4.22)$$

$$J_{k,j} = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \in \mathbb{C}^{m_{k,j} \times m_{k,j}}. \quad (4.23)$$

Without loss of generality, we may assume that  $m_{k,1} \geq m_{k,2} \geq \dots \geq m_{k,\mathcal{M}_k}$ .

Let  $m_k = \max\{m_{k,1}, m_{k,2}, \dots, m_{k,\mathcal{M}_k}\}$ . Since  $m_{k,1} \geq m_{k,2} \geq \dots \geq m_{k,\mathcal{M}_k}$ , it is obvious that  $m_k = m_{k,1}$ . With the help of the Jordan canonical form of  $A$  as in (4.17), we get a proposition for the sequence  $\{f_n\}_0^\infty$  in (4.13).

**Proposition 4.3.1** (Akaiwa et al. [A2]). *Let  $\mathbf{u}$  be the vector given by a linear combination of the eigenvectors and the generalized eigenvectors of  $A$ , namely, for some constants  $\kappa_{k,j,i}$ ,*

$$\mathbf{u} = \sum_{k=1}^N \sum_{j=1}^{\mathcal{M}_k} \sum_{i=1}^{m_{k,j}} \kappa_{k,j,i} \mathbf{v}_{k,j}(i). \quad (4.24)$$

Moreover, for a vector  $\mathbf{w}$ , let

$$c_{k,i} = \sum_{j=1}^{\mathcal{M}_k} \sum_{i'=i}^{m_{k,j}} \kappa_{k,j,i'} \mathbf{v}_{k,j}^H(i' - i + 1) \mathbf{w}. \quad (4.25)$$

Then, the sequence  $\{f_n\}_0^\infty$  in (4.13) can be expressed by

$$f_n = \sum_{k=1}^N \sum_{i=1}^{m_k} \binom{n}{i-1} c_{k,i} \lambda_k^{n-i+1}, \quad (4.26)$$

where the binomial coefficients are 0 if  $n < i - 1$ . Also, for some  $\mathbf{u}$  and  $\mathbf{w}$ , it holds that

$$c_{k,i} \neq 0, \quad i = 1, 2, \dots, m_k, \quad k = 1, 2, \dots, N. \quad (4.27)$$

*Proof.* From  $V^{-1}AV = J$  in (4.17), it holds that  $A^n = VJ^nV^{-1}$ . By combining it with (4.13) and (4.24), we derive

$$\begin{aligned} f_n &= \mathbf{w}^H V J^n V^{-1} \mathbf{u} \\ &= \sum_{k=1}^N \sum_{j=1}^{\mathcal{M}_k} \sum_{i=1}^{m_{k,j}} \kappa_{k,j,i} \mathbf{w}^H V J^n V^{-1} \mathbf{v}_{k,j}(i). \end{aligned} \quad (4.28)$$

Let  $\rho_{k,j,i}$  be the column indexed in which  $\mathbf{v}_{k,j}(i)$  in  $V$ . Then it is obvious that  $V^{-1} \mathbf{v}_{k,j}(i) = \mathbf{e}_{k,j}(i)$  where  $\mathbf{e}_{k,j}(i)$  denotes a unit vector such that the  $\rho_{k,j,i}$ th entry is 1 and the others are 0. Thus, it follows that

$$f_n = \sum_{k=1}^N \sum_{j=1}^{\mathcal{M}_k} \sum_{i=1}^{m_{k,j}} \kappa_{k,j,i} \mathbf{w}^H V J^n \mathbf{e}_{k,j}(i). \quad (4.29)$$

Since  $J$  is the block diagonal matrix, the matrix  $J^n$  and its small blocks  $(J_k)^n$  are also so. It also turns out that  $(J_{k,j})^n$  is upper triangle. So, it is

worth noting that  $J^n \mathbf{e}_{k,j}(i)$  becomes the  $\rho_{k,j,i}$ th column vector of  $J^n$  and its zero-entries are arranged in all rows except for its  $\rho_{k,j,1}$ th,  $\rho_{k,j,2}$ th,  $\dots$ ,  $\rho_{k,j,i}$ th rows. The Jordan blocks  $J_{k,j}$  can be decomposed as

$$J_{k,j} = \lambda_k I_{m_{k,j}} + E_{m_{k,j}}, \quad (4.30)$$

$$E_{m_{k,j}} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{R}^{m_{k,j} \times m_{k,j}}. \quad (4.31)$$

It is emphasized that  $E_{m_{k,j}}$  is a nilpotent matrix whose  $i'$ th power becomes the zero-matrix  $O$  for  $i' \geq m_{k,j}$ . Thus,  $(J_{k,j})^n$  can be expressed as

$$(J_{k,j})^n = \sum_{i'=1}^{m_{k,j}} \binom{n}{i'-1} \lambda_k^{n-i'+1} E_{m_{k,j}}^{i'-1}, \quad (4.32)$$

where  $(E_{m_{k,j}})^0 = I_{m_{k,j}}$ . Let us introduce an  $m_{k,j}$ -dimensional unit vector  $\mathbf{e}(i)$  which is regarded as a part of  $\mathbf{e}_{k,j}(i)$ . Then, by taking account that  $E_{m_{k,j}}^{i'-1} \mathbf{e}(i) = \mathbf{e}(i - i' + 1)$  in (4.32), we derive

$$J^n \mathbf{e}_{k,j}(i) = \sum_{i'=1}^i \binom{n}{i'-1} \lambda_k^{n-i'+1} \mathbf{e}_{k,j}(i - i' + 1). \quad (4.33)$$

Since it holds that  $V \mathbf{e}_{k,j}(i - i' + 1) = \mathbf{v}_{k,j}(i - i' + 1)$ , by combining it with (4.29) and (4.33), we therefore have

$$\begin{aligned} f_n &= \sum_{k=1}^N \sum_{j=1}^{\mathcal{M}_k} \sum_{i=1}^{m_{k,j}} \sum_{i'=1}^i \kappa_{k,j,i} \mathbf{w}^H \mathbf{v}_{k,j}(i - i' + 1) \\ &\quad \times \binom{n}{i'-1} \lambda_k^{n-i'+1}. \end{aligned} \quad (4.34)$$

By writing down two summations, we get

$$\begin{aligned}
f_n = & \sum_{k=1}^N \sum_{j=1}^{\mathcal{M}_k} \left[ \kappa_{k,j,1} \mathbf{w}^H \mathbf{v}_{k,j}(1) \binom{n}{0} \lambda_k^n \right. \\
& + \kappa_{k,j,2} \left( \mathbf{w}^H \mathbf{v}_{k,j}(2) \binom{n}{0} \lambda_k^n + \mathbf{w}^H \mathbf{v}_{k,j}(1) \binom{n}{1} \lambda_k^{n-1} \right) \\
& + \kappa_{k,j,3} \left( \mathbf{w}^H \mathbf{v}_{k,j}(3) \binom{n}{0} \lambda_k^n + \mathbf{w}^H \mathbf{v}_{k,j}(2) \binom{n}{1} \lambda_k^{n-1} \right. \\
& \quad \left. + \mathbf{w}^H \mathbf{v}_{k,j}(1) \binom{n}{2} \lambda_k^{n-2} \right) \\
& + \dots \\
& + \kappa_{k,j,m_{k,j}} \left( \mathbf{w}^H \mathbf{v}_{k,j}(m_{k,j}) \binom{n}{0} \lambda_k^n \right. \\
& \quad + \mathbf{w}^H \mathbf{v}_{k,j}(m_{k,j}-1) \binom{n}{1} \lambda_k^{n-1} \\
& \quad \left. + \dots + \mathbf{w}^H \mathbf{v}_{k,j}(1) \binom{n}{m_{k,j}-1} \lambda_k^{n-m_{k,j}+1} \right) \left. \right].
\end{aligned}$$

Moreover, by paying our attention to the binomial coefficients, we can rewrite

$f_n$  as

$$\begin{aligned}
f_n &= \sum_{k=1}^N \left[ \binom{n}{0} \lambda_k^n \sum_{j=1}^{\mathcal{M}_k} \left( \kappa_{k,j,1} \mathbf{w}^H \mathbf{v}_{k,j}(1) + \kappa_{k,j,2} \mathbf{w}^H \mathbf{v}_{k,j}(2) \right. \right. \\
&\quad \left. \left. + \cdots + \kappa_{k,j,m_{k,j}} \mathbf{w}^H \mathbf{v}_{k,j}(m_{k,j}) \right) \right. \\
&\quad \left. + \binom{n}{1} \lambda_k^{n-1} \sum_{j=1}^{\mathcal{M}_k} \left( \kappa_{k,j,2} \mathbf{w}^H \mathbf{v}_{k,j}(j, 1) + \kappa_{k,j,3} \mathbf{w}^H \mathbf{v}_{k,j}(j, 2) \right. \right. \\
&\quad \left. \left. + \cdots + \kappa_{k,j,m_{k,j}} \mathbf{w}^H \mathbf{v}_{k,j}(m_{k,j} - 1) \right) \right. \\
&\quad \left. + \cdots \right. \\
&\quad \left. + \binom{n}{m_{k,j} - 1} \lambda_k^{n-m_{k,j}+1} \sum_{j=1}^{\mathcal{M}_k} \kappa_{k,j,m_{k,j}} \mathbf{w}^H \mathbf{v}_{k,j}(1) \right] \\
&= \sum_{k=1}^N \left[ \binom{n}{0} \lambda_k^n \sum_{j=1}^{\mathcal{M}_k} \sum_{i=1}^{m_{k,j}} \kappa_{k,j,i} \mathbf{w}^H \mathbf{v}_{k,j}(i - 1 + 1) \right. \\
&\quad \left. + \binom{n}{1} \lambda_k^{n-1} \sum_{j=1}^{\mathcal{M}_k} \sum_{i=2}^{m_{k,j}} \kappa_{k,j,i} \mathbf{w}^H \mathbf{v}_{k,j}(i - 2 + 1) \right. \\
&\quad \left. + \cdots \right. \\
&\quad \left. + \binom{n}{m_{k,j} - 1} \lambda_k^{n-m_{k,j}+1} \right. \\
&\quad \left. \times \sum_{j=1}^{\mathcal{M}_k} \sum_{i=m_{k,j}}^{m_{k,j}} \kappa_{k,j,i} \mathbf{w}^H \mathbf{v}_{k,j}(i - m_{k,j} + 1) \right].
\end{aligned}$$

From  $m_k \geq m_{k,j}$  and  $\mathbf{w}^H \mathbf{v}_{k,j}(i - i' + 1) = \mathbf{v}_{k,j}^H(i - i' + 1) \mathbf{w}$ , it follows that

$$\begin{aligned}
f_n &= \sum_{k=1}^N \sum_{i'=1}^{m_k} \left( \sum_{j=1}^{\mathcal{M}_k} \sum_{i=i'}^{m_{k,j}} \kappa_{k,j,i} \mathbf{v}_{k,j}^H(i - i' + 1) \mathbf{w} \right) \\
&\quad \times \binom{n}{i' - 1} \lambda_k^{n-i'+1}.
\end{aligned} \tag{4.35}$$

The exchange of  $i$  and  $i'$  in (4.35) brings us to (4.25) and (4.26).

For example, let us consider the case where the constants  $\kappa_{k,j,i}$  are all 1. Then  $\mathbf{u}$  becomes the sum of all the eigenvectors and generalized eigenvectors. Moreover, let  $\mathbf{w} = V^{-H}\boldsymbol{\alpha}$  in (4.25) where  $\boldsymbol{\alpha}$  is an  $M$ -dimensional vector with all the entries 1. Then it holds that  $\kappa_{k,j,i}\mathbf{v}_{k,j}^H(i'-i+1)\mathbf{w} = \mathbf{e}_{k,j}^\top(i'-i+1)\boldsymbol{\alpha} = 1$ . Thus, it is concluded that  $c_{k,i} \neq 0$ . The above discussion suggests that there exists at least one pair of  $\mathbf{u}$  and  $\mathbf{w}$  satisfying (4.27).  $\square$

Proposition 4.3.1 leads to a theorem concerning the generating function  $F(z)$  with the moments  $f_n = \mathbf{w}^H A^n \mathbf{u}$ .

**Theorem 4.3.2** (Akaiwa et al. [A2]). *Let  $F(z)$  be the generating function with the moments  $f_n = \mathbf{w}^H A^n \mathbf{u}$ . Then,  $F(z)$  converges absolutely in the disk  $D : |z| < |\lambda_1|^{-1}$ , and  $F(z)$  is expressed as*

$$F(z) = \sum_{k=1}^N \sum_{i=1}^{m_k} \frac{c_{k,i} z^{i-1}}{(1 - \lambda_k z)^i}. \quad (4.36)$$

*Epecially, if  $\lambda_N = 0$ , then  $F(z)$  is expressed as*

$$\begin{aligned} F(z) = & c_{N,1} + c_{N,2}z + \cdots + c_{N,m_N}z^{m_N-1} \\ & + \sum_{k=1}^{N-1} \sum_{i=1}^{m_k} \frac{c_{k,i} z^{i-1}}{(1 - \lambda_k z)^i}. \end{aligned} \quad (4.37)$$

*Let us assume that (4.27) holds for some  $\mathbf{u}$  and  $\mathbf{w}$ . If  $\lambda_N \neq 0$ , then  $F(z)$  has the finite poles  $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_N^{-1}$  of orders  $m_1, m_2, \dots, m_N$ , respectively, and the sum of the orders is  $m = m_1 + m_2 + \cdots + m_N$ . If  $\lambda_N = 0$ , then  $F(z)$  has pole of order  $m_N - 1$  at infinity and the finite poles  $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_{N-1}^{-1}$  of orders  $m_1, m_2, \dots, m_N$ , respectively, and the sum of the orders of all the finite poles is  $m - m_N$ .*

*Proof.* By substituting  $f_n$  in (4.26) into  $F(z)$  in (4.2), we get

$$\begin{aligned} F(z) &= \sum_{k=1}^N \sum_{i=1}^{m_k} c_{k,i} \left( \sum_{n=0}^{\infty} z^n \binom{n}{i-1} \lambda_k^{n-i+1} \right) \\ &= \sum_{k=1}^N \sum_{i=1}^{m_k} c_{k,i} \left( \sum_{n=i-1}^{\infty} z^n \binom{n}{i-1} \lambda_k^{n-i+1} \right). \end{aligned} \quad (4.38)$$

By letting  $n = n' + i - 1$ , we derive

$$F(z) = \sum_{k=1}^N \sum_{i=1}^{m_k} c_{k,i} z^{i-1} \left( \sum_{n'=0}^{\infty} \binom{n' + i - 1}{i - 1} (\lambda_k z)^{n'} \right). \quad (4.39)$$

It is noted that, for  $|z| < 1$ ,

$$\sum_{n'=0}^{\infty} \binom{n' + i - 1}{i - 1} z^{n'} = \frac{1}{(1 - z)^i}. \quad (4.40)$$

From (4.39) and (4.40), it turns out that  $F(z)$  converges absolutely in the disk  $D : |z| < |\lambda_1|^{-1}$ . Simultaneously, we have (4.36) for  $z \in D$ . It is obvious that (4.36) with  $\lambda_N = 0$  becomes (4.37). Moreover, (4.36) and (4.37) immediately lead to the latter statement concerning the poles of  $F(z)$ .  $\square$

Let  $\psi_A(z)$  be the polynomial with minimal degree such that  $\psi_A(A) = O$ . Here  $\psi_A(z)$  is called the minimal polynomial of  $A$ . Let us recall here that the maximal dimension of the Jordan blocks  $J_{k,1}, J_{k,2}, \dots, J_{k,\mathcal{M}_k}$  corresponding to  $\lambda_k$  is  $m_k$ . So,  $\psi_A(z)$  is expressed as

$$\psi_A(z) = (z - \lambda_1)^{m_1} (z - \lambda_2)^{m_2} \dots (z - \lambda_N)^{m_N}. \quad (4.41)$$

Therefore, we have the main theorem in this section for the relationship between the minimal polynomial a general matrix  $A$  and the characteristic polynomial of a tridiagonal matrix  $T_l^{(n)}$ .

**Theorem 4.3.3** (Akaiwa et al. [A2]). *Let  $F(z)$  be given by the generating function with the moments  $f_n = \mathbf{w}^H A^n \mathbf{u}$ . Let us assume that (4.6) and (4.27) hold for some  $\mathbf{u}$  and  $\mathbf{w}$ . If  $\lambda_1 \neq 0, \lambda_2 \neq 0, \dots, \lambda_N \neq 0$ , then it holds that*

$$\det(zI_m - T_m^{(n)}) = \psi_A(z), \quad n = 0, 1, \dots, \quad (4.42)$$

otherwise,

$$\det(zI_{m-m_N} - T_{m-m_N}^{(n)}) = \frac{\psi_A(z)}{z^{m_N}}, \quad n = 0, 1, \dots. \quad (4.43)$$

*Proof.* It is remarkable that three integers  $L, l, l_k$  and a complex  $z_k$  associated with the tridiagonal matrix  $T_l^{(n)}$  in Theorem 4.2.2 are given in terms of three

integers  $N, m, m_k$  and a complex  $\lambda_k$  associated with the general matrix  $A$ . If  $\lambda_N \neq 0$ , then it follows from the latter statement of Theorem 4.3.2 with  $L = N$ ,  $l = m$ ,  $l_0, l_1 = m_1, l_2 = m_2, \dots, l_N = m_N$  and  $z_k = \lambda_k^{-1}$ . So, from (4.11) and (4.41), we derive (4.42). Similarly, if  $\lambda_N = 0$ , then  $L = N - 1$ ,  $l = m - m_N$ ,  $l_0 = m_N - 1, l_1 = m_1, l_2 = m_2, \dots, l_{N-1} = m_{N-1}$  and  $z_k = \lambda_k^{-1}$ . Thus (4.11) and (4.41) lead to (4.43).  $\square$

Incidentally, the editors in [54, pp. 444–445] give a simple example with short comments concerning the minimal polynomial, the Jordan canonical form of  $A$  and the multiple poles of  $F(z)$ .

## 4.4 Minimal polynomial of tridiagonal matrix

In this section, with the help of the Jordan canonical form, we clarify the relationship of the characteristic polynomial with the tridiagonal matrix  $T_l^{(n)}$  to the minimal polynomial of  $T_l^{(n)}$ .

For simplicity, let us here adopt the following abbreviations for matrices  $T_s^{(n)}$ ,

$$T_s = \begin{pmatrix} u_1 & v_1 & & & \\ 1 & u_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & 1 & u_s \\ & & & & v_{s-1} \end{pmatrix}, \quad s = 0, 1, \dots, l, \quad (4.44)$$

where  $l = m$  if  $\lambda_N \neq 0$  or  $l = m - m_N$  if  $\lambda_N = 0$ . Let  $p_0(z) = 1$  and  $p_s(z) = \det(zI_s - T_s)$  for  $s = 1, 2, \dots, l$ . Then  $p_l(z)$  is just the characteristic polynomial of  $T_l$ , namely,

$$\phi_T(z) = (z - \lambda_1)^{m_1} (z - \lambda_2)^{m_2} \dots (z - \lambda_L)^{m_L}, \quad (4.45)$$

where  $L = N$  if  $\lambda_N \neq 0$  or  $L = N - 1$  if  $\lambda_N = 0$ . The following proposition gives the Jordan canonical form of the tridiagonal matrix  $T_l$ .

**Proposition 4.4.1** (Akaiwa et al. [A2]). *There exists a nonsingular matrix  $P$  such that*

$$P^{-1}(T_l)^\top P = \hat{J}, \quad (4.46)$$

$$\hat{J} = \text{diag}(J_{1,1}, J_{2,1}, \dots, J_{L,1}) \in \mathbb{C}^{l \times l}, \quad (4.47)$$

where  $J_{1,1}, J_{2,1}, \dots, J_{L,1}$  are of the same form as (4.23).

*Proof.* The characteristic polynomials  $p_0(z), p_1(z), \dots, p_l(z)$  satisfy

$$\begin{cases} zp_0(z) = u_1 p_0(z) + p_1(z), \\ zp_s(z) = v_s p_{s-1}(z) + u_{s+1} p_s(z) + p_{s+1}(z), \\ s = 1, 2, \dots, l-1. \end{cases} \quad (4.48)$$

This is easily derived from the expansion of  $\det(zI_s - T_s)$  into minors by the  $s$ th row. By taking the 0th, the 1st,  $\dots$ , the  $(m_k - 1)$ th derivatives with respect to  $z$  in (4.48), we get

$$\begin{cases} zD^i p_0(z) + iD^{i-1} p_0(z) = u_1 D^i p_0(z) + D^i p_1(z), \\ i = 0, 1, \dots, m_k - 1, \\ zD^i p_s(z) + iD^{i-1} p_s(z) = v_s D^i p_{s-1}(z) + u_{s+1} D^i p_s(z) + D^i p_{s+1}(z), \\ i = 0, 1, \dots, m_k - 1, \quad s = 1, 2, \dots, l-1, \end{cases} \quad (4.49)$$

where  $D^i p_s(z)$  denotes the  $i$ th derivative of  $p_s(z)$  with respect to  $z$ . Let  $\mathbf{p}_{k,i} = (D^i p_0(\lambda_k), D^i p_1(\lambda_k), \dots, D^i p_{l-1}(\lambda_k))^\top \in \mathbb{C}^l$ . Then, by substituting  $z = \lambda_k$  in (4.49) and by taking account that  $D^i p_l(\lambda_k) = D^i(z - \lambda_1)^{m_1}(z - \lambda_2)^{m_2} \dots (z - \lambda_l)^{m_l}|_{z=\lambda_k} = 0$  for  $i = 0, 1, \dots, m_k - 1$ , we obtain

$$\lambda_k \mathbf{p}_{k,i} + i \mathbf{p}_{k,i-1} = (T_l)^\top \mathbf{p}_{k,i}, \quad i = 0, 1, \dots, m_k - 1. \quad (4.50)$$

Moreover, it follows that

$$\begin{cases} (T_l)^\top P_{k,0} = \lambda_k P_{k,0}, \\ (T_l)^\top P_{k,i} = \lambda_k P_{k,i} + P_{k,i-1}, \quad i = 1, 2, \dots, m_k - 1, \end{cases} \quad (4.51)$$

where  $P_{k,i} = (1/i!) \mathbf{p}_{k,i}$ . Thus, by letting  $P = (P_{1,0} P_{1,1} \dots P_{1,m_1-1} | P_{2,0} P_{2,1} \dots P_{2,m_2-1} | \dots | P_{L,0} P_{L,1} \dots P_{L,m_L-1}) \in \mathbb{C}^{l \times l}$ , we have  $(T_l)^\top P = P \hat{J}$ .

Here, it remains to prove that  $P$  is nonsingular. Of course,  $P_{k,i} \neq O$  since the  $(i+1)$ th row of  $P_{k,i}$  is  $D^i p_i(\lambda_k)/i! = 1$ . Let  $W_{k,i} = \text{Ker}((T_l)^\top - \lambda_k I_l)^i$  for  $i = 1, 2, \dots, m_k - 1$ , which indicates the generalized eigenspace of  $(T_l)^\top$  corresponding to  $\lambda_k$ . Then it is obvious from (4.51) that  $((T_l)^\top - \lambda_k I_l) P_{k,0} = O$  and  $P_{k,0} \in W_{k,1}$ . Equation (4.51) with  $i = 1$  also leads to that  $((T_l)^\top - \lambda_k I_l)^2 P_{k,1} = O$  and  $P_{k,1} \in W_{k,2}$ . Simultaneously, it is observed that  $P_{k,1} \notin$

$W_{k,1}$ . Let us assume that  $P_{k,1} \in W_{k,1}$ , namely,  $(T_l)^\top P_{k,1} = \lambda_k P_{k,1}$ . Then, from (4.51), we derive  $P_{k,0} = O$ , which contradicts with  $P_{k,0} \neq O$ . Thus, it follows that  $P_{k,1} \notin W_{k,1}$ . Similarly, by induction for  $i = 2, 3, \dots, m_k - 1$  in  $P_{k,i}$ , we have

$$P_{k,i} \notin W_{k,1}, W_{k,2}, \dots, W_{k,i}, \quad P_{k,i} \in W_{k,i+1}, \quad i = 1, 2, \dots, m_k - 1. \quad (4.52)$$

From (4.52), it turns out that  $P_{k,i}$  for  $i = 0, 1, \dots, m_k - 1$  and  $k = 1, 2, \dots, L$  are linearly independent. Therefore, it is concluded that  $P$  is nonsingular and the Jordan canonical form of  $(T_l)^\top$  is given by (4.46).  $\square$

Proposition 4.4.1 implies that the minimal polynomial of  $(T_l)^\top$  becomes

$$\psi_T(z) = (z - \lambda_1)^{m_1} (z - \lambda_2)^{m_2} \cdots (z - \lambda_L)^{m_L}, \quad (4.53)$$

which is equal to the characteristic polynomial of  $T_l$  in (4.45). If  $m_1 = m_2 = \cdots = m_L = 1$ , then it is obvious that  $T_l$  is diagonalizable. Otherwise,  $T_l$  is not diagonalizable. This is because multiplicity of roots in minimal polynomial coincides with maximal size of the Jordan blocks. To sum up, we have a theorem for the properties of the tridiagonal matrix  $T_l$ .

**Theorem 4.4.2** (Akaiwa et al. [A2]). *The minimal polynomial of  $T_l$  is equal to the characteristic polynomial of  $T_l$ . Also,  $T_l$  is a diagonalizable tridiagonal matrix if and only if it has no multiple eigenvalues.*

## 4.5 Procedure for constructing tridiagonal matrix and its examples

In this section, based on the discussions in the previous sections, we first design a procedure for constructing a tridiagonal matrix with specified multiple eigenvalues. We next give four kinds of examples for demonstrating that the designed procedure can provide tridiagonal matrices with multiple eigenvalues. Examples have been carried out with a computer with OS: Mac OS X 10.8.5, CPU: Intel Core i7 2 GHz, RAM: 8 GB. We also use the scientific computing software Wolfram Mathematica 9.0. In every example, all the entries of  $\mathbf{u}$  are simply set to 1 and those of  $\mathbf{w}$  are not artificial. The readers will realize that the settings of  $\mathbf{u}$  and  $\mathbf{w}$  are not so difficult for satisfying (4.6) and (4.27).

Let us here consider the relationship of five theorems in the previous sections. Theorem 4.2.2 shows that the eigenvalues of  $T_l^{(n)}$  in the tridiagonal form as (4.10) are equal to the poles of the generating function  $F(z)$  and the multiplicity of the eigenvalues coincide with those of the poles of  $F(z)$ . Theorems 4.3.2 and 4.3.3 claim that the minimal polynomial of a general matrix  $A$ , denoted by  $\psi_A(z)$ , is just the denominator of  $F(z)$  involving  $f_n = \mathbf{w}^H A^n \mathbf{u}$ , and it coincides with the characteristic polynomial of  $T_l^{(n)}$  denoted by  $\phi_T(z)$ , except for the factor corresponding to zero-eigenvalues. With the help of Theorem 4.2.1, we thus realize that the nonzero eigenvalues of  $T_l^{(0)}$  with the entries involving  $q_1^{(0)}, q_2^{(0)}, \dots, q_l^{(0)}$  and  $e_1^{(0)}, e_2^{(0)}, \dots, e_{l-1}^{(0)}$  become roots of the minimal polynomial  $\psi_A(z)$  in the case where  $q_1^{(0)}, q_2^{(0)}, \dots, q_l^{(0)}$  and  $e_1^{(0)}, e_2^{(0)}, \dots, e_{l-1}^{(0)}$  are given by the qd formula (4.4) under the initial condition  $e_0^{(n)} = 0$  and  $q_1^{(n)} = f_{n+1}/f_n$  with  $f_n = \mathbf{w}^H A^n \mathbf{u}$ . See also Figure 4.1 for the diagram for getting  $q_s^{(n)}$  and  $e_s^{(n)}$  by the qd formula (4.4). A procedure for constructing  $T = T_l^{(0)}$  with the same nonzero eigenvalues as  $A$  is therefore as follows.

- 1: Set  $l = m$  if  $\lambda_N \neq 0$  and  $l = m - m_N$  if  $\lambda_N = 0$ .
- 2: Choose  $\mathbf{u}$  and  $\mathbf{w}$  as in (4.6) and (4.27).
- 3: Compute  $f_n = \mathbf{w}^H A^n \mathbf{u}$  for  $n = 0, 1, \dots, 2l - 1$ .
- 4: Set  $e_0^{(n)} = 0$  for  $n = 0, 1, \dots, 2l - 3$ .
- 5: Compute  $q_1^{(n)} = f_{n+1}/f_n$  for  $n = 0, 1, \dots, 2l - 2$ .
- 6: Repeat (a) and (b) for  $s = 2, 3, \dots, l$ .
  - (a) Compute  $e_{s-1}^{(n)} = q_{s-1}^{(n+1)} + e_{s-2}^{(n+1)} - q_{s-1}^{(n)}$  for  $n = 0, 1, \dots, 2l - 2s + 1$ .
  - (b) Compute  $q_s^{(n)} = q_{s-1}^{(n+1)} e_{s-1}^{(n+1)} / e_{s-1}^{(n)}$  for  $n = 0, 1, \dots, 2l - 2s$ .
- 7: Construct a tridiagonal matrix by arranging  $q_1^{(0)}, q_2^{(0)}, \dots, q_l^{(0)}$  and  $e_1^{(0)}, e_2^{(0)}, \dots, e_{l-1}^{(0)}$ .

According to Theorem 4.4.2, the minimal and the characteristic polynomials of the resulting tridiagonal matrix  $T$  are equal to each other. Moreover,  $T$  is diagonalizable if and only if it has no multiple eigenvalues.

It is necessary to control the eigenvalues of  $A$  for getting  $T$  as a tridiagonal matrix with prescribed eigenvalues. It is easy to specify the eigenvalues of diagonal matrices and the Jordan matrices.

First, in the procedure, let us consider the case where

$$A = \text{diag}(2, 2, 2, 1, 1, 1) \in \mathbb{R}^{6 \times 6}$$

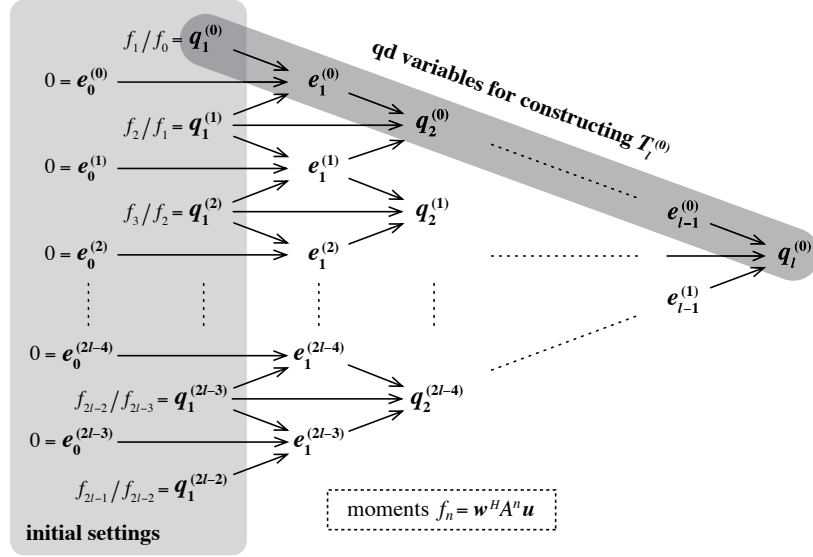


Figure 4.1: The qd diagram for a tridiagonal matrix construction.

which is a diagonal matrix with two eigenvalues 1 and 2 of multiplicities 3. Obviously, the characteristic and the minimal polynomials are factorized as  $(z - 1)^3(z - 2)^3$  and  $(z - 1)(z - 2)$ , respectively. So, the integers  $l$  and  $m$  are immediately determined as  $l = 2$  and  $m = 6$ . Moreover, by letting  $\mathbf{u} = (1, 1, 1, 1, 1, 1)^\top$  and  $\mathbf{w} = (1, 1, 1, 1, 1, 1)^\top$ , we derive a tridiagonal matrix as

$$T = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ & 3 \\ 1 & 2 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$$

whose characteristic and minimal polynomials are both factorized as  $(z - 1)(z - 2)$ . The tridiagonal matrix  $T$  is diagonalizable matrix with the distinct eigenvalues 1 and 2.

Next, let us adopt a bidiagonal matrix, which can be regarded as the

Jordan matrix, with eigenvalues 2 of multiplicity 6 as  $A$ , namely,

$$A = \begin{pmatrix} 2 & 1 & & & & \\ & 2 & 1 & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & 1 \\ & & & & & 2 \end{pmatrix} \in \mathbb{R}^{6 \times 6},$$

in the procedure. Since the characteristic polynomial of  $A$  are equal to the minimal one, the integer  $l$  and  $m$  are determined as  $l = m = 6$ . Then the procedure with  $\mathbf{u} = (1, 1, 1, 1, 1, 1)^\top$  and  $\mathbf{w} = (1, 1, 0, 1, 0, 1)^\top$  constructs a tridiagonal matrix, which can not be symmetrized,

$$T = \begin{pmatrix} \frac{11}{4} & \frac{3}{16} & & & & \\ 1 & \frac{11}{12} & -\frac{4}{9} & & & \\ & 1 & \frac{10}{3} & 3 & & \\ & & 1 & 0 & -8 & \\ & & & 1 & \frac{29}{8} & -\frac{1}{64} \\ & & & & 1 & \frac{11}{8} \end{pmatrix} \in \mathbb{R}^{6 \times 6}.$$

The characteristic and the minimal polynomials of  $A$  and  $T$  are all the same polynomial with respect to  $z$ , which is factored as  $(z-2)^6$ . So, the tridiagonal matrix  $T$  is not diagonalizable.

Let us prepare the Jordan matrix

$$A = \begin{pmatrix} 3 & 1 & & & & \\ & 3 & 1 & & & \\ & & 3 & & & \\ & & & 3 & 1 & \\ & & & & 3 & \\ & & & & & 3 \\ & & & & & & 2 & 1 \\ & & & & & & & 2 \end{pmatrix} \in \mathbb{R}^{8 \times 8}.$$

The matrix  $A$  has multiple eigenvalues such as  $\lambda_1 = 3$ ,  $\lambda_2 = 3$ ,  $\lambda_3 = 3$ ,  $\lambda_4 = 3$ ,  $\lambda_5 = 3$ ,  $\lambda_6 = 3$ ,  $\lambda_7 = 2$ ,  $\lambda_8 = 2$ . It is noted that  $|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = |\lambda_5| = |\lambda_6| > |\lambda_7| = |\lambda_8| > 0$ . The characteristic and the minimal polynomials of  $A$  are factorized as  $(z - 2)^2(z - 3)^6$  and  $(z - 2)^2(z - 3)^3$ , respectively. So, let  $l = 5$  and  $m = 8$  in the procedure. Then, the setting  $\mathbf{u} = (1, 1, 1, 1, 1, 1, 1, 1)^\top$  and  $\mathbf{w} = (1, 1, 1, 1, 1, 1, 1, 1)^\top$  brings us to a tridiagonal matrix, which can not be symmetrized,

$$T = \begin{pmatrix} \frac{13}{4} & \frac{1}{16} & & & & \\ & \frac{17}{4} & -\frac{13}{2} & & & \\ & 1 & -\frac{11}{26} & \frac{116}{169} & & \\ & & 1 & -\frac{1232}{377} & -\frac{13}{841} & \\ & & & 1 & \frac{77}{29} & \\ & & & & & \end{pmatrix} \in \mathbb{R}^{5 \times 5}$$

whose characteristic and minimal polynomials are both factorized as  $(z - 2)^2(z - 3)^3$ , which is just equal to the minimal one of  $A$ . The tridiagonal matrix  $T$  is not a diagonalizable matrix with eigenvalues 2 and 3 of multiplicities 2 and 3, respectively.

Finally, let  $A$  be set as the Jordan matrix with complex eigenvalues  $2 + i$  and  $2 - i$  of multiplicities 2 and distinct real eigenvalues 1 and 2, namely,

$$A = \begin{pmatrix} 2+i & 1 & & & & \\ & 2+i & & & & \\ & & 2-i & 1 & & \\ & & & 2-i & & \\ & & & & 2 & \\ & & & & & 1 \end{pmatrix} \in \mathbb{C}^{6 \times 6},$$

in this procedure. By taking account that the characteristic and the minimal polynomial are equal to each other, the characteristic polynomial of  $A$  are equal to the minimal one, let  $l = m = 6$  in the procedure. Under the setting  $\mathbf{u} = (1, 1, 1, 1, 1, 1)^\top$  and  $\mathbf{w} = (1, 1, 1, 1, 1, 1)^\top$ , the resulting matrix  $T$  is a

real tridiagonal matrix, which can not be symmetrized,

$$T = \begin{pmatrix} \frac{13}{6} & -\frac{19}{36} & & & & \\ 1 & \frac{443}{114} & -\frac{1920}{361} & & & \\ & 1 & -\frac{363}{760} & -\frac{209}{1600} & & \\ & & 1 & \frac{1187}{440} & -\frac{240}{121} & \\ & & & 1 & \frac{37}{66} & \frac{11}{36} \\ & & & & 1 & \frac{13}{6} \end{pmatrix} \in \mathbb{R}^{6 \times 6}.$$

The characteristic and the minimal polynomials of  $A$  and  $T$  are all the same polynomials with respect to  $z$ , which is factored as  $(z - 2 + i)^2(z - 2 - i)^2(z - 2)(z - 1)$ . So, the tridiagonal matrix  $T$  is not a diagonalizable matrix with the same complex multiple eigenvalues and real distinct ones as  $A$ .

## Chapter 5

# A finite-step construction of Hessenberg-type TN matrices with prescribed eigenvalues based on the discrete hungry Toda equation

Entry-wise nonnegative matrices whose minors are all nonnegative are called totally nonnegative (TN) matrices. TN matrices appear in many mathematical branches and applications. However, there is no practical method for inverse eigenvalue problems (IEPs) for TN matrices because of the difficulty of satisfying the TN property.

The discrete hungry Toda equation is a kind of extension of the discrete Toda equation. The numerical algorithm for computing eigenvalues of TN matrices of Hessenberg form is developed from the discrete hungry Toda equation.

In this chapter, an IEP for Hessenberg TN matrices is discussed from the viewpoint of application of the discrete hungry Toda equation to numerical algorithm. A finite-step procedure for constructing Hessenberg TN matrices is designed based on the discrete hungry Toda equation.

## 5.1 TN matrices and inverse eigenvalue problems

The real-valued nonnegative inverse eigenvalue problem (RNIEP) is a type of inverse eigenvalue problems. Chu and Golub's book [8], which is a fundamental reference in the area of inverse eigenvalue problems, notes that the subjects of RNIEP cover the constructions of an entry-wise nonnegative matrix  $A \in \mathbb{R}^{m \times m}$  with prescribed real eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ . Totally nonnegative (TN) matrices are nonnegative matrices whose minors are restricted to be all nonnegative. TN matrices appear in various branches of mathematics, such as stochastic processes, probability and combinatorics [2, 14, 40, 48]. TN matrices also contain inner totally positive (ITP) matrices and an inverse eigenvalue problem for an ITP matrix has an application to a vibrating beam in flexure [21]. If eigenvalues are not prescribed, then TN matrices with banded forms are easily given by preparing TN matrices with bidiagonal forms and computing their matrix products. Generally speaking, however, we cannot prescribe eigenvalues for the resulting banded TN matrix in this method.

The bidiagonal factorization theorem in [7, 13] states that TN matrices can be decomposed into products of lower and upper bidiagonal TN matrices with at most one nonzero offdiagonal entry. In this chapter, we consider a sequence of  $m$ -by- $m$  banded matrices with a positive integer  $M$ ,

$$A^{(n)} := L^{(n)} R^{(n+M-1)} \dots R^{(n+1)} R^{(n)}, \quad n = 0, 1, \dots, \quad (5.1)$$

where  $L^{(n)}$  and  $R^{(n)}$  are, respectively, lower and upper bidiagonal matrices

$$L^{(n)} := \begin{pmatrix} 1 & & & \\ E_1^{(n)} & 1 & & \\ & \ddots & \ddots & \\ & & E_{m-1}^{(n)} & 1 \end{pmatrix}, \quad R^{(n)} := \begin{pmatrix} Q_1^{(n)} & 1 & & \\ & Q_2^{(n)} & \ddots & \\ & & \ddots & 1 \\ & & & Q_m^{(n)} \end{pmatrix}, \quad (5.2)$$

$n = 0, 1, \dots$

The superscripts and subscripts distinguish matrices and identify entries, respectively. The banded matrices in  $\{A^{(n)}\}_{n=0,1,\dots}$  are matrices with the band width  $\min(M+2, m+1)$ . Note that in the case where  $M \geq m-1$  these banded matrices are Hessenberg matrices. If the bidiagonal entries of

$L^{(n)}$  and  $R^{(n)}$  are all positive, then  $L^{(n)}$  and  $R^{(n)}$  are all TN. According to Gasca and Micchelli [20], banded matrices given by products of bidiagonal TN matrices are themselves TN. Though, in fact,  $L^{(n)}$  and  $R^{(n)}$  differ from the bidiagonal matrices appearing in the bidiagonal factorization theorem, they can be decomposed into such bidiagonal matrices. Thus, the banded matrices in  $\{A^{(n)}\}_{n=0,1,\dots}$  can be also decomposed using the bidiagonal factorization theorem.

A sequence of  $LR$  transformations,

$$L^{(n+1)}R^{(n+M)} = R^{(n)}L^{(n)}, \quad n = 0, 1, \dots, \quad (5.3)$$

generates a similarity transformation from  $A^{(n)}$  to  $A^{(n+1)}$  such that  $A^{(n+1)} = R^{(n)}A^{(n)}(R^{(n)})^{-1}$ . The  $LR$  transformations in (5.3) exist even if  $A^{(n)}$  is not TN, and  $A^{(n)}$  can be transformed to  $A^{(n+1)}$  without changing eigenvalues. Focusing on the matrix entries in (5.3), we obtain the recursion formula for the similarity transformation from  $A^{(n)}$  to  $A^{(n+1)}$ ,

$$\begin{cases} Q_k^{(n+M)} + E_{k-1}^{(n+1)} = Q_k^{(n)} + E_k^{(n)}, & k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \\ Q_k^{(n+M)} E_k^{(n+1)} = Q_{k+1}^{(n)} E_k^{(n)}, & k = 1, 2, \dots, m-1, \quad n = 0, 1, \dots \end{cases} \quad (5.4)$$

Equation (5.4) is the discrete hungry Toda (dhToda) equation, which is originally derived from an inverse ultra-discretization of the numbered ball and box system [61]. The subscript  $k$  and superscript  $n$  denote the discrete spatial variable and discrete time variable, respectively. The dhToda equation (5.4) with  $M = 1$  is equal to the discrete-time Toda equation [32, 33, 34], which is a discrete analogue of the continuous-time Toda equation [57]. The bidiagonal matrices  $L^{(n)}$  and  $R^{(n)}$  also appear in the matrix representation, called the Lax representation, of the dhToda equation (5.4) [61].

The discrete Toda equation, the dhToda equation (5.4) with  $M = 1$ , is equal to the recursion formula of the quotient-difference (qd) algorithm. The qd algorithm is a well-known algorithm for computing eigenvalues of tridiagonal matrices, which are often called Jacobi matrices. As  $n \rightarrow \infty$ , the tridiagonal matrix  $A^{(n)}$  with  $M = 1$  converges to an upper bidiagonal matrix; specifically, the diagonal entries of  $A^{(n)}$  with  $M = 1$  converge to the eigenvalues of  $A^{(0)}$  with  $M = 1$ . By Fukuda et al. [18], it is shown that the dhToda equation (5.4) can be applied to compute eigenvalues of  $A^{(0)}$  with arbitrary values of  $M$ . The banded TN matrix  $A^{(n)}$  tends to an upper

triangle matrix whose diagonal entries are eigenvalues of  $A^{(0)}$  as  $n \rightarrow \infty$ . Subsequently, an algorithm based on the dhToda equation (5.4) for solving an eigenvalue problem for the banded TN matrices is designed. A shift of origin by accelerating the convergence is introduced, and then an error analysis for it is presented by Fukuda et al. [19, 17].

Henrici's book [30] briefly comments on the construction of tridiagonal matrices with prescribed eigenvalues by employing the qd recursion formula. The relationship between the inverse eigenvalue problem for tridiagonal matrices and the Toda lattice is demonstrated in the book [8] by Chu and Golub. Several papers deal with the construction of tridiagonal matrices and, apart from the qd algorithm, a few effective algorithms have been proposed [5, 26]. In Chapter 4 [A2], the case of tridiagonal matrices with multiple eigenvalues is discussed.

In this chapter, by investigating an inverse eigenvalue problem for banded TN matrices, we clarify how to construct banded TN matrices with prescribed eigenvalues. The key point is to examine the evolution not from  $n$  to  $n + 1$  via the dhToda equation (5.4) but from  $k$  to  $k + 1$  via

$$\begin{cases} E_k^{(n)} = Q_k^{(n+M)} - Q_k^{(n)} + E_{k-1}^{(n+1)}, & k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \\ Q_{k+1}^{(n)} = \frac{E_k^{(n+1)}}{E_k^{(n)}} Q_k^{(n+M)}, & k = 1, 2, \dots, m-1, \quad n = 0, 1, \dots \end{cases} \quad (5.5)$$

which is easily obtained from the dhToda equation (5.4). Hereinafter, we refer to (5.5) as the dhToda equation, without distinguishing it from (5.4).

The remainder of this chapter is organized as follows. In Section 5.2, by extending the Hankel determinants appearing in the determinant solution to the discrete Toda equation, we derive a determinant solution to the dhToda equation (5.5). In Section 5.3, we present the relationships among extended Hadamard polynomials, where each extended Hadamard polynomial is defined by the ratio of an extended Hankel determinant to its associated polynomial. In Section 5.4, we observe the eigenpairs of banded matrices in terms of the extended Hadamard polynomials. In Section 5.5, we restrict the banded matrices to banded TN matrices. In Section 5.6, we present a finite-step procedure for constructing banded TN matrices with prescribed eigenvalues, and then we give an example to demonstrate the procedure.

## 5.2 The determinant solution to the dhToda equation

Hankel determinants are useful for analyzing the qd recursion formula [30]. In this section, we begin by extending the Hankel determinants and considering their properties. Some readers may find this intuitive, as the dhToda equation (5.5) is an extension of the qd recursion formula. We will then present the determinant solution to the dhToda equation (5.5) using the extended Hankel determinants.

Let us define a sequence of extended Hankel determinants by

$$\begin{aligned} \tau_{-1}^{(n)} &\equiv 0, \quad \tau_0^{(n)} \equiv 1, \\ \tau_k^{(n)} &:= \begin{vmatrix} f_n & f_{n+M} & \cdots & f_{n+(k-1)M} \\ f_{n+1} & f_{n+M+1} & \cdots & f_{n+(k-1)M+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+k-1} & f_{n+M+k-1} & \cdots & f_{n+(k-1)(M+1)} \end{vmatrix}, \\ k &= 1, 2, \dots, m+1, \end{aligned} \tag{5.6}$$

where  $\{f_n\}_{n=0,1,\dots}$  is an arbitrary sequence. Moreover, let us assume that  $k, m, n$  and  $M$  are the same as in the dhToda equation (5.5). The extended Hankel determinants (5.6) differ from standard Hankel determinants in that the subscripts in the entries increases by  $M$  as the column index increases by 1. Of course, the extended Hankel determinants (5.6) with  $M = 1$  become standard Hankel determinants. By considering Jacobi's identity of determinants, we derive the following relationship among the extended Hankel determinants (5.6).

**Lemma 5.2.1.** *The extended Hankel determinants (5.6) satisfy*

$$\begin{aligned} \tau_k^{(n+M+1)} \tau_k^{(n)} &= \tau_k^{(n+1)} \tau_k^{(n+M)} + \tau_{k+1}^{(n)} \tau_{k-1}^{(n+M+1)}, \\ k &= 0, 1, \dots, m, \quad n = 1, 2, \dots \end{aligned} \tag{5.7}$$

*Proof.* Let us begin by briefly explaining the Laplace expansion for the determinant of a  $(2k+2)$ -by- $(2k+2)$  matrix  $X$ . Let  $X(i_1, i_2, \dots, i_{k+1} | j_1, j_2, \dots, j_{k+1})$  denote the  $(k+1)$ -by- $(k+1)$  submatrix consisting of the intersections of the  $i_1$ th,  $i_2$ th,  $\dots$ ,  $i_{k+1}$ th rows and the  $j_1$ th,  $j_2$ th,  $\dots$ ,  $j_{k+1}$ th columns of  $X$ . Moreover, let  $\bar{X}(i_1, i_2, \dots, i_{k+1} | j_1, j_2, \dots, j_{k+1})$  be the  $(k+1)$ -by- $(k+1)$

submatrix obtained by deleting the  $i_1$ th,  $i_2$ th,  $\dots$ ,  $i_{k+1}$ th rows and the  $j_1$ th,  $j_2$ th,  $\dots$ ,  $j_{k+1}$ th columns from  $X$ . Then, for fixed  $i_1, i_2, \dots, i_{k+1}$  with  $1 \leq i_1 < \dots < i_{k+1} \leq 2k+2$ , it holds that

$$\begin{aligned} \det X = & \sum_{1 \leq j_1 < j_2 < \dots < j_{k+1} \leq 2k+2} (-1)^{i_1+i_2+\dots+i_{k+1}+j_1+j_2+\dots+j_{k+1}} \\ & \times \det X(i_1, i_2, \dots, i_{k+1} | j_1, j_2, \dots, j_{k+1}) \\ & \times \det \bar{X}(i_1, i_2, \dots, i_{k+1} | j_1, j_2, \dots, j_{k+1}). \end{aligned} \quad (5.8)$$

Equation (5.8) is called the Laplace expansion of  $\det X$ .

For the  $(k+1)$ -dimensional column vectors  $\mathbf{e}_1 := (1, 0, \dots, 0, 0)^\top$ ,  $\mathbf{e}_{k+1} := (0, 0, \dots, 0, 1)^\top$  and  $\mathbf{f}_n := (f_n, f_{n+1}, \dots, f_{n+k})^\top$ , let

$$X := \left( \begin{array}{ccccc|cccc} \mathbf{e}_1 & \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-1)M} & \mathbf{f}_{n+kM} & \mathbf{f}_n & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{e}_{k+1} \\ \mathbf{e}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{f}_{n+kM} & \mathbf{f}_n & \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-1)M} & \mathbf{e}_{k+1} \end{array} \right). \quad (5.9)$$

Let us consider the case where  $i_1 = 1, i_2 = 2, \dots, i_{k+1} = k+1$  in (5.8). The minors vanish except in the six cases where  $\{j_1, j_2, \dots, j_{k+1}\} = \{1, 2, \dots, k, k+1\}, \{2, \dots, k, k+2, 2k+2\}, \{1, \dots, k, k+2\}, \{2, \dots, k+1, 2k+2\}, \{2, 3, \dots, k+2\}, \{1, \dots, k, 2k+2\}$ . Taking these into account, we derive

$$\begin{aligned} \det X = & 2|\mathbf{e}_1 \mathbf{f}_{n+M} \cdots \mathbf{f}_{n+(k-1)M} \mathbf{f}_{n+kM}| |\mathbf{f}_n \mathbf{f}_{n+M} \cdots \mathbf{f}_{n+(k-1)M} \mathbf{e}_{k+1}| \\ & - 2|\mathbf{e}_1 \mathbf{f}_n \mathbf{f}_{n+M} \cdots \mathbf{f}_{n+(k-1)M}| |\mathbf{f}_{n+M} \cdots \mathbf{f}_{n+(k-1)M} \mathbf{f}_{n+kM} \mathbf{e}_{k+1}| \\ & - 2|\mathbf{f}_n \mathbf{f}_{n+M} \cdots \mathbf{f}_{n+(k-1)M} \mathbf{f}_{n+kM}| |\mathbf{e}_1 \mathbf{f}_{n+M} \cdots \mathbf{f}_{n+(k-1)M} \mathbf{e}_{k+1}|. \end{aligned}$$

By applying cofactor expansions, we can rewrite as

$$\det X = 2(\tau_k^{(n+M+1)} \tau_k^{(n)} - \tau_k^{(n+1)} \tau_k^{(n+M)} - \tau_{k+1}^{(n)} \tau_{k-1}^{(n+M+1)}), \quad (5.10)$$

since  $\tau_{k+1}^{(n)} = |\mathbf{f}_n \mathbf{f}_{n+M} \cdots \mathbf{f}_{n+kM}|$ .

Moreover, by subtracting the last  $k+1$  rows from the corresponding first  $k+1$  rows in  $X$ , we easily obtain

$$\det X = \left| \begin{array}{ccccc|cccc} \mathbf{0} & \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-1)M} & \mathbf{0} & \mathbf{0} & -\mathbf{f}_{n+M} & \cdots & -\mathbf{f}_{n+(k-1)M} & \mathbf{0} \\ \mathbf{e}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{f}_{n+kM} & \mathbf{f}_n & \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-1)M} & \mathbf{e}_{k+1} \end{array} \right|. \quad (5.11)$$

Similarly, it follows from the Laplace expansion for (5.11) that  $\det X = 0$ . Thus, by combining this with (5.10), we have (5.7).  $\square$

With the help of Lemma 5.2.1, we can obtain the following theorem for the determinant solution to the dhToda equation (5.5).

**Theorem 5.2.2** (Akaiwa et al. [A3]). *Let us assume that  $Q_1^{(n)}$  and  $E_0^{(n)}$  satisfy the boundary conditions*

$$Q_1^{(n)} = \frac{f_{n+1}}{f_n}, \quad n = 0, 1, \dots, \quad (5.12)$$

$$E_0^{(n)} = 0, \quad n = 0, 1, \dots \quad (5.13)$$

*Then, the solution to the dhToda equation (5.5) is expressed by using the extended Hankel determinants (5.6) as*

$$Q_k^{(n)} = \frac{\tau_{k-1}^{(n)} \tau_k^{(n+1)}}{\tau_k^{(n)} \tau_{k-1}^{(n+1)}}, \quad k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \quad (5.14)$$

$$E_k^{(n)} = \frac{\tau_{k+1}^{(n)} \tau_{k-1}^{(n+M)}}{\tau_k^{(n)} \tau_k^{(n+M)}}, \quad k = 0, 1, \dots, m, \quad n = 0, 1, \dots \quad (5.15)$$

*Moreover, if  $\{f_n\}_{n=0,1,\dots}$  satisfies the linear equations*

$$f_{n+m} + a_1 f_{n+m-1} + \dots + a_m f_n = 0, \quad n = 0, 1, \dots, \quad (5.16)$$

*with arbitrary constants  $f_0, f_1, \dots, f_{m-1}$  and  $a_1, a_2, \dots, a_m$ , then it holds that*

$$E_m^{(n)} = 0, \quad n = 0, 1, \dots \quad (5.17)$$

*Proof.* Since  $\tau_{-1}^{(n)} = 0$ ,  $\tau_0^{(n)} = 1$ ,  $\tau_1^{(n)} = f_n$  and  $\tau_1^{(n+1)} = f_{n+1}$ , it is obvious that  $Q_1^{(n)} = \tau_0^{(n)} \tau_1^{(n+1)} / (\tau_1^{(n)} \tau_0^{(n+1)})$  and  $E_0^{(n)} = \tau_1^{(n)} \tau_{-1}^{(n+M)} / (\tau_0^{(n)} \tau_0^{(n+M)})$  satisfy the boundary conditions (5.12) and (5.13), respectively. The solution to the dhToda equation (5.5) is easily checked by substituting  $Q_k^{(n)}$  in (5.14) and  $E_k^{(n)}$  in (5.15) for the dhToda equation (5.5) and by using Lemma 5.2.1. By taking (5.16) into account and applying elementary transformations of determinants to the  $(m+1)$ th row of  $\tau_{m+1}^{(n)}$ , we easily derive

$$\tau_{m+1}^{(n)} = 0, \quad n = 0, 1, \dots \quad (5.18)$$

Moreover, by combining (5.18) with (5.15), we have (5.17).  $\square$

The solution to the original dhToda equation (5.4) has also been presented with other boundary conditions [61]. It is different from (5.14) and (5.15), which is the solution to the dhToda equation (5.4) with the boundary conditions (5.12) and (5.13), in entries of extended Hankel determinants.

### 5.3 Extended Hadamard polynomials

The ratio of the Hankel polynomial to the Hankel determinant is often referred to as the Hadamard determinant [30]. Following Henrici [30], in this section, we consider extended Hadamard polynomials in terms of the extended Hankel determinants (5.6) and their associated polynomials. We then clarify the properties of the extended Hadamard polynomials.

Let us introduce polynomials with respect to  $z$ ,

$$\begin{aligned} \varrho_{-1}^{(n)}(z) &\equiv 0, \quad \varrho_0^{(n)}(z) \equiv 1, \\ \varrho_k^{(n)}(z) &:= \begin{vmatrix} f_n & f_{n+M} & \cdots & f_{n+(k-1)M} & 1 \\ f_{n+1} & f_{n+M+1} & \cdots & f_{n+(k-1)M+1} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n+k-1} & f_{n+M+k-1} & \cdots & f_{n+(k-1)(M+1)} & z^{k-1} \\ f_{n+k} & f_{n+M+k} & \cdots & f_{n+(k-1)M+k} & z^k \end{vmatrix}, \\ k &= 1, 2, \dots, m, \end{aligned} \tag{5.19}$$

as extended Hankel polynomials associated with the extended Hankel determinants. The meanings of  $k, m, n$  and  $M$  are the same as in (5.6). It is noted here that the extended Hankel determinant  $\tau_k^{(n)}$  is the  $k$ th order leading principal minor of the extended Hankel determinant  $\varrho_k^{(n)}(z)$ , whose degree is  $k$ . The extended Hankel polynomial  $\varrho_k^{(n)}(z)$  with  $M = 1$  coincides with the standard Hankel polynomial. Similarly to Henrici [30], we adopt the extended Hadamard polynomials as

$$\begin{aligned} \phi_{-1}^{(n)}(z) &\equiv 0, \quad \phi_0^{(n)}(z) \equiv 1, \\ \phi_k^{(n)}(z) &= \frac{\varrho_k^{(n)}(z)}{\tau_k^{(n)}}, \quad k = 1, 2, \dots, m. \end{aligned} \tag{5.20}$$

The following two propositions describe a relationship among the extended Hadamard polynomials.

**Proposition 5.3.1** (Akaiwa et al. [A3]). *The extended Hadamard polynomials (5.20) satisfy*

$$z\phi_k^{(n+1)}(z) = Q_{k+1}^{(n)}\phi_k^{(n)}(z) + \phi_{k+1}^{(n)}(z), \quad k = 0, 1, \dots, m-1. \tag{5.21}$$

*Proof.* This proof is similar to that of Lemma 5.2.1. The vectors  $\mathbf{e}_1, \mathbf{e}_{k+1}, \mathbf{f}_n$  are the same as in the proof of Lemma 5.2.1. Let us introduce a  $(k+1)$ -dimensional column vector  $\mathbf{z} = (1, z, \dots, z^k)^\top$ . Then, by expanding

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{f}_n & \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-2)M} & \mathbf{f}_{n+(k-1)M} \\ \mathbf{e}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{f}_{n+(k-1)M} \end{vmatrix} \begin{vmatrix} \mathbf{z} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{e}_{k+1} \\ \mathbf{z} & \mathbf{f}_n & \cdots & \mathbf{f}_{n+(k-2)M} & \mathbf{e}_{k+1} \end{vmatrix} \quad (5.22)$$

and considering that  $\varrho_k^{(n)} = |\mathbf{f}_n \ \mathbf{f}_{n+M} \ \cdots \ \mathbf{f}_{n+(k-1)M} \ \mathbf{z}|$ , we obtain

$$\tau_k^{(n+1)} \varrho_{k-1}^{(n)}(z) - z \tau_k^{(n)} \varrho_{k-1}^{(n+1)}(z) + \tau_{k-1}^{(n+1)} \varrho_k^{(n)}(z) = 0. \quad (5.23)$$

By dividing both sides of (5.23) by  $\tau_k^{(n)} \tau_{k-1}^{(n+1)}$ , we derive

$$\frac{\tau_k^{(n+1)} \tau_{k-1}^{(n)}}{\tau_k^{(n)} \tau_{k-1}^{(n+1)}} \frac{\varrho_{k-1}^{(n)}(z)}{\tau_{k-1}^{(n)}} - z \frac{\varrho_{k-1}^{(n+1)}(z)}{\tau_{k-1}^{(n+1)}} + \frac{\varrho_k^{(n)}(z)}{\tau_k^{(n)}} = 0,$$

which implies (5.21).  $\square$

**Proposition 5.3.2** (Akaiwa et al. [A3]). *The extended Hadamard polynomials (5.20) satisfy*

$$\phi_k^{(n)}(z) = \phi_k^{(n+M)}(z) + E_k^{(n)} \phi_{k-1}^{(n+M)}(z), \quad k = 0, 1, \dots, m. \quad (5.24)$$

*Proof.* Similarly to the proofs of Lemma 5.2.1 and Proposition 5.3.1, by expanding

$$\begin{vmatrix} \mathbf{f}_n & \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-1)M} & \mathbf{e}_{k+1} \\ \mathbf{f}_n & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{e}_{k+1} \end{vmatrix} \begin{vmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{f}_{n+kM} & \mathbf{z} \\ \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-1)M} & \mathbf{f}_{n+kM} & \mathbf{z} \end{vmatrix}, \quad (5.25)$$

we derive

$$\tau_k^{(n)} \varrho_k^{(n+M)}(z) - \tau_k^{(n+M)} \varrho_k^{(n)}(z) + \tau_{k+1}^{(n)} \varrho_{k-1}^{(n+M)}(z) = 0. \quad (5.26)$$

By dividing both sides of (5.30) by  $\tau_k^{(n)} \tau_k^{(n+M)}$ , we obtain

$$\frac{\varrho_k^{(n+M)}(z)}{\tau_k^{(n+M)}} - \frac{\varrho_k^{(n)}(z)}{\tau_k^{(n)}} + \frac{\tau_{k+1}^{(n)} \tau_{k-1}^{(n+M)}}{\tau_k^{(n)} \tau_k^{(n+M)}} \frac{\varrho_{k-1}^{(n+M)}(z)}{\tau_{k-1}^{(n+M)}} = 0,$$

which immediately leads to (5.28).  $\square$

## 5.4 Associated eigenvalue problems

In this section, we observe the eigenvalue problem of  $A^{(n)}$  by focusing on the extended Hadamard polynomials (5.20).

Let us introduce a monic polynomial with respect to  $z$ ,

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \quad (5.27)$$

where  $a_1, a_2, \dots, a_m$  are the same constants as in (5.16). The following proposition shows a relationship between an extended Hadamard polynomial in (5.20) and the monic polynomial  $p(z)$ .

**Proposition 5.4.1** (Akaiwa et al. [A3]). *The extended Hadamard polynomials (5.20) satisfy*

$$\phi_k^{(n)}(z) = \phi_k^{(n+M)}(z) + E_k^{(n)} \phi_{k-1}^{(n+M)}(z), \quad k = 0, 1, \dots, m. \quad (5.28)$$

*Proof.* Similarly to the proofs of Lemma 5.2.1 and Proposition 5.3.1, by expanding

$$\begin{vmatrix} \mathbf{f}_n & \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-1)M} & \mathbf{e}_{k+1} \\ \mathbf{f}_n & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{e}_{k+1} \end{vmatrix} \begin{vmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{f}_{n+kM} & z \\ \mathbf{f}_{n+M} & \cdots & \mathbf{f}_{n+(k-1)M} & \mathbf{f}_{n+kM} & z \end{vmatrix}, \quad (5.29)$$

we derive

$$\tau_k^{(n)} \varrho_k^{(n+M)}(z) - \tau_k^{(n+M)} \varrho_k^{(n)}(z) + \tau_{k+1}^{(n)} \varrho_{k-1}^{(n+M)}(z) = 0. \quad (5.30)$$

By dividing both sides of (5.30) by  $\tau_k^{(n)} \tau_k^{(n+M)}$ , we obtain

$$\frac{\varrho_k^{(n+M)}(z)}{\tau_k^{(n+M)}} - \frac{\varrho_k^{(n)}(z)}{\tau_k^{(n)}} + \frac{\tau_{k+1}^{(n)} \tau_{k-1}^{(n+M)}}{\tau_k^{(n)} \tau_k^{(n+M)}} \frac{\varrho_{k-1}^{(n+M)}(z)}{\tau_{k-1}^{(n+M)}} = 0,$$

which immediately leads to (5.28).  $\square$

In this section, we observe the eigenvalue problem of  $A^{(n)}$  by focusing on the extended Hadamard polynomials (5.20).

Let us introduce a monic polynomial with respect to  $z$ ,

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \quad (5.31)$$

where  $a_1, a_2, \dots, a_m$  are the same constants as in (5.16). The following proposition shows the relationship between an extended Hadamard polynomial in (5.20) and the monic polynomial  $p(z)$ .

**Proposition 5.4.2** (Akaiwa et al. [A3]). *Let us assume that  $Q_1^{(n)}$  and  $E_0^{(n)}$  for  $n = 0, 1, \dots$  satisfy the boundary conditions in (5.12) and (5.13), respectively. If  $\{f_n\}_{n=0,1,\dots}$  in  $\{\phi_m^{(n)}(z)\}_{n=0,1,\dots}$  satisfies (5.16), then it holds that*

$$\phi_m^{(n)}(z) = p(z), \quad n = 0, 1, \dots \quad (5.32)$$

*Proof.* For the extended Hankel determinant  $\varrho_m^{(n)}$  in (5.19), by adding  $a_m$  times the 1st row,  $a_{m-1}$  times the 2nd row, ...,  $a_1$  times the  $m$ th row to the  $(m+1)$ th row, we get

$$\varrho_m^{(n)}(z) = \begin{vmatrix} f_n & f_{n+M} & \cdots & f_{n+(m-1)M} & 1 \\ f_{n+1} & f_{n+M+1} & \cdots & f_{n+(m-1)M+1} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n+m-1} & f_{n+M+m-1} & \cdots & f_{n+(m-1)(M+1)} & z^{m-1} \\ \sum_{i=0}^m a_i f_{n+m-i} & \sum_{i=0}^m a_i f_{n+M+m-i} & \cdots & \sum_{i=0}^m a_i f_{n+(m-1)M-i} & \sum_{i=0}^m a_i z^{m-i} \end{vmatrix},$$

where  $a_0 \equiv 1$ . Considering (5.16) and (5.31), we realize that the  $(m+1, 1)$ ,  $(m+1, 2)$ , ...,  $(m+1, m)$  entries are all 0 and the  $(m+1, m+1)$  entry is  $p(z)$ . Thus, it follows that  $\varrho_m^{(n)}(z) = \tau_m^{(n)} p(z)$ . Since  $\phi_m^{(n)}(z) = \varrho_m^{(n)}(z) / \tau_m^{(n)}$  in (5.20), we therefore obtain (5.32).  $\square$

Now, let us assume that  $p(z)$  has  $m$  distinct roots  $\sigma_1, \sigma_2, \dots, \sigma_m$ , namely,

$$p(z) = (z - \sigma_1)(z - \sigma_2) \cdots (z - \sigma_m). \quad (5.33)$$

Then, from Propositions 5.3.1, 5.4.1 and 5.4.2, we derive the relationships between the Lax matrices  $L^{(n)}$  and  $R^{(n)}$  in (5.2) and an  $m$ -dimensional vector  $\Phi_i^{(n)} := (\phi_0^{(n)}(\sigma_i), \phi_1^{(n)}(\sigma_i), \dots, \phi_{m-1}^{(n)}(\sigma_i))^\top$ .

**Proposition 5.4.3** (Akaiwa et al. [A3]). *Let us assume that  $Q_1^{(n)}$  and  $E_0^{(n)}$  for  $n = 0, 1, \dots$  satisfy the boundary conditions (5.12) and (5.13), respectively. If  $\{f_n\}_{n=0,1,\dots}$  in  $\{\Phi_i^{(n)}(z)\}_{n=0,1,\dots}$  satisfies (5.16), then it holds that, for distinct  $\sigma_1, \sigma_2, \dots, \sigma_m$  in (5.33),*

$$L^{(n)} \Phi_i^{(n+M)} = \Phi_i^{(n)}, \quad i = 1, 2, \dots, m, \quad n = 0, 1, \dots, \quad (5.34)$$

$$R^{(n)} \Phi_i^{(n)} = \sigma_i \Phi_i^{(n+1)}, \quad i = 1, 2, \dots, m, \quad n = 0, 1, \dots \quad (5.35)$$

*Proof.* Let us recall here that  $p(\sigma_i) = 0$ . Combining this with Proposition 5.4.2, we see that

$$\phi_m^{(n)}(\sigma_i) = 0, \quad i = 1, 2, \dots, m, \quad n = 0, 1, \dots \quad (5.36)$$

Propositions 5.3.1 and 5.4.1 with (6.39) immediately lead to

$$\begin{pmatrix} \phi_0^{(n)}(\sigma_i) \\ \phi_1^{(n)}(\sigma_i) \\ \vdots \\ \phi_{m-1}^{(n)}(\sigma_i) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ E_1^{(n)} & 1 & & \\ & E_2^{(n)} & 1 & \\ & & \ddots & \ddots \\ & & & E_{m-1}^{(n)} & 1 \end{pmatrix} \begin{pmatrix} \phi_0^{(n+M)}(\sigma_i) \\ \phi_1^{(n+M)}(\sigma_i) \\ \vdots \\ \phi_{m-1}^{(n+M)}(\sigma_i) \end{pmatrix}, \quad (5.37)$$

$$\sigma_i \begin{pmatrix} \phi_0^{(n+1)}(\sigma_i) \\ \phi_1^{(n+1)}(\sigma_i) \\ \vdots \\ \phi_{m-1}^{(n+1)}(\sigma_i) \end{pmatrix} = \begin{pmatrix} Q_1^{(n)} & 1 & & \\ & Q_2^{(n)} & \ddots & \\ & & \ddots & 1 \\ & & & Q_m^{(n)} \end{pmatrix} \begin{pmatrix} \phi_0^{(n)}(\sigma_i) \\ \phi_1^{(n)}(\sigma_i) \\ \vdots \\ \phi_{m-1}^{(n)}(\sigma_i) \end{pmatrix}, \quad (5.38)$$

which are equivalent to (5.34) and (5.35), respectively.  $\square$

Let  $\lambda_1, \lambda_2, \dots, \lambda_m$  be constants given in terms of  $\sigma_1, \sigma_2, \dots, \sigma_m$  by

$$\lambda_1 = \sigma_1^M, \quad \lambda_2 = \sigma_2^M, \quad \dots, \quad \lambda_m = \sigma_m^M. \quad (5.39)$$

Of course,  $\lambda_1, \lambda_2, \dots, \lambda_m$  are distinct from each other. We thus have an important theorem for the eigenpairs of the banded matrix  $A^{(n)}$  defined by (5.1).

**Theorem 5.4.4** (Akaiwa et al. [A3]). *Let us assume that  $Q_1^{(n)}$  and  $E_0^{(n)}$  for  $n = 0, 1, \dots$  satisfy the boundary conditions (5.12) and (5.13), respectively. Moreover, let  $\sigma_1, \sigma_2, \dots, \sigma_m$  in (5.33) be distinct. If  $\{f_n\}_{n=0,1,\dots}$  in  $\{\Phi_i^{(n)}(z)\}_{n=0,1,\dots}$  satisfies (5.16), then, for distinct  $\lambda_1, \lambda_2, \dots, \lambda_m$  in (5.39),  $\{\lambda_1, \Phi_1^{(n)}\}, \{\lambda_2, \Phi_2^{(n)}\}, \dots, \{\lambda_m, \Phi_m^{(n)}\}$  are eigenpairs of  $A^{(n)} = L^{(n)} R^{(n+M-1)} \dots R^{(n+1)} R^{(n)}$ .*

*Proof.* From Proposition 5.4.3, it follows that

$$\begin{aligned}
A^{(n)}\Phi_i^{(n)} &= L^{(n)}R^{(n+M-1)} \dots R^{(n+1)}(R^{(n)}\Phi_i^{(n)}) \\
&= \sigma_i L^{(n)}R^{(n+M-1)} \dots (R^{(n+1)}\Phi_i^{(n+1)}) \\
&= \sigma_i^2 L^{(n)}R^{(n+M-1)} \dots (R^{(n+2)}\Phi_i^{(n+2)}) \\
&\vdots \\
&= \sigma_i^M (L^{(n)}\Phi_i^{(n+M)}) \\
&= \sigma_i^M \Phi_i^{(n)} \\
&= \lambda_i \Phi_i^{(n)}, \quad i = 1, 2, \dots, m,
\end{aligned}$$

which implies that  $\{\lambda_1, \Phi_1^{(n)}\}, \{\lambda_2, \Phi_2^{(n)}\}, \dots, \{\lambda_m, \Phi_m^{(n)}\}$  are eigenpairs of  $A^{(n)}$ .  $\square$

Theorem 5.4.4 claims that  $\det(zI - A^{(n)}) = (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_m)$ . It is obvious that  $p(z) = \phi_m^{(n)}(z) = (z - \sigma_1)(z - \sigma_2) \dots (z - \sigma_m)$  is not the characteristic polynomial of  $A^{(n)}$  with  $M \neq 1$ . It is, however, remarkable that  $p(z) = \phi_m^{(n)}(z)$  yields the roots of the characteristic polynomial of  $A^{(n)}$ . Of course,  $p(z) = \phi_m^{(n)}(z)$  coincides with the characteristic polynomial of  $A^{(n)}$  if  $M = 1$ , that is,  $A^{(n)}$  is a tridiagonal matrix.

## 5.5 The TN property

The sequence  $\{f_n\}_{n=0,1,\dots}$  satisfying (5.16) is uniquely determined by given values of  $f_0, f_1, \dots, f_{m-1}$ . The dhToda variables  $Q_k^{(n)}$  and  $E_k^{(n)}$  appearing in the entries of  $A^{(n)}$  are similarly uniquely determined. This is because  $Q_k^{(n)}$  and  $E_k^{(n)}$  are expressed by using the extended Hankel determinants (5.6) with respect to  $\{f_n\}_{n=0,1,\dots}$ . Setting  $f_0, f_1, \dots, f_{m-1}$  thus plays a key role in realizing the TN property of  $A^{(n)}$ . We first express the extended Hankel determinants (5.6) in terms of distinct  $\sigma_1, \sigma_2, \dots, \sigma_m$ , which are shown in Theorem 5.4.4 to be the  $M$ -roots of the eigenvalues of  $A^{(n)}$ . Next, we clarify what setting of  $f_0, f_1, \dots, f_{m-1}$  enables  $A^{(n)}$  to have the TN property.

It is worth noting here that  $\{f_n\}_{n=0,1,\dots}$  satisfying (5.16) can be expressed as

$$f_n = c_1 \sigma_1^n + c_2 \sigma_2^n + \dots + c_m \sigma_m^n, \quad n = 0, 1, \dots, \quad (5.40)$$

where  $c_1, c_2, \dots, c_m$  are constants given by  $\sigma_1, \sigma_2, \dots, \sigma_m$  and  $f_0, f_1, \dots, f_{m-1}$ . From (5.40), we derive a proposition for an expansion of the extended Hankel determinants (5.6) using  $\sigma_1, \sigma_2, \dots, \sigma_m$  and  $c_1, c_2, \dots, c_m$ .

**Proposition 5.5.1** (Akaiwa et al. [A3]). *If  $\{f_n\}_{n=0,1,\dots}$  satisfies (5.40), then the extended Hankel determinants (5.6) are expressed as*

$$\begin{aligned} \tau_k^{(n)} &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq m} (c_{i_1} c_{i_2} \dots c_{i_k}) (\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k})^n \\ &\quad \times \prod_{1 \leq j < l \leq k} (\sigma_{i_l} - \sigma_{i_j}) \prod_{1 \leq j < l \leq k} (\sigma_{i_l}^M - \sigma_{i_j}^M), \quad k = 1, 2, \dots, m. \end{aligned} \quad (5.41)$$

*Proof.* By applying (5.40) to the  $(j+1)$ th column of  $\tau_k^{(n)}$ , denoted by  $\mathbf{f}_{n+jM} = (f_{n+jM}, f_{n+jM+1}, \dots, f_{n+jM+k-1})^\top$ , we easily obtain

$$\mathbf{f}_{n+jM} = \sum_{i=1}^m c_i \sigma_i^n \boldsymbol{\sigma}_i^{(j)},$$

where  $\boldsymbol{\sigma}_i^{(j)} := (\sigma_i^{jM}, \sigma_i^{jM+1}, \dots, \sigma_i^{jM+k-1})^\top$ . Thus, it follows that

$$\begin{aligned} \tau_k^{(n)} &= \begin{vmatrix} \mathbf{f}_n & \mathbf{f}_{n+M} & \dots & \mathbf{f}_{n+(k-1)M} \end{vmatrix} \\ &= \begin{vmatrix} \sum_{i_1=1}^m c_{i_1} \sigma_{i_1}^n \boldsymbol{\sigma}_{i_1}^{(0)} & \sum_{i_2=1}^m c_{i_2} \sigma_{i_2}^n \boldsymbol{\sigma}_{i_2}^{(1)} & \dots & \sum_{i_k=1}^m c_{i_k} \sigma_{i_k}^n \boldsymbol{\sigma}_{i_k}^{(k-1)} \end{vmatrix} \\ &= \sum_{i_1=1}^m \sum_{i_2=1}^m \dots \sum_{i_k=1}^m c_{i_1} c_{i_2} \dots c_{i_k} (\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k})^n \begin{vmatrix} \boldsymbol{\sigma}_{i_1}^{(0)} & \boldsymbol{\sigma}_{i_2}^{(1)} & \dots & \boldsymbol{\sigma}_{i_k}^{(k-1)} \end{vmatrix}. \end{aligned}$$

This can be rewritten by reframing the summations as

$$\begin{aligned} \tau_k^{(n)} &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq m} c_{i_1} c_{i_2} \dots c_{i_k} (\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k})^n \\ &\quad \times \sum_{j_1=1}^k \sum_{j_2=1}^k \dots \sum_{j_k=1}^k \begin{vmatrix} \boldsymbol{\sigma}_{i_{j_1}}^{(0)} & \boldsymbol{\sigma}_{i_{j_2}}^{(1)} & \dots & \boldsymbol{\sigma}_{i_{j_k}}^{(k-1)} \end{vmatrix}. \end{aligned}$$

By taking into account that

$$\begin{aligned}
& \sum_{j_1=1}^k \sum_{j_2=1}^k \cdots \sum_{j_k=1}^k \left| \sigma_{i_{j_1}}^{(0)} \sigma_{i_{j_2}}^{(1)} \cdots \sigma_{i_{j_k}}^{(k-1)} \right| \\
&= \left| \sum_{j_1=1}^k \sigma_{i_{j_1}}^{(0)} \sum_{j_2=1}^k \sigma_{i_{j_2}}^{(1)} \cdots \sum_{j_k=1}^k \sigma_{i_{j_k}}^{(k-1)} \right| \\
&= \left| \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sigma_{i_1} & \sigma_{i_2} & \cdots & \sigma_{i_k} \\ \vdots & \vdots & & \vdots \\ \sigma_{i_1}^{k-1} & \sigma_{i_2}^{k-1} & \cdots & \sigma_{i_k}^{k-1} \end{pmatrix} \begin{pmatrix} 1 & \sigma_{i_1}^M & \cdots & \sigma_{i_1}^{(k-1)M} \\ 1 & \sigma_{i_2}^M & \cdots & \sigma_{i_2}^{(k-1)M} \\ \vdots & \vdots & & \vdots \\ 1 & \sigma_{i_k}^M & \cdots & \sigma_{i_k}^{(k-1)M} \end{pmatrix} \right| \\
&= \prod_{1 \leq j < l \leq k} (\sigma_{i_l} - \sigma_{i_j}) \prod_{1 \leq j < l \leq k} (\sigma_{i_l}^M - \sigma_{i_j}^M),
\end{aligned}$$

we therefore have (5.41).  $\square$

Let us consider a sufficient condition for  $A^{(n)}$  being a TN matrix. From Proposition 5.5.1, we find  $\tau_k^{(n)} > 0$  if  $\sigma_1 > \sigma_2 > \cdots > \sigma_m > 0$  and  $c_1 > 0, c_2 > 0, \dots, c_m > 0$ . It is obvious from Theorem 5.2.2 that if  $\tau_k^{(n)} > 0$ , then  $Q_k^{(n)} > 0$  and  $E_k^{(n)} > 0$ . Thus,  $L^{(n)}, R^{(n)}, R^{(n+1)}, \dots, R^{(n+M-1)}$  are all TN if  $\sigma_1 > \sigma_2 > \cdots > \sigma_m > 0$  and  $c_1 > 0, c_2 > 0, \dots, c_m > 0$ , since all of their minors are 0, 1, or some positive constants given in terms of  $Q_k^{(n)}$  and  $E_k^{(n)}$  [2, 48]. According to Gasca and Micchelli [20, p. 110], if  $L^{(n)}, R^{(n)}, R^{(n+1)}, \dots, R^{(n+M-1)}$  are TN, then their products  $L^{(n)} R^{(n)} R^{(n+1)} \cdots R^{(n+M-1)} = A^{(n)}$  are so. If  $\sigma_1 > \sigma_2 > \cdots > \sigma_m > 0$  in (5.39), then it holds that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0. \quad (5.42)$$

Thus, we have a theorem for  $A^{(n)}$  being a TN matrix.

**Theorem 5.5.2** (Akaiwa et al. [A3]). *Let us assume that  $Q_1^{(n)}$  and  $E_0^{(n)}$  for  $n = 0, 1, \dots$  satisfy the boundary conditions (5.12) and (5.13), respectively. Let  $\sigma_1, \sigma_2, \dots, \sigma_m$ , which are the  $M$ th roots of  $\lambda_1, \lambda_2, \dots, \lambda_m$ , respectively, be distinct positive with  $\sigma_1 > \sigma_2 > \cdots > \sigma_m$  in (5.40). Moreover, let  $c_1, c_2, \dots, c_m$  be positive in (5.40). Then,  $A^{(n)} = L^{(n)} R^{(n+M-1)} \cdots R^{(n+1)} R^{(n)}$  for  $n = 0, 1, \dots$ , which are composed of  $\{f_n\}_{n=0,1,\dots}$ , are TN.*

It is emphasized here that  $\sigma_1, \sigma_2, \dots, \sigma_m$  are not restricted except for  $\sigma_1 > \sigma_2 > \dots > \sigma_m > 0$ . In other words, the eigenvalues of  $A^{(n)}$  denoted by  $\lambda_1, \lambda_2, \dots, \lambda_m$  must only be positive and distinct to each other. Thus, Theorem 5.5.2 also implies that TN matrices can have any set of distinct positive eigenvalues.

## 5.6 A finite-step construction procedure

In this section, we describe how to construct TN matrices with prescribed eigenvalues by making full use of the dhToda equation (5.5). We also give an example to demonstrate the resulting procedure.

It is easy to check that the dhToda equation (5.5) can generate  $A^{(0)}$  under the boundary conditions (5.12) and (5.13). Strictly speaking, the boundary conditions are necessary only for  $n = 0, 1, \dots, (M+1)(m-1) + M - 1$  and not for all  $n = 0, 1, \dots$ . According to Theorem 5.5.2, eigenvalues of  $A^{(0)}$  then become  $\lambda_1 = \sigma_1^M, \lambda_2 = \sigma_2^M, \dots, \lambda_m = \sigma_m^M$  if  $\{f_n\}_{n=0,1,\dots}$  in (5.12) is given in terms of  $c_1, c_2, \dots, c_m$  and  $\sigma_1, \sigma_2, \dots, \sigma_m$  by (5.40). It is noted that  $\lambda_1, \lambda_2, \dots, \lambda_m$  do not depend on the values of  $c_1, c_2, \dots, c_m$ . To summarize, we can derive a procedure for constructing the TN matrix  $A^{(0)}$  with prescribed eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ , as follows.

- 1: Select positive integers  $m$  and  $M$  where  $m$  is the matrix size and  $\min\{m+1, M+2\}$  designates the band width.
- 2: Specify  $m$  distinct positive eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ .
- 3: Set  $\sigma_1, \sigma_2, \dots, \sigma_m$  as the positive  $M$ -roots of  $\lambda_1, \lambda_2, \dots, \lambda_m$ , respectively.
- 4: Choose  $c_1, c_2, \dots, c_m$  as positive constants.
- 5: **for**  $n = 0$  to  $(M+1)(m-1) + M$  **do**
- 6:    $f_n = c_1\sigma_1^n + c_2\sigma_2^n + \dots + c_m\sigma_m^n$
- 7: **end for**
- 8: **for**  $n = 0$  to  $(M+1)(m-1) + M - 1$  **do**
- 9:    $E_0^{(n)} = 0$  and  $Q_1^{(n)} = f_{n+1}/f_n$
- 10: **end for**
- 11: **for**  $k = 1$  to  $m - 1$  **do**
- 12:   **for**  $n = 0$  to  $(M+1)(m-k-1) + M$  **do**
- 13:      $E_k^{(n)} = Q_k^{(n+M)} - Q_k^{(n)} + E_{k-1}^{(n+1)}$
- 14:   **end for**
- 15: **for**  $n = 0$  to  $(M+1)(m-k-1) + M$  **do**

```

16:    $Q_{k+1}^{(n)} = E_k^{(n+1)} Q_k^{(n+M)} / E_k^{(n)}$ 
17: end for
18: end for
19: Construct  $L^{(0)}$  and  $R^{(0)}, R^{(1)}, \dots, R^{(M-1)}$  as in (5.2).
20:  $A^{(0)} = L^{(0)} R^{(M-1)} \dots R^{(1)} R^{(0)}$ 

```

We observe that the procedure requires  $O(Mm^2)$  finite arithmetic. Moreover, by using the original dhToda equation (5.4) not (5.5), we can obtain  $A^{(1)}, A^{(2)}, \dots$  which are similar, but are not equal, to  $A^{(0)}$ .

We demonstrate the construction of 5-by-5 TN matrices with band width 6. Let  $m = 5$ ,  $M = 5$ , and  $\lambda_1 = 5^5 = 3125$ ,  $\lambda_2 = 4^5 = 1024$ ,  $\lambda_3 = 3^5 = 243$ ,  $\lambda_4 = 2^5 = 32$  and  $\lambda_5 = 1^5 = 1$  in the procedure. The positive constants  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  and  $\sigma_5$  are set as  $\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3, \sigma_4 = 4$  and  $\sigma_5 = 5$ . Let us recall here that  $c_1, c_2, c_3, c_4$  and  $c_5$  must be positive. Therefore, let us consider the two cases where  $c_1 = 1, c_2 = 1, c_3 = 1, c_4 = 1$  and  $c_5 = 1$ , and  $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4$  and  $c_5 = 5$ . Numerical construction were carried out on a computer with the following specifications: OS: Mac OS X Ver. 10.9.5, CPU: 2.7 GHz 12-Core Intel Xeon E5, Compiler: C compiler Ver. clang-600.0.54 with the multiple-precision floating-point arithmetic libraries, GNU GMP Ver. 6.0.0 [22] and GNU MPFR Ver. 3.1.2 [23]. We used 53-bit, 64-bit and 96-bit precision arithmetic for generating the TN matrix  $A^{(0)}$ . The resulting matrices using 53-bit precision arithmetic were

$$A^{(0)} = \begin{pmatrix} 885.000 & 961.070 & 442.988 & 109.221 & 15.0000 \\ 1448.00 & 1957.38 & 1222.37 & 435.067 & 95.0000 \\ & 687.288 & 1082.01 & 701.610 & 253.198 \\ & & 290.531 & 427.490 & 264.537 \\ & & & 56.1535 & 73.1240 \end{pmatrix},$$

$$A^{(0)} = \begin{pmatrix} 1367.67 & 1163.89 & 481.448 & 112.174 & 15.0000 \\ 1440.22 & 1705.04 & 1049.47 & 378.985 & 85.0000 \\ & 663.777 & 952.493 & 596.957 & 214.141 \\ & & 252.800 & 349.068 & 208.832 \\ & & & 41.5420 & 50.7303 \end{pmatrix},$$

in the cases where  $c_1 = 1, c_2 = 1, c_3 = 1, c_4 = 1$  and  $c_5 = 1$ , and  $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4$  and  $c_5 = 5$ , respectively. All of the entries are rounded to the nearest 6-digit numbers. In 6-digit representations,  $A^{(0)}$  using 53-bit precision arithmetic is equal to  $A^{(0)}$  using 64-bit and 96-bit precision arithmetic, although they are distinct to each other in more than 6-digit representations.

Table 5.1: The eigenvalues  $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$  and  $\hat{\lambda}_5$  of  $A^{(0)}$  that were computed with 53-bit, 64-bit and 96-bit precision arithmetic under the settings  $c_1 = 1, c_2 = 1, c_3 = 1, c_4 = 1$  and  $c_5 = 1$ , and  $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4$  and  $c_5 = 5$ .

(a) $c_1 = 1, c_2 = 1, c_3 = 1, c_4 = 1, c_5 = 1$ .			
	53-bit	64-bit	96-bit
$\hat{\lambda}_1$	3124.99999999996	3125.000000000000	3125.000000000000
$\hat{\lambda}_2$	1023.99999999352	1024.000000000000	1024.000000000000
$\hat{\lambda}_3$	242.999999838420	243.0000000000051	243.000000000000
$\hat{\lambda}_4$	31.9999992918544	32.00000000002290	32.00000000000000
$\hat{\lambda}_5$	0.99999982866470	1.00000000005619	1.000000000000000
(b) $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4, c_5 = 5$ .			
	53-bit	64-bit	96-bit
$\hat{\lambda}_1$	3124.99999999994	3125.000000000000	3125.000000000000
$\hat{\lambda}_2$	1023.99999998810	1024.000000000000	1024.000000000000
$\hat{\lambda}_3$	242.999999606751	243.0000000000018	243.000000000000
$\hat{\lambda}_4$	31.9999973784169	32.0000000000423	32.00000000000000
$\hat{\lambda}_5$	0.99999868220200	0.99999999997805	1.000000000000000

We find that setting  $c_1, c_2, c_3, c_4$  and  $c_5$  affects the entries of  $A^{(0)}$ , but not the eigenvalues of  $A^{(0)}$ . From the positivity of all the bidiagonal nonzero entries of  $L^{(0)}, R^{(0)}, R^{(1)}, R^{(2)}, R^{(3)}$  and  $R^{(4)}$ , it is easy to check that the above  $A^{(0)}$  matrices are both TN matrices in the procedure for obtaining  $A^{(0)}$ . Table 5.1 shows the eigenvalues  $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$  and  $\hat{\lambda}_5$  of  $A^{(0)}$  that were computed using the  $QR$  algorithm [49] with 1024-bit precision arithmetic and then rounded to 16-digit numbers. As the number of bits grows larger,  $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$  and  $\hat{\lambda}_5$  get linearly closer to  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  and  $\lambda_5$ , respectively. In 16-digit representations,  $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$  and  $\hat{\lambda}_5$  using 96-bit precision arithmetic coincide with  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  and  $\lambda_5$ , respectively. With respect to the numerical accuracy of eigenvalues, similar properties are observed in other examples.

## Chapter 6

# An extension of totally nonnegative matrices with prescribed eigenvalues from Hessenberg to dense

In Chapter 5 [A3], an inverse eigenvalue problem for Hessenberg TN matrices is discussed. In this chapter, target TN matrices are extended from Hessenberg matrices to dense ones. The recursion formula for constructing dense TN matrices is derived as an extension of the discrete hungry Toda equation. A finite-step construction of dense TN matrices with prescribed eigenvalues is developed based on a discrete integrable system to be derived. A relationship between the proposed procedure and the numerical algorithms based on discrete Toda-type equations is also mentioned.

### 6.1 Inverse eigenvalue problems for TN matrices

An interesting topic in inverse eigenvalue problems is to construct a matrix with prescribed eigenvalues. Such problem is classified according to the forms and the properties of a matrix [8]. The real-valued nonnegative inverse eigenvalue problem (RNIIEP) is a problem to construct an entry-wise nonnegative matrix with prescribed real eigenvalues. The totally nonnegative inverse eigenvalue problem (TNIIEP) is a subclass of RNIIEP. A totally

nonnegative (TN) matrix is an entry-wise nonnegative matrix such that all of its minors are nonnegative. As is shown in [40, 2, 14, 48], TN matrices play important roles in several fields, such as stochastic processes, probability and combinatorics. In [7, 13], it is proved that any TN matrix can be factorized into products of lower and upper bidiagonal TN matrices with at most one nonzero off-diagonal entry. In Chapter 5 [A3], it is proposed to construct Hessenberg-type TN matrices with prescribed eigenvalues. The proposed algorithm generates a sequence  $\{\tilde{A}^{(n)} \mid n = 0, 1, \dots\}$  of  $m$ -by- $m$  matrices defined by

$$\tilde{A}^{(n)} = \tilde{L}^{(n)} \tilde{R}^{(n+\mathcal{M}-1)} \tilde{R}^{(n+\mathcal{M}-2)} \dots \tilde{R}^{(n+1)} \tilde{R}^{(n)}, \quad n = 0, 1, \dots, \quad (6.1)$$

where  $\tilde{M}$  is a positive integer and  $\tilde{L}^{(n)}$  and  $\tilde{R}^{(n)}$  are lower and upper bidiagonal  $m$ -by- $m$  matrices defined by

$$\tilde{L}^{(n)} = \begin{pmatrix} 1 & & & & \\ E_1^{(n)} & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & E_{m-1}^{(n)} & 1 \end{pmatrix}, \quad \tilde{R}^{(n)} = \begin{pmatrix} Q_1^{(n)} & 1 & & & \\ & Q_2^{(n)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & Q_m^{(n)} \end{pmatrix} \quad (6.2)$$

for  $n = 0, 1, \dots$ , respectively. The entries of  $\tilde{L}^{(n)}$  and  $\tilde{R}^{(n)}$  are generated by recursion formula,

$$\begin{cases} E_k^{(n)} = Q_k^{(n+\mathcal{M})} - Q_k^{(n)} + E_{k-1}^{(n+1)}, & k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \\ Q_{k+1}^{(n)} = Q_k^{(n+\mathcal{M})} \frac{E_k^{(n+1)}}{E_k^{(n)}}, & k = 1, 2, \dots, m-1, \quad n = 0, 1, \dots, \end{cases} \quad (6.3)$$

under some boundary condition. It is proved in Chapter 5 [A3] that the eigenvalues of  $\tilde{A}^{(n)}$  coincides with prescribed eigenvalues. Moreover, the bidiagonal entries of  $\tilde{A}^{(n)}$  are all positive under some boundary condition, so that all of  $\tilde{A}^{(n)}$  are TN matrices. The matrices  $\tilde{A}^{(n)}$  are banded matrices, whose band widths are  $1 + \min(m, \mathcal{M} + 1)$ .

The purpose of this chapter is to propose a method for constructing general dense matrices, which are constrained to be TN, with prescribed eigenvalues. We consider two positive integers  $M$  and  $N$ . Let us introduce a

sequence  $\{A^{(n)} \mid n = 0, 1, \dots\}$  of  $m$ -by- $m$  matrices defined by

$$A^{(n)} = \mathcal{L}^{(n)} \mathcal{R}^{(n)}, \quad n = 0, 1, \dots, \quad (6.4)$$

where  $\mathcal{L}^{(n)}$  and  $\mathcal{R}^{(n)}$  are lower and upper banded  $m$ -by- $m$  matrices defined by

$$\mathcal{L}^{(n)} = L^{(n)} L^{(n+M)} L^{(n+2M)} \dots L^{(n+(N-1)M)}, \quad n = 0, 1, \dots, \quad (6.5)$$

$$\mathcal{R}^{(n)} = R^{(n+(M-1)N)} \dots R^{(n+2N)} R^{(n+N)} R^{(n)}, \quad n = 0, 1, \dots, \quad (6.6)$$

respectively. Here,  $L^{(n)}$  and  $R^{(n)}$  are lower and upper bidiagonal  $m$ -by- $m$  matrices defined by

$$L^{(n)} = \begin{pmatrix} 1 & & & & \\ e_1^{(n)} & 1 & & & \\ & & \ddots & \ddots & \\ & & & e_{m-1}^{(n)} & 1 \end{pmatrix}, \quad R^{(n)} = \begin{pmatrix} q_1^{(n)} & 1 & & & \\ & q_2^{(n)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & q_m^{(n)} \end{pmatrix} \quad (6.7)$$

for  $n = 0, 1, \dots$ , respectively. Since the band widths of  $\mathcal{L}^{(n)}$  and  $\mathcal{R}^{(n)}$  are  $\min(m, N+1)$  and  $\min(m, M+1)$ , respectively, the band widths of  $A^{(n)}$  are

$$\min(m, N+1) + \min(m, M+1) - 1 \quad (6.8)$$

for  $n = 0, 1, \dots$ . If  $N = M = 1$ , then  $A^{(n)}$  are tridiagonal matrices which called Jacobi matrices. Several algorithms for constructing tridiagonal matrices with prescribed eigenvalues are discussed in [30, 5, 26, 8] and Chapter 4 [A2]. If  $N = 1$ , then  $A^{(n)}$  are upper banded Hessenberg matrices which are equivalent to  $\tilde{A}^{(n)}$ . For any  $N \geq 1, M \geq 1$ , in this chapter, we show how to construct TN matrices  $A^{(n)}$  with prescribed eigenvalues.

The recursion formula (6.3), which is called the discrete hungry Toda (dhToda) equation, is originated from the study of a box and ball system [61]. If  $M = 1$ , then the dhToda equation reduces to the discrete Toda (dToda) equation [32, 33, 34]. The dToda equation is equivalent to the well-known quotient-difference (qd) algorithm for computing eigenvalues of Jacobi matrices [34]. An application of the dhToda equation is also proposed in [17, 18, 19] for computing eigenvalues of given TN matrices  $\tilde{A}^{(0)}$ . In this chapter, we derive an extension of the dhToda equation, and develop a method for constructing TN matrices  $A^{(n)}$  with prescribed eigenvalues.

The remainder of this chapter is organized as follows. In Section 6.2, we begin with investigating a sequence generated by linear combinations of powers of prescribed distinct eigenvalues  $\lambda_1 > 0, \lambda_2 > 0, \dots, \lambda_m > 0$ . In Section 6.3, we consider an extension of Hankel determinants whose entries are given by such a sequence, and then clarify the positivity of the extended Hankel determinants thorough expanding them. In Section 6.4, we introduce an extension of Hadamard polynomials associated with the extended Hankel determinants, and then derive the recursion formula from the extended Hadamard polynomials involving the extended Hankel determinants. In Section 6.5, by making use of the extended Hankel determinants and the extended Hadamard polynomials, we prove that some of  $A^{(n)}$  are TN matrices with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ . In Section 6.6, we design a finite-step procedure for constructing dense TN matrices without explicitly using the extended Hankel determinants and the extended Hadamard polynomials. Numerical examples are given for demonstrating the proposed procedure.

## 6.2 Sequence associated with prescribed eigenvalues

In Chapter 5 [A3], a felicitous sequence plays a key role for relating  $m$  distinct positive eigenvalues of  $m$ -by- $m$  Hessenberg TN matrices to the dhToda equation (6.3). The viewpoint of sequences is also useful in the case where the TN matrix is extended from Hessenberg to dense. Readers will presume that the number of parameters in a sequence increases according to the extension of the matrix form. In this section, we consider such a sequence associated with distinct positive eigenvalues of  $m$ -by- $m$  dense TN matrices, denoted by  $\lambda_1, \lambda_2, \dots, \lambda_m$  where  $\lambda_1 > \lambda_2 > \dots > \lambda_m$ , and then clarify relationships of entries in the sequence.

For positive integers  $M$  and  $N$ , let us introduce

$$\sigma_1 := \sqrt[MN]{\lambda_1}, \quad \sigma_2 := \sqrt[MN]{\lambda_2}, \quad \dots, \quad \sigma_m := \sqrt[MN]{\lambda_m}. \quad (6.9)$$

The constants  $\sigma_1, \sigma_2, \dots, \sigma_m$  with  $N = 1$  in (6.9) just coincide with those in Chapter 5 [A3]. Moreover, let us prepare a sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  from the constants  $\sigma_1, \sigma_2, \dots, \sigma_m$  and arbitrary constants  $c_1, c_2, \dots, c_m$  by

$$f_n = c_1 \sigma_1^n + c_2 \sigma_2^n + \dots + c_m \sigma_m^n, \quad n = 0, 1, \dots \quad (6.10)$$

Then, we obtain the following proposition for linear combinations of entries of the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$ .

**Proposition 6.2.1** (Akaiwa et al. [A5]). *For any  $c_1, c_2, \dots, c_m$ , the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  satisfies*

$$f_{n+mN} + a_1 f_{n+(m-1)N} + a_2 f_{n+(m-2)N} + \dots + a_{m-1} f_{n+N} + a_m f_n = 0, \quad (6.11)$$

$$n = 0, 1, \dots,$$

where  $a_1, a_2, \dots, a_m$  are constants given through expanding a monic polynomial  $p(z) = (z - \sigma_1^N)(z - \sigma_2^N) \dots (z - \sigma_m^N)$  as

$$p(z) = z^m + a_1 z^{m-1} + a_2 z^{m-2} + \dots + a_{m-1} z + a_m. \quad (6.12)$$

*Proof.* By using (6.10), we can rewrite the left hand side of (6.11) as

$$\sum_{j=0}^n \sum_{i=1}^m c_i \sigma_i^n \sum_{j=0}^n a_j (\sigma_i^N)^{m-j} = \sum_{i=1}^m c_i \sigma_i^n p(\sigma_i^N). \quad (6.13)$$

Thus, by taking into account that  $p(\sigma_i^N) = 0$ , we have (6.11).  $\square$

### 6.3 Extended Hankel determinants and their positivity

In this section, we consider an extension of Hankel determinants in terms of the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$ , and then discuss the positivity of the extended Hankel determinants.

Let us introduce determinants of  $k$ -by- $k$  matrices

$$H_{-1}^{(n)} \equiv 0, \quad H_0^{(n)} \equiv 1,$$

$$H_k^{(n)} := \det \begin{pmatrix} \mathbf{f}_k^{(n)} & \mathbf{f}_k^{(n+M)} & \mathbf{f}_k^{(n+2M)} & \dots & \mathbf{f}_k^{(n+(k-1)M)} \end{pmatrix}, \quad (6.14)$$

$$k = 1, 2, \dots, m+1, \quad n = 0, 1, \dots,$$

where  $k$ -dimensional column vectors are defined by

$$\mathbf{f}_k^{(n)} := \begin{pmatrix} f_n \\ f_{n+N} \\ \vdots \\ f_{n+(k-1)N} \end{pmatrix}, \quad k = 1, 2, \dots, m+1, \quad n = 0, 1, \dots \quad (6.15)$$

If  $N = 1$  in (6.14) with (6.15), then  $H_k^{(n)}$  have the same form as the extended Hankel determinants  $\tau_k^{(n)}$  in Chapter 5 [A3]. So,  $\tau_k^{(n)}$  can be regarded as a specialization of  $H_k^{(n)}$  in (6.14). Since it is not necessary to distinguish  $\tau_k^{(n)}$  from  $H_k^{(n)}$  in (6.14) with (6.15), we also call  $H_k^{(n)}$  in (6.14) with (6.15) the extended Hankel determinants in this chapter. Of course, if  $M = N = 1$  in (6.14) with (6.15), then  $H_k^{(n)}$  are equal to ordinary Hankel determinants. The extended Hankel determinants  $H_k^{(n)}$  in (6.14) with (6.15) can be also expressed as

$$H_k^{(n)} = \det \begin{pmatrix} \mathbf{g}_k^{(n)} \\ \mathbf{g}_k^{(n+N)} \\ \vdots \\ \mathbf{g}_k^{(n+(k-1)N)} \end{pmatrix}, \quad k = 1, 2, \dots, m+1, \quad n = 0, 1, \dots, \quad (6.16)$$

in terms of  $k$ -dimensional row vectors

$$\begin{aligned} \mathbf{g}_k^{(n)} &:= (f_n, f_{n+M}, f_{n+2M}, \dots, f_{n+(k-1)N}), \\ k &= 1, 2, \dots, m+1, \quad n = 0, 1, \dots \end{aligned} \quad (6.17)$$

The following proposition gives boundary conditions of the extended Hankel determinants  $H_k^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  in (6.14) with (6.15).

**Proposition 6.3.1** (Akaiwa et al. [A5]). *Let us assume that the extended Hankel determinants  $H_k^{(n)}$  are given by the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  in (6.14) with (6.15) satisfying (6.10). Then, it holds that*

$$H_{m+1}^{(n)} = 0, \quad n = 0, 1, \dots \quad (6.18)$$

*Proof.* Proposition 6.2.1 immediately leads to

$$\begin{aligned} \mathbf{g}_k^{(n+mN)} + a_1 \mathbf{g}_k^{(n+(m-1)N)} + \dots + a_{m-1} \mathbf{g}_k^{(n+N)} + a_m \mathbf{g}_k^{(n)} &= 0, \\ k = 1, 2, \dots, m+1, \quad n = 0, 1, \dots \end{aligned} \quad (6.19)$$

Equation (6.19) implies that the  $(m+1)$  vectors  $\mathbf{g}_{m+1}^{(n)}, \mathbf{g}_{m+1}^{(n+N)}, \dots, \mathbf{g}_{m+1}^{(n+mN)}$  are linearly dependent. Noting that  $\mathbf{g}_{m+1}^{(n)}, \mathbf{g}_{m+1}^{(n+N)}, \dots, \mathbf{g}_{m+1}^{(n+mN)}$  are just the row vectors of  $H_{m+1}^{(n)}$ , we have (6.18).  $\square$

For the purpose of examining the positivity of the extended Hankel determinants  $H_k^{(n)}$  in (6.14) with (6.15), we present the following theorem for expansions of them.

**Theorem 6.3.2** (Akaiwa et al. [A5]). *Let  $S_k := \{\{s_1, s_2, \dots, s_k\} \mid s_1, s_2, \dots, s_k \in \mathbb{N}, 1 \leq s_1 < s_2 < \dots < s_k \leq m\}$  and  $\xi_k := \{x_1, x_2, \dots, x_k\} \in S_k$ . Then, the extended Hankel determinants  $H_k^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  in (6.14) with (6.15) can be expressed as*

$$H_k^{(n)} = \sum_{\xi_k \in S_k} c_{x_1} c_{x_2} \cdots c_{x_k} (\sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_k})^n V_{\xi_k}^{(M)} V_{\xi_k}^{(N)}, \quad (6.20)$$

$$k = 1, 2, \dots, m, \quad n = 0, 1, \dots,$$

where  $V_{\xi_k}^{(\ell)} := \prod_{1 \leq j < i \leq k} (\sigma_{x_i}^\ell - \sigma_{x_j}^\ell)$  for  $\ell = M, N$ .

*Proof.* Substituting (6.10) for (6.14) with (6.15), we can rewrite the extended Hankel determinants  $H_k^{(n)}$  as

$$H_k^{(n)} = \det \left( \Lambda_k^{(N)} D^{(n)} (\Lambda_k^{(M)})^\top \right), \quad k = 1, 2, \dots, m+1, \quad n = 0, 1, \dots, \quad (6.21)$$

where  $\Lambda_k^{(\ell)}$  and  $D^{(n)}$  are respectively  $k$ -by- $m$  matrices and  $m$ -by- $m$  diagonal matrices given by

$$\Lambda_k^{(\ell)} := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^\ell & \sigma_2^\ell & \cdots & \sigma_m^\ell \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{(k-1)\ell} & \sigma_2^{(k-1)\ell} & \cdots & \sigma_m^{(k-1)\ell} \end{pmatrix}, \quad k = 1, 2, \dots, m+1, \quad (6.22)$$

$$D^{(n)} := \text{diag} (c_1 \sigma_1^n, c_2 \sigma_2^n, \dots, c_m \sigma_m^n), \quad n = 0, 1, \dots \quad (6.23)$$

For  $\alpha := \{\alpha_1, \alpha_2, \dots, \alpha_k\}$  and  $\beta := \{\beta_1, \beta_2, \dots, \beta_k\}$ , let  $(X)_\beta^\alpha$  denotes the submatrix obtained from  $X$  by deleting all rows and columns except for the  $\alpha_1$ th,  $\alpha_2$ th,  $\dots$ ,  $\alpha_k$ th rows and the  $\beta_1$ th,  $\beta_2$ th,  $\dots$ ,  $\beta_k$ th columns. Then, Cauchy-Binet formula [37] enable us to transform (6.21) into

$$H_k^{(n)} = \sum_{\xi_k \in S_k} \det \left( (\Lambda_k^{(N)})_{\xi_k}^{\kappa_k} \right) \det \left( (D^{(n)} (\Lambda_k^{(M)})^\top)_{\kappa_k}^{\xi_k} \right), \quad (6.24)$$

where  $\kappa_k := \{1, 2, \dots, k\}$ . Noting that  $(D^{(n)}(\Lambda_k^{(M)})^\top)_{\kappa_k}^{\xi_k} = (D^{(n)})_{\kappa_m}^{\xi_k} (\Lambda_k^{(M)})^\top$  and applying Cauchy-Binet formula to  $\det((D^{(n)})_{\kappa_m}^{\xi_k} (\Lambda_k^{(M)})^\top)$ , we thus obtain

$$H_k^{(n)} = \sum_{\xi_k \in S_k} \det \left( (\Lambda_k^{(N)})_{\xi_k}^{\kappa_k} \right) \left[ \sum_{\eta_k \in S_k} \det \left( (D^{(n)})_{\eta_k}^{\xi_k} \right) \det \left( ((\Lambda_k^{(M)})^\top)_{\kappa_k}^{\eta_k} \right) \right]. \quad (6.25)$$

Since  $D^{(n)}$  is a diagonal matrix, it holds that  $\det((D^{(n)})_{\eta_k}^{\xi_k}) = 0$  for  $\eta_k \neq \xi_k$ . Moreover, it follows from (6.22) that  $(\Lambda_k^{(M)})_{\eta_k}^{\kappa_k} = V_{\eta_k}^{(M)}$  and  $(\Lambda_k^{(N)})_{\xi_k}^{\kappa_k} = V_{\xi_k}^{(N)}$ . Therefore, by considering them in (6.25), we have (6.20).  $\square$

It is obvious from (6.9) that  $\sigma_1 > \sigma_2 > \dots > \sigma_m > 0$ . Simultaneously, it follows that  $V_{\xi_k}^{(M)} > 0$  and  $V_{\xi_k}^{(N)} > 0$ . Consequently, by combining them with Theorem 6.3.2, we see that  $H_k^{(n)} > 0$  for  $k = 1, 2, \dots, m$  if  $c_k > 0$  for  $k = 1, 2, \dots, m$ .

## 6.4 Extended Hadamard polynomials and their relationships

In this section, we consider the extended Hadamard polynomials associated with the extended Hankel determinants  $H_k^{(n)}$ , and then clarify relationships among them with the help of the Jacobi identity and the Plücker relation for determinants.

As polynomials associated with the extended Hankel determinants  $H_k^{(n)}$  in (6.14) with (6.15), let us introduce polynomials of order  $k$  with respect to  $z$  given by using the column vectors  $\mathbf{f}_{k+1}^{(n)}, \mathbf{f}_{k+1}^{(n+M)}, \dots, \mathbf{f}_{k+1}^{(n+(k-1)M)}$  and  $\mathbf{z}_k := (1, z, z^2, \dots, z^k)^\top$  as

$$\begin{cases} H_{-1}^{(n)}(z) \equiv 0, & H_0^{(n)}(z) \equiv 1, \\ H_k^{(n)}(z) := \det \begin{pmatrix} \mathbf{f}_{k+1}^{(n)} & \mathbf{f}_{k+1}^{(n+M)} & \dots & \mathbf{f}_{k+1}^{(n+(k-1)M)} & \mathbf{z}_k \end{pmatrix}, \\ k = 1, 2, \dots, m, & n = 0, 1, \dots \end{cases} \quad (6.26)$$

Moreover, let  $\mathcal{H}_k^{(n)}(z)$  be monic polynomials of order  $k$  with respect to  $z$  given

as

$$\begin{aligned}\mathcal{H}_{-1}^{(n)}(z) &\equiv 0, \quad \mathcal{H}_0^{(n)}(z) \equiv 1, \\ \mathcal{H}_k^{(n)}(z) &:= \frac{H_k^{(n)}(z)}{H_k^{(n)}}, \quad k = 1, 2, \dots, m, \quad n = 0, 1, \dots\end{aligned}\quad (6.27)$$

As is shown in Section 6.2, the denominators  $H_k^{(n)}$  in (6.27) associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  are nonzero if the constants  $c_k$ , which are composed of entries of  $H_k^{(n)}$ , are positive. The polynomials  $\mathcal{H}_k^{(n)}(z)$  with  $N = 1$  are equivalent to the extended Hadamard polynomials  $\phi_k^{(n)}$  appearing in Chapter 5 [A3]. Similarly to the case of the extended Hankel determinants  $H_k^{(n)}$  in (6.14) with (6.15), we hereinafter call  $\mathcal{H}_k^{(n)}(z)$  in (6.27) the extended Hadamard polynomials. The extended Hadamard polynomials  $\mathcal{H}_k^{(n)}(z)$  with  $M = 1$  and  $N = 1$  become ordinary Hadamard polynomials.

Reconsidering (6.19), we obtain the following proposition for a relationship of the Hadamard polynomial  $\mathcal{H}_k^{(n)}(z)$  with  $p(z) = z^m + a_1 z^{m-1} + a_2 z^{m-2} + \dots + a_{m-1} z + a_m$ .

**Proposition 6.4.1** (Akaiwa et al. [A5]). *The Hadamard polynomials  $\mathcal{H}_k^{(n)}(z) = H_k^{(n)}(z)/H_k^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  with  $c_1 > 0, c_2 > 0, \dots, c_m > 0$  satisfy*

$$\mathcal{H}_m^{(n)}(z) = p(z), \quad n = 0, 1, \dots \quad (6.28)$$

*Proof.* By using the row vectors  $\mathbf{g}_m^{(n)}, \mathbf{g}_m^{(n+N)}, \dots, \mathbf{g}_m^{(n+mN)}$ , we can rewrite  $H_m^{(n)}(z)$  in (6.26) as

$$H_m^{(n)}(z) = \det \begin{pmatrix} \mathbf{g}_m^{(n)} & 1 \\ \mathbf{g}_m^{(n+N)} & z \\ \vdots & \vdots \\ \mathbf{g}_m^{(n+mN)} & z^m \end{pmatrix}. \quad (6.29)$$

Let us add the 1st, 2nd,  $\dots$ ,  $m$ th rows in  $H_m^{(n)}(z)$  multiplied by  $a_m, a_{m-1}, \dots, a_1$ , respectively, to the  $(m+1)$ th row. Taking account of (6.19) with  $k = m$ , we can express the  $(m+1)$ th row as

$$\left( \sum_{j=0}^m a_j \mathbf{g}_m^{(n+(m-j)N)} \quad \sum_{j=0}^m a_j z^{m-j} \right) = (\mathbf{0}_m \quad p(z)), \quad (6.30)$$

where  $a_0 \equiv 1$  and  $\mathbf{0}_m$  denotes the  $m$ -dimensional column zero vector. Replacing the  $(m+1)$ th row on the right hand side of (6.29) with (6.30) and considering (6.16), we thus derive  $H_m^{(n)}(z) = H_m^{(n)} p(z)$  which immediately leads to (6.28).  $\square$

The remainder of this section presents two relationships among the extended Hadamard polynomials  $\mathcal{H}_k^{(n)}(z)$ . We derive the following proposition for the evolution of  $n$  to  $n+1$  with respect to the extended Hadamard polynomials  $\mathcal{H}_k^{(n)}(z)$ .

**Proposition 6.4.2** (Akaiwa et al. [A5]). *The extended Hadamard polynomials  $\mathcal{H}_k^{(n)}(z) = H_k^{(n)}(z)/H_k^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  with  $c_1 > 0, c_2 > 0, \dots, c_m > 0$  satisfy*

$$\begin{aligned} z\mathcal{H}_{k-1}^{(n+N)}(z) &= q_k^{(n)}\mathcal{H}_{k-1}^{(n)}(z) + \mathcal{H}_k^{(n)}(z), \\ k &= 1, 2, \dots, m-1, \quad n = 0, 1, \dots, \end{aligned} \quad (6.31)$$

where  $q_k^{(n)}$  are given by using the extended Hankel determinants  $H_k^{(n)}$  in (6.14) with (6.15) as

$$q_k^{(n)} = \frac{H_{k-1}^{(n)} H_k^{(n+N)}}{H_k^{(n)} H_{k-1}^{(n+N)}}, \quad k = 1, 2, \dots, m, \quad n = 0, 1, \dots \quad (6.32)$$

*Proof.* For the polynomials  $H_k^{(n)}(z)$ , let us consider the Jacobi identity [30, 36] (see also Chapter 5 [A3])

$$\begin{aligned} &\left(H_k^{(n)}(z)\right)_{\kappa_{k+1} \setminus \{j_1\}}^{\kappa_{k+1} \setminus \{i_1\}} \left(H_k^{(n)}(z)\right)_{\kappa_{k+1} \setminus \{j_2\}}^{\kappa_{k+1} \setminus \{i_2\}} \\ &= \left(H_k^{(n)}(z)\right)_{\kappa_{k+1} \setminus \{j_2\}}^{\kappa_{k+1} \setminus \{i_1\}} \left(H_k^{(n)}(z)\right)_{\kappa_{k+1} \setminus \{j_1\}}^{\kappa_{k+1} \setminus \{i_2\}} + H_k^{(n)}(z) \left(H_k^{(n)}(z)\right)_{\kappa_{k+1} \setminus \{j_1, j_2\}}^{\kappa_{k+1} \setminus \{i_1, i_2\}}. \end{aligned} \quad (6.33)$$

By observing the extended Hankel determinants  $H_k^{(n)}$  in (6.14) with (6.15)

and the polynomials  $H_k^{(n)}(z)$  in (6.26) with (6.15), we see that

$$\begin{aligned} & \left( H_k^{(n)}(z) \right)_{\kappa_{k+1} \setminus \{k+1\}}^{\kappa_{k+1} \setminus \{k+1\}} \\ &= \det \begin{pmatrix} \mathbf{f}_k^{(n)} & \mathbf{f}_k^{(n+M)} & \cdots & \mathbf{f}_k^{(n+(k-1)M)} \end{pmatrix} = H_k^{(n)}, \end{aligned} \quad (6.34)$$

$$\begin{aligned} & \left( H_k^{(n)}(z) \right)_{\kappa_{k+1} \setminus \{k\}}^{\kappa_{k+1} \setminus \{1\}} \\ &= \det \begin{pmatrix} \mathbf{f}_k^{(n+N)} & \mathbf{f}_k^{(n+2N)} & \cdots & \mathbf{f}_k^{(n+N+(k-2)M)} & z\mathbf{z}_{k-1} \end{pmatrix} = zH_{k-1}^{(n+N)}(z), \end{aligned} \quad (6.35)$$

$$\begin{aligned} & \left( H_k^{(n)}(z) \right)_{\kappa_{k+1} \setminus \{k\}}^{\kappa_{k+1} \setminus \{k+1\}} \\ &= \det \begin{pmatrix} \mathbf{f}_k^{(n)} & \mathbf{f}_k^{(n+M)} & \cdots & \mathbf{f}_k^{(n+(k-2)M)} & \mathbf{z}_{k-1} \end{pmatrix} = H_{k-1}^{(n)}(z), \end{aligned} \quad (6.36)$$

$$\begin{aligned} & \left( H_k^{(n)}(z) \right)_{\kappa_{k+1} \setminus \{k+1\}}^{\kappa_{k+1} \setminus \{1\}} \\ &= \det \begin{pmatrix} \mathbf{f}_k^{(n+N)} & \mathbf{f}_k^{(n+N+M)} & \cdots & \mathbf{f}_k^{(n+N+(k-1)M)} \end{pmatrix} = H_k^{(n+N)}, \end{aligned} \quad (6.37)$$

$$\begin{aligned} & \left( H_k^{(n)}(z) \right)_{\kappa_{k+1} \setminus \{k, k+1\}}^{\kappa_{k+1} \setminus \{1, k+1\}} \\ &= \det \begin{pmatrix} \mathbf{f}_{k-1}^{(n+N)} & \mathbf{f}_{k-1}^{(n+N+M)} & \cdots & \mathbf{f}_{k-1}^{(n+N+(k-2)M)} \end{pmatrix} = H_{k-1}^{(n+N)}. \end{aligned} \quad (6.38)$$

By letting  $i_1 = k + 1$ ,  $i_2 = 1$ ,  $j_1 = k + 1$  and  $j_2 = k$  in (6.33), we thus derive

$$z \frac{H_{k-1}^{(n+N)}(z)}{H_{k-1}^{(n+N)}} = \frac{H_{k-1}^{(n)} H_k^{(n+N)}}{H_k^{(n)} H_{k-1}^{(n+N)}} \frac{H_{k-1}^{(n)}(z)}{H_{k-1}^{(n)}} + \frac{H_k^{(n)}(z)}{H_k^{(n)}}. \quad (6.39)$$

Therefore, by using the extended Hadamard polynomials  $\mathcal{H}_k^{(n)}(z)$  in (6.27) and the variables  $q_k^{(n)}$  in (6.32), we can rewrite (6.39) as (6.31).  $\square$

Similarly, the following proposition gives the evolution from  $n + M$  to  $n$  with respect to the extended Hadamard polynomials  $\mathcal{H}_k^{(n+M)}$ .

**Proposition 6.4.3** (Akaiwa et al. [A5]). *The extended Hadamard polynomials  $\mathcal{H}_k^{(n)}(z) = H_k^{(n)}(z)/H_k^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  with  $c_1 > 0$ ,  $c_2 > 0$ ,  $\dots$ ,  $c_m > 0$  satisfy*

$$\begin{aligned} \mathcal{H}_k^{(n)}(z) &= \mathcal{H}_k^{(n+M)}(z) + e_k^{(n)} \mathcal{H}_{k-1}^{(n+M)}(z), \\ k &= 0, 1, \dots, m, \quad n = 0, 1, \dots, \end{aligned} \quad (6.40)$$

where  $e_k^{(n)}$  are given by using the extended Hankel determinants  $H_k^{(n)}$  as

$$e_k^{(n)} := \frac{H_{k+1}^{(n)} H_{k-1}^{(n+M)}}{H_k^{(n)} H_k^{(n+M)}}, \quad k = 0, 1, \dots, m-1, \quad n = 0, 1, \dots, \quad (6.41)$$

$$e_m^{(n)} \equiv 0, \quad n = 0, 1, \dots \quad (6.42)$$

*Proof.* Let us introduce a  $(k+1)$ -by- $(k+3)$  matrix

$$F := \begin{pmatrix} \mathbf{f}_{*,k+1}^{(n+M)} & \mathbf{f}_{*,k+1}^{(n+2M)} & \dots & \mathbf{f}_{*,k+1}^{(n+(k-1)M)} & \mathbf{f}_{*,k+1}^{(n+kM)} & \mathbf{f}_{*,k+1}^{(n)} & \mathbf{z}_k & \mathbf{e}_{k+1} \end{pmatrix}, \quad (6.43)$$

where  $\mathbf{e}_{k+1}$  denotes the  $(k+1)$ -dimensional column unit vector whose  $(k+1)$ th entry is 1. Then, we derive the Plücker relation [30, 36] (see also Chapter 5 [A3]) concerning  $F$ ,

$$\begin{aligned} & \det \left( (F)_{\kappa_{k+3} \setminus \{k+2, k+3\}}^{\kappa_{k+1}} \right) \det \left( (F)_{\kappa_{k+3} \setminus \{k, k+1\}}^{\kappa_{k+1}} \right) \\ & - \det \left( (F)_{\kappa_{k+3} \setminus \{k+1, k+3\}}^{\kappa_{k+1}} \right) \det \left( (F)_{\kappa_{k+3} \setminus \{k, k+2\}}^{\kappa_{k+1}} \right) \\ & + \det \left( (F)_{\kappa_{k+3} \setminus \{k+1, k+2\}}^{\kappa_{k+1}} \right) \det \left( (F)_{\kappa_{k+3} \setminus \{k, k+3\}}^{\kappa_{k+1}} \right) = 0. \end{aligned} \quad (6.44)$$

The cofactor expansions or interchanges of columns of determinants enable us to relate  $\det((F)_{\kappa_{k+3} \setminus \{i, j\}}^{\kappa_{k+1}})$  to the extended Hankel determinants  $H_k^{(n)}$  in

(6.14) with (6.15) or the associated polynomials  $H_k^{(n)}(z)$  in (6.26) as follows,

$$\begin{aligned} & \det \left( (F)^{\kappa_{k+1}}_{\kappa_{k+3} \setminus \{k+2, k+3\}} \right) \\ &= (-1)^k \det \begin{pmatrix} \mathbf{f}_{k+1}^{(n)} & \mathbf{f}_{k+1}^{(n+M)} & \cdots & \mathbf{f}_{k+1}^{(n+kM)} \end{pmatrix} = (-1)^k H_{k+1}^{(n)}, \end{aligned} \quad (6.45)$$

$$\begin{aligned} & \det \left( (F)^{\kappa_{k+1}}_{\kappa_{k+3} \setminus \{k, k+1\}} \right) \\ &= \det \begin{pmatrix} \mathbf{f}_k^{(n+M)} & \mathbf{f}_k^{(n+2M)} & \cdots & \mathbf{f}_k^{(n+(k-1)M)} & \mathbf{z}_{k-1} \end{pmatrix} = H_{k-1}^{(n+M)}(z), \end{aligned} \quad (6.46)$$

$$\begin{aligned} & \det \left( (F)^{\kappa_{k+1}}_{\kappa_{k+3} \setminus \{k+1, k+3\}} \right) \\ &= \det \begin{pmatrix} \mathbf{f}_{k+1}^{(n+M)} & \mathbf{f}_{k+1}^{(n+2M)} & \cdots & \mathbf{f}_{k+1}^{(n+kM)} & \mathbf{z}_k \end{pmatrix} = H_k^{(n+M)}(z), \end{aligned} \quad (6.47)$$

$$\begin{aligned} & \det \left( (F)^{\kappa_{k+1}}_{\kappa_{k+3} \setminus \{k, k+2\}} \right) \\ &= (-1)^{k-1} \det \begin{pmatrix} \mathbf{f}_k^{(n)} & \mathbf{f}_k^{(n+M)} & \cdots & \mathbf{f}_k^{(n+(k-1)M)} \end{pmatrix} = (-1)^{k-1} H_k^{(n)}, \end{aligned} \quad (6.48)$$

$$\begin{aligned} & \det \left( (F)^{\kappa_{k+1}}_{\kappa_{k+3} \setminus \{k+1, k+2\}} \right) \\ &= \det \begin{pmatrix} \mathbf{f}_k^{(n+M)} & \mathbf{f}_k^{(n+2M)} & \cdots & \mathbf{f}_k^{(n+kM)} \end{pmatrix} = H_k^{(n+M)}, \end{aligned} \quad (6.49)$$

$$\begin{aligned} & \det \left( (F)^{\kappa_{k+1}}_{\kappa_{k+3} \setminus \{k, k+3\}} \right) \\ &= (-1)^{k-1} \det \begin{pmatrix} \mathbf{f}_{k+1}^{(n)} & \mathbf{f}_{k+1}^{(n+M)} & \cdots & \mathbf{f}_{k+1}^{(n+(k-1)M)} & \mathbf{z}_k \end{pmatrix} = (-1)^{k-1} H_k^{(n)}(z). \end{aligned} \quad (6.50)$$

Substituting (6.45)–(6.50) for (6.44) and considering the positivity of the extended Hankel determinants  $H_k^{(n)}$  in (6.14) with (6.15), we thus obtain

$$\frac{H_{k+1}^{(n)} H_{k-1}^{(n+M)}}{H_k^{(n)} H_k^{(n+M)}} \frac{H_{k-1}^{(n+M)}(z)}{H_{k-1}^{(n+M)}} + \frac{H_k^{(n+M)}(z)}{H_k^{(n+M)}} - \frac{H_k^{(n)}(z)}{H_k^{(n)}} = 0. \quad (6.51)$$

Equation (6.51) with the extended Hadamard polynomials  $\mathcal{H}_k^{(n)}(z)$  in (6.27) and the variables  $e_k^{(n)}$  in (6.41) therefore leads to (6.40) for  $k = 1, 2, \dots, m-1$ . From Proposition 6.3.1, it also turns out that, for  $k = m$ , (6.51) is equivalent to (6.40) with (6.42).  $\square$

## 6.5 TN dense matrices with prescribed eigenvalues

In this section, we first grasp eigenpairs of dense TN matrices  $A^{(n)} = L^{(n)} L^{(n+M)} \dots L^{(n+(N-1)M)} R^{(n+(M-1)N)} R^{(n+(M-2)N)} \dots R^{(n)}$  in terms of the extended Hadamard polynomials  $\mathcal{H}_k^{(n)}(z)$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  with  $c_1 > 0, c_2 > 0, \dots, c_m > 0$ . We next derive a key recursion formula for constructing dense TN matrices with prescribed eigenvalues. Finally, we discuss the case where constructed TN matrices are restricted to be Hessenberg form.

Let us introduce  $m$ -dimensional vectors defined by

$$\mathbf{h}_i^{(n)} = \begin{pmatrix} \mathcal{H}_0^{(n)}(\sigma_i^N) \\ \mathcal{H}_1^{(n)}(\sigma_i^N) \\ \vdots \\ \mathcal{H}_{m-1}^{(n)}(\sigma_i^N) \end{pmatrix}, \quad i = 1, 2, \dots, m, \quad n = 0, 1, \dots \quad (6.52)$$

By using Propositions 6.4.1–6.4.3, we obtain the following proposition for the TN property of the bidiagonal matrices  $L^{(n)}$  and  $R^{(n)}$  in (6.7) and their relationships to the vectors  $\mathbf{h}_i^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  with  $c_1 > 0, c_2 > 0, \dots, c_m > 0$  in (6.52).

**Proposition 6.5.1.** *The bidiagonal matrices  $L^{(n)}$  and  $R^{(n)}$  in (6.7), involving  $e_k^{(n)} = H_{k+1}^{(n)} H_{k-1}^{(n+M)} / (H_k^{(n)} H_k^{(n+M)})$  and  $q_k^{(n)} = H_{k-1}^{(n)} H_k^{(n+N)} / (H_k^{(n)} H_{k-1}^{(n+N)})$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  are TN matrices. Moreover, for the associated vectors  $\mathbf{h}_i^{(n)}$ ,*

$$R^{(n)} \mathbf{h}_i^{(n)} = \sigma_i^N \mathbf{h}_i^{(n+N)}, \quad i = 1, 2, \dots, m, \quad n = 0, 1, \dots, \quad (6.53)$$

$$L^{(n)} \mathbf{h}_i^{(n+M)} = \mathbf{h}_i^{(n)}, \quad i = 1, 2, \dots, m, \quad n = 0, 1, \dots \quad (6.54)$$

*Proof.* As is shown in Section 6.3, the extended Hankel determinants  $H_k^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  are positive if  $c_1 > 0, c_2 > 0, \dots, c_m > 0$ . Thus, it is obvious that  $q_k^{(n)} = H_{k-1}^{(n)} H_k^{(n+N)} / (H_k^{(n)} H_{k-1}^{(n+N)})$  and  $e_k^{(n)} = H_{k+1}^{(n)} H_{k-1}^{(n+M)} / (H_k^{(n)} H_k^{(n+M)})$  are also so. Minors of the bidiagonal matrices  $L^{(n)}$  and  $R^{(n)}$  are 0, 1 or products of some of  $q_k^{(n)}$  and  $e_k^{(n)}$ , respectively. Since minors of  $L^{(n)}$  and  $R^{(n)}$  are nonnegative if  $q_k^{(n)} > 0$  and  $e_k^{(n)} > 0$ , it turns out that  $L^{(n)}$  and  $R^{(n)}$  are TN.

From Proposition 6.4.1, it follows that

$$\mathcal{H}_m^{(n)}(\sigma_i^N) = 0, \quad i = 1, 2, \dots, m, \quad n = 0, 1, \dots \quad (6.55)$$

By letting  $z = \sigma_i^N$  in Proposition 6.4.2 and by considering (6.55), we derive

$$\begin{cases} \sigma_i^N \mathcal{H}_0^{(n+N)}(\sigma_i^N) = q_1^{(n)} \mathcal{H}_0^{(n)}(\sigma_i^N) + \mathcal{H}_1^{(n)}(\sigma_i^N), \\ \vdots \\ \sigma_i^N \mathcal{H}_{m-2}^{(n+N)}(\sigma_i^N) = q_{m-1}^{(n)} \mathcal{H}_{m-2}^{(n)}(\sigma_i^N) + \mathcal{H}_{m-1}^{(n)}(\sigma_i^N), \\ \sigma_i^N \mathcal{H}_{m-1}^{(n+N)}(\sigma_i^N) = q_m^{(n)} \mathcal{H}_{m-1}^{(n)}(\sigma_i^N). \end{cases} \quad (6.56)$$

Thus, by observing (6.56) in terms of  $R^{(n)}$  and  $\mathbf{h}_k^{(n)}$ , we have (6.53).

Similarly, Proposition 6.4.3 with  $z = \sigma_i^N$  and  $\mathcal{H}_{-1}^{(n)}(z) = 0$  lead to

$$\begin{cases} \mathcal{H}_0^{(n)}(\sigma_i^N) = \mathcal{H}_0^{(n+M)}(\sigma_i^N), \\ \mathcal{H}_1^{(n)}(\sigma_i^N) = \mathcal{H}_1^{(n+M)}(\sigma_i^N) + e_1^{(n)} \mathcal{H}_0^{(n+M)}(\sigma_i^N), \\ \vdots \\ \mathcal{H}_{m-1}^{(n)}(\sigma_i^N) = \mathcal{H}_{m-1}^{(n+M)}(\sigma_i^N) + e_{m-1}^{(n)} \mathcal{H}_{m-2}^{(n+M)}(\sigma_i^N) \end{cases} \quad (6.57)$$

whose matrix representation coincides with (6.54).  $\square$

Proposition 6.5.1 yields the following theorem for the TN property and eigenpairs of  $A^{(n)} = L^{(n)} L^{(n+M)} \dots L^{(n+(N-1)M)} R^{(n+(M-1)N)} \dots R^{(n+N)} R^{(n)}$ .

**Theorem 6.5.2** (Akaiwa et al. [A5]). *The matrices  $A^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  with  $c_1 > 0, c_2 > 0, \dots, c_m > 0$  are TN matrices and their eigenpairs are  $(\lambda_i, \mathbf{h}_i^{(n)})$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  with  $c_1 > 0, c_2 > 0, \dots, c_m > 0$ , namely,*

$$A^{(n)} \mathbf{h}_i^{(n)} = \lambda_i \mathbf{h}_i^{(n)}, \quad i = 1, 2, \dots, m, \quad n = 0, 1, \dots \quad (6.58)$$

*Proof.* It is shown in [20] that any products of TN matrices are TN. Thus, by combining Proposition 6.5.2 with it, we see that  $A^{(n)}$  are TN under the assumption  $c_1 > 0, c_2 > 0, \dots, c_m > 0$ .

Equation (6.55) in Proposition 6.5.1 simplifies  $(R^{(n+(M-1)N)} \dots R^{(n+2N)} R^{(n+N)} R^{(n)}) \mathbf{h}_i^{(n)}$  as

$$\begin{aligned}
& R^{(n+(M-1)N)} \dots R^{(n+2N)} R^{(n+N)} (R^{(n)} \mathbf{h}_i^{(n)}) \\
&= \sigma_i^N [R^{(n+(M-1)N)} \dots R^{(n+2N)} R^{(n+N)} (R^{(n)} \mathbf{h}_i^{(n+N)})] \\
&\vdots \\
&= \sigma_i^{(M-1)N} (R^{(n+(M-1)N)} \mathbf{h}_i^{(n+(M-1)N)}) \\
&= \sigma_i^{MN} \mathbf{h}_i^{(n+MN)}. \tag{6.59}
\end{aligned}$$

It is obvious from (6.9) that  $\sigma_i^{MN} = \lambda_i$ . By combining it with (6.59), we obtain

$$R^{(n+(M-1)N)} R^{(n+(M-2)N)} \dots R^{(n)} \mathbf{h}_i^{(n)} = \lambda_i \mathbf{h}_i^{(n+MN)}. \tag{6.60}$$

Similarly, it follows from (6.65) that

$$L^{(n)} L^{(n+M)} \dots L^{(n+M(N-2))} L^{(n+M(N-1))} \mathbf{h}_i^{(n+MN)} = \mathbf{h}_i^{(n)}. \tag{6.61}$$

Thus, by taking account that  $A^{(n)} \mathbf{h}_i^{(n)} = (L^{(n)} L^{(n+M)} \dots L^{(n+M(N-2))} L^{(n+M(N-1))}) (R^{(n+(M-1)N)} R^{(n+2N)} \dots R^{(n)}) \mathbf{h}_i^{(n)}$  and by using (6.59) and (6.61), we have (6.58).  $\square$

From Proposition 6.5.2, we also obtain the following theorem concerning  $L^{(n)}$  and  $R^{(n)}$  in (6.7), which forms  $A^{(n)} = L^{(n)} L^{(n+M)} \dots L^{(n+(N-1)M)} R^{(n+(M-1)N)} R^{(n+(M-2)N)} \dots R^{(n)}$ , and their entries.

**Theorem 6.5.3** (Akaiwa et al. [A5]). *For  $L^{(n)}$  and  $R^{(n)}$  associated with the sequence  $\{f_n(c_1, c_2, \dots, c_m; \sigma_1, \sigma_2, \dots, \sigma_m)\}_{n=0,1,\dots}$  with  $c_1 > 0, c_2 > 0, \dots, c_m > 0$ , it holds that*

$$L^{(n+N)} R^{(n+M)} = R^{(n)} L^{(n)}, \quad n = 0, 1, \dots \tag{6.62}$$

Moreover,  $q_k^{(n)}$  and  $e_k^{(n)}$ , appearing in entries of  $L^{(n)}$  and  $R^{(n)}$  satisfy

$$\begin{cases} e_k^{(n)} = q_k^{(n+M)} - q_k^{(n)} + e_{k-1}^{(n+N)}, & k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \\ q_{k+1}^{(n)} = q_k^{(n+M)} \frac{e_k^{(n+N)}}{e_k^{(n)}}, & k = 1, 2, \dots, m-1, \quad n = 0, 1, \dots \end{cases} \tag{6.63}$$

*Proof.* From Proposition 6.5.1, it follows that

$$\sigma_i^N \mathbf{h}_i^{(n+N)} = R^{(n)} L^{(n)} \mathbf{h}_i^{(n+M)}. \quad (6.64)$$

It is obvious from (6.54) in Proposition 6.5.1 that  $\mathbf{h}_i^{(n+N)} = L^{(n+N)} \mathbf{h}_i^{(n+M+N)}$ . Multiplying both hand sides of it by  $\sigma_i^N$  and by using (6.53) in Proposition 6.5.1, we can rewrite  $\sigma_i^N \mathbf{h}_i^{(n+N)}$  as

$$\sigma_i^N \mathbf{h}_i^{(n+N)} = L^{(n+N)} (\sigma_i^N \mathbf{h}_i^{(n+N+M)}) = L^{(n+N)} R^{(n+M)} \mathbf{h}_i^{(n+M)}. \quad (6.65)$$

Thus, from (6.65) and (6.64) we derive (6.62). By observing the equality in each entry in (6.64), we also have (6.63).  $\square$

Theorem 6.5.3 enables us to give similarity transformations of TN matrices  $A^{(n)} L^{(n)} = L^{(n+M)} \dots L^{(n+(N-1)M)} R^{(n+(M-1)N)} R^{(n+(M-2)N)} \dots R^{(n)}$  under evolutions with respect to  $n$ .

**Proposition 6.5.4** (Akaiwa et al. [A5]). *For the sequence  $\{A^{(n)}\}_{n=0,1,\dots}$  associated with the sequences  $\{L^{(n)}\}_{n=0,1,\dots}$ ,  $\{R^{(n)}\}_{n=0,1,\dots}$  satisfying (6.62), it holds that*

$$A^{(n+N)} = R^{(n)} A^{(n)} (R^{(n)})^{-1}, \quad n = 0, 1, \dots, \quad (6.66)$$

and

$$A^{(n+M)} = (L^{(n)})^{-1} A^{(n)} L^{(n)}, \quad n = 0, 1, \dots. \quad (6.67)$$

*Proof.* Applying (6.62) in Theorem 6.5.2 to  $R^{(n)} L^{(n)} L^{(n+M)} \dots L^{(n+(N-2)M)} L^{(n+(N-1)M)}$  repeatedly, we derive

$$\begin{aligned} & R^{(n)} L^{(n)} L^{(n+M)} \dots L^{(n+(N-1)M)} \\ &= (R^{(n)} L^{(n)}) L^{(n+M)} \dots L^{(n+(N-2)M)} L^{(n+(N-1)M)} \\ &= L^{(n+N)} (R^{(n+M)} L^{(n+M)}) \dots L^{(n+(N-2)M)} L^{(n+(N-1)M)} \\ &= L^{(n+N)} L^{(n+M+N)} R^{(n+2M)} \dots L^{(n+(N-2)M)} L^{(n+(N-1)M)} \\ &\vdots \\ &= L^{(n+N)} L^{(n+M+N)} L^{(n+2M+N)} \dots (R^{(n+(N-1)M)} L^{(n+(N-1)M)}) \\ &= L^{(n+N)} L^{(n+M+N)} L^{(n+2M+N)} \dots L^{(n+(N-1)M+N)} R^{(n+MN)} \\ &= L^{(n+N)} L^{(n+M+N)} \dots L^{(n+(N-1)M+N)} R^{(n+MN)}, \quad n = 0, 1, \dots \end{aligned} \quad (6.68)$$

Equation (6.68) gives the evolution from  $n$  to  $n + N$  of  $A^{(n)}$ ,

$$R^{(n)} A^{(n)} = A^{(n+N)} R^{(n)}, \quad n = 0, 1, \dots \quad (6.69)$$

Since  $\det R^{(n)} = q_1^{(n)} q_2^{(n)} \cdots q_m^{(n)} \geq 0$ , the inverse matrices  $(R^{(n)})^{-1}$  exist. Thus, by combining it with (6.69) we have (6.66).

Similarly to (6.68), it follows that

$$\begin{aligned} & R^{(n+(M-1)N)} R^{(n+(M-2)N)} \cdots R^{(n)} L^{(n)} \\ &= L^{(n+MN)} R^{(n+M)} R^{(n+2M)} \cdots R^{(n+NM)}. \end{aligned} \quad (6.70)$$

Equation (6.70) generates the evolution from  $n$  to  $n + M$  of  $A^{(n)}$ ,

$$A^{(n)} L^{(n)} = L^{(n)} A^{(n+M)}, \quad n = 0, 1, \dots \quad (6.71)$$

Thus, by noting  $\det L^{(n)} = 1$ , we have (6.67).  $\square$

The remainder of this section describes some restrictions on two integers  $M$  and  $N$ . If  $M \equiv 0$  modulo  $N$  in the recursion formula (6.63), then the recursion formula (6.63) is essentially equivalent to the dhToda equation (6.3). This is easily checked by replacing  $q_k^{(Nn)}$  and  $e_k^{(Nn)}$  with  $Q_k^{(n)}$  and  $E_k^{(n)}$  in the recursion formula (6.63), respectively. The recursion formula (6.63) with  $N = 1$  also simply becomes the dhToda equation (6.3). Moreover, if  $M = N$  in the recursion formula (6.63), then by replacing  $q_k^{(Nn)}$  and  $e_k^{(Nn)}$  with  $\hat{q}_k^{(n)}$  and  $\hat{e}_k^{(n)}$ , respectively, we can rewrite the recursion formula as

$$\begin{cases} \hat{e}_k^{(n)} = \hat{q}_k^{(n+1)} - \hat{q}_k^{(n)} + \hat{e}_{k-1}^{(n+1)}, & k = 1, 2, \dots, m, \quad n = 0, 1, \dots, \\ \hat{q}_{k+1}^{(n)} = \hat{q}_k^{(n+1)} \frac{\hat{e}_k^{(n+1)}}{\hat{e}_k^{(n)}}, & k = 1, 2, \dots, m-1, \quad n = 0, 1, \dots, \end{cases} \quad (6.72)$$

which is just the famous discrete Toda equation. Of course, the discrete Toda equation (6.72) is a specialization of the dhToda equation (6.3). Thus, we can regard the recurrence formula (6.63) as an extension of the dhToda equation (6.3). The matrix structure of  $A^{(n)}$  with special  $M$  and  $N$  are shown in the following proposition.

The multiple dqd algorithm for computing the eigenvalues of dense TN matrices  $A^{(n)}$  is proposed by Yamamoto and Fukaya [65]. It is emphasized that the recursion formula of the multiple dqd algorithm is not equivalent to the extended dhToda equation (6.63). The recursion formula of the multiple dqd algorithm coincides with the extended dhToda equation (6.63) only in the case where  $M = N = 1$ , namely,  $A^{(n)} = L^{(n)} R^{(n)}$ .

**Proposition 6.5.5** (Akaiwa et al. [A5]). *Let  $\mathcal{L}^{(n,j)}$  and  $\mathcal{R}^{(n,j)}$  be truncated matrices of  $L^{(n)}L^{(n+M)} \dots L^{(n+(N-1)M)}$  in (6.5) and  $R^{(n+(M-1)N)}R^{(n+(M-2)N)} \dots R^{(n)}$  in (6.6), namely,*

$$\mathcal{L}^{(n,j)} := L^{(n)}L^{(n+M)}L^{(n+2M)} \dots L^{(n+(j-1)M)}, \quad j = 1, 2, \dots, N, \quad (6.73)$$

$$\mathcal{R}^{(n,j)} := R^{(n+(j-1)N)} \dots R^{(n+2N)}R^{(n+N)}R^{(n)}, \quad j = 1, 2, \dots, M. \quad (6.74)$$

Moreover, let  $\mathcal{M}$  and  $\mathcal{N}$  be integers such that  $M = \mathcal{M}N$  and  $N = \mathcal{N}N$ , respectively. Then  $A^{(n)}$  with  $M$  and  $N$  satisfying  $M \equiv 0 \pmod{N}$  can be expressed as

$$A^{(n)} = (L^{(n)}\mathcal{R}^{(n,\mathcal{M})})^N, \quad n = 0, 1, \dots \quad (6.75)$$

It also holds that

$$A^{(Nn)} = (\mathcal{A}^{(n)})^N. \quad (6.76)$$

If  $N \equiv 0 \pmod{M}$ , then it holds that

$$A^{(n)} = (\mathcal{L}^{(n,\mathcal{N})}R^{(n)})^M, \quad n = 0, 1, \dots, \quad (6.77)$$

$$A^{(Mn)} = (\mathcal{A}^{(n)})^M. \quad (6.78)$$

*Proof.* For some positive integer  $j$ , let us assume that

$$(L^{(n)}\mathcal{R}^{(n,\mathcal{M})})^j = \mathcal{L}^{(n,j)}\mathcal{R}^{(n,j\mathcal{M})}. \quad (6.79)$$

Considering that  $\mathcal{L}^{(n,1)} = L^{(n)}$ , we can easily check that (6.79) with  $j = 1$  holds. Multiplying the both hand sides of (6.79) by  $L^{(n)}\mathcal{R}^{(n,\mathcal{M})}$ , we derive

$$(L^{(n)}\mathcal{R}^{(n,\mathcal{M})})^{j+1} = \mathcal{L}^{(n,j)}\mathcal{R}^{(n,j\mathcal{M})}L^{(n)}\mathcal{R}^{(n,\mathcal{M})}. \quad (6.80)$$

Similarly to (6.68) and (6.70) in the proof of Proposition 6.5.4, with the help of (6.62) in Theorem 6.5.3, we obtain

$$\mathcal{R}^{(n,j\mathcal{M})}L^{(n)} = L^{(n+jM)}\mathcal{R}^{(n+M,j\mathcal{M})}. \quad (6.81)$$

From (6.80) and (6.81), it follows that

$$(L^{(n)}\mathcal{R}^{(n,\mathcal{M})})^{j+1} = \mathcal{L}^{(n,j)}L^{(n+jM)}\mathcal{R}^{(n+M,j\mathcal{M})}\mathcal{R}^{(n,\mathcal{M})}. \quad (6.82)$$

Taking into account that  $\mathcal{L}^{(n,j)}L^{(n+jM)} = \mathcal{L}^{(n,j+1)}$  and  $\mathcal{R}^{(n+M,j\mathcal{M})}\mathcal{R}^{(n,\mathcal{M})} = \mathcal{R}^{(n,(j+1)\mathcal{M})}$  in (6.82), we thus obtain

$$(L^{(n)}\mathcal{R}^{(n,\mathcal{M})})^{j+1} = \mathcal{L}^{(n,j+1)}\mathcal{R}^{(n,(j+1)\mathcal{M})}. \quad (6.83)$$

By induction for  $j = 1, 2, \dots$ , we conclude that (6.79) holds for  $j = 1, 2, \dots, N$ . Since the right hand side of (6.79) with  $j = N$  becomes just  $A^{(n)}$ , we therefore have (6.75).

Let  $q_k^{(Nn)} = Q_k^{(n)}$  and  $e_k^{(Nn)} = E_k^{(n)}$  for  $n = 0, 1, \dots$ . If  $M \equiv 0 \pmod{N}$ , then it holds that  $L^{(Nn)}\mathcal{R}^{(Nn,\mathcal{M})} = \mathcal{A}^{(n)}$ . Thus, by combining it with (6.75), we obtain (6.76) if  $M \equiv 0 \pmod{N}$ . Similarly, we also have (6.77) and (6.78).  $\square$

## 6.6 A finite-step construction of TN dense matrices

In this section, we present a procedure for constructing of dense TN matrices with finite computation cost. In Theorem 6.5.2, we show that the dense TN matrices  $A^{(n)}$ , which are products of bidiagonal TN matrices  $L^{(n)}$  and  $R^{(n)}$ , have the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$  if the initial values  $e_0^{(n)} = 0$  and  $q_1^{(n)} = f_{n+N}/f_n$  are given by a sequence  $\{f_n\}$  satisfying (6.10) with positive constants  $\sigma_1, \sigma_2, \dots, \sigma_m$  and  $c_1, c_2, \dots, c_m$ . In Theorem 6.5.3, we proved a relationship concerning the evolution with respect to  $n$  in the product of  $L^{(n)}$  and  $R^{(n)}$ . The relationship gives the recursion formula (6.63), which is referred to as the extended dhToda equation, with respect to  $q_k^{(n)}$  and  $e_k^{(n)}$ . Thus, dense TN matrices  $A^{(n)}$  with prescribed eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$  are constructed by using the extended dhToda equation (6.63) with initial values  $e_0^{(n)}$  and  $q_1^{(n)}$  given by positive constants in (6.10). The procedure is as follows.

- 1: Specify the matrix size  $m$ .
- 2: Specify a positive integer  $M$ , which indicates the upper band width.
- 3: Specify a positive integer  $N$ , which indicates the lower band width.
- 4: Prescribe eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$  such that  $\lambda_1 > \lambda_2 > \dots > \lambda_m > 0$ .
- 5: Choose  $c_1, c_2, \dots, c_m$  as positive constants.
- 6: Compute  $\sigma_i = \sqrt[MN]{\lambda_i}$  for  $i = 1, 2, \dots, m$ .
- 7: **for**  $n = 0$  to  $(M + N)(m - 1) + MN$  **do**

```

8:    $f_n = c_1\sigma_1^n + c_2\sigma_2^n + \dots + c_m\sigma_m^n$ 
9: end for
10: for  $n = 0$  to  $(M + N)(m - 1) + (N - 1)M$  do
11:    $e_0^{(n)} = 0$ 
12: end for
13: for  $n = 0$  to  $(M + N)(m - 1) + (M - 1)N$  do
14:    $q_1^{(n)} = f_{n+N}/f_n$ 
15: end for
16: for  $n = 0$  to  $(M + N)(m - 1) + (N - 1)M - N$  do
17:    $e_1^{(n)} = q_1^{(n+M)} - q_1^{(n)}$ 
18: end for
19: for  $k = 1$  to  $m - 2$  do
20:   for  $n = 0$  to  $(M + N)(m - k - 1) + (M - 1)N$  do
21:      $q_{k+1}^{(n)} = q_k^{(n+M)} e_k^{(n+N)} / e_k^{(n)}$ 
22:   end for
23:   for  $n = 0$  to  $(M + N)(m - k - 1) + (N - 1)M$  do
24:      $e_{k+1}^{(n)} = q_{k+1}^{(n+M)} - q_{k+1}^{(n)} + e_k^{(n+N)}$ 
25:   end for
26: end for
27: for  $n = 0$  to  $(M - 1)N$  do
28:    $q_m^{(n)} = q_{m-1}^{(n+M)} e_{m-1}^{(n+N)} / e_{m-1}^{(n)}$ 
29: end for
30: for  $n = 0$  to  $M - 1$  do
31:   Compute  $R^{(Nn)} = \begin{pmatrix} q_1^{(Nn)} & 1 & & \\ & q_2^{(Nn)} & \ddots & \\ & & \ddots & 1 \\ & & & q_m^{(Nn)} \end{pmatrix}$ 
32: end for
33: for  $n = 0$  to  $N - 1$  do
34:   Compute  $L^{(Mn)} = \begin{pmatrix} 1 & & & \\ e_1^{(Mn)} & 1 & & \\ & & \ddots & \ddots \\ & & & e_{m-1}^{(Mn)} & 1 \end{pmatrix}$ 
35: end for
36:  $A^{(0)} = L^{(0)} L^{(M)} \dots L^{((N-1)M)} R^{((M-1)N)} \dots R^{(N)} R^{(0)}.$ 

```

This finite-step procedure needs the computational cost  $O((M + N)m^2)$ .

We give a numerical example of construction of 5-by-5 dense TN matrix. Let us set parameters  $M = 4$ ,  $N = 3$  and  $m = 5$ . Moreover, let us prescribe the eigenvalues  $\lambda_1 = 5$ ,  $\lambda_2 = 4$ ,  $\lambda_3 = 3$ ,  $\lambda_4 = 2$  and  $\lambda_5 = 1$  and choose parameters  $\sigma_1 = \sqrt[12]{5}$ ,  $\sigma_2 = \sqrt[12]{4}$ ,  $\sigma_3 = \sqrt[12]{3}$ ,  $\sigma_4 = \sqrt[12]{2}$  and  $\sigma_5 = \sqrt[12]{1} = 1$ , and  $c_1 = c_2 = \dots = c_5 = 1$ . Numerical construction was carried out on a computer, OS: Mac OS X (ver. 10.10.2), CPU: Intel Core i7 2.7 GHz, Compiler: GNU C Compiler (ver. clang-600.0.56). We employed multi precision floating point libraries, GNU GMP Library [22] (ver. 13.0.0) and GNU MPFR Library [23] (ver. 6.2.0).

The resulting matrix

$$A^{(0)} = \begin{pmatrix} 3.000000 & 8.010292 & 9.460189 & 5.063102 & 1 \\ 0.2431408 & 2.618892 & 7.867681 & 9.599262 & 5.131875 \\ 0.005522672 & 0.1703295 & 2.856828 & 8.698981 & 10.29791 \\ 0.00002480504 & 0.002333276 & 0.1093556 & 3.125130 & 9.520207 \\ 0 & 0.000004689172 & 0.0006481098 & 0.05348237 & 3.399150 \end{pmatrix}$$

was computed by using 53-bit, 64-bit and 96-bit precision arithmetic. The entries of  $A^{(0)}$  are written in 6-digit representation. Note that all the case of 53-bit, 64-bit and 96-bit precision arithmetic resulted in the same 6-digit representation.

Table 6.3 shows the computed eigenvalues  $\hat{\lambda}_1$ ,  $\hat{\lambda}_2$ ,  $\hat{\lambda}_3$ ,  $\hat{\lambda}_4$  and  $\hat{\lambda}_5$  of  $A^{(n)}$  by using the  $QR$  algorithm [49] with 1024-bit precision arithmetic for 53-bit, 64-bit and 96-bit precision arithmetic, respectively. The computed eigenvalues are rounded in 15-digit numbers. Table 6.3 implies that the computed eigenvalues  $\hat{\lambda}_1$ ,  $\hat{\lambda}_2$ ,  $\hat{\lambda}_3$ ,  $\hat{\lambda}_4$  and  $\hat{\lambda}_5$  get linearly closer to the prescribed eigenvalues  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ ,  $\lambda_4$  and  $\lambda_5$  as the number of bits become larger.

Tables 6.1 and 6.2 show entries  $q_k^{(n)}$  and  $e_k^{(n)}$  of  $R^{(n)}$  and  $L^{(n)}$  which compose the resulting  $A^{(0)}$ , respectively.

Table 6.1: The computed entries  $q_1^{(3n)}, q_2^{(3n)}, \dots, q_5^{(3n)}$  of  $R^{(3n)}$  for  $n = 0, 1, 2, 3$ .

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$q_k^{(0)}$	1.282969	1.185928	1.242699	1.299143	1.347404
$q_k^{(3)}$	1.306709	1.182876	1.233479	1.292734	1.342877
$q_k^{(6)}$	1.328202	1.183186	1.223977	1.285887	1.338142
$q_k^{(9)}$	1.347294	1.186712	1.214453	1.278552	1.333179

Table 6.2: The computed entries  $e_1^{(4n)}, e_2^{(4n)}, e_3^{(4n)}$  and  $e_4^{(4n)}$  of  $L^{(4n)}$  for  $n = 0, 1, 2$ .

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$e_k^{(0)}$	$3.116606 \times 10^{-2}$	$2.480502 \times 10^{-2}$	$1.369851 \times 10^{-2}$	$5.824590 \times 10^{-3}$
$e_k^{(4)}$	$2.706313 \times 10^{-2}$	$2.645286 \times 10^{-2}$	$1.473892 \times 10^{-2}$	$6.172281 \times 10^{-3}$
$e_k^{(8)}$	$2.281775 \times 10^{-2}$	$2.773123 \times 10^{-2}$	$1.593554 \times 10^{-2}$	$6.565911 \times 10^{-3}$

Table 6.3: The eigenvalues  $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$  and  $\hat{\lambda}_5$  of  $A^{(0)}$  that were computed with 53-bit, 64-bit and 96-bit precision arithmetic under setting  $c_1 = c_2 = \dots = c_5 = 1$ .

	53bit	64bit	96bit
$\hat{\lambda}_1$	5.00000010393680	5.00000000008894	5.00000000000000
$\hat{\lambda}_2$	4.00000089084733	4.000000000074033	4.00000000000000
$\hat{\lambda}_3$	3.00000095209525	3.000000000076930	3.00000000000000
$\hat{\lambda}_4$	2.00000016636331	2.000000000013204	2.00000000000000
$\hat{\lambda}_5$	1.00000000287182	1.00000000000235	1.00000000000000

# Chapter 7

## Concluding remarks

Each chapter of the thesis is summarized as follows.

In Chapter 2 [A1], it is shown that a sequence  $\{f_j^{(n)}\}_{j=0,1,\dots,2m-1}$ , appearing in the determinant solution to the discrete Lotka-Volterra (dLV) system, becomes an  $m$ -step Fibonacci sequence if  $\{f_j^{(0)}\}_{j=0,1,\dots,2m-1}$  is an  $m$ -step Fibonacci sequence and  $H_k^{(0)} \neq 0$ ,  $\hat{H}_k^{(0)} \neq 0$  for  $k = 1, 2, \dots, m$  where  $n$  indicates a discrete-time variable. It is also proved that, as  $n \rightarrow \infty$ , one of special solutions to the dLV system converges to the special constant that is the ratio of two successive  $m$ -step Fibonacci numbers. In the case where  $m = 2, 3, 4, 5$ , the convergence of the special solutions to the constants is demonstrated through numerical examples.

In Chapter 3 [A4], the asymptotic convergence of the quotient-difference (qd) algorithm for tridiagonal matrices with multiple eigenvalues is clarified from the viewpoint of the determinant expressions of the qd variables. First, the expressions of entries of the Hankel determinants in terms of eigenvalues are given in Theorem 3.3.2 where the Hankel determinants have the same form as in the case where eigenvalues of tridiagonal matrices are real and distinct. Next, in Theorem 3.4.7, the asymptotic expansions of the Hankel determinants are presented in terms of multiple eigenvalues of tridiagonal matrices. Finally, in Theorem 3.5.1, it is proved that the qd variables  $q_k^{(n)}$  and  $e_k^{(n)}$  converge to eigenvalues of tridiagonal matrices and 0 as  $n \rightarrow \infty$ , respectively, independently of multiplicity of eigenvalues.

By Ferreira and Parlett [12], the convergence of the  $LR$  algorithm, which is a generalization of the qd algorithm, for tridiagonal matrices is shown in the case where eigenvalues are all the same. The result in Chapter 3 [A4] is

considered to be obtained by restricting a special case of the  $LR$  algorithm.

In Chapter 4 [A2], it is clarified that the qd recursion formula is applicable to construct tridiagonal matrices with prescribed multiple eigenvalues. First, the denominator of the generating function associated with the sequence given from two suitable vectors and the powers of a general matrix  $A$  is presented through considering the Jordan canonical form of  $A$ . Accordingly, it is observed that the minimal polynomial of  $A$  coincides with the characteristic polynomial of a tridiagonal matrix  $T$ , denoted by  $\phi_T(z)$ , or the polynomial  $z^{m_L}\phi_T(z)$  for the multiplicity  $m_L$  of the zero-eigenvalues of  $A$ . Next, by taking account of the Jordan canonical form of  $T$ , it is shown that the characteristic and the minimal polynomials of  $T$  are equal to each other. Finally, a procedure for constructing tridiagonal matrices with prescribed multiple eigenvalues is proposed, and then four examples for the resulting procedure are given.

In Chapter 5 [A3], based on the integrable discrete hungry Toda (dhToda) equation, an inverse eigenvalue problem for Hessenberg-type banded totally nonnegative (TN) matrices which can be expressed by products of several bidiagonal TN matrices is discussed. The determinant solution to the dhToda equation with certain boundary conditions is firstly presented in Theorem 5.2.2. Next, in Theorem 5.4.4, the eigenpairs of banded matrices associated with the dhToda equation are clarified. In Theorem 5.5.2, it is proved that, under a restriction of boundary conditions in the dhToda equation, the banded matrices become TN with arbitrary prescribed eigenvalues. It is also shown from Theorem 5.5.2 that TN matrices can have any distinct positive eigenvalues. Finally, a finite-step construction with  $O(Mm^2)$  of the banded TN matrices is designed in Section 5.6 where  $M$  and  $m$  denote the upper band width and the matrix size, respectively.

In Chapter 6 [A5], based on an extension of the dhToda equation, a finite-step construction of dense TN matrices given as products of bidiagonal TN matrices  $A^{(n)}$  is developed. First, in Section 6.2, a sequence determined by arbitrary parameters and prescribed eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$  is given, and a polynomial in terms of parameters corresponding to  $\lambda_1, \lambda_2, \dots, \lambda_m$  is defined. Next, in Section 6.3, the extended Hankel determinants are introduced, and their positivity is discussed by expanding the extended Hankel determinants. In Section 6.4, the extended Hadamard polynomials given by the extended Hankel determinants and associated polynomials are defined. The recurrence relation of the extended Hadamard polynomials and expressions of entries of bidiagonal TN matrices which compose dense TN matrices are derived. In

Section 6.5, it is shown that the eigenpairs of  $A^{(n)}$  are  $\lambda_1, \lambda_2, \dots, \lambda_m$  and vectors whose elements are the Hadamard polynomials. The extended dhToda equation is given from the relationship of the bidiagonal matrices. Finally, a finite-step procedure with  $O((M+N)m^2)$  for constructing dense TN matrices is proposed based on the extended dhToda equation in Section 6.6 where  $M$ ,  $N$  and  $m$  denote the upper band width, the lower band width and the matrix size, respectively. Numerical construction of a dense TN matrix  $A^{(0)}$  through the proposed procedure is also given.

In the thesis, through the linear sequence  $\{f_n\}_{n=0,1,\dots}$  appearing in discrete integrable systems and the integrable algorithms, the analysis of the integrable algorithms and a new application of discrete integrable systems are discussed. Especially, discrete integrable systems enable us to solve an inverse eigenvalue problem for TN matrices that has been considered to be difficult. Few studies for the eigenvalue and inverse eigenvalue solvers have been published, although TN matrices appear in many fields, such as oscillation in mechanical systems, stochastic processes in mathematical biology, statistical computing, computer-aided geometric design [14, 20, 40]. An inverse eigenvalue problem for a symmetric pentadiagonal inner totally positive (ITP) matrix, which is decomposed as products of bidiagonal matrices, has practical applications to a vibrating beam in flexure [21]. It is expected that a symmetric pentadiagonal ITP matrix could be constructed through using the construction procedure, since ITP matrices are included in a subclass of TN matrices. Moreover, it might be said that numerical algorithms for inverse eigenvalue problems based on discrete integrable systems are novel and meaningful.

Approaches to inverse eigenvalue problems for not only TN matrices but also other structured matrices by using discrete integrable systems should be the most important works in the future. The proposed procedure could be more widely applicable to such problems. It is necessary to estimate the numerical error of the procedure for solving inverse eigenvalue problems to improve the numerical stability and investigate suitable implementation. One of the other future work is to investigate the asymptotic behavior of the integrable algorithms. The key point might be that the recursion formula of the qd algorithm is equivalent to the discrete Toda equation. It is also expected to be able to analyze the convergence of the integrable algorithms such as the dhToda algorithm and the dhLV algorithm [18], in the case where target matrices are extended from TN matrices to matrices with multiple eigenvalues.

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# List of the Author's Papers and Works Related to the Thesis

## Original Papers

- [A1] K. Akaiwa, M. Iwasaki, On m-step Fibonacci sequence in discrete Lotka-Volterra system, *J. Appl. Math. Comput.* **38** (2012) 429–442. (Chapter 2)  
The original publication is available at [www.springerlink.com](http://www.springerlink.com)  
(<http://link.springer.com/article/10.1007%2Fs12190-011-0488-x>).
- [A2] K. Akaiwa, M. Iwasaki, K. Kondo, Y. Nakamura, A tridiagonal matrix construction by the quotient difference recursion formula in the case of multiple eigenvalues, *Pacific J. Math. Indust.* **6** (2014) 21–29. (Chapter 4)
- [A3] K. Akaiwa, Y. Nakamura, M. Iwasaki, H. Tsutsumi, K. Kondo, A finite-step construction of totally nonnegative matrices with specified eigenvalues, *Numer. Algor.* (2015) DOI: 10.1007/s11075-015-9957-x. (Chapter 5)
- [A4] K. Akaiwa, M. Iwasaki, K. Kondo, Y. Nakamura, Convergence of the quotient difference algorithm for the Jacobi matrix with multiple eigenvalues, in preparation. (Chapter 3)
- [A5] K. Akaiwa, Y. Nakamura, M. Iwasaki, A. Yoshida, K. Kondo, An extension of totally nonnegative matrices with prescribed eigenvalues from Hessenberg to dense, in preparation. (Chapter 6)