Title	Surface/state correspondence as a generalized holography
Author(s)	Miyaji, Masamichi; Takayanagi, Tadashi
Citation	Progress of Theoretical and Experimental Physics (2015), 2015(7)
Issue Date	2015-07-15
URL	http://hdl.handle.net/2433/202764
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Type	Journal Article
Textversion	publisher

PTEP

Prog. Theor. Exp. Phys. **2015**, 073B03 (19 pages)
DOI: 10.1093/ptep/ptv089

Surface/state correspondence as a generalized holography

Masamichi Miyaji[1] and Tadashi Takayanagi[1,2,*]

[1] *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*
[2] *Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8582, Japan*
*E-mail: takayana@yukawa.kyoto-u.ac.jp

Received April 1, 2015; Accepted May 29, 2015; Published July 15, 2015

We propose a new duality relation between codimension-two space-like surfaces in gravitational theories and quantum states in dual Hilbert spaces. This surface/state correspondence largely generalizes the idea of holography such that we do not need to rely on the existence of boundaries in gravitational spacetimes. The present idea is motivated by the recent interpretation of anti-de Sitter space/conformal field theory (AdS/CFT) in terms of the tensor networks, so-called MERA (multi-scale entanglement renormalization ansatz). Moreover, we study this correspondence from the viewpoint of entanglement entropy and an information metric. The Cramer–Rao bound in quantum estimation theory implies that the quantum fluctuations of the radial coordinate of the AdS are highly suppressed in the large-N limit.

Subject Index B21, B24

1. Introduction

Recent progress in string theory strongly suggests that the idea of holography [1–3] will play a crucial role in constructing a complete theory of quantum gravity. The holographic principle argues that gravitational theories are equivalent to non-gravitational theories, which are defined as quantum many-body systems or quantum field theories. As in the AdS/CFT correspondence [4], which is the best known example of holography, a dual non-gravitational theory lives on the boundary of its original gravitational spacetime.

In the AdS/CFT, a dual conformal field theory (CFT) lives on a time-like boundary of an anti-de Sitter space (AdS). Since the boundary includes the time direction, the dual CFT is a dynamical theory in a Lorentzian space. We can also perform a Wick rotation of both sides and this leads to a duality between a CFT on a Euclidean space and a gravity on a hyperbolic space. As has been confirmed by numerous papers, the AdS/CFT works perfectly in both signatures.

On the other hand, if we turn to other spacetimes, such as de Sitter spaces (dS), the idea of holography, which assumes a theory on the boundary, gets much more complicated and subtle. For example, in de Sitter spaces, the only available boundaries are space-like ones. One possibility of its holography is known as the dS/CFT correspondence [5], which argues that gravitational theories on de Sitter spaces are dual to some Euclidean CFTs on their space-like boundaries. However, the AdS/CFT already argues that Euclidean CFTs are dual to gravity on hyperbolic spaces and thus we need to better understand how the dS/CFT works compared with the AdS/CFT. Moreover, if we perform a Wick rotation to find a Euclidean space, we obtain a sphere that has no boundaries.

Funded by SCOAP[3]

Therefore these motivate us to consider a generalization of the holographic principle without referring to dual theories on boundaries. For this purpose, we need a new framework-of-correspondence principle and the main aim of this paper is to propose one such possibility.

In this paper, we would like to propose a new framework of correspondence between structures of gravitational theories on any spacetimes and those of quantum states in quantum many-body systems. In particular, we argue that each codimension-two convex surface in gravitational spacetime is dual to a certain quantum state; we call this surface/state correspondence. This leads to an understanding that gravitational spacetimes emerge from various distributions of quantum states. As a particular example, our formulation provides a refined structure of AdS/CFT correspondence.

It has been conjectured in Ref. [6] (see also Ref. [7] for a modified version of this conjecture) that the AdS/CFT can be interpreted as a real-space renormalization flow called entanglement renormalization or MERA (multi-scale entanglement renormalization ansatz) [8–11]. Refer to, e.g., Refs. [12–18] for more progress in this direction. Our proposal in this paper is highly motivated by this and the continuum version (called cMERA [19]) of this conjecture, studied in Refs. [20–22]. In the cMERA interpretation of the AdS/CFT, we can consider dual quantum states for each value of the radial coordinate z of the AdS, corresponding to each step of the real-space renormalization flow. We would also like to refer to earlier interesting arguments [23,24] and [25,26], where the mechanism of the emergence of gravitational spacetimes has been discussed from the viewpoint of quantum entanglement.

This paper is organized as follows. In Sect. 2, we will present our main proposal of surface/state correspondence. In Sect.3, we will extend our proposed correspondence to the analysis of an information metric. We will also comment on an implication of quantum estimation theory in the final subsection. In Sect.4, we will discuss several examples. In Sect.5, we will summarize our conclusions and discuss future problems.

2. Surface/state correspondence proposal

Consider a gravity on an arbitrary $d+2$-dimensional spacetime M_{d+2}. Below we would like to present our proposal, which describes the gravity on M_{d+2} in terms of quantum many-body systems. We assume that this gravitational theory is approximated by the Einstein gravity coupled to various matter fields, such as supergravity theories. However, we expect that our proposal described below can be generalized to any gravitational theories by taking quantum effects appropriately into account.

2.1. A basic principle of surface/state correspondence

We start with a very large Hilbert space $\mathcal{H}_{\text{tot}}$, associated with the total spacetime M_{d+2}. Since $\mathcal{H}_{\text{tot}}$ is often given by an infinite-dimensional Hilbert space of a (generically non-local) quantum field theory, it is useful to introduce a UV cutoff ϵ, interpreted as a lattice constant. For example, the Hilbert space is identified with that of a $d+1$-dimensional CFT if M_{d+2} is given by the $d+2$-dimensional AdS spacetime following AdS/CFT. However, our construction is much more general and we do not need even time-like boundaries of M_{d+2} as long as we can assume the presence of the total Hilbert space $\mathcal{H}_{\text{tot}}$.

Now we take a codimension-two (i.e., d-dimensional) surface Σ in M_{d+2}. In this paper, we always require that this surface Σ is convex. Here we call a surface convex if an extremal surface γ that ends at an arbitrarily chosen $d-1$-dimensional submanifold in Σ is always included inside the

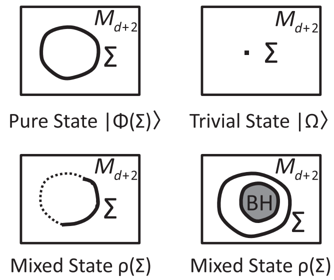

Fig. 1. Schematic images of surface/state correspondence.

region[1] surrounded by Σ. In other words, there always exists a $d+1$-dimensional space-like surface N_Σ that ends on Σ such that the extremal surface γ is completely included in N_Σ. This condition becomes important in defining the entanglement entropy in the next subsection.

First, let us focus on the case where Σ is a closed convex surface that is topologically trivial, i.e., is homologous to a point. In this case, our basic principle starts by arguing that there exists a pure quantum state $|\Phi(\Sigma)\rangle \in \mathcal{H}_{\text{tot}}$ that corresponds to the surface Σ (see the upper-left panel in Fig. 1):

$$|\Phi(\Sigma)\rangle \in \mathcal{H}_{\text{tot}} \longleftrightarrow \Sigma \in M_{d+2} \ \ \text{(topologically trivial)}. \tag{2.1}$$

More generally, if Σ is a topologically non-trivial surface, then its corresponding state is given by a mixed state $\rho(\Sigma)$ in a Hilbert space $\mathcal{H}_\Sigma$, which is a subspace of $\mathcal{H}_{\text{tot}}$ (see the lower-right panel in Fig. 1):

$$\rho(\Sigma) \in \text{End}(\mathcal{H}_\Sigma) \longleftrightarrow \Sigma \in M_{d+2} \ \ \text{(topologically non-trivial)}. \tag{2.2}$$

This is reduced to (2.1) if the surface is topologically trivial by setting $\rho(\Sigma) = |\Phi(\Sigma)\rangle\langle\Phi(\Sigma)|$. This subspace $\mathcal{H}_\Sigma$ only depends on the topological class of Σ and does not change under continuous deformation of Σ in the sense of homology. In the case of the AdS eternal black hole [27], $\mathcal{H}_{\text{tot}}$ is given by a product of two copies of CFT Hilbert space based on the thermofield construction. If Σ is wrapped on the black hole horizon, then $\mathcal{H}_\Sigma$ is given by one of the two CFT Hilbert spaces. If Σ is a topologically trivial surface, then we have $\mathcal{H}_\Sigma = \mathcal{H}_{\text{tot}}$.

So far we have assumed that Σ is closed. If Σ has its boundary $\partial\Sigma$, the dual quantum state becomes a mixed state again as in (2.2), depicted in the lower-left panel in Fig. 1. When Σ is a submanifold of a closed convex surface $\tilde{\Sigma}$ in M_{d+2}, then its mixed state is given by tracing out the Hilbert space corresponding to the complement of Σ in $\tilde{\Sigma}$, denoted by $\mathcal{H}_{\tilde{\Sigma}/\Sigma}$:

$$\rho(\Sigma) = \text{Tr}_{\mathcal{H}_{\tilde{\Sigma}/\Sigma}}\left[\rho(\tilde{\Sigma})\right]. \tag{2.3}$$

It is also useful to consider the zero size limit of a topologically trivial surface Σ. We argue that it corresponds to the state $|\Omega\rangle$ in $\mathcal{H}_{\text{tot}}$ with no real-space entanglement:

$$\lim_{A(\Sigma)\to 0} |\Phi(\Sigma)\rangle \longrightarrow |\Omega\rangle, \tag{2.4}$$

[1] If Σ is an open surface, then we define the region to be surrounded by Σ and the extremal surface that connects the boundary $\partial\Sigma$.

where $A(\Sigma)$ denotes the area of a topologically trivial manifold Σ, depicted in the upper-right panel in Fig. 1. This identification comes from an additional principle on the interpretation of the surface area as the sum of real-space entanglement, which will be explained in the next subsection. In a recent paper [22], such a state was identified with boundary states (or Cardy states) of a given CFT.

When two surfaces Σ_1 and Σ_2 are connected by a smooth deformation preserving convexity, we can describe this deformation by an integral of infinitesimal unitary transformations:

$$|\Phi(\Sigma_1)\rangle = U(s_1, s_2)|\Phi(\Sigma_2)\rangle, \tag{2.5}$$

$$U(s_1, s_2) \equiv P \cdot \exp\left[-i\int_{s_2}^{s_1} \hat{M}(s)ds\right], \tag{2.6}$$

where P denotes the path-ordering and $\hat{M}(s)$ is a Hermitian operator; the parameter s describes the continuous deformation such that $s = s_1$ and $s = s_2$ correspond to Σ_1 and Σ_2, respectively. In the expression (2.5) we assume that the surfaces are topologically trivial so that they are dual to pure states.

When two surfaces Σ_1 and Σ_2 share the same boundaries $\partial\Sigma_1 = \partial\Sigma_2$ and are related to each other by a smooth deformation preserving convexity, the corresponding density matrices are related by unitary transformation

$$\rho(\Sigma_1) = U(s_1, s_2)\rho(\Sigma_2)U^{-1}(s_1, s_2), \tag{2.7}$$

as long as there is no extremal surface between Σ_1 and Σ_2. This requirement of the absence of extremal surface is because of the requirement of convexity (for more details refer to the next subsection). The claim (2.7) can be naturally understood if we note that the deformation of Σ_1 with fixed endpoints acts non-trivially only for quantum entanglement inside Σ_1 and not for entanglement between Σ_1 and its complement.

Finally, let us comment that the precise definition of topological (non-)triviality is sometimes subtle. It is clearly defined if we can Wick-rotate M_{d+2} into a regular Euclidean geometry by using the homology as in the case of holographic entanglement entropy [28]. For example, in a typical example of black holes, the non-trivial surfaces correspond to the black hole horizons. However, such a Wick rotation is not always possible, as in time-dependent black holes. We would like to refer to recent discussions [29–31] and future developments in the context of holographic entanglement entropy for more details and we will not get into this subtle problem in this paper.

2.2. *Entanglement entropy and effective entropy*

A basic physical quantity that we can associate with a given (pure or mixed) quantum state for a d-dimensional (closed or open) surface Σ is the entanglement entropy. For this purpose, we would like to consider a division of Σ into two subregions: Σ_A and Σ_B such that $\Sigma = \Sigma_A \cup \Sigma_B$ and $\Sigma_A \cap \Sigma_B = \phi$. Following the basic principle (2.2), the surface Σ_A corresponds to a reduced density matrix ρ_A. The choice of subregions corresponds to a factorization of Hilbert space:

$$\mathcal{H}_\Sigma = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{2.8}$$

The reduced density matrix ρ_A^Σ is defined by

$$\rho_A^\Sigma = \mathrm{Tr}_{\mathcal{H}_B}\rho(\Sigma). \tag{2.9}$$

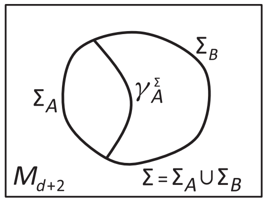

Fig. 2. Calculations of entanglement entropy in surface/state correspondence.

Then we conjecture that the von Neumann entropy of ρ_A^Σ or equally the entanglement entropy S_A^Σ (with respect to the quantum state $\rho(\Sigma)$) is given by the area formula:

$$S_A^\Sigma = \frac{A\left(\gamma_A^\Sigma\right)}{4G_N}, \tag{2.10}$$

where $A\left(\gamma_A^\Sigma\right)$ is the area of the extremal surface γ_A^Σ in M_{d+2} (refer to Fig. 2). Also, G_N is the Newton constant of the $d+2$-dimensional gravity on M_{d+2}. This extremal surface γ_A^Σ is defined such that its boundary coincides with that of Σ_A and that it is included in the region surrounded by Σ. The latter condition requires that the surface Σ should be convex, as we have mentioned.

In this formulation, the true dimension of the density matrix Σ_A (or equally $\mathcal{H}_A$) is invariant under a smooth deformation with the two boundaries of Σ_A fixed because they are related by the unitary transformation as in (2.7). Indeed, the von Neumann entropy S_A^Σ does not change under this deformation, as is clear from (2.10), which is consistent with the unitary evolution. Note that this unitary deformation of Σ_A (denoted by $\hat{\Sigma}_A$) is terminated when it reaches the extremal surface γ_A^Σ. This is because we need to keep the closed surface $\hat{\Sigma}_A \cup \Sigma_B$ convex in order to define the reduced density matrix $\rho(\hat{\Sigma}_A)$. We can also argue that $\rho(\Sigma_A)$ does not change if we deform the surface Σ_B with the same constraint.

Note that if we apply these claims to the AdS/CFT correspondence and take Σ to be the AdS boundary, then (2.10) is reduced to the holographic entanglement entropy formula [32–36]. Therefore our proposal (2.10) can be regarded as a generalization of holographic entanglement entropy. For example, we can prove the strong subadditivity in the same way as that in the holographic entanglement entropy [28,37].

Now it is also intriguing to ask what the quantum interpretation of the area of Σ itself is. Even though Σ is not an extremal surface in general, we can divide it into infinitely many small subregions, which are all well approximated by extremal surfaces. In such a small region, the geometry is approximated by a flat space and thus the extremal surfaces are given by flat planes. This consideration and the proposed correspondence (2.10) lead to the following relation:

$$\sum_i S_{A_i}^\Sigma = \frac{A(\Sigma)}{4G_N}, \tag{2.11}$$

where A_i describes the infinitesimally small portions of Σ such that $\Sigma = \cup_i A_i$ and $A_i \cap A_j = \phi$. $S_{A_i}^\Sigma$ is the entanglement when we trace out the complement of A_i inside Σ. It is useful to note that the left-hand side of (2.11) is always larger than or equal to the total von Neumann entropy for $\rho(\Sigma)$ owing to the subadditivity relation.

We would like to call the left-hand side of (2.11) the effective entropy $S_{\text{eff}}(\Sigma)$. This quantity can be interpreted as the log of the effective dimension of the Hilbert space $\mathcal{H}_\Sigma$, written as $\log[\dim\mathcal{H}_\Sigma^{\text{eff}}]$. Note that this effective dimension $\dim\mathcal{H}_\Sigma^{\text{eff}}$ counts the dimension of effective degrees of freedom

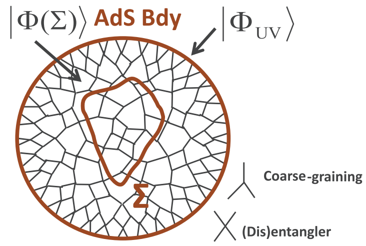

Fig. 3. Tensor network of MERA and surface/state correspondence in AdS/CFT. In the standard AdS/CFT correspondence, the CFT state (UV state) is given by $|\Phi_{\rm UV}\rangle$ and is defined by the quantum state realized at the boundary (Bdy) of the above MERA network. The black lines describe the flow of quantum states on discretized lattice points, which are acted on by the coarse-graining and (dis)entangler operations. For a given convex closed surface Σ, we define the corresponding pure quantum state $|\Phi(\Sigma)\rangle$ by contracting the indices of the tensors starting from the UV state $|\Phi_{\rm UV}\rangle$, following the tensor network as Σ is homologous to the AdS boundary. Note that we can add a dummy trivial state $|0\rangle$ for each coarse-graining operator so that both $|\Phi(\Sigma)\rangle$ and $|\Phi_{\rm UV}\rangle$ live in the same Hilbert space $\mathcal{H}_{\rm tot}$. Equally, we can start from any point inside the region surrounded by Σ and expand into Σ to eventually find the state $|\Phi(\Sigma)\rangle$. Therefore, we can apply this correspondence to any network, even those without boundaries.

that participate in the entanglement between A_i and its complement and is smaller than the actual dimension of the Hilbert space $\dim\mathcal{H}_\Sigma$. We expect that this quantity $S_{\rm eff}(\Sigma)$ is of the same order as the number of links in the tensor network representation that intersect with Σ. For this, refer to Fig. 3 and the arguments in the next subsection.

Then the relation (2.11) can be written as[2]

$$S_{\rm eff}(\Sigma) = \log\left[\dim\mathcal{H}_\Sigma^{\rm eff}\right] = \frac{A(\Sigma)}{4G_N}. \tag{2.12}$$

If we apply this to the AdS/CFT and choose Σ to be the AdS boundary, then this estimation of effective dimension is reduced to the holographic bound [47]. Note also that the quantity (2.12) does not depend on the details of the decomposition of Σ into infinitesimally small pieces.

As the simplest example, consider the case where we shrink the closed surface Σ to zero size. Then all S_A and the effective dimension become vanishing. This means that the corresponding pure state $|\Phi(\Sigma)\rangle$ does not have any real-space entanglement, in spite of the fact that this state is defined in the infinitely large Hilbert space $\mathcal{H}_{\rm tot}$ and this justifies our previous identification (2.4).

If a closed or open surface Σ is an extremal surface, then the entanglement entropy S_A^Σ coincides with the effective entropy $S_{\rm eff}(\Sigma)$ for the surface Σ_A. This means that the state $\rho(\Sigma_A)$ saturates the subadditivity for any choices of subsystems in Σ_A. Therefore the density matrix $\rho(\Sigma_A)$ is a direct product of density matrices at each point: $\rho(\Sigma_A) = \otimes_i \rho(\Sigma_{A_i})$. This means that there is no true quantum entanglement within Σ_A. Indeed, a typical example of such a closed surface is a black hole horizon. If Σ_A is an open surface as in Fig. 2, S_A^Σ and $S_{A_i}^\Sigma$ are non-trivial and all come from the entanglement between Σ_A and Σ_B, not from the entanglement inside Σ_A.

[2] The areas of generic surfaces as in the right-hand side of (2.12) have recently been interpreted as an interesting quantity called differential entropy in the dual CFT [38–44]. Also, another intriguing interpretation of the same quantity in terms of entanglement of gravitational theories has been conjectured in Refs. [45,46]. Note that our interpretation (2.12) is apparently different from these in that our basic principle introduces quantum states for each surface.

2.3. Relation to (c)MERA and tensor networks

In the setup of AdS/CFT correspondence, our surface/state correspondence can be understood from the framework of real-space renormalization called multi-scale entanglement renormalization ansatz (MERA), introduced in Refs. [8–11]. Our construction is closer to its continuum version, called continuous MERA (cMERA) [19]. In Ref. [6], it has been argued that the mechanism of AdS/CFT can be understood as that of MERA if we consider discretizations (or lattice versions) of CFTs. Also, to realize continuum quantum field theoretic descriptions of AdS/CFT, we can relate cMERA to AdS/CFT [20] to eliminate lattice artifacts. We would like to refer the reader to these references for detailed explanations of the conjectured equivalence between (c)MERA and AdS/CFT. Below we will give a brief summary of the argument.[3]

MERA gives a scheme of real-space renormalization in terms of wave functions. This is rather different from the familiar Wilsonian approach, where the renormalization-group flow in momentum space is studied in terms of effective actions. Suppose we are interested in the ground state of a quantum spin chain with a complicated Hamiltonian. We can coarse-grain the original spin system by combining two spins into a single spin according to an appropriate linear map (so-called isometry). We can repeat this process an arbitrary number of times until we reach just a single spin. However, if we literally do this, the short-range entanglement of the coarse-grained quantum state is not necessarily removed. Therefore, we need to cut out the short-range entanglement just after each coarse-graining process by a unitary transformation, called a disentangler. These whole processes that modify the original spin state into a single spin state can be regarded as a network of spins, which is a particular example of a tensor network (see Fig. 3). The conjecture in Ref. [6] argues that the tensor network of MERA can be identified with the AdS spacetime, which can be qualitatively confirmed, e.g., from evaluations of entanglement entropy. For reviews on tensor networks refer to, e.g., Refs. [55–57].

In MERA or, more generally, tensor networks, we can pick up a convex closed surface in a given network and define a quantum state defined on the boundary of that part, by contracting indices of matrices of disentanglers and coarse-grainings (see the surface Σ in Fig. 3). This consideration naturally leads to our surface/state correspondence, assuming that the tensor networks are equivalent to gravitational theories. For this connection, as already mentioned in Ref. [20], we do not need the presence of any actual boundaries in a gravitational spacetime, as is usually required in holography.

It is also useful to note that the number of intersections between Σ and the links in the tensor estimate the effective entropy $S(\Sigma)$. On the other hand, the entanglement entropy S^{Σ}_{A} is estimated by the minimum number of intersections for curves that are homologous to A. However, we are not arguing for a precise match for these estimations as each link in a tensor network is not necessarily maximally entangled.

In the cMERA formulation [19], we focus on a one-parameter family of quantum states $|\Psi(u)\rangle$. The parameter u ($-\infty < u < 0$) denotes the scale of renormalization such that the momentum cutoff is given by $|k| \leq e^{u}/\epsilon$, where ϵ is the original UV cutoff (lattice spacing) of the theory that we consider. The UV limit corresponds to $u = 0$ and its state $|\Phi(0)\rangle$ coincides with the vacuum of a given CFT. On the other hand, $u = -\infty$ corresponds to the IR limit and, with an appropriate IR cutoff (i.e., a very

[3] For recent progress on different approaches to the construction of bulk theory from the boundary via renormalization-group (RG) flow, refer to, e.g., Refs. [48–54].

small mass term), this IR state is given by the state $|\Omega\rangle$ with no real-space entanglement. We can identify such states with boundary states [22] in CFTs with an appropriate regularization.

Since the vacuum state $|\Psi(0)\rangle$ is highly entangled, we can construct this state by adding quantum entanglement to the trivial state $|\Omega\rangle$ as

$$|\Psi(u)\rangle = Pe^{-i\int_{u_{\rm IR}}^{u}(K(s)+L)ds}|\Omega\rangle, \tag{2.13}$$

where P denotes the path-ordering [19]. The Hermitian operators $K(s)$ and L describe the disentangler and the coarse-graining procedure. For our purpose it is useful to introduce another (equivalent) formulation [20] given in terms of the state

$$|\Phi(u)\rangle \equiv e^{iuL}|\Psi(u)\rangle = Pe^{-i\int_{u_{\rm IR}}^{u}\hat{K}(s)ds}|\Omega\rangle, \tag{2.14}$$

where $\hat{K}(s)$ is defined by the disentangler in the interaction picture

$$\hat{K}(u) = e^{iuL}K(u)e^{-iuL}. \tag{2.15}$$

Note that we perform a scale transformation after disentangling for the states $|\Psi(u)\rangle$, while we do not for $|\Phi(u)\rangle$.

One important idea of cMERA is to keep the dimension of the total Hilbert space the same. This can be realized by combining the coarse-graining procedure with adding trivial dummy states (e.g., $|0\rangle$ at each lattice point) to keep the dimension the same, so that the coarse-graining is described by a unitary transformation. We can do the same procedure for the coarse-graining operator in the network of Fig. 3 so that the states $|\Phi(\Sigma)\rangle$ live in the same Hilbert space $\mathcal{H}_{\rm tot}$. Therefore, any states $|\Phi(u)\rangle$ belong to states in the large Hilbert space $\mathcal{H}_{\rm tot}$ defined in the UV theory. This trick was also borne in mind in our discussions of Sect. 2, where we argued that the Hilbert space does not change as long as the deformation of a surface preserves the homology class. For our purpose, we have in mind a generalization of cMERA such that we can choose the parameter u for any inhomogeneous coarse-graining procedure (or inhomogeneous RG flow).

From the AdS/CFT viewpoint, this parameter u corresponds to the coordinate of extra dimension defined by the following metric of Poincaré AdS$_{d+2}$:

$$ds^2 = R^2du^2 + \frac{R^2}{\epsilon^2}e^{2u}dx_\mu dx^\mu, \tag{2.16}$$

where R is the AdS radius and ϵ is the UV cutoff of the dual CFT.

It is also possible to extend the cMERA formalism to finite-temperature CFTs, which has been done in Ref. [21] for free scalar field theories. The MERA for quantum many-body systems at finite temperature was recently formulated in Ref. [18] by improving the construction in Refs. [6,14]. The finite-temperature (c)MERA can be nicely interpreted as the geometry of a double-sided AdS black hole.

3. Information metric

Consider two different closed surfaces Σ and Σ', which are both topologically trivial. According to (2.1), we can associate them with pure states in the common Hilbert space $\mathcal{H}_{\rm tot}$. This allows us to define the inner product $\langle\Phi(\Sigma)|\Phi(\Sigma')\rangle$ between them and provides us with additional information on the structure of quantum states. In this section, we will focus on the inner product when the two surfaces Σ and Σ' are very close to each other by extracting the information metric and will study its dual gravity description. We will also comment on an implication of quantum estimation theory in the final subsection.

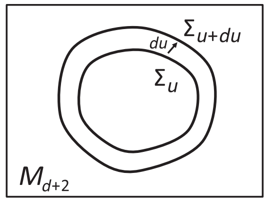

Fig. 4. Infinitesimal deformations of surfaces for the information metric.

3.1. Information metric and gravity dual

Consider a one-parameter family of codimension-two closed surfaces Σ_u and assume that the surface Σ_u for any u is topologically trivial such that it corresponds to the pure state $|\Phi(\Sigma_u)\rangle$. We are interested in an infinitesimal shift du of u and in the inner product $\langle\Phi(\Sigma_u)|\Phi(\Sigma_{u+du})\rangle$ (refer to Fig. 4). Note that this inner product approaches 1 in the limit $du \to 0$ and it is straightforward to confirm that it always behaves as $1 - \langle\Phi(\Sigma_u)|\Phi(\Sigma_{u+du})\rangle = O\big(du^2\big)$. Therefore we will define the quantity $G^{(B)}_{uu}$ as follows in the limit $du \to 0$:

$$1 - |\langle\Phi(\Sigma_u)|\Phi(\Sigma_{u+du})\rangle| = G^{(B)}_{uu} du^2. \tag{3.17}$$

This quantity $G^{(B)}_{uu}$ is called the Fisher information metric (in terms of Bures distance) [58–62] and measures the distance between two different quantum states.[4] We will give a more general definition of this metric in a later subsection including mixed states.

We describe the metric of M_{d+2} by using the Gaussian normal coordinate as follows:

$$ds^2 = R^2 du^2 + g_{\mu\nu}(x, u) dx^\mu\, dx^\nu, \quad (\mu, \nu = 0, \ldots, d), \tag{3.18}$$

and consider the shift of u. Here R is a constant with the length dimension so that u becomes dimensionless. Our formulation is covariant on the choice of R.

In this setup, we conjecture that the information metric $G^{(B)}_{uu}$ defined by (3.17) is expressed in terms of the gravity on M_{d+2} in the following form:

$$G^{(B)}_{uu} = \frac{1}{G_N} \int_{\Sigma_u} dx^d \sqrt{g(x)} \int_{\Sigma_u} dy^d \sqrt{g(y)} \cdot P_{\mu\nu\xi\eta}(x, y, u) \left(\frac{\partial g^{\mu\nu}(x, u)}{\partial u}\right) \left(\frac{\partial g^{\xi\eta}(y, u)}{\partial u}\right), \tag{3.19}$$

where G_N is the Newton constant of the gravity on M_{d+2} and $P_{\mu\nu\xi\eta}$ is a certain function that is proportional to the degrees of freedom (such as the central charges in CFTs). Note that, even though x_μ and y_μ are originally coordinates of $d+1$-dimensional space, we restricted to the d-dimensional space-like surface Σ_u in the integrals in (3.19).

If we assume that the metric $g_{\mu\nu}(x, u)$ does not depend on the transverse coordinate x, then we argue that (3.19) gets simplified into

$$G^{(B)}_{uu} = \frac{1}{G_N} \left(\int_{\Sigma_u} dx^d \sqrt{g}\right) \cdot \hat{P}_{\mu\nu\xi\eta}(u) \left(\frac{\partial g^{\mu\nu}(u)}{\partial u}\right) \left(\frac{\partial g^{\xi\eta}(u)}{\partial u}\right). \tag{3.20}$$

In order to match the cMERA results, as we will discuss briefly later, the tensor $\hat{P}_{\mu\nu\xi\eta}$ should scale like $g_{\mu\nu} g_{\xi\eta}$ under the Weyl scaling. The fact that the information metric is non-negative suggests

[4] As we will mention in Sect. 3.4, there is another definition of the Fisher information metric $G^{(S)}_{uu}$ based on the relative entropy. These two metrics $G^{(B)}_{uu}$ and $G^{(S)}_{uu}$ are equivalent only for classical states. However, in this paper, we will not distinguish them seriously and assume that the results in this paper can be applied to both metrics.

that $\hat{P}$ is also non-negative. From this estimation we can find that the tensor P in (3.19) scales as $P_{\mu\nu\xi\eta}(x, y, u) \sim \delta^d(x-y)g_{\mu\nu}g_{\xi\eta}/\sqrt{g} + \cdots$, where we have abbreviated non-local terms.

To understand the form (3.19), it is useful to think of an artificial Hamiltonian $H(u)$ such that the state $|\Phi(\Sigma_u)\rangle$ is the ground state of $H(u)$ for each u. Note that in general the real-time evolution by the time t can be described by another true Hamiltonian $H_{\text{true}}(u)$. In particular, in the cMERA description of a CFT ground state, $H(u)$ is given by the Hamiltonian after entanglement renormalization and coincides with the true Hamiltonian $H_{\text{true}}(u)$ because the time evolution is trivial.

Then, by using the standard second-order perturbation theory of quantum mechanics (assuming no ground-state degeneracy), we can derive

$$1 - |\langle\Phi(\Sigma_u)|\Phi(\Sigma_{u+du})\rangle| = (du)^2 \cdot \sum_{m\neq 0} \frac{|\langle m|\partial_u H(u)|0\rangle|^2}{(\Delta E_m)^2}, \tag{3.21}$$

where the ground state $|0\rangle$ should be regarded as $|\Phi(\Sigma_u)\rangle$ and $|m\rangle$ denote all of its excited states; ΔE_m is the energy difference between $|m\rangle$ and $|0\rangle$ with respect to the Hamiltonian $H(u)$. The infinitesimal change $\delta g_{\mu\nu}$ in (3.18) linearly affects the Hamiltonian $\delta H(u) = \int_{\Sigma_u} dx^d \delta g^{\mu\nu}(x) O_{\mu\nu}(x)$, where $O_{\mu\nu}$ is a certain operator that is analogous to the energy momentum tensor $T_{\mu\nu}$. Because we have $\delta g_{\mu\nu} = du \cdot \frac{\partial g_{\mu\nu}}{\partial u}$ for the infinitesimal change of u, we reproduce the expression (3.19). Note also that (3.21) shows the well known fact that the information metric $G^{(B)}_{uu}$ is non-negative for any unitary theory.

To estimate the normalization of P and $\hat{P}$ in (3.19) and (3.20), consider the cMERA description of a $d+1$-dimensional CFT ground state. The information metric in cMERA was computed in Ref. [20] for surfaces Σ_u defined by a fixed u in the AdS space (2.16) and the result is given by

$$G^{(B)}_{uu} = N_{\text{deg}} \cdot \frac{V_d}{\epsilon^d} e^{du} \sim S_{\text{eff}}(\Sigma_u), \tag{3.22}$$

where N_{deg} estimates the number of fields and is proportional the central charge. It is also useful to notice that the right-hand side of (3.22) is identical to the effective entropy (2.12) up to an order-one factor.

Note that, even though this behavior (3.22) was derived for a free massless scalar field theory, we naturally expect it to also be applicable to any CFTs due to the scaling property, where it is proportional to the effective volume of phase space. We can indeed confirm that (3.22) can be reproduced from (3.20) by substituting the metric of pure AdS (2.16) and noting that $\hat{P}_{\mu\nu\xi\eta}\partial_u g^{\mu\nu}\partial_u g^{\xi\eta}$ is an $O(1)$ constant and that the ratio R^d/G_N is proportional to the central charge of the CFT.

So far we have only discussed the information metric for topologically trivial closed surfaces, which correspond to pure states. However, it is natural to expect that our expressions (3.19) and (3.20) can be applied to open surfaces and topologically non-trivial closed surfaces, which are dual to mixed states. Indeed, we can define the Fisher information metric for mixed states, as we will explain in Sect. 3.4.

Finally, we would like to suggest a plausible argument to fix the form of the tensor $\hat{P}$. For simplicity, let us assume that our space is static. Consider an AdS/CFT setup when Σ_u becomes extremal (or equally minimal) at $u = u_0$. As we mention, the dual state is very special in that there is no real-space entanglement inside Σ_{u_0} as the subadditivity is saturated. As we move from the AdS boundary to the bulk, the information metric G_{uu} is expected to decrease[5] and eventually it becomes zero. However, we know that G_{uu} is non-negative and thus this flow is terminated at this point. Indeed, according to our arguments in the previous section, we cannot move the surface across the extremal surface (see our argument in section 2.2). Therefore, this implies that we have $G_{uu} = 0$ at $u = u_0$. As the condition of the extremal surface is given by the vanishing of the trace of extrinsic curvature, $K_u = g_{\bar{\mu}\bar{\nu}}\partial_u g^{\bar{\mu}\bar{\nu}} = 0$, we find that the tensor $\hat{P}$ should only have spacial components and is given by the form $\hat{P}_{\bar{\mu}\bar{\nu}\bar{\xi}\bar{\eta}} = C_P \cdot g_{\bar{\mu}\bar{\nu}} g_{\bar{\xi}\bar{\eta}}$, where $(\bar{\mu}, \bar{\nu}, \bar{\xi}, \bar{\eta})$ denotes the d-dimensional spacial part of the Lorentzian $d+1$-dimensional indices (μ, ν, ξ, η). The coefficient C_P is proportional to the degrees of freedom of the CFT. We can extend this argument to the time-dependent case as our present argument tells us that G_{uu} is proportional to the integral $\frac{1}{G_N}\int_{\Sigma_u} dx^d \sqrt{g} K_u^2$, where K_u is the trace of extrinsic curvature in the direction of our surface deformation.

3.2. *Analysis of the information metric in AdS/CFT*

Here we would like to present an explicit analysis of the information metric by concentrating on the AdS/CFT example by applying the cMERA construction. We focus on the state $|\Psi(u)\rangle$ defined in (2.13) instead of our standard state $|\Phi(u)\rangle$. Note that they are related by the scale transformation as in (2.14). Thus, the information metric that we compute in this subsection is different from the original one (3.17). Nevertheless, it may still be helpful to know the general behavior of the information metric in CFTs.

Below, we will evaluate the Fisher information metric for the state $|\Psi(u)\rangle$. We would like to postpone general studies of the Fisher information metric in quantum field theories to future work (M. Miyaji et al., work in progress). Note that the change of u is simply interpreted as the standard renormalization-group flow in the relativistic field theories. Refer also to Ref. [63] for interesting results of the inner products under quantum quenches.

Consider the ground states $|0 : V_i\rangle$ $(i = 1, 2)$ for the theories whose Euclidean actions are given by $S_{\mathrm{CFT}} + \int dt V_i(t)$, where $V_i(t)$ describes a infinitesimally small perturbation around a CFT defined by the action S_{CFT}; t is now a Euclidean time. Then the overlap between these vacuum states is given by the following formula (M. Miyaji et al., work in progress):

$$\langle 0 : V_2 | 0 : V_1\rangle = \frac{Z\left[\int dt\theta(-t)V_1(t) + \int dt\theta(t)V_2(t)\right]}{\sqrt{Z\left[\int dt V_1(t)\right] Z\left[\int dt V_2(t)\right]}}, \tag{3.23}$$

where $Z[V]$ denotes the partition function in the presence of the interaction V. If V_i are infinitesimal, we can expand (3.23) as

$$\langle 0 : V_1 | 0 : V_2\rangle = 1 - \frac{1}{2}\left\langle \int_{\epsilon}^{\infty} dt(V_1(t) - V_2(t)) \int_{-\infty}^{-\epsilon} dt'\big(V_1(t') - V_2(t')\big)\right\rangle + O\big(V^3\big), \tag{3.24}$$

where we have assumed time-reversal symmetry of the interaction $V_i(t)$. Note also that we have introduced the UV cutoff ϵ to remove the divergent contributions when the positions of two interaction vertices collide. Moreover, we find that the inner product (3.24) is real valued by considering the case where V_i are Hermitian operators.

[5] This looks analogous to the c-theorem. We would like to come back to this in more detail in future work (M. Miyaji et al., work in progress).

We express the gravitational spacetime metric as in the Fefferman–Graham coordinate

$$ds^2 = R^2 du^2 + \frac{R^2}{\epsilon^2} e^{2u} \hat{g}_{\mu\nu}(x,u) dx^\mu dx^\nu \quad (\mu, \nu = 0, 1, 2, \ldots, d), \tag{3.25}$$

where R is the AdS radius and ϵ is the UV cutoff of the dual CFT. The rescaled metric $\hat{g}$ corresponds to the metric in the CFT.

We assume that the metric perturbation is time independent, so that the action for the Euclidean theory on Σ_u is given by

$$S_{\mathrm{CFT}} + \int dt\, d^d x\, \delta \hat{g}_{\mu\nu}(x,u) T^{\mu\nu}(x,t), \tag{3.26}$$

where we have defined $\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \delta \hat{g}_{\mu\nu}$.

In this setup, we can show that[6]

$$1 - |\langle \Psi(u) | \Psi(u+du) \rangle| = \frac{(du)^2}{2} \int d^d x\, d^d y \left(\frac{\partial \hat{g}^{\mu\nu}(x,u)}{\partial u} \right) \left(\frac{\partial \hat{g}^{\sigma\rho}(y,u)}{\partial u} \right) Q_{\mu\nu\sigma\rho}(x-y), \tag{3.27}$$

where we have introduced the tensor Q as follows:

$$Q_{\mu\nu\sigma\rho}(x-y) = \int_\epsilon^\infty dt \int_{-\infty}^{-\epsilon} dt' \langle T_{\mu\nu}(x,t) T_{\sigma\rho}(y,t') \rangle. \tag{3.28}$$

We can calculate this by using the explicit expression [64]:

$$\langle T_{\mu\nu}(x,t) T_{\sigma\rho}(y,t') \rangle = \frac{C_T}{\left((x-y)^2 + (t-t')^2\right)^{d+1}} R_{\mu\alpha}(x,t:y,t') R_{\nu\beta}(x,t;y,t') \mathcal{E}^T_{\alpha\beta:\sigma\rho}, \tag{3.29}$$

where

$$\mathcal{E}^T_{\mu\nu:\eta\xi} = \frac{1}{2}\left(\delta_{\mu\eta}\delta_{\nu\xi} + \delta_{\mu\xi}\delta_{\nu\eta}\right) - \frac{1}{d+1}\delta_{\mu\nu}\delta_{\xi\eta} \tag{3.30}$$

$$R_{\mu\nu}(x,t:y,t') = \delta_{\mu\nu} - 2\frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2 + (t-t')^2}. \tag{3.31}$$

Note that, in even-dimensional CFTs, the coefficient C_T is proportional to the central charge of the dual CFT in the Einstein gravity description of AdS/CFT.

For simplicity, let us set $d = 1$, i.e., a 2D CFT, and compute the tensor $Q_{\mu\nu\sigma\rho}$ explicitly. To simplify the expressions, we define the components (a, b, c) so that they denote pairs of components (tt, xx, tx). Note that in this convention the tensor Q is written as a symmetric 3×3 matrix. We obtain

$$\begin{aligned} Q_{aa} = Q_{bb} = -Q_{ab} = -Q_{cc} &= C_T \frac{3\epsilon^4 - 6\epsilon^2 (x-y)^2 - (x-y)^4}{12\left((x-y)^2 + \epsilon^2\right)^3}, \\ Q_{ac} = -Q_{bc} &= C_T \frac{2\epsilon^3 (x-y)}{3\left((x-y)^2+\epsilon^2\right)^3} = -\frac{\pi}{12} C_T \delta'(x-y), \end{aligned} \tag{3.32}$$

where we have taken the $\epsilon \to 0$ limit in the final expression.

[6] Strictly speaking, the deformation in (3.26) corresponds only to the shift $\hat{g}_{\mu\nu}(x,u) \to \hat{g}_{\mu\nu}(x, u+du)$ with the warp factor e^{2u} unchanged in the metric (3.25). However, we can find the same result even if we do the infinitesimal shift u in e^{2u}, owing to the fact that the scale invariance of our CFT leads to the identity $Q^{\mu}_{\mu\sigma\rho} = Q_{\mu\nu}{}^{\rho}_{\rho} = 0$.

It is also useful to perform the Fourier transformation $Q_{\mu\nu\rho\sigma}(x-y)=\int_{-\infty}^{\infty}dkQ_{\mu\nu\rho\sigma}(k)e^{-ikx}$, which leads to

$$\begin{aligned} Q_{aa}&=Q_{bb}=-Q_{ab}=-Q_{cc}=\tfrac{1}{24}C_T|k|,\\ Q_{ac}&=-Q_{bc}=\tfrac{1}{24}iC_Tk. \end{aligned} \tag{3.33}$$

After we goes back to the original Lorentz signature by the Wick rotation $t\to it$, we get

$$\begin{aligned} Q_{aa}&=Q_{bb}=Q_{ab}=Q_{cc}=\tfrac{1}{24}C_T|k|,\\ Q_{ac}&=Q_{bc}=\tfrac{1}{24}C_Tk. \end{aligned} \tag{3.34}$$

In particular, if we assume that the metric $\hat{g}$ does not depend on the coordinate x^μ, we simply find that $|\langle\Psi(\Sigma_u)|\Psi(\Sigma_{u+du})\rangle|=1$, i.e., the information metric is trivial, as follows from (3.32) and (3.34).

In the presence of a more general x-dependent metric perturbation, we can evaluate (3.27) as follows:

$$\begin{aligned} 1-|\langle\Psi(u)|\Psi(u+du)\rangle| = \frac{\pi^2}{6}C_T(du)^2\int_{-\infty}^{\infty}dk|k|\Big(&|\partial_u\hat{g}^{tt}(k)|^2+2\partial_u\hat{g}^{tt}(k)\partial_u\hat{g}^{xx}(-k)\\ &+|\partial_u\hat{g}^{xx}(k)|^2+4|\partial_u\hat{g}^{tx}(k)|^2\Big). \end{aligned} \tag{3.35}$$

We can easily confirm that the information metric $G^{(B)}_{uu}$ is indeed non-negative.

The relation between the CFT metric $\hat{g}_{\mu\nu}$ defined by (3.25) and the $g_{\mu\nu}$ one in (3.18) is given by $g_{\mu\nu}=\frac{R^2}{\epsilon^2}e^{2u}\hat{g}_{\mu\nu}$. Note also the familiar relation $C_T\propto R/G_N$ and remember that the UV cutoff in the momentum integral in (3.35) will be $1/\epsilon$. Since the transformation from the information metric for $|\Psi(u)\rangle$ to that for $|\Phi(u)\rangle$ corresponds to removing the scale factor e^{2u} in (d-dimensional) spacial metric components, it is natural to get the form (3.19) and (3.20) in the $|\Phi(u)\rangle$ frame.

3.3. *Distance between quantum mixed states*

Here we would like to briefly review the information metric. First we should admit that the definition of the information metric is not unique in quantum theory. For example, we can define two different Fisher information metrics $G^{(B)}_{\theta\theta}$ and $G^{(S)}_{\theta\theta}$ as follows (see, e.g., Refs. [58–60]):

$$\begin{aligned} G^{(B)}_{\theta\theta}du^2&=B(\rho_{\theta+d\theta},\rho_\theta),\\ 2G^{(S)}_{\theta\theta}du^2&=S(\rho_{\theta+d\theta}||\rho_\theta), \end{aligned} \tag{3.36}$$

where we have introduced two different measures of distance between two quantum states given by the density matrix: ρ and σ

$$\begin{aligned} B(\rho,\sigma)&=1-\mathrm{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}],\\ S(\rho||\sigma)&=\mathrm{Tr}[\rho(\log\rho-\log\sigma)]. \end{aligned} \tag{3.37}$$

The quantity $B(\rho,\sigma)$ is called the Bures distance, while $S(\rho||\sigma)$ is known as the relative entropy, both of which measure entropic distance between two quantum states. Notice that, in particular, for two pure states $|u\rangle\langle u|$ and $|v\rangle\langle v|$, we can show

$$B\left(|u\rangle\langle u|,|v\rangle\langle v|\right)=1-|\langle u|v\rangle|. \tag{3.38}$$

It is known that these two different metrics $G^{(B)}_{\theta\theta}$ and $G^{(S)}_{\theta\theta}$ coincide for classical states, while they do not for generic quantum states. In this paper, we simply focus on the former, $G^{(B)}_{\theta\theta}$, for our convenience, assuming that the difference is not important for our purpose. We would also like to note that the distance (3.17) in our conjectured surface/state correspondence can be naturally generalized for that between two mixed states, corresponding to topologically non-trivial surfaces.

3.4. Quantum estimation theory and the information metric

The information metric plays an important role in (quantum) estimation theory. Assume that the density matrix is parameterized by a real value $\theta \in$R, denoted by ρ_θ. Now we would like to estimate the value of θ based on the so-called positive-operator valued measure. We define X as an operation that describes the measurement of θ such that its expectation value $\langle X \rangle$ for ρ_θ coincides with θ. Then we would like to estimate the mean-square error $\langle(\delta\theta)^2\rangle = \langle(X-\theta)^2\rangle$. In this setup, it is known that this error $\langle(\delta\theta)^2\rangle$ is bounded by the inverse of the Fisher information metric as follows:

$$\left\langle(\delta\theta)^2\right\rangle \geq \frac{1}{8G^{(B)}_{\theta\theta}}. \tag{3.39}$$

In classical information theory, this is known as the Cramer–Rao bound, which has been generalized to that in quantum information theory [61,62,65]. Intuitively, this inequality means that, if the density matrix ρ_θ changes more radically when we vary θ, then the estimation gets easier.

If we apply this bound to a one-parameter family of codimension-two surfaces Σ_u in our surface/state correspondence, we immediately obtain the inequality $\left\langle(\delta u)^2\right\rangle \geq \frac{1}{8G^{(B)}_{uu}}$ from (3.39). The variance $\left\langle(\delta u)^2\right\rangle$ describes an error of the value of the coordinate u in the gravitational spacetime M_{d+2}.

In particular, if we consider the AdS/CFT setup and identify u with the radial coordinate as in (2.16), then we get the following bound:

$$\left\langle\left(\frac{\delta z}{z}\right)^2\right\rangle = \left\langle(\delta u)^2\right\rangle \geq \frac{1}{8G^{(B)}_{uu}} \sim \frac{z^d}{N_{\rm deg}\cdot V_d} \sim \frac{G_N}{\mathrm{A}(\Sigma_u)}, \tag{3.40}$$

where we have introduced the standard radial coordinate $z = \epsilon e^{-u}$ of the Poincaré AdS. Also, $\mathrm{A}(\Sigma_u) = R^d V_d z^{-d}$ denotes the actual area of the surface Σ_u. For large-N gauge theories, we have $N_{\rm deg} \sim N^2$ and thus the error gets suppressed in the large-N limit.[7] This seems to be consistent with the standard expectation in the AdS/CFT that classical geometries appear in the large-N limit. Moreover, we can also interpret (3.40) in terms of effective entropy as follows:

$$\left\langle\left(\frac{\delta z}{z}\right)^2\right\rangle \gtrsim \frac{1}{S_{\rm eff}(\Sigma_u)} \sim \frac{1}{\#\,\text{links intersecting with } \Sigma_u}, \tag{3.41}$$

where, in the final qualitative relation, we have employed the holographic interpretation of tensor networks. It is suggestive to rewrite this inequality in the following way:

$$\left\langle\, (\delta\mathrm{A}(\Sigma_u))^2 \,\right\rangle \gtrsim G_N \cdot \mathrm{A}(\Sigma_u), \tag{3.42}$$

in terms of the area $\mathrm{A}(\Sigma_u)$ of Σ_u. It will be a very intriguing future problem to understand the precise physical meaning of such a bound in quantum estimation theory from the viewpoint of AdS/CFT correspondence.

4. Several examples

Here we briefly study several explicit examples to see how our proposals work.

[7] If we take Σ_u to be a d-dimensional plane with u fixed, then V_d is already infinite. However, even if we choose Σ_u to be an open manifold with a finite volume at the same value of u, we expect the same evaluation (3.40).

4.1. Pure AdS

Consider the AdS space in the Poincaré coordinate given by the metric (2.16). If we take the surface Σ_u to be a constant u slice with the time $t\left(=x^0\right)$ fixed, then (3.22) agrees with the cMERA result for the dual CFT, as we have already mentioned in the previous section.

It is also intriguing to choose the one-parameter family Σ_{x^i} of surfaces by taking a constant x^i slice, where i can be one of $1, 2, \ldots, d$, with the time fixed. In this case the proposed formula (3.20) leads to $G^{(B)}_{uu} = 0$. This is consistent with the dual quantum state calculation because there is translational invariance in the x^i direction and the quantum state, which is actually a mixed state, is invariant under the shift of x^a.

4.2. AdS black hole vs AdS soliton

The AdS black hole and AdS soliton are related to each other by a Wick rotation and their Euclidean spaces include a common cigar-like geometry, whose polar coordinate is defined as (r, θ) [66]. At $r = 0$, the θ circle shrinks smoothly to zero size.

In the AdS black hole, the Euclidean time is taken in the θ direction. A one-parameter family of surfaces Σ_r is defined by fixing r with $\theta = 0$, which is a point in the cigar. Therefore it is not possible to contract Σ_r to zero size by a smooth deformation. Our conjecture tells us that the corresponding state is a mixed state $\rho(\Sigma_r)$, as expected from the thermal nature of the black hole. The von Neumann entropy of $\rho(\Sigma_r)$ coincides with the black hole entropy S_{BH} for any r, while we can have matching for the effective entropy: $S_{\text{eff}}(\Sigma_r) = S_{\text{BH}}$ only for $r = 0$.

On the other hand, in the AdS soliton, the time direction is not included in the cigar. We define Σ_r by fixing the value of r with the time fixed. Then Σ is wrapped on the θ circle. However, this circle can be smoothly contractible to zero size and thus we can conclude that the corresponding state is a pure state $|\Phi(\Sigma_r)\rangle$, as expected.

4.3. Flat spaces

If we consider the flat spacetime $R^{1,d+1}$ and choose u to be one of the Cartesian coordinates:

$$ds^2 = du^2 + dx_\mu\, dx^\mu, \tag{4.43}$$

then we immediately find that the right-hand side of the information metric (3.20) is simply vanishing. On the other hand, the translation symmetry in the u direction implies that $|\Phi(\Sigma_{u+du})\rangle = |\Phi(\Sigma_u)\rangle$ and this clearly explains why the left-hand side of (3.19) is also vanishing. Note that Σ_u is itself an extremal surface and therefore its entanglement entropy S_A for a finite-area part of A satisfies the volume law and is equal to the effective entropy $S_{\text{eff}}(A)$. This shows that the quantum state dual to $\rho_A^{\Sigma_u}$ is non-locally entangled, while the entanglement between finitely separated regions is vanishing.[8] This is consistent with the analysis of flat-space holography in Refs. [67,68]. See also Ref. [69] for an interesting analysis of entanglement entropy in flat-space holography from a different viewpoint.

We can generalize this argument to the class of metric

$$ds^2 = du^2 + g_{\mu\nu}(x)dx^\mu\, dx^\nu. \tag{4.44}$$

[8] This can be understood clearly by compactifying Σ_u, e.g., regarding Σ_u as a limit of a large spherical surface.

In this case, we again have translational symmetry in the u direction; we find that the information metric (3.19) should be trivial and this agrees with the dual description by quantum states.

4.4. Discussion: de Sitter spaces

A much more non-trivial example is de Sitter spaces. A holography for de Sitter spaces, called dS/CFT, has been proposed in Ref. [5] and it has been suggested that if gravity theories on de Sitter spaces can be dual to CFTs, then they are non-unitary CFTs [70] (see also Refs. [71–74]).

A global metric of $d + 2$-dimensional de Sitter space is given by

$$ds^2 = R^2\big(-dt^2 + \cosh^2 t \, d\Omega_{d+1}^2\big), \tag{4.45}$$

where $d\Omega_{d+1}^2$ is the metric of S^{d+1} with the unit radius. First let us define Σ_t as an equator of S^{d+1} on a constant t slice. At $t = 0$, it is clear that the surface $\Sigma_{t=0}$, which is given by a d-dimensional round sphere S^d, is an extremal surface. Therefore we find the saturation of subadditivity $S_{\text{eff}}(\Sigma_A) = S_A^{\Sigma_{t=0}}$ for any Σ_A whose size is less than half of $\Sigma_{t=0}$. From this fact, we can conclude that each point in $\Sigma_{t=0}$ is only entangled with its antipodal point and there is no other entanglement. This property of quantum entanglement is far from that of CFT vacua. It is also interesting to note that the total effective entropy $S_{\text{eff}}(\Sigma_{t=0})$ coincides with the de Sitter entropy, as follows from the relation (2.12).

On the other hand, for the surfaces Σ_t at $t \neq 0$, we find that their interpretation looks puzzling. It is obvious that their effective entropy grows exponentially: $S_{\text{eff}}(\Sigma_t) \propto (\cosh(t))^d$. However, if we consider the entanglement entropy when we trace some part of Σ_t, it turns out that its corresponding space-like extremal surface does not always exist. For example, in the late time limit $t \gg 1$, but only if we choose the size of subsystem Σ_A to be as small as $O(e^{-t})$, we can find an extremal surface that computes to the holographic entanglement entropy $S_A^{\Sigma_t}$. One possibility to resolve this problem might be to pick up extremal surfaces by complexifying the spacetime coordinates as in Refs. [73,74]. This trick leads to negative- or complex-valued entanglement entropy, which might be due to the non-unitary nature of dual CFTs. Another possibility is to dismiss such surfaces without proper extremal surfaces and to focus on the other class of surfaces. We would like to leave these problems to future studies.

5. Conclusions

In this paper, we have proposed a new duality: surface/state correspondence, between codimension-two space-like convex surfaces in gravity theory and quantum states in an infinitely large Hilbert space. Quantum states dual to topologically trivial closed surfaces are pure states, while those dual to open surfaces and topologically non-trivial closed surfaces are dual to mixed states. This proposal generalizes and refines the idea of holography and is highly motivated by the conjectured equivalence between AdS/CFT and tensor networks. It will be interesting to see more close connections between our formulation and the tensor network description. For example, our consideration of extremal surfaces for the calculation of entanglement entropy leads to the requirement of convexity, and it will be important to understand its tensor network counterpart. This looks related to the irreversibility of coarse-graining operations.

Moreover, we have studied some properties of these states, such as entanglement entropy, effective entropy, and the Fisher information metric. The entanglement entropy, which is a generalization of holographic entanglement entropy, is given by the area of an extremal surface. The effective entropy is simply given by the area of the surface. The information metric is related to an integral of a square of extrinsic curvatures. Using our evaluation of the information metric, we applied the Cramer–Rao bound in quantum estimation theory and showed that quantum fluctuations in the estimated value of the radial coordinate of AdS space are suppressed in the large-N limit. It would be nice if we could find the precise form of the tensor P in the information-metric expression from a direct calculation. Also, it might be intriguing to study other quantities related to quantum entanglement, such as the entanglement density introduced in Refs. [75–77] (see also Ref. [78] for a closely related work), and discuss how the Einstein equation appears in our formulation.

The most attractive feature of our proposal is that it does not rely on the existence of boundaries, as opposed to the standard holography. In principle, we can apply our proposal to understanding the holography for de Sitter spaces, as we have briefly mentioned in this paper and we would like to come back to this problem in future publications. Also, in string theory, we often encounter internal compact spaces such as S^5 in the type IIB background $\mathrm{AdS}_5 \times S^5$. Even though interpretations of internal spaces have been discussed in Refs. [79,80] from the viewpoint of quantum entanglement, their realizations in tensor networks and in our surface/state correspondence also remain as future problems.

Acknowledgements

We are grateful to Pawel Caputa, Veronika Hubeny, Esperanza Lopez, Mukund Rangamani, and Joan Simon for useful discussions, and especially to Horacio Casini, Bartlomiej Czech, and Xiao-Liang Qi for important comments and stimulating discussions on holographic interpretations of tensor networks. We would like to thank Masahiro Nozaki, Tokiro Numasawa, Shinsei Ryu, Noburo Shiba, and Kento Watanabe for closely related collaborations and helpful discussions. T.T. also acknowledges the organizers and participants of the IFT workshop "Entangle This: Space, Time and Matter" for giving him the opportunity to present the results of this paper and for useful feedback. T.T. is supported by a JSPS Grant-in-Aid for Scientific Research (B) No. 25287058, a JSPS Grant-in-Aid for Challenging Exploratory Research No. 24654057, and also by the World Premier International Research Center Initiative (WPI Initiative) from the Japan Ministry of Education, Culture, Sports, Science and Technology (MEXT).

Funding

Open Access funding: SCOAP^3.

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