Analytic Extension of the Birkhoff Normal Forms for the Free Rigid Body Dynamics on $SO(3)$

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Abstract

Birkhoff normal form is a power series expansion associated with the local behavior of the Hamiltonian systems near a critical point. It is known that around the critical point one can take a convergent canonical transformation which puts the Hamiltonian into Birkhoff normal form for integrable systems under some non-degeneracy conditions. By means of an expression of the derivative for the inverse of Birkhoff normal form by a period integral, analytic continuation of the Birkhoff normal forms is considered for the free rigid body dynamics on $SO(3)$. It is shown that the monodromy of the analytic continuation for the derivative of the inverse for the Birkhoff normal forms coincides with that of an elliptic fibration which naturally arises from the dynamics.

1 Introduction

In analytical mechanics, the motions of rigid bodies are basic problems. Among them, the free rigid body, which stands for the rigid body under no external force, is the simplest example. Its complete integrability and the stability of its equilibria are understood well through the long history of researches. In particular, geometric mechanics provides a well-organized description of this dynamical system. (See [38, 1, 25].)

The free rigid body dynamics should first be defined as a Hamiltonian system on the cotangent bundle of the rotation group $SO(3)$. Because of the left-invariance of this system, it is essentially described by the so-called Euler equation posed on the angular momentum, which can be justified by the Lie-Poisson reduction procedure. Moreover, by means of the Marsden-Weinstein reduction, one can reduce the original system onto the level surface of the norm of the angular momentum in the space of angular momenta, which is a two-dimensional sphere. The reduced system is of one degree of freedom, and therefore completely integrable in the sense of Liouville. It is well known that there are generically six equilibria for the reduced systems on the sphere, four of which are elliptic and the other two are hyperbolic. (cf. [25].)

Around a stationary point of a Hamiltonian system, it is possible to consider the normal form of the Hamiltonian. Historically, Birkhoff introduced the notion of Birkhoff normal forms as formal

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series and discussed the relation with the stability [5, 6]. It is known that there exists a Darboux coordinate system which makes the Hamiltonian in Birkhoff normal form, in a neighbourhood of a non-degenerate stationary point by a result of Vey [37] for analytic completely integrable Hamiltonian systems. The differentiable case of class $\mathcal{C}^\infty$ was studied by Eliasson [12]. The convergence of canonical transformation which makes the Hamiltonian into Birkhoff normal form for analytic integrable systems including the degenerate case was shown by Ito [21] under the non-resonance condition. The convergence of the canonical transformation which puts the Hamiltonian into Birkhoff normal form for the systems of one degree of freedom was essentially shown by Siegel [34]. In [41], Nguyen Tien Zung developed another more geometric approach for the analytic completely integrable Hamiltonian systems, on the basis of the direct proof of the analytic extension through the period integrals.

The detailed structure of the Birkhoff normal form has been studied recently for the pendulum [15] and for the free rigid body [16]. In [16], the Birkhoff normal forms both for the elliptic and hyperbolic stationary points are considered by using the method of relative cohomology, and the properties of the Birkhoff normal forms and those of the inverse of the Birkhoff normal forms are discussed. There are also the papers [11, 31] which calculate the Birkhoff normal forms for the spherical pendulum and for the free rigid body in order to study the semi-global symplectic invariants introduced by Vũ Ngọc San [33] for these specific examples.

On the other hand, the free rigid body dynamics is closely related to complex algebraic geometry because of its complete integrability. In fact, the integral curve of the free rigid body can be described as an intersection of two quadric surfaces in three-dimensional Euclidean space, which is a (real) elliptic curve. From the viewpoint of complex algebraic geometry, it is natural to complexify and to compactify all the settings, which is also helpful to understand the deep geometric structure of the free rigid body dynamics. In view of this, several elliptic fibrations arising from the free rigid bodies have been considered in [29]. In this paper, the fibrations are considered over the base space which includes not only the values of the Hamiltonian but also the principal axes of the inertia tensor, as their base coordinates. This even allows a geometric description of the bifurcation phenomena of the free rigid body dynamics as described in [29], where the singular fibres of these elliptic fibrations are classified in relation to such bifurcation phenomena. In the present paper, one mainly studies the so-called naive elliptic fibration in [29], which is the family of complexified and compactified integral curves.

Although the Birkhoff normal form is by definition a local object associated to a stationary point for a Hamiltonian system, it is possible to enquire its analytic extension in the integrable case. In the present paper, the Birkhoff normal forms of the equilibria for the free rigid body dynamics and their analytic continuation are considered in relation to the naive elliptic fibration which is discussed in [29]. The key to this relation is an expression of the derivative of the inverse for the Birkhoff normal form in terms of period integrals, which is closely related to a special Gauß hypergeometric differential equation. This makes the concrete calculation rather easy.

The main results of the present paper are as follows:
First, as to the global properties of Birkhoff normal forms for the free rigid body dynamics, it is shown that the derivative of the inverse for the Birkhoff normal forms is analytically extended as multi-valued functions over the regular locus of the base space of the naive elliptic fibration. One finds that these analytic continuations have the monodromy in a function theoretic sense and the naive elliptic fibration has also the monodromy structure as one of its geometric aspects. It is shown that these two kinds of monodromy structures coincide with each other. Moreover, the explicit calculation of the global monodromy is given by means of the analysis of a Gauß hypergeometric differential equation and the topology of the complement of a hyperplane arrangement given by the singular locus of the naive elliptic fibration.

Recently, several researches have been done on the relation between elliptic fibrations and the compactifications of string theory in view of the F-theory. Among them, [13] deals with the elliptic fibrations modeled by quadrics intersections in $P_3(\mathbb{C})$, called D_5 elliptic fibrations, and puts emphasis on the appearance of non-Kodaira singular fibres. The non-Kodaira singular fibre of D_5 elliptic fibrations is, in [13], shown to consist of four smooth rational curves intersecting at one point and it is labeled as I_0^{*-} fibre. It is to be noted that such a singular fibre is also observed in the modification of elliptic fibrations arising from the free rigid body dynamics in [29], although the aim of [29] is to give the elliptic fibration whose singular fibres are in the list of singular fibres of elliptic surfaces by Kodaira for the elliptic fibrations arising from the free rigid body dynamics. See Subsection 5.3 (in particular pp.392-393) in [29] for detail. It is of much interest that one can observe the importance of elliptic fibrations modeled by quadrics intersections in $P_3(\mathbb{C})$ even in the simple example of classical mechanics, the free rigid body dynamics, in [29] and the present paper.

The structure of the present paper is as follows:
In Section 2, the expression of the derivative for the inverse of Birkhoff normal forms both around the elliptic and hyperbolic stationary points for an arbitrary Hamiltonian system of one degree of freedom is given in terms of period integrals.

After a brief explanation about Euler equation for the $SO(3)$ free rigid body dynamics, Section 3 describes the application of the arguments in Section 2 to the case of the free rigid body dynamics. In this case, the explicit expression of the derivative of the inverse for the Birkhoff normal forms is given in terms of the complete elliptic integral of the first kind. From this expression, the derivative of the inverse for the Birkhoff normal forms is connected to a special Gauß hypergeometric differential equation.

Section 4 deals with the relation between the result in Section 3 and the elliptic fibration arising from the free rigid bodies which is considered in [29]. The singular fibres of this elliptic fibration are classified in [29], but the global monodromy is not discussed there. One computes explicitly the cocycles of the regular fibre in a neighobourhood of singular fibres on an irreducible component of the singular locus, which form a basis of its first cohomology group, and it is shown that they are written in terms of the period integrals appearing in Section 3.

Moreover, in Section 5, the global monodromy of the elliptic fibration in [29] is calculated. First, using the symmetry of this elliptic fibration and the period integrals in Section 3, as well as the results in Section 4, one computes the basis of the first cohomology group of regular fibres around the singular fibres on each irreducible component of the singular locus for the elliptic fibration. Second, the local monodromy of the elliptic fibration is calculated around each irreducible component of the singular locus. Third, the fundamental group of the regular locus of the fibration in [29] is determined explicitly, by using the techniques of the topology of the complement of hyperplane arrangements. Finally, on the basis of these results, the global monodromy of the elliptic fibration is computed. As the main results of the present paper, the monodromy representation of the fundamental group of the regular locus of the base space for the naive elliptic fibration is explicitly determined and it is shown that this monodromy coincides with that of the analytic continuation for the derivatives of the inverse of the Birkhoff normal forms for the free rigid body dynamics. The results in Sections 4 and 5 are compared with the results of [16]. In [16, §VI], it is already emphasised that the n-th coefficient of the inverse for the Birkhoff normal form, which is viewed as power series in the action variable, around an elliptic equilibrium on the axis corresponding to I_3 is a symmetric polynomial P_n in the parameter $r^2 = \left(\frac{1}{I_1} - \frac{1}{I_2}\right) \Big/ \left(\frac{1}{I_3} - \frac{1}{I_2}\right)$, where I_1, I_2, I_3 stand for the principal axes of the inertia tensor of the rigid body, in the sense that

$$P_n(r^2) = \sum_{j=0}^{n} F_j r^{2j}, \qquad F_j = F_{n-j}.$$

Note that it is assumed that $I_3 < I_1 < I_2$ in [16]. It is shown here that this property is a consequence of a covariance of the analytic extensions of the derivative of the inverse for the Birkhoff normal forms, relative to the symmetry group $\mathfrak{S}_4$ of the base space of the naive elliptic fibration arising from the free rigid body dynamics. This covariance is explained in the formulae (5.1). On the top of these covariance formulae, another formula is derived in the spirit of the connection formulae of the Gauß hypergeometric differential equation (cf. Proposition 5.4 and [9]).

A similar study on the simple pendulum dynamics is dealt with in [36]. The relation to the rigid body dynamics might be clarified in view of [20] and [22] in a future work.

2 Birkhoff normal forms for a system of one degree of freedom and period integrals

In this section, we discuss the expressions for the derivative of the inverse of Birkhoff normal forms both around elliptic and hyperbolic equilibria for a real analytic Hamiltonian system of one degree of freedom by certain period integrals. Note that the derivative for the inverse of Birkhoff normal form around an elliptic equilibrium is physically the period of the integral curve. We consider the derivative for the inverse of Birkhoff normal form because it is more convenient to calculate the monodromy of its analytic extension than the inverse of Birkhoff normal form itself, as we see in the free rigid body case.

Take a real analytic symplectic manifold (M, ω) of dimension two and consider a real analytic Hamiltonian H. For the systems with one degree of freedom, the convergence of the canonical transformation which puts the Hamiltonian into Birkhoff normal form for an equilibrium was essentially known from the study by Siegel [34]. Let (x, y) be a Darboux coordinates of (M, ω) such that $\omega = \mathsf{d}x \wedge \mathsf{d}y$. Assume that the origin $(x, y) = (0, 0)$ is an elliptic equilibrium, where $H = 0$, and that the Hamiltonian H is in Birkhoff normal form $H = \mathcal{H}\left(\dfrac{x^2 + y^2}{2}\right)$, where $\mathcal{H}$ is an invertible analytic function in one variable around the origin. Denote the inverse of the function $\mathcal{H}$ by Φ. Suppose that there is a one-form η defined on $U \setminus \{(0, 0)\}$, where U is a neighbourhood of the origin $(x, y) = (0, 0)$, such that $\omega = \eta \wedge \mathsf{d}H$. Then, we have the following theorem.

Theorem 2.1. *The derivative of the inverse Φ of the Birkhoff normal form $\mathcal{H}$ around an elliptic stationary point, where $H = 0$, writes*

$$\Phi'(h) = -\frac{1}{2\pi} \int_{H=h} \eta.$$

Here, the integral path is taken as the integral curve of the energy level $H = h$.

Theorem 2.1 can be proved by performing a straightforward integration of some explicit expression of the one-form η in the above coordinates (x, y), so we omit the proof. As to the integral path, we can take another real closed arc homotopic to the original real integral curve in the complexification of the real integral curve $H = h$. See the explanation below.

Remark 2.1. The one-form η can be replaced by $\eta + f\mathsf{d}H$, where f is a function on $U \setminus \{(0, 0)\}$. In fact, if η' satisfies $\eta' \wedge \mathsf{d}H = \omega$, we have $(\eta - \eta') \wedge \mathsf{d}H = 0$. Thus, we have $\eta - \eta' = f\mathsf{d}H$, for a suitable function f, so that $\int_{H=h} \eta = \int_{H=h} \eta'$. See [10] and [37, §4] for the general case. In fact, the one-form η is an example of the so-called Gelfand-Leray form, which can be found in [2, Chapter 7]. An application of Gelfand-Leray form in dynamical systems is given in [17]. Lemma 1 *loc. cit.* guarantees the existence of η in Theorem 2.1.

In a parallel manner, we consider an expression of the derivative of the inverse for Birkhoff normal form around a hyperbolic stationary point. In this case, we can take a Darboux coordinates (X, Y) with the origin at a hyperbolic equilibrium, such that $\omega = \mathsf{d}X \wedge \mathsf{d}Y$ and the Hamiltonian

H with $H(0,0) = 0$ is in Birkhoff normal form $H = \mathcal{H}(XY)$, where $\mathcal{H}$ is an invertible analytic function in one variable whose inverse is again denoted by Φ. We consider the complexification $M^{\mathbb{C}}$ of M where the symplectic form ω and the Hamiltonian H are extended as a holomorphic two-form and as a holomorphic function. Taking a suitable complex neighbourhood $U^{\mathbb{C}} \subset M^{\mathbb{C}}$ of $(X,Y) = (0,0)$, we choose a holomorphic one-form $\eta^{\mathbb{C}}$ on $U^{\mathbb{C}} \setminus \{(0,0)\}$, such that $\eta^{\mathbb{C}} \wedge \mathsf{d}H = \omega$. We can assume that (X,Y) are holomorphic coordinates on $U^{\mathbb{C}}$. The real integral curve is naturally complexified to a complex one-dimensional curve $H(XY) = h$, $(X,Y) \in U^{\mathbb{C}} \subset M^{\mathbb{C}}$, as a level curve of the holomorphic function H. On the complex curve $H = h$ in $M^{\mathbb{C}}$, we consider the real closed arc

$$\gamma :\ X = \sqrt{\epsilon} e^{\sqrt{-1}\theta},\ Y = \sqrt{\epsilon} e^{-\sqrt{-1}\theta},\ \theta : 0 \to 2\pi,$$

where $\epsilon := \Phi(h) = \mathcal{H}^{-1}(h)$.

Theorem 2.2. *The derivative of the inverse Φ of the Birkhoff normal form $\mathcal{H}$ for the Hamiltonian H around a hyperbolic equilibrium where $H = 0$ writes*

$$\Phi'(h) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \eta^{\mathbb{C}}.$$

Remark 2.2. Theorem 2.2 can be proved from Theorem 2.1 through the imaginary transformation

$$X = \frac{x + \sqrt{-1}y}{\sqrt{2}}, \quad Y = \frac{x - \sqrt{-1}y}{\sqrt{2}}$$

of $U^{\mathbb{C}}$. In fact, if we regard (x,y) as real coordinates and restrict the symplectic form ω and the Hamiltonian H, which are holomorphic on $U^{\mathbb{C}}$, to the real two-dimensional space $U' = U^{\mathbb{C}} \cap \{(x,y) \in \mathbb{R}^2\}$, then we can use Theorem 2.1 for the one-form $\eta' = \eta^{\mathbb{C}}|_{U'}$. Theorem 2.2 follows by restricting the complexification on $U^{\mathbb{C}}$ of this result to U.

Now, we make a comment about the above theorems from the viewpoint of complex analytic geometry and we discuss the choice of the integral paths of the period integrals. We take a real analytic Hamiltonian system (M, ω, H) of one degree of freedom as above. Assume that there is an isolated non-degenerate equilibrium $x_0 \in M$, either elliptic or hyperbolic, where $H(x_0) = 0$. Since all the settings are real analytic, we can complexify the phase space M to a complex two-dimensional manifold $M^{\mathbb{C}}$, where ω and H are defined as a holomorphic two-form and as a holomorphic function. The holomorphic function H induces a fibration $\pi : N \to B$ of a neighbourhood $N \subset M^{\mathbb{C}}$ of the singular level set $H^{-1}(0) (= \pi^{-1}(0))$ in $M^{\mathbb{C}}$ to a neighbourhood $B \subset \mathbb{C}$ of the origin. Then, the singular fibre $\pi^{-1}(0)$ has an A_1-singularity at $x_0 \in \pi^{-1}(0)$ and the fibration $\pi : N \to B$ can be regarded as a deformation of the A_1-singularity. Clearly, the complexification in $M^{\mathbb{C}}$ of the level set $H = h$ can be seen as a fibre of the fibration π. Now, the real integral curve in Theorem 2.1, which we denote γ_0, and the real closed arc γ in Theorem 2.2 are both vanishing cycles in the fibre $\pi^{-1}(h)$ which shrink into the A_1-singularity x_0, as $h \to 0$. From such a kind of viewpoint, we can combine the statements of Theorems 2.1 and 2.2 as follows:

Theorem 2.3. *The period integral of the holomorphic one-form η defined on a neighbourhood N of the complexified phase space $M^{\mathbb{C}}$, such that $\eta \wedge \mathsf{d}H = \omega$, along a vanishing cycle ν in the fibre $\pi^{-1}(h) \subset N$ as the integral path is proportional to the derivative Φ of the inverse for Birkhoff normal form respectively around the elliptic or hyperbolic equilibrium $x_0 \in N$ as*

$$\Phi'(h) = -\frac{c_{x_0}}{2\pi} \int_{\nu} \eta,$$

where $c_{x_0} = 1$ when x_0 is elliptic and $c_{x_0} = \sqrt{-1}$ when x_0 is hyperbolic.

For the application, it is to be noted that the real closed arc γ around the hyperbolic equilibrium in Theorem 2.2 is in general hard to find explicitly since we do not necessarily know the Darboux coordinates (X, Y) concretely, but we can rather easily take another vanishing cycle in an explicit manner, as we do in the free rigid body case in the next section.

Remark 2.3. A similar formula for the inverse of Birkhoff normal form of Hamiltonian system of one degree of freedom is given in [14, Theorem 1], but they have considered the period integral of a one-form ξ such that $\mathsf{d}\xi = \omega$, which is different from η, as $\Phi(h) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\nu} \xi$. One can also prove Theorems 2.1 and 2.2 on the basis of this formula. The idea to use period integrals over vanishing cycles in the complexification of the real integral curve is also considered in [11] to compute certain action variables of spherical pendulum around hyperbolic equilibria, but the formula in Theorem 2.2 does not appear explicitly.

3 Derivative of inverse of Birkhoff normal forms for free rigid body dynamics

In this section, we apply the formulae in the previous section to the free rigid body dynamics. The motion of a free rigid body can essentially be described by the Euler equation

$$\frac{\mathsf{d}}{\mathsf{d}t} P = P \times \left(\mathcal{I}^{-1}(P)\right),$$

where $P = (p_1, p_2, p_3) \in \mathbb{R}^3$ is the angular momentum and $\mathcal{I} : \mathbb{R}^3 \to \mathbb{R}^3$ is the inertia tensor which is a symmetric positive-definite linear operator with respect to the standard inner product $\cdot$ of $\mathbb{R}^3$. Without loss of generality, we can assume $\mathcal{I} = \operatorname{diag}(I_1, I_2, I_3)$ and $I_1 < I_2 < I_3$. Then, the Euler equation writes

$$\begin{aligned} \frac{\mathsf{d}}{\mathsf{d}t} p_1 &= -\left(\frac{1}{I_2} - \frac{1}{I_3}\right) p_2 p_3, \\ \frac{\mathsf{d}}{\mathsf{d}t} p_2 &= -\left(\frac{1}{I_3} - \frac{1}{I_1}\right) p_3 p_1, \\ \frac{\mathsf{d}}{\mathsf{d}t} p_3 &= -\left(\frac{1}{I_1} - \frac{1}{I_2}\right) p_1 p_2. \end{aligned} \tag{3.1}$$

An important property of the Euler equation is that the functions $H(P) = \dfrac{1}{2} P \cdot \mathcal{I}^{-1}(P)$ and $L(P) = \dfrac{1}{2} P \cdot P$ are first integrals, so that the system can be restricted to the level surface of L. On the level surface $L = \dfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) = \ell$, where ℓ is a positive constant, there are six equilibria on the p_1-, p_2-, p_3-axes, and those four on the p_1- and p_3-axes are elliptic, while the other two on the p_2-axis are hyperbolic. The restricted system on $L(P) = \ell$ is a Hamiltonian system for the Hamiltonian $H|_{\{L=\ell\}}$ with respect to the symplectic form

$$\omega = \frac{\mathsf{d}p_1 \wedge \mathsf{d}p_2}{3p_3} = \frac{\mathsf{d}p_2 \wedge \mathsf{d}p_3}{3p_1} = \frac{\mathsf{d}p_3 \wedge \mathsf{d}p_1}{3p_2}. \tag{3.2}$$

See [25] for more detail on the free rigid body dynamics.

We consider the derivative of the inverse for the Birkhoff normal form around the elliptic equilibrium $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$, where (p_2, p_3) serves as the local coordinate system on

$L=\ell$. As is mentioned in the previous section, the derivative for the inverse of Birkhoff normal form around an elliptic equilibrium is the period of the integral curve. By considering such a period in the free rigid body case, one can more easily consider the analytic extension than the inverse for Birkhoff normal form itself because of the relation between the derivative of it and a Gauß hypergeometric differential equation as is described below. On this coordinate neighbourhood, we consider the one-form

$$\eta_s := (1-s)\frac{\mathsf{d}p_2}{3\left(\dfrac{1}{I_3}-\dfrac{1}{I_1}\right)p_3p_1} + s\frac{\mathsf{d}p_3}{3\left(\dfrac{1}{I_1}-\dfrac{1}{I_2}\right)p_1p_2}, \tag{3.3}$$

where s is an arbitrary parameter. It is easy to verify that $\eta_s \wedge \mathsf{d}H = \omega$ for any s.

Denote the inverse function of the Birkhoff normal form $H - \dfrac{\ell}{I_1} = \mathcal{H}_1$ around the elliptic stationary point $\left(\sqrt{2\ell}, 0, 0\right)$ by Φ_1. We have an expression of Φ_1' in terms of a period integral as follows:

Theorem 3.1. *The derivative of the inverse for the Birkhoff normal form around* $(p_1,p_2,p_3) = \left(\sqrt{2\ell},0,0\right)$ *writes*

$$\Phi_1'(h) = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-c)(a-b)}}\mathcal{K}\left(\frac{(d-a)(b-c)}{(d-c)(b-a)}\right). \tag{3.4}$$

Here, $a=\dfrac{1}{I_1}, b=\dfrac{1}{I_2}, c=\dfrac{1}{I_3}, d=\dfrac{h}{\ell}$ and $\mathcal{K}(\lambda) := \displaystyle\int_0^1 \frac{\mathsf{d}x}{\sqrt{(1-x^2)(1-\lambda x^2)}}$ is the complete elliptic integral of the first kind. Denote the right hand side of (3.4) by $S(a,b,c,d)$.

Proof. By Theorem 2.1, we have the expression $\Phi_1'(h) = -\dfrac{1}{2\pi}\displaystyle\int_{H=h}\eta_s$. The integral path is given by

$$\begin{cases} p_1^2+p_2^2+p_3^2 = 2\ell, \\ \dfrac{p_1^2}{I_1}+\dfrac{p_2^2}{I_2}+\dfrac{p_3^2}{I_3} = 2h. \end{cases} \tag{3.5}$$

Using these equations, we have

$$\begin{aligned} \int_{H=h}\eta_s &= 2\left\{(1-s)\int_{-\sqrt{2\ell\frac{\frac{1}{I_1}-\frac{h}{\ell}}{\frac{1}{I_1}-\frac{1}{I_2}}}}^{\sqrt{2\ell\frac{\frac{1}{I_1}-\frac{h}{\ell}}{\frac{1}{I_1}-\frac{1}{I_2}}}}\frac{\mathsf{d}p_2}{3\left(\frac{1}{I_3}-\frac{1}{I_1}\right)p_3p_1} + s\int_{-\sqrt{2\ell\frac{\frac{1}{I_1}-\frac{h}{\ell}}{\frac{1}{I_1}-\frac{1}{I_3}}}}^{\sqrt{2\ell\frac{\frac{1}{I_1}-\frac{h}{\ell}}{\frac{1}{I_1}-\frac{1}{I_3}}}}\frac{\mathsf{d}p_3}{3\left(\frac{1}{I_1}-\frac{1}{I_2}\right)p_1p_2}\right\} \\ &= (1-s)\sqrt{\frac{2}{\ell}}\frac{1}{3\sqrt{(d-c)(a-b)}}\mathcal{K}\left(\frac{(d-a)(b-c)}{(d-c)(b-a)}\right) + s\sqrt{\frac{2}{\ell}}\frac{1}{3\sqrt{(d-b)(a-c)}}\mathcal{K}\left(\frac{(d-a)(c-b)}{(d-b)(c-a)}\right) \\ &= -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-c)(a-b)}}\mathcal{K}\left(\frac{(d-a)(b-c)}{(d-c)(b-a)}\right). \end{aligned} \tag{3.6}$$

Here, we have used the formula (8.128.1) in [18] which displays

$$\mathcal{K}\left(\frac{\lambda}{\lambda-1}\right) = \sqrt{1-\lambda}\,\mathcal{K}(\lambda), \quad \text{for } \mathrm{Im}\sqrt{\lambda}<0. \tag{3.7}$$

Note that $\dfrac{(d-a)(c-b)}{(d-b)(c-a)} < 0$ near the elliptic stationary point $\left(\sqrt{2\ell}, 0, 0\right)$ on the p_1-axis. ■

Here, we mention some covariance properties of the Euler equation (3.1). First, obviously, the following transformations preserve the Euler equation:

$$\delta_1 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto \begin{bmatrix} -p_1 \\ p_2 \\ p_3 \end{bmatrix}, \; t \mapsto -t; \quad \delta_2 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto \begin{bmatrix} p_1 \\ -p_2 \\ p_3 \end{bmatrix}, \; t \mapsto -t; \quad \delta_3 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto \begin{bmatrix} p_1 \\ p_2 \\ -p_3 \end{bmatrix}, \; t \mapsto -t.$$

It is clear that δ_1, δ_2, δ_3 generate a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. On the other hand, the following transformations, where t is not transformed, preserve the Euler equation:

$$\epsilon_1 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto - \begin{bmatrix} p_1 \\ p_3 \\ p_2 \end{bmatrix}, \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \mapsto \begin{bmatrix} I_1 \\ I_3 \\ I_2 \end{bmatrix}; \quad \epsilon_2 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto - \begin{bmatrix} p_3 \\ p_2 \\ p_1 \end{bmatrix}, \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \mapsto \begin{bmatrix} I_3 \\ I_2 \\ I_1 \end{bmatrix};$$
$$\epsilon_3 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto - \begin{bmatrix} p_2 \\ p_1 \\ p_3 \end{bmatrix}, \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \mapsto \begin{bmatrix} I_2 \\ I_1 \\ I_3 \end{bmatrix}. \tag{3.8}$$

These involutions ϵ_1, ϵ_2, ϵ_3 generate another group isomorphic to the symmetric group $\mathfrak{S}_3$ of degree three. Needless to say that the first integral L is invariant with respect to the above transformations. Note that the transformations δ_1, δ_2, δ_3, ϵ_1, ϵ_2, ϵ_3 act on the symplectic form ω and the Hamiltonian H as $\delta_j^*\omega = -\omega$, $\delta_j^* H = H$ and $\epsilon_j^*\omega = -\omega$, $\epsilon_j^* H = H$, $j = 1, 2, 3$.

By the transformation δ_1 of the Euler equation, the equilibrium $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$ is mapped to $\left(-\sqrt{2\ell}, 0, 0\right)$. The integral curves around the two equilibrium points $\left(\pm\sqrt{2\ell}, 0, 0\right)$ are mapped to each other, but their orientations are opposite, since the time is reversed by δ_1. As to the period integral, the integrand η_s is transformed as $\delta_1^*\eta_s = -\eta_s$. Thus, the integral $\displaystyle\int_{H=h} \eta_s$ is invariant with respect to δ_1. As a result, we have the following corollary.

Corollary 3.2. *The derivative of the inverse of the Birkhoff normal form around* $(p_1, p_2, p_3) = \left(-\sqrt{2\ell}, 0, 0\right)$ *is given by* (3.4).

Similarly, the transformation ϵ_2 maps the equilibrium $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$ to $\left(0, 0, -\sqrt{2\ell}\right)$. It maps the integral curves around each equilibrium to each other by ϵ_2, reversing their orientations. The integrand η_s of the period integral is transformed as $\epsilon_2^*\eta_s = -\eta_s$ and the period integral itself is transformed from (3.4) by the permutation $(ac) \in \mathfrak{S}_4$ with respect to ϵ_2.

Corollary 3.3. *The derivative of the inverse* Φ_3 *for the Birkhoff normal form around* $(p_1, p_2, p_3) = \left(0, 0, \pm\sqrt{2\ell}\right)$ *writes*

$$\Phi_3'(h) = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}} \frac{1}{\sqrt{(d-a)(c-b)}} \mathcal{K}\left(\frac{(d-c)(b-a)}{(d-a)(b-c)}\right) = S(c, b, a, d). \tag{3.9}$$

Remark 3.1. In [24, §3], the period of the integral curve around the elliptic equilibria for the Euler equation of the free rigid body dynamics is considered, although the precise formulae as (3.4) in Theorem 3.1 and (3.9) in Corollary 3.3 are not given. We need these precise expression in order to consider the analytic continuation of the derivative of the inverse for Birkhoff normal forms.

Next, we consider the derivative of the inverse for the Birkhoff normal form around the hyperbolic stationary points $(p_1, p_2, p_3) = \left(0, \pm\sqrt{2\ell}, 0\right)$. Around the stationary point $(p_1, p_2, p_3) = \left(0, \sqrt{2\ell}, 0\right)$, we can use (p_3, p_1) as a local coordinate system. Take the one-form

$$\eta'_s := (1-s)\frac{\mathsf{d}p_3}{3\left(\dfrac{1}{I_1} - \dfrac{1}{I_2}\right)p_1p_2} + s\frac{\mathsf{d}p_1}{3\left(\dfrac{1}{I_2} - \dfrac{1}{I_3}\right)p_2p_3},$$

where s is an arbitrary parameter as before, and the real closed arc

$$\gamma:\ p_3 = \sqrt{2\ell\frac{\frac{1}{I_2} - \frac{h}{\ell}}{\frac{1}{I_2} - \frac{1}{I_3}}}\cos\theta,\ p_1 = \sqrt{2\ell\frac{\frac{1}{I_2} - \frac{h}{\ell}}{\frac{1}{I_2} - \frac{1}{I_1}}}\sin\theta, \quad \theta : 0 \to 2\pi.$$

Note that γ is a real closed arc contained in the complexification of the real integral curve given by the affine curve (3.5), where $(p_1, p_2, p_3) \in \mathbb{C}^3$ are regarded as complex affine coordinates.

We consider the period integral $\displaystyle\int_\gamma \eta'_s$. Note that the equilibrium $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$ is mapped to $\left(0, \sqrt{2\ell}, 0\right)$ by the transformation $\epsilon_3 \circ \delta_1$, which maps the integral curves around $\left(\sqrt{2\ell}, 0, 0\right)$ to those represented by γ. Therefore, using Theorem 2.2, we can calculate the derivative for the inverse of the Birkhoff normal form around $\left(0, \sqrt{2\ell}, 0\right)$. As before, this is invariant with respect to the transformation δ_2, which maps $(p_1, p_2, p_3) = \left(0, \sqrt{2\ell}, 0\right)$ to $\left(0, -\sqrt{2\ell}, 0\right)$.

Theorem 3.4. *The derivative of the inverse Φ_2 of the Birkhoff normal forms around $(p_1, p_2, p_3) = \left(0, \pm\sqrt{2\ell}, 0\right)$ writes*

$$\Phi'_2(h) = \frac{\sqrt{-1}}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-c)(b-a)}}\mathcal{K}\left(\frac{(d-b)(a-c)}{(d-c)(a-b)}\right) = -\sqrt{-1}S(b, a, c, d). \tag{3.10}$$

Remark 3.2. In [31], they consider the inverse for the Birkhoff normal form around a hyperbolic equilibrium of the free rigid body dynamics in terms of a period integral over a vanishing cycle in the complexification of the integral curve in order to compute the semi-global symplectic invariant, although the formula (3.10) is not given there.

Before closing this section, we give the following formulae for $S(a, b, c, d)$ obtained through the transformations $\delta_1, \delta_2, \delta_3, \epsilon_1, \epsilon_2, \epsilon_3$:

$$S(a, b, c, d) = S(a, c, b, d),\ S(b, a, c, d) = S(b, c, a, d),\ S(c, b, a, d) = S(c, a, b, d). \tag{3.11}$$

These three formulae follow from (3.7) in the proof of the previous theorem and the covariance with respect to δ_j, ϵ_j, $j = 1, 2, 3$. The list can be regarded as the description of the action of the symmetric group $\mathfrak{S}_3$ of degree three on $S(a, b, c, d)$. We explain the action by the symmetric group $\mathfrak{S}_4$ of degree four in Section 5.

4 Elliptic fibration and explicit calculation of the cocycles

In this section, we discuss the naive elliptic fibration considered in [29] and calculate the cocycles of the first cohomology group of regular fibres in relation to the period integrals in Section 3. The main results of this section are the explicit expressions (4.3) and (4.6), in terms of the analytic extension of the function S, for a basis of the first cohomology group.

We start with a basic description of the naive elliptic fibration. As we have seen, the integral curve of the Euler equation is given by the intersection of the two quadrics (3.5). From the viewpoint of algebraic or analytic geometry, it is natural to complexify and to compactify the integral curve by the complex projective curve

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = 0, \\ ax^2 + by^2 + cz^2 + dw^2 = 0, \end{cases} \tag{4.1}$$

where $(x : y : z : w) \in P_3(\mathbb{C})$ are the coordinates given by $p_1 = \sqrt{-2\ell}\dfrac{x}{w}, p_2 = \sqrt{-2\ell}\dfrac{y}{w}, p_3 = \sqrt{-2\ell}\dfrac{z}{w}$, and where $a, b, c, d \in \mathbb{C}$ are parameters given by $\dfrac{1}{I_1} = a, \dfrac{1}{I_2} = b, \dfrac{1}{I_3} = c, \dfrac{h}{\ell} = d$. The following proposition is fundamental to the geometric understanding of this projective curve.

Proposition 4.1. *If a, b, c, d are distinct, then the variety C defined by the above equations* (4.1) *is a smooth elliptic curve, which has four branch points a, b, c, d as a double covering of the projective line $P_1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$.*

For the proof, see [29]. Because of this proposition, we have a natural elliptic fibration. In fact, denoting by F the algebraic variety in $P_3(\mathbb{C}) \times P_3(\mathbb{C}) : ((x : y : z : w), (a : b : c : d))$ defined by the equation (4.1), we consider the projection $\pi_F : F \ni ((x : y : z : w), (a : b : c : d)) \mapsto (a : b : c : d) \in P_3(\mathbb{C})$ to the second component of the product space, which is an elliptic fibration called the naive elliptic fibration in [29]. Here, an elliptic fibration means a smooth holomorphic mapping of a complex space onto another complex space whose regular fibres are elliptic curves. As basic facts of the naive elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$, it is known that the total space F is smooth rational variety and that the fibration π_F is non-flat, i.e. there is a two-dimensional fibre of π_F. In fact, the singular fibres of π_F are classified in [29] as follows:

Classification of the singular fibres of π_F

1. If only two of the parameters a, b, c, d are equal, the fibre consists of two smooth rational curves intersecting at two points. This is a singular fibre of type I_2 in Kodaira's notation [23, 4]. Topologically, these singular fibres are double pinched tori of real dimension two.

2. If two of a, b, c, d are equal and the other two are also equal without further coincidence, the fibre consists of four smooth rational curves intersecting cyclically. This is a singular fibre of type I_4 in Kodaira's notation. Topologically, these singular fibres are quadruple pinched tori of real dimension two.

3. If three of a, b, c, d are equal without further coincidence, the fibre is a smooth rational curve, i.e. a two-dimensional sphere as a point set, but with multiplicity two. This singular fibre is not in the list of singular fibres of elliptic surfaces by Kodaira.

4. If $a = b = c = d$, the fibre is a space quadric surface $x^2 + y^2 + z^2 + w^2 = 0$, which is isomorphic to $P_1(\mathbb{C}) \times P_1(\mathbb{C})$.

It is also known that the fibration π_F has no (meromorphic) section. On the total space F, there is an action of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ which respects the fibration π_F and which has no fixed point on the regular fibres. Taking the quotient, we have an elliptic fibration bimeromorphic to a Weierstraß normal form, which is flat and which admits only singular fibres included in the Kodaira's list, after suitable modifications. See [29] for the detailed discussion.

In order to think about the monodromy of the naive elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$, we introduce the following locally constant sheaf G on the base space $P_3(\mathbb{C})$. As is known from the description of the singular fibres, the singular locus of the elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$ is given by the divisor $D : \{a = b\} + \{a = c\} + \{a = d\} + \{b = c\} + \{b = d\} + \{c = d\}$ on $P_3(\mathbb{C}) : (a : b : c : d)$. For $(a : b : c : d) \in P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$, the fibre $\pi^{-1}(a : b : c : d)$ is a smooth complex torus, so that the first homology group $H_1\left(\pi_F^{-1}(a : b : c : d); \mathbb{Z}\right)$ forms a locally constant sheaf of $\mathbb{Z}_{P_3(\mathbb{C})\setminus\mathrm{Supp}(D)}$-module over $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$:

$$G' = \bigsqcup_{(a:b:c:d)\in P_3(\mathbb{C})\setminus\mathrm{Supp}(D)} H_1\left(\pi_F^{-1}(a : b : c : d); \mathbb{Z}\right).$$

For a point $(a : b : c : d) \in \mathrm{Supp}(D)$, we consider its sufficiently small polydisc neighbourhood $\mathcal{U}$ and set $\mathcal{U}' := \mathcal{U} \setminus \mathrm{Supp}(D)$. The group $\Gamma(\mathcal{U}', G')$ of sections of G' is determined independently from the choice of $\mathcal{U}$. Regarding $\Gamma(\mathcal{U}', G')$ as the stalk $G_{(a:b:c:d)}$ on $(a : b : c : d) \in \mathrm{Supp}(D)$, we consider the locally constant sheaf

$$G = G' \sqcup \bigsqcup_{(a:b:c:d)\in\mathrm{Supp}(D)} G_{(a:b:c:d)}$$

of $\mathbb{Z}_{P_3(\mathbb{C})}$-module over the base space $P_3(\mathbb{C})$. Taking the dual of each stalk, we have the locally constant sheaf of $\mathbb{Z}_{P_3(\mathbb{C})}$-module

$$G^* = G^{*\prime} \sqcup \bigsqcup_{(a:b:c:d)\in\mathrm{Supp}(D)} G^*_{(a:b:c:d)},$$

where

$$G^{*\prime} = \bigsqcup_{(a:b:c:d)\in P_3(\mathbb{C})\setminus\mathrm{Supp}(D)} H^1\left(\pi_F^{-1}(a : b : c : d); \mathbb{Z}\right)$$

and $G^*_{(a:b:c:d)}$ for $(a : b : c : d) \in \mathrm{Supp}(D)$ is determined in the same manner as $G_{(a:b:c:d)}$. We call G and G^* as the homological and cohomological invariant of π_F.

Let p_0 be a point in $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$ and $\gamma : t \mapsto \gamma(t)$, $0 \le t \le 1$, a closed path with the reference point at p_0 in $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$. Taking a basis $\sigma_{1,0}, \sigma_{2,0}$ of $H_1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$, we consider the elements $\sigma_1(t), \sigma_2(t)$ of $H_1\left(\pi_F^{-1}(\gamma(t)), \mathbb{Z}\right)$ which continuously depend on t and $\sigma_i(0) = \sigma_{i,1}$, $i = 1, 2$. Then, there is a matrix $A_{[\gamma]} \in SL(2, \mathbb{Z})$ such that

$$\begin{bmatrix}\sigma_1(1)\\ \sigma_2(1)\end{bmatrix} = A_{[\gamma]} \begin{bmatrix}\sigma_{1,0}\\ \sigma_{2,0}\end{bmatrix}.$$

Note that $A_{[\gamma]}$ depends only on the homotopy class $[\gamma] \in \pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D); p_0\right)$ of γ. The homomorphism $\rho : \pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D); p_0\right) \ni [\gamma] \mapsto A_{[\gamma]} \in SL(2, \mathbb{Z})$ is called the monodromy representation of $\pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D); p_0\right)$. Similarly, taking the dual basis $\sigma^*_{1,0}, \sigma^*_{2,0}$ of $H^1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$ such that $\sigma^*_{i,0} \cdot \sigma_{j,0} = \delta_{ij}$, we have the representation $\rho^* : \pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D); p_0\right) \ni [\gamma] \mapsto A^{\mathrm{T}}_{[\gamma]} \in SL(2, \mathbb{Z})$. We call it also monodromy representation. Clearly, the homological invariant G, as well as the cohomological invariant G^*, is determined by the monodromy representation ρ or ρ^*.

We, now, consider the period integrals which appeared in the last section from the viewpoint of the (co)homological invariant G (or G^*) and discuss the global monodromy, namely the monodromy representation itself. We start with the extension of the one-form η such that $\omega = \eta \wedge \mathsf{d}H$ to the complex elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$. On the level surface $L(P) = \dfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) = \ell$, the symplectic form ω writes as (3.2). On the coordinate neighbourhood with the coordinates (p_2, p_3), the one-form η_s in (3.3) satisfies $\omega = \eta_s \wedge \mathsf{d}H$ for arbitrary s. With this in mind, we consider the following one-form η:

$$\eta = \frac{1}{\sqrt{2\ell}} \frac{w^2 \mathsf{d}\left(\dfrac{y}{w}\right)}{3(c-a)zx}.$$

According to the notation in Section 2, η should be written as $\eta^{\mathbb{C}}$, but we use η also for the holomorphic form for the brevity. Because of the transformation of the coordinates $p_1 = \sqrt{-2\ell}\dfrac{x}{w}, p_2 = \sqrt{-2\ell}\dfrac{y}{w}, p_3 = \sqrt{-2\ell}\dfrac{z}{w}$ and $a = \dfrac{1}{I_1}, b = \dfrac{1}{I_2}, c = \dfrac{1}{I_3}, d = \dfrac{h}{\ell}$, we see that η and η_s induce the same section in $\Omega^1_{\pi_F} := \Omega^1_F / \pi_F^* \Omega^1_{P_3(\mathbb{C})}$, which is the sheaf over F of germs of relative differential forms with respect to the fibration $\pi_F : F \to P_3(\mathbb{C})$. See e.g. [19] for details on relative differential forms. More precisely, we have the following proposition.

Proposition 4.2. *The one-form η induces a meromorphic section of $\Omega^1_{\pi_F}$ on F, which is a holomorphic non-zero one-form on $\pi_F^{-1}\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)\right)$.*

Proof. On the total space F of the elliptic fibration π_F, we have

$$\begin{aligned}
\eta = \;& \frac{1}{\sqrt{-2\ell}} \frac{w^2 \mathsf{d}\left(\frac{x}{w}\right)}{3(b-c)yz} = \frac{1}{\sqrt{-2\ell}} \frac{w^2 \mathsf{d}\left(\frac{y}{w}\right)}{3(c-a)zx} = \frac{1}{\sqrt{-2\ell}} \frac{w^2 \mathsf{d}\left(\frac{z}{w}\right)}{3(a-b)xy} \\
= \;& \frac{1}{\sqrt{-2\ell}} \frac{x^2 \mathsf{d}\left(\frac{y}{x}\right)}{3(c-d)zw} = \frac{1}{\sqrt{-2\ell}} \frac{x^2 \mathsf{d}\left(\frac{z}{x}\right)}{3(d-b)wy} = \frac{1}{\sqrt{-2\ell}} \frac{x^2 \mathsf{d}\left(\frac{w}{x}\right)}{3(b-c)yz} \\
= \;& \frac{1}{\sqrt{-2\ell}} \frac{y^2 \mathsf{d}\left(\frac{z}{y}\right)}{3(d-a)wx} = \frac{1}{\sqrt{-2\ell}} \frac{y^2 \mathsf{d}\left(\frac{w}{y}\right)}{3(a-c)xz} = \frac{1}{\sqrt{-2\ell}} \frac{y^2 \mathsf{d}\left(\frac{x}{y}\right)}{3(c-d)zw} \\
= \;& \frac{1}{\sqrt{-2\ell}} \frac{z^2 \mathsf{d}\left(\frac{w}{z}\right)}{3(a-b)xy} = \frac{1}{\sqrt{-2\ell}} \frac{z^2 \mathsf{d}\left(\frac{x}{z}\right)}{3(b-d)yw} = \frac{1}{\sqrt{-2\ell}} \frac{z^2 \mathsf{d}\left(\frac{y}{z}\right)}{3(d-a)wx},
\end{aligned} \tag{4.2}$$

where η, as well as other one-forms, is regarded as a (local) meromorphic section of $\Omega^1_{\pi_F}$. Clearly, η is holomorphic and non-zero over $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$ from (4.2). ■

Note that the integral $\int_\sigma \eta$ of the one-form η over a cycle σ included in a fibre of π_F depends only on the class of the relative differential one-form to which η belongs. Thus, the correspondence

$$\int_\cdot \eta : G' \supset H_1\left(\pi_F^{-1}(a:b:c:d), \mathbb{Z}\right) \ni \sigma \mapsto \int_\sigma \eta \in \mathbb{C}$$

can be regarded as a linear functional over G' with respect to $\mathbb{C}_{P_3(\mathbb{C})\setminus\mathrm{Supp}(D)}$, so that $\int_\cdot \eta \in {G^*}'^{\mathbb{C}} := {G^*}' \otimes_{\mathbb{Z}_{P_3(\mathbb{C})\setminus\mathrm{Supp}(D)}} \mathbb{C}_{P_3(\mathbb{C})\setminus\mathrm{Supp}(D)}$. Note that η can be regarded as a holomorphic (hence closed) one-form on each regular fibre, so that $\displaystyle\int_\sigma \eta$ depends only on the homology class of σ. It is clear that $\int_{\sigma_{1,0}} \eta$ and $\int_{\sigma_{2,0}} \eta$ form a basis of ${{G^*}'^{\mathbb{C}}}_{p_0} = H^1\left(\pi_F^{-1}(p_0), \mathbb{C}\right)$ and it suffices to consider the monodromy of these period integrals in order to calculate the monodromy representation of $\pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D), p_0\right)$ with respect to the fibration $\pi_F : F \to P_3(\mathbb{C})$.

We choose a basis of the first homology group $H_1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$ of the regular fibre near the elliptic and hyperbolic stationary points of the free rigid body dynamics. Assuming the same condition $I_1 < I_2 < I_3$ for the inertia tensor $\mathcal{I} = \mathrm{diag}(I_1, I_2, I_3)$ as in Section 3, we start with a regular fibre around the elliptic stationary points on the p_1-axis. Note that these two points are in the same intersection of two quadrics $H = h, L = \ell$, such that $\frac{h}{\ell} = \frac{1}{I_1}$, i.e. $a = d$. For the simplicity, we assume that $\frac{1}{I_1} > \frac{h}{\ell} > \frac{1}{I_2} > \frac{1}{I_3}$. The real integral curves are parameterized as

$$\sigma_\pm : (p_1, p_2, p_3) = \left(\pm \sqrt{ -\frac{2\ell\left(\frac{1}{I_2} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_2}} \left\{ 1 - \frac{\left(\frac{h}{\ell} - \frac{1}{I_1}\right)\left(\frac{1}{I_3} - \frac{1}{I_2}\right)}{\left(\frac{h}{\ell} - \frac{1}{I_2}\right)\left(\frac{1}{I_3} - \frac{1}{I_1}\right)} \sin^2\theta \right\} }, \right.$$
$$\left. \sqrt{\frac{2\ell\left(\frac{1}{I_1} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_2}}} \cos\theta, \sqrt{\frac{2\ell\left(\frac{1}{I_1} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_3}}} \sin\theta \right),$$

where $\theta : 0 \to 2\pi$, near $(p_1, p_2, p_3) = \left(\pm\sqrt{2\ell}, 0, 0\right)$. If h is near to $\frac{\ell}{I_1}$, i.e. if d is near to a, these real closed arcs are vanishing cycles around two different A_1 singular points and are included in the same fibre of $\pi_F : F \to P_3(\mathbb{C})$. By Theorem 3.1 and Corollary 3.2, the period integrals of η along $\sigma_\pm$ are the same:

$$\int_{\sigma_\pm} \eta = S(a, b, c, d). \tag{4.3}$$

This reflects the fact that $\sigma_\pm$ are homologous to each other in $H_1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$. Here, $S(a, b, c, d)$ is considered as a multi-valued holomorphic function over $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$. We take this period integral as one of the basis of ${G^{*\prime}}^{\mathbb{C}}_{p_0} = H^1\left(\pi_F^{-1}(p_0), \mathbb{C}\right)$.

To obtain the other basis element, we consider the following real one-dimensional arcs in the same regular fibre. We take the arcs

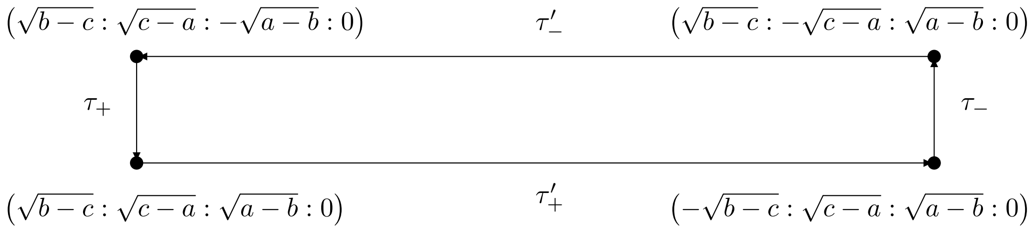
$$\tau_\pm : (p_1, p_2, p_3) = \left(\pm \sqrt{ -\frac{2\ell\left(\frac{1}{I_2} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_2}} \left\{ 1 + \frac{\left(\frac{h}{\ell} - \frac{1}{I_1}\right)\left(\frac{1}{I_3} - \frac{1}{I_2}\right)}{\left(\frac{h}{\ell} - \frac{1}{I_2}\right)\left(\frac{1}{I_3} - \frac{1}{I_1}\right)} \sinh^2\varphi \right\} }, \right.$$
$$\left. \sqrt{\frac{2\ell\left(\frac{1}{I_1} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_2}}} \cosh\varphi, \pm\sqrt{-1}\sqrt{\frac{2\ell\left(\frac{1}{I_1} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_3}}} \sinh\varphi \right),$$

where φ moves from $-\infty$ to $+\infty$. It is easy to check that these two arcs are included in the same fibre of $\pi_F : F \to P_3(\mathbb{C})$ near the singular locus $a = d$. Note that $\sigma_\pm$ meet $\tau_\pm$ respectively at single points $\left(\pm\sqrt{b-d} : \sqrt{d-a} : 0 : \sqrt{a-b}\right)$, while $\sigma_\pm$ do not meet $\tau_\mp$. We also need to take the following arcs:

$$\tau'_\pm : (p_1, p_2, p_3) = \left(-\sqrt{-1}\sqrt{\frac{2\ell\left(\frac{1}{I_2} - \frac{h}{\ell}\right)}{\frac{1}{I_2} - \frac{1}{I_1}}} \sinh\varphi, \right.$$
$$\left. \pm\sqrt{ -\frac{2\ell\left(\frac{1}{I_3} - \frac{h}{\ell}\right)}{\frac{1}{I_2} - \frac{1}{I_3}} \left\{ 1 + \frac{\left(\frac{h}{\ell} - \frac{1}{I_2}\right)\left(\frac{1}{I_1} - \frac{1}{I_3}\right)}{\left(\frac{h}{\ell} - \frac{1}{I_3}\right)\left(\frac{1}{I_1} - \frac{1}{I_2}\right)} \sinh^2\varphi \right\} }, \pm\sqrt{-1}\sqrt{\frac{2\ell\left(\frac{1}{I_2} - \frac{h}{\ell}\right)}{\frac{1}{I_2} - \frac{1}{I_3}}} \cosh\varphi \right),$$

where φ moves from $-\infty$ to $+\infty$. It is easy to see that $\tau'_\pm$ are included in the same regular fibre of $\pi_F : F \to P_3(\mathbb{C})$ near the singular locus $a = d$ as $\tau_\pm$. By considering the behavior of these arcs when the parameters approaches infinity, we can check that the arcs $\tau_\pm$, $\tau'_\pm$ are connected at these infinity points as in Figure 1.

Figure 1. The arcs $\tau_\pm$ and $\tau'_\pm$



Note that there is no other intersection among $\tau_\pm$ and $\tau'_\pm$ than these four points and that $\tau'_\pm$ do not meet $\sigma_\pm$. We denote the multiplication $\tau_+ \cdot \tau'_+ \cdot \tau_- \cdot \tau'_-$ by τ. In particular, τ is a closed arc in the same regular fibre of $\pi_F : F \to P_3(\mathbb{C})$ as $\sigma_\pm$. Moreover, the two cycles σ_+, homologous to σ_-, and τ form a basis of $H_1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right) = G'_{p_0}$. Thus, calculating the period integrals $\displaystyle\int_\tau \eta$, together with (4.3), we can obtain a concrete basis of $H^1\left(\pi_F^{-1}(p_0), \mathbb{C}\right) = {G'^*}^{\mathbb{C}}_{p_0}$.

In fact, we can express the period integral $\displaystyle -\frac{1}{2\pi}\int_\tau \eta$ in terms of the analytic continuation of the function S as in Theorem 4.4. For the proof, we need the following proposition.

Proposition 4.3. *The following formula holds for the analytic continuation of the function S:*

$$S(a,b,c,d) + S(b,a,c,d) = S(c,b,a,d).$$

Proof. For simplicity, we write as $S_1 = S(a,b,c,d)$, $S_2 = S(b,a,c,d)$, $S_3 = S(c,b,a,d)$. The proof of the proposition can be performed by using the connection formulae of a special Gauß hypergeometric differential equation. It is known that the complete elliptic integral of the first kind is a Gauß hypergeometric function $\mathcal{K}(\lambda) = \dfrac{\pi}{2} F\left(\dfrac{1}{2}, \dfrac{1}{2}, 1; \lambda\right)$, which satisfies the Gauß hypergeometric differential equation

$$(1-\lambda)\lambda \frac{\mathsf{d}^2 f}{\mathsf{d}\lambda^2} + (1-2\lambda)\frac{\mathsf{d} f}{\mathsf{d}\lambda} - \frac{1}{4} f = 0. \tag{4.4}$$

See e.g. [39, Ch. 12. §12.5, p.494]. According to the description of the connection formulae for this equation by [9, 7.405-7.406, pp.167-169], we consider the function $F(\lambda) := F\left(\dfrac{1}{2}, \dfrac{1}{2}, 1; \lambda\right)$ and $F^*(\lambda) = F^*\left(\dfrac{1}{2}, \dfrac{1}{2}, 1; \lambda\right)$ (*loc. cit.*), which are holomorphic around $\lambda = 0$. Then, we set

$$\begin{aligned} &\varphi_1 = F(\lambda),\ \varphi_3 = F(1-\lambda),\ \varphi_5 = \frac{1}{\sqrt{-\lambda}} F\left(\frac{1}{\lambda}\right),\ \varphi_2^* = F(\lambda)\log\lambda + F^*(\lambda), \\ &\varphi_4^* = F(1-\lambda)\log(1-\lambda) + F^*(1-\lambda),\ \varphi_6^* = \frac{1}{\sqrt{-\lambda}}\left\{F\left(\frac{1}{\lambda}\right)\log(-\lambda) - F^*\left(\frac{1}{\lambda}\right)\right\}. \end{aligned} \tag{4.5}$$

The pairs (φ_1, φ_2^*), (φ_3, φ_4^*), and (φ_5, φ_6^*) give bases of the solution space of the Gauß hypergeometric differential equation (4.4) around $\lambda = 0$, $\lambda = 1$, and $\lambda = \infty$, respectively. Between two of them, we have the following connection formulae ([9, 7.405-7.406, pp.167-169]):

$$\begin{cases} \varphi_1 = \dfrac{1}{\pi}\left(\varphi_3 \log 16 - \varphi_4^*\right), \\ \varphi_2^* = \dfrac{1}{\pi}\left\{\left((\log 16)^2 - \pi^2\right)\varphi_3 - \varphi_4^* \log 16\right\} = \varphi_1 \log 16 - \pi\varphi_3, \end{cases}$$

$$\begin{cases}\varphi_1 = \dfrac{1}{\pi}\left(\varphi_5 \log 16 + \varphi_6^*\right), \\ \varphi_2^* = \dfrac{1}{\pi}\left\{\left(\left(\log 16 + \pi\sqrt{-1}\right)\log 16 - \pi^2\right)\varphi_5 + \varphi_6^*\left(\log 16 + \pi\sqrt{-1}\right)\right\} = \varphi_1\left(\log 16 + \pi\sqrt{-1}\right) - \pi\varphi_5.\end{cases}$$

Thus, we have $\varphi_1\sqrt{-1} + \varphi_3 = \varphi_5$, which shows $S_1 + S_2 = S_3$. Note that, for $\lambda = \dfrac{(d-a)(b-c)}{(d-c)(b-a)}$,

$$S_1 = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{\varphi_1(\lambda)}{\sqrt{(d-c)(a-b)}},\ S_2 = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{\varphi_3(\lambda)}{\sqrt{(d-c)(b-a)}},\ S_3 \quad = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{\varphi_5(\lambda)}{\sqrt{(d-c)(b-a)}}.$$

■

Theorem 4.4. *The period integral of the one-form η along the cycle τ is calculated as*

$$-\frac{1}{2\pi}\int_\tau \eta = S(c,b,a,d), \tag{4.6}$$

the right hand side of which is a branch of the analytic continuation of $S(c,b,a,d)$.

Proof. We calculate the integrals of the one-form $\eta = \dfrac{\mathsf{d}p_2}{3\left(\frac{1}{I_3} - \frac{1}{I_1}\right)p_3p_1}$ along the real arcs $\tau_\pm$:

$$\begin{aligned}-\frac{1}{2\pi}\int_{\tau_+}\eta = &-\frac{1}{\pi}\int_{\sqrt{\frac{2\ell\left(\frac{1}{I_1}-\frac{h}{\ell}\right)}{\frac{1}{I_1}-\frac{1}{I_2}}}}^{+\infty}\frac{\mathsf{d}p_2}{3\left(\frac{1}{I_3}-\frac{1}{I_1}\right)p_3p_1} \\ =&-\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\left\{\frac{1}{\sqrt{(d-a)(b-c)}}\mathcal{K}\left(\frac{(d-b)(c-a)}{(d-a)(c-b)}\right) + \frac{1}{\sqrt{(d-b)(a-c)}}\mathcal{K}\left(\frac{(d-a)(c-b)}{(d-b)(c-a)}\right)\right\} \\ =&\frac{1}{2}\left(S(b,c,a,d) + S(a,c,b,d)\right) = \frac{1}{2}\left(S(b,a,c,d) + S(a,b,c,d)\right) = \frac{1}{2}S(c,b,a,d).\end{aligned}$$

Here, we used (3.11) and Proposition 4.3. Clearly, we have $\int_{\tau_+}\eta = \int_{\tau_-}\eta$ and $\int_{\tau'_+}\eta = -\int_{\tau'_-}\eta$ from the choice of the arcs. Thus, we have $-\dfrac{1}{2\pi}\int_\tau \eta = -\dfrac{1}{\pi}\int_{\tau_+}\eta = S(c,b,a,d)$. ■

5 Monodromy and global behavior of Birkhoff normal forms

In this section, we calculate the global monodromy of the naive elliptic fibration π_F, which is connected to the global behavior of Birkhoff normal forms. Starting with the symmetry of the naive elliptic fibration and of the function S, whose relation to the cohomology group of regular fibres is discussed in Section 4, we first calculate the local monodromy with respect to the basis associated to the analytic continuation of S, by using the connection formulae of the Gauß hypergeometric differential equation (4.4). Next, we give a description of the fundamental group of the regular locus of the base space $P_3(\mathbb{C})$ for the naive elliptic fibration π_F, by using the arguments of the topology of the complement of hyperplane arrangements. Combining these results by the formula in Proposition 4.3, we finally obtain the explicit global monodromy of the naive elliptic fibration in Theorem 5.4. This theorem implies that the monodromy of the analytic continuation for the derivative of the inverse of Birkhoff normal forms coincides with the global monodromy of the elliptic fibration in [29].

The two period integrals $-\frac{1}{2\pi}\int_{\sigma_\pm}\eta = S(a,b,c,d)$ and $-\frac{1}{2\pi}\int_{\tau}\eta = S(c,b,a,d)$ of the one-form η along the basis $\sigma_\pm$ and τ of $H_1\left(\pi_F^{-1}(p_0),\mathbb{Z}\right)$ for a regular fibre $\pi_F^{-1}(p_0)$, where p_0 is near to the singular locus $a=d$, form a basis of $H^1\left(\pi_F^{-1}(p_0),\mathbb{C}\right)$. We take such a basis of the cohomology group $H^1\left(\pi_F^{-1}(p_0),\mathbb{C}\right)$ for a regular fibre $\pi_F^{-1}(p_0)$ over p_0 which is near to each irreducible component of the singular locus $D:\{a=b\}+\{a=c\}+\{a=d\}+\{b=c\}+\{b=d\}+\{c=d\}$. To do this, we use the symmetry of the naive elliptic fibration and that of the function $S(a,b,c,d)$ with respect to the symmetric group $\mathfrak{S}_4$ acting on the base space $P_3(\mathbb{C}):(a:b:c:d)$ as the permutations of the four letters a,b,c,d. In fact, we have the following list of equalities among the analytic continuation of $S\left(\sigma(a,b,c,d)\right)$, for $\sigma\in\mathfrak{S}_4$, which can be obtained by using the equalities (4.5) of the Gauß hypergeometric function F:

$$\begin{array}{lllll} S_1 & =S(a,b,c,d) & =S(a,c,b,d) & =S(b,a,d,c) & =S(b,d,a,c) \\ & =S(c,a,d,b) & =S(c,d,a,b) & =S(d,b,c,a) & =S(d,c,b,a), \\ S_2 & =S(b,a,c,d) & =S(b,c,a,d) & =S(a,b,d,c) & =S(a,d,b,c) \\ & =S(c,b,d,a) & =S(c,d,b,a) & =S(d,a,c,b) & =S(d,c,a,b), \\ S_3 & =S(c,b,a,d) & =S(c,a,b,d) & =S(b,c,d,a) & =S(b,d,c,a) \\ & =S(a,c,d,b) & =S(a,d,c,b) & =S(d,a,b,c) & =S(d,b,a,c). \end{array} \tag{5.1}$$

This means that the function $S(a,b,c,d)$ is invariant with respect to the dihedral group generated by (bc) and $(abdc)$, which is isomorphic to $\mathbb{Z}_2\ltimes\mathbb{Z}_4$.

The naive elliptic fibration $\pi_F:F\to P_3(\mathbb{C})$ is invariant with respect to $\mathfrak{S}_4$ acting on $P_3(\mathbb{C})\times P_3(\mathbb{C}):((x:y:z:w),(a:b:c:d))$ as

$$\sigma\left((x:y:z:w),(a:b:c:d)\right)=\left((\sigma x:\sigma y:\sigma z:\sigma w),(\sigma a:\sigma b:\sigma c:\sigma d)\right),$$

for $\sigma\in\mathfrak{S}_4$. The irreducible components $b=c$, $b=d$, $a=c$, $c=d$, $a=b$ of the singular locus D can be obtained from $a=d$ for instance by the action of $(ab)\cdot(cd)$, (abc), (acd), (acb), (abd), respectively. The basis $S_3=S(c,b,a,d)$, $S_1=S(a,b,c,d)$ of $H^1\left(\pi_F^{-1}(p_0),\mathbb{C}\right)$ for p_0 near to $a=d$ is mapped by the action of $(ab)\cdot(cd)$, (abc), (acd), (acb), (abd) to the bases $S_3=S(d,a,b,c)$, $S_1=S(b,a,d,c)$; $S_1=S(a,c,b,d)$, $S_2=S(b,c,a,d)$; $S_1=S(d,b,c,a)$, $S_2=S(c,b,d,a)$; $S_2=S(b,a,c,d)$, $S_3=S(c,a,b,d)$; $S_2=S(c,d,b,a)$, $S_3=S(b,d,c,a)$ of the first cohomology group of the regular fibres near the components $b=c$, $b=d$, $a=c$, $c=d$, $a=b$, respectively.

We calculate the local monodromy of the fibration $\pi_F:F\to P_3(\mathbb{C})$ around each irreducible component of the singular locus D with respect to the above bases.

- Around the singular fibres sitting over the irreducible component $\{a=d\}$ or $\{b=c\}$ of D, we have the basis S_3,S_1 of $H^1\left(\pi_F^{-1}(p_0),\mathbb{C}\right)$ over $p_0\in P_3(\mathbb{C})\setminus\mathrm{Supp}(D)$ near these components. We take a real closed arc α_1 in $P_3(\mathbb{C})\setminus\mathrm{Supp}(D)$, which is homotopic to the arc $a=d_0+\epsilon e^{\sqrt{-1}\theta}$, $b=b_0$, $c=c_0$, $d=d_0$ or the one $a=a_0$, $b=c_0+\epsilon e^{\sqrt{-1}\theta}$, $c=c_0$, $d=d_0$. Then, the lambda-function $\lambda=\frac{(d-a)(b-c)}{(d-c)(b-a)}$ can be seen to move as $\lambda=\epsilon e^{\sqrt{-1}\theta}\lambda_0$, $\lambda_0=\frac{(d_0-a_0)(b_0-c_0)}{(d_0-c_0)(b_0-a_0)}$, where $\theta:0\to 2\pi$. Here, a_0, b_0, c_0, d_0 are suitable fixed complex numbers. The formulae (4.5) implies that the basis φ_1, φ_2^* of the solutions of the Gauß hypergeometric differential equation (4.4) is analytically continued along the above arc as

$$\begin{bmatrix}\varphi_1(\lambda)\\ \varphi_2^*(\lambda)\end{bmatrix}\mapsto\begin{bmatrix}\varphi_1(\lambda)\\ 2\pi\sqrt{-1}\varphi_1(\lambda)+\varphi_2^*(\lambda)\end{bmatrix}.$$

Using the connection formula $\varphi_2^* = \varphi_1 \left(\log 16 + \pi\sqrt{-1}\right) - \pi\varphi_5$, we can see that the basis S_3, S_1 is analytically continued along α_1 as

$$\begin{bmatrix} S_3 \\ S_1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S_3 \\ S_1 \end{bmatrix}.$$

- Around the singular fibres over the irreducible component $\{b = d\}$ or $\{a = c\}$ of D, we take the basis S_1, S_2 of $H^1\left(\pi_F^{-1}(p_0), \mathbb{C}\right)$ over $p_0 \in P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$ near these components. Consider a real closed arc α_2 in $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$, which is homotopic to the arc $a = a_0,\ b = d_0 + \epsilon e^{\sqrt{-1}\theta},\ c = c_0,\ d = d_0$ or the one $a = c_0 + \epsilon e^{\sqrt{-1}\theta},\ b = b_0,\ c = c_0,\ d = d_0$. Then, $1 - \lambda = \dfrac{(d-b)(a-c)}{(d-c)(a-b)}$ can be assumed to move as $1 - \lambda = \epsilon e^{\sqrt{-1}\theta}$, $\lambda_0 = \dfrac{(d_0-b_0)(a_0-c_0)}{(d_0-c_0)(a_0-b_0)}$, where $\theta : 0 \to 2\pi$. From (4.5), the basis φ_3, φ_4^* of the solution space of the Gauß hypergeometric differential equation is analytically continued along the above arc as

$$\begin{bmatrix} \varphi_3(\lambda) \\ \varphi_4^*(\lambda) \end{bmatrix} \mapsto \begin{bmatrix} \varphi_3(\lambda) \\ 2\pi\sqrt{-1}\varphi_3(\lambda) + \varphi_4^*(\lambda) \end{bmatrix}.$$

From the connection formula $\varphi_1 = \dfrac{1}{\pi}\left(\varphi_3 \log 16 - \varphi_4^*\right)$, we see that the basis S_1, S_2 is analytically continued along α_2 as

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}.$$

- Around the singular fibres over the irreducible component $\{c = d\}$ or $\{a = b\}$ of D, we have the basis S_2, S_3 of $H^1\left(\pi_F^{-1}(p_0), \mathbb{C}\right)$ over $p_0 \in P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$ near these components. Take a real closed arc α_3 in $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$, which is homotopic to the arc $a = a_0$, $b = b_0$, $c = d_0 + \epsilon e^{\sqrt{-1}\theta}$, $d = d_0$ or the one $a = b_0 + \epsilon e^{\sqrt{-1}\theta}$, $b = b_0$, $c = c_0$, $d = d_0$. Then, $\dfrac{1}{\lambda} = \dfrac{(d-c)(b-a)}{(d-a)(b-c)}$ can be assumed to move as $\dfrac{1}{\lambda} = \epsilon e^{\sqrt{-1}\theta}\dfrac{1}{\lambda_0}$, $\lambda_0 = \dfrac{(d_0-a_0)(b_0-c_0)}{(d_0-c_0)(b_0-a_0)}$, where $\theta : 0 \to 2\pi$. From (4.5), the basis φ_5, φ_6^* of the solution space of the Gauß hypergeometric differential equation is analytically continued along the above arc as

$$\begin{bmatrix} \varphi_5(\lambda) \\ \varphi_6^*(\lambda) \end{bmatrix} \mapsto \begin{bmatrix} -\varphi_5(\lambda) \\ -2\pi\sqrt{-1}\varphi_5(\lambda) - \varphi_6^*(\lambda) \end{bmatrix}.$$

From the connection formulae, we have $\varphi_6^* = \pi\sqrt{-1}\varphi_3 + \left(-\pi\sqrt{-1} + \log 16\right)\varphi_5$. Taking into account the fact that the analytic continuation of $\sqrt{(d-c)(b-a)}$ along the above arc α_3 is given by the multiplication of -1, we find the analytic continuation of S_2, S_3 as

$$\begin{bmatrix} S_2 \\ S_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S_2 \\ S_3 \end{bmatrix}.$$

Remark 5.1. The conjugacy class of the monodromy matrices of the singular fibres of type I_2, which are topologically a double pinched torus, was found by Kodaira [23] together with all the types of singular fibres of elliptic surfaces. In the context of integrable systems theory, the conjugacy class of the monodromy around the double pinched torus is mentioned in [27] and [40].

Next, we describe the fundamental group of the regular locus $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$ of the fibration $\pi_F : F \to P_3(\mathbb{C})$. Note that the equations $a = b$, $a = c$, $a = d$, $b = c$, $b = d$, $c = d$ of the

singular locus D form the arrangement of affine hyperplanes in $V = \mathbb{C}^4 : (a, b, c, d)$ of type A_3. As to the complement of the affine hyperplane arrangement, it is known that its fundamental group is the colored braid group [8]. Applying this general argument to our setting, we can calculate $\pi_1 (P_3 (\mathbb{C}) \setminus \mathrm{Supp}(D))$ as follows:
On V, there is the action of the Weyl group $\mathcal{W} = \mathfrak{S}_4$ whose elements permute the coordinates a, b, c, d. Obviously, its fixed-point-set is the union of the six hyperplanes $a = b$, $a = c$, $a = d$, $b = c$, $b = d$, $c = d$. Set

$$Y := V \setminus (\{a = b\} \cup \{a = c\} \cup \{a = d\} \cup \{b = c\} \cup \{b = d\} \cup \{c = d\}), \ X := Y/\mathcal{W}.$$

The quotient mapping $Y \to X$ is an unramified covering and we have the exact sequence

$$1 \to \pi_1(Y) \to \pi_1(X) \to \mathcal{W} \to 1.$$

Further, the fundamental group $\pi_1(X)$ of the quotient space is the braid group generated by the generators g_1, g_2, g_3 with the relations $g_1g_2g_1 = g_2g_1g_2$, $g_2g_3g_2 = g_3g_2g_3$, $g_1g_3 = g_3g_1$ (cf. [7, 8]). In the sense of geometry of braids, g_1, g_2, g_3 describe the simplest strands between the first and the second strings, the second and the third strings, the third and the fourth strings, respectively, for the braids with four strings. The group homomorphism $\pi_1(X) \to \mathcal{W} = \mathfrak{S}_4$ is given by the natural correspondence

$$g_1 \mapsto (12), \quad g_2 \mapsto (23), \quad g_3 \mapsto (34).$$

By the above short exact sequence, we can realize $\pi_1(Y)$ as the kernel of this homomorphism. More precisely, putting $h_{12} = g_1^2$, $h_{23} = g_2^2$, $h_{34} = g_3^2$, $h_{13} = g_1g_2^2g_1^{-1} = g_2^{-1}g_1^2g_2$, $h_{14} = g_1g_2g_3^2g_2^{-1}g_1^{-1} = g_3^{-1}g_1g_2^2g_1^{-1}g_3 = g_3^{-1}g_2^{-1}g_1^2g_2g_1$, $h_{24} = g_2g_3^2g_2^{-1} = g_3^{-1}g_2^2g_3$, we can describe $\pi_1(Y)$ as the group generated by h_{12}, h_{23}, h_{34}, h_{13}, h_{14}, h_{24} with the relations

$$\begin{gathered} h_{12}h_{23}h_{13} = h_{23}h_{13}h_{12} = h_{13}h_{12}h_{23}, \quad h_{23}h_{34}h_{24} = h_{34}h_{24}h_{23} = h_{24}h_{23}h_{34}, \\ h_{12}h_{24}h_{14} = h_{24}h_{14}h_{12} = h_{14}h_{12}h_{24}, \quad h_{34}h_{14}h_{13} = h_{14}h_{13}h_{34} = h_{13}h_{34}h_{14}, \\ h_{12}h_{34} = h_{34}h_{12}, \quad h_{13}h_{23}^{-1}h_{24}h_{23} = h_{23}^{-1}h_{24}h_{23}h_{13}, \quad h_{23}h_{14} = h_{14}h_{23}. \end{gathered} \tag{5.2}$$

By taking the quotient of Y through the fixed-point-free action of the group $\mathbb{C}^*$, we have the complement $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$:

$$\begin{array}{ccc} \mathbb{C}^* & \to & Y \\ & & \downarrow \\ & & P_3(\mathbb{C}) \setminus \mathrm{Supp}(D) \end{array}$$

The homotopy exact sequence (cf. [35]) of this fibre bundle writes

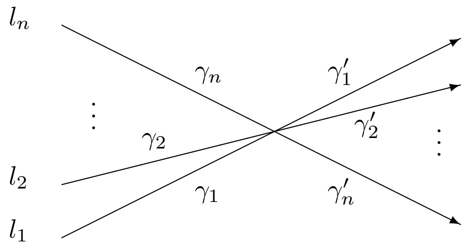
$$1 \to \pi_1(\mathbb{C}^*) \to \pi_1(Y) \to \pi_1 (P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)) \to 1,$$

so that $\pi_1 (P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)) \cong \pi_1(Y)/\pi_1(\mathbb{C}^*)$. In order to determine the fundamental group $\pi_1 (P_3(\mathbb{C}) \setminus \mathrm{Supp}(D))$, we have to find a generator of $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$ in $\pi_1(Y)$. Since all the hyperplanes $a = b$, $a = c$, $a = d$, $b = c$, $b = d$, $c = d$ in V pass through the origin, the generators of $\pi_1(\mathbb{C}^*)$ can be realized as a multiple of the six elements in $\pi_1(Y)$ presented by closed arcs which enclose the hyperplanes. Further, the generators of $\pi_1(\mathbb{C}^*)$ are in the centre of $\pi_1(Y)$. This follows from the following theorem, by means of Zariski's Theorem (cf. [30, Chapter 5, §5.3]).

Theorem 5.1 (Randell)**.** *Let $l_1, \ldots, l_n$ be affine complex lines in $\mathbb{C}^2$ which are defined by linear equations with real coefficients and which pass through the origin. Figure 2 describes the real section of the line arrangement. Denote the generators of $\pi_1 \left(\mathbb{C}^2 \setminus \cup_{j=1}^n l_j\right)$ represented by closed arcs around l_j by γ_j and γ_j' as in Figure 2. Then, we have*

$$\gamma_1 \cdots \gamma_n = \gamma_2 \cdots \gamma_n \gamma_1 = \cdots = \gamma_n \gamma_1 \cdots \gamma_{n-1}, \quad \gamma_j' = \gamma_n^{-1} \cdots \gamma_{j-1}^{-1} \gamma_j \gamma_{j+1} \cdots \gamma_n.$$

Figure 2. Lines in $\mathbb{C}^2$ passing through the origin



For the proof, see [32] or [30], although [30] deals with the fundamental group of the complement of more general hyperplane arrangements. From this theorem, we see that $\gamma = \gamma_1 \cdots \gamma_n$ commute with γ_j, since $\gamma\gamma_j = \gamma_j\gamma_{j+1}\cdots\gamma_n\gamma_1\cdots\gamma_{j-1}\gamma_j = \gamma_j\gamma$. The centre of $\pi_1(Y)$ is generated by $(g_1g_2g_3)^4$ (cf. [3]) and, further, we have the expression

$$(g_1g_2g_3)^4 = h_{13}h_{12}h_{23}h_{34}h_{24}h_{14}.$$

Thus, we see that $\pi_1(\mathbb{C}^*)$ is generated by $h_{13}h_{12}h_{23}h_{34}h_{24}h_{14}$.

Proposition 5.2. *The fundamental group* $\pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)\right)$ *can be realized as the group generated by* $h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}$ *with the relations* (5.2) *and* $h_{13}h_{12}h_{23}h_{34}h_{24}h_{14} = 1$.

Here, we make comments on the relation between $\pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)\right)$ and the fundamental group of the complement of the line arrangement on the (hyper)plane $E : a+b+c+d=0$, which is isomorphic to $P_2(\mathbb{C})$, given by the same equations as D: $a=b$, $a=c$, $a=d$, $b=c$, $b=d$, $c=d$. Denote the induced divisor on the plane $E : a+b+c+d=0$ by $\overline{D}$.

Proposition 5.3. *The following isomorphism holds:*

$$\pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)\right) \cong \pi_1\left(E \setminus \mathrm{Supp}\left(\overline{D}\right)\right).$$

Proof. We consider the blowing-up of $P_3(\mathbb{C}) : (a:b:c:d)$ with the centre at $a=b=c=d$ as [29]: $\Phi_B : B \to P_3(\mathbb{C})$. The hyperplane E can be identified with the exceptional divisor through Φ_B. Clearly, we have $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D) \cong B \setminus \left(E \cup \mathrm{Supp}\left(\widetilde{D}\right)\right)$, where $\widetilde{D}$ is the proper transform of D through Φ_B. On the other hand, identifying all the points on a line in $P_3(\mathbb{C})$ which passes through the point $a=b=c=d$, we have the mapping $\tau_B : B \to E$, which is in fact a $P_1(\mathbb{C})$-fibre bundle. This structure of $P_1(\mathbb{C})$-fibre bundle is inherited to the complement $B\setminus\left(E \cup \mathrm{Supp}\left(\widetilde{D}\right)\right)$, so that we have the $P_1(\mathbb{C})$-fibre bundle $B\setminus\left(E \cup \mathrm{Supp}\left(\widetilde{D}\right)\right) \to E\setminus\mathrm{Supp}\left(\overline{D}\right)$. By the homotopy exact sequence of fibre bundles [35], we have the exact sequence $\pi_1(P_1(\mathbb{C})) \to \pi_1\left(B \setminus \left(E \cup \mathrm{Supp}\left(\widetilde{D}\right)\right)\right) \to \pi_1\left(E \setminus \mathrm{Supp}\left(\overline{D}\right)\right) \to 1$. The simple connectedness of $P_1(\mathbb{C})$ proves the proposition. ■

The real section of the arrangement $\overline{D}$ on $E \cong P_2(\mathbb{C})$ is drawn as in Figure 3. The six lines $a=b$, $a=c$, $a=d$, $b=c$, $b=d$, $c=d$ on (the real part of) the plane E form an A_3 configuration.

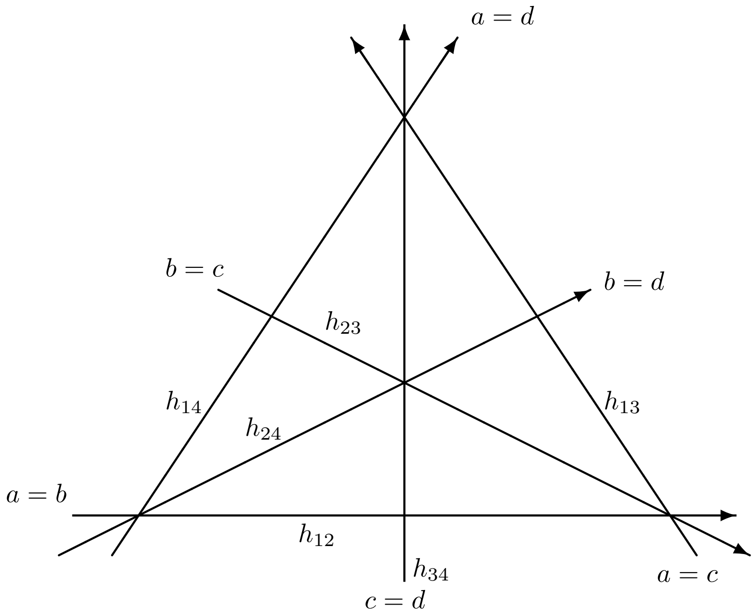


Figure 3. A_3 configuration on E

In relation to the description of $\pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)\right)$ by the generators h_{12}, h_{23}, h_{34}, h_{13}, h_{14}, h_{24}, the corresponding closed arcs can be chosen as indicated in Figure 3. Note that the closed arc around a line is chosen as follows:
Let l be the line in $\mathbb{C}^2 : (u, v)$ given by $v = \alpha u$, where α is a real positive constant. We fix the orientation of l as indicated in Figure 4, which describes the real section. Corresponding to this line with the orientation, we assign the closed arc $u = \epsilon e^{\sqrt{-1}\theta}, v = 0$, where $\theta : 0 \to 2\pi$ and $\epsilon > 0$ is a sufficiently small real constant.

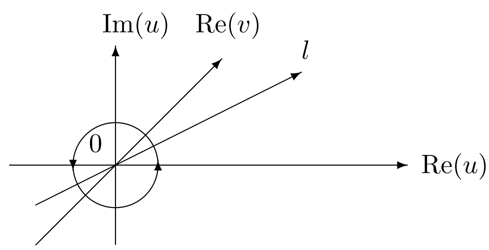


Figure 4. Closed arc around l

We can calculate the global monodromy, by the results of local monodromy and Proposition 4.3, which means

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} S_3 \\ S_1 \end{bmatrix}, \quad \begin{bmatrix} S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S_3 \\ S_1 \end{bmatrix}.$$

Theorem 5.4. *The basis of the first cohomology group for the regular fibre of the fibration* $\pi_F : F \to P_3(\mathbb{C})$ *is given by branches of the analytic continuations of* S_3 *and of* S_1*, which are proportional to the derivative of the inverse for the Birkhoff normal forms around the* p_3*- and* p_1*-axes, respectively. The monodromy of the fibration* π_F *with respect to* S_3 *and* S_1 *is given by the correspondence of the generators* h_{12}*,* h_{13}*,* h_{14}*,* h_{23}*,* h_{24}*,* h_{34} *of the fundamental group* $\pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)\right)$ *to the* *matrices in* $SL(2,\mathbb{Z})$ *as follows:*

$$h_{14}, h_{23} \mapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, h_{13}, h_{24} \mapsto \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}, h_{12}, h_{34} \mapsto \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \tag{5.3}$$

Remark 5.2. In [29], several different elliptic fibrations are considered besides the naive elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$ in relation to the free rigid body dynamics. As before, we consider the blowing-up $\Phi_B : B \to P_3(\mathbb{C})$ of $P_3(\mathbb{C}) : (a : b : c : d)$ with the centre at $a = b = c = d$, and the projection $\tau_B : B \to E \subset P_3(\mathbb{C})$ to the exceptional set of Φ_B. In fact, we have the following commutative diagram of elliptic fibrations:

$$\begin{array}{ccccccccccccccc} F & \leftarrow & \Phi_B^* F & \cong & \tau_B^* \overline{F} & \to & \overline{F} & \overset{4:1}{-\cdots\to} & T & \leftarrow & \tau_B^* T & \cong & \Phi_B^* W & \to & W \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_3(\mathbb{C}) & \overset{\Phi_B}{\leftarrow} & B & = & B & \overset{\tau_B}{\to} & E & = & E & \overset{\tau_B}{\leftarrow} & B & = & B & \overset{\Phi_B}{\to} & P_3(\mathbb{C}) \end{array} \tag{5.4}$$

The arrows down from top to bottom are elliptic fibrations: $\pi_F : F \to P_3(\mathbb{C})$, $\pi_{\Phi_B^* F} : \Phi_B^* F \to B$, $\pi_{\tau_B^* \overline{F}} : \tau_B^* \overline{F} \to B$, $\pi_{\overline{F}} : \overline{F} \to E$, $\pi_T : T \to E$, $\pi_{\tau_B^* T} : \tau_B^* T \to B$, $\pi_{\Phi_B^* W} : \Phi_B^* W \to B$, $\pi_W : W \to P_3(\mathbb{C})$. The horizontal arrows are bimeromorphisms except $\overline{F} - \cdots \to T$, which is a $4:1$ meromorphic mapping, and $\tau_B : B \to E$, $\tau_B^* \overline{F} \to \overline{F}$, $\tau_B^* T \to T$, which are projections. The fibration $\pi_W : W \to P_3(\mathbb{C})$ and $\pi_T : T \to E$ are in Weierstraß normal form, while $\pi_{\overline{F}} : \overline{F} \to E$ is not in Weierstraß normal form. The fibration $\pi_{\overline{F}} : \overline{F} \to E$, which was not explicitly considered in [29], is naturally induced on E, since the fibration π_F has the same fibre along the fibre of τ_B. The singular loci of these elliptic fibrations are as follows:

- For $\pi_F : F \to P_3(\mathbb{C})$ and $\pi_W : W \to P_3(\mathbb{C})$, the singular fibres are over the divisor $D = \{a = b\} + \{a = c\} + \{a = d\} + \{b = c\} + \{b = d\} + \{c = d\}$ on $P_3(\mathbb{C}) : (a : b : c : d)$.
- For $\pi_{\Phi_B^* F} : \Phi_B^* F \to B$, $\pi_{\Phi_B^* W} : \Phi_B^* W \to B$, the singular locus is the total transform of D through $\Phi_B : B \to P_3(\mathbb{C})$, i.e. $\widetilde{D} + E$, where $\widetilde{D}$ is the proper transform of D through Φ_B.
- For $\pi_{\tau_B^* \overline{F}} : \tau_B^* \overline{F} \to B$, $\pi_{\tau_B^* T} : \tau_B^* T \to B$, the singular fibres are sitting over the proper transform $\widetilde{D}$.
- For $\pi_{\overline{F}} : \overline{F} \to E$, $\pi_T : T \to E$, the singular locus is given by the divisor $\overline{D} = \{a = b\} + \{a = c\} + \{a = d\} + \{b = c\} + \{b = d\} + \{c = d\}$ on $E = \{a + b + c + d = 0\} \subset P_3(\mathbb{C}) : (a : b : c : d)$.

We set $P_3(\mathbb{C})^* = P_3(\mathbb{C}) \setminus \{(1 : 1 : 1 : 1)\}$, $B^* = B \setminus E$, $B' = B \setminus \mathrm{Supp}\left(\widetilde{D}\right)$, $B'' = B \setminus \mathrm{Supp}\left(\widetilde{D} + E\right)$, $P_3(\mathbb{C})' = P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$, $E' = E \setminus \mathrm{Supp}\left(\widetilde{D}\right)$. Then, we have $B'' \subset B'$, $B' \setminus B'' \cong E'$. Using the homotopy exact sequence of fibre bundles, we can show that these regular loci of B', B'', E' for the four elliptic fibrations π_F, π_W, $\pi_{\Phi_B^* F}$, $\pi_{\Phi_B^* W}$; for the two fibrations $\pi_{\tau_B^* \overline{F}}$, $\pi_{\tau_B^* T}$; and for the two fibrations $\pi_{\overline{F}}$, π_T, respectively, have the isomorphic fundamental groups:

$$\pi_1(B', *) \cong \pi_1(B'', *) \cong \pi_1(E', *).$$

Moreover, the $4:1$ meromorphic mapping $\overline{F} - \cdots \to T$ induces an isogeny on each regular fibres. By means of these facts, we can show that the monodromy representations of all the elliptic fibrations which appeared in (5.4) are equivalent.

Remark 5.3. In [29, Theorem 2], the bimeromorphism between $\pi_W : W \to P_3(\mathbb{C})$ and $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$ should be understood as a biholomorphic mapping between the Zariski open set consisting of regular fibres of π_W and that of $\pi_{\widehat{W}}$. For the Zariski open set of $\widehat{W}$, we need to subtract the fibres over the proper transforms of the exceptional set E through Φ_B.

Remark 5.4. In [28], the confluence of singular fibres in elliptic fibrations is discussed from the viewpoint of monodromy. In particular, the monodromy matrices for the confluence of three singular fibres of type I_2 to that of type I_0^* in Kodaira's notation are determined in [28, §8, Table 5] as

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix},$$

up to simultaneous conjugations by $SL(2,\mathbb{Z})$. This is checked by an easy computation as follows:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

On the other hand, the modification of the total and the base spaces of the fibration $\pi_T : T \to E$ and $\pi_{\Phi_B^* W} : \Phi_B^* W \to B$ are performed to obtain smooth and flat fibrations which admit only singular fibres in Kodaira's list of singular fibres for elliptic surfaces in [29]. For the fibration $\pi_T : T \to E$, the blowing-up $\widehat{E} \to E$ with the separate centres at the four points $(a : b : c : d) = (-3 : 1 : 1 : 1), (1 : -3 : 1 : 1), (1 : 1 : -3 : 1), (1 : 1 : 1 : -3)$, where three of the six lines intersect, is important to obtain such a desired fibration. See Figure 3. For the finally obtained elliptic fibration $\pi_{\widehat{T}} : \widehat{T} \to \widehat{E}$, the singular fibres on the exceptional sets through the blowing-up $\widehat{E} \to E$ are in general of type I_0^* in Kodaira's notation. Note that the monodromy matrix for the singular fibre of type I_0^* is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This configuration of singular fibres is compatible with the result in Theorem 5.4. In fact, we have seen in Theorem 5.4 that the global monodromy is given by the correspondence (5.3). A similar comparison can be performed also on the fibration $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$.

Remark 5.5. According to [16, §VI], we consider the Birkhoff normal form around the p_2-axis under the condition $I_3 < I_1 < I_2$. (Note that we have used another order $I_1 < I_2 < I_3$, but the result coincides if we make the permutation (abc) or $(I_1 I_2 I_3)$.) We take the normalized Hamiltonian

$$Z = \frac{1}{4r^2}\left(\frac{H}{\ell} - b\right)\Big/(c-b) = \frac{1}{4r^2}\left(\frac{H}{\ell} - \frac{1}{I_2}\right)\Big/\left(\frac{1}{I_3} - \frac{1}{I_2}\right) = \frac{1}{4}F\left(x, r^2\right), \quad r^2 = \frac{a-b}{c-b},$$

which is a power series in an action variable x with the parameter r^2. Denote the inverse as $x = Z\widehat{C}(Z)^2$, $\widehat{C}(Z)^2 = \sum_{n=0}^{\infty} \frac{P_n(r^2)}{n+1} Z^n$, where P_n is a polynomial in r^2 of degree n. It is shown in [16] that the roots of P_n are on the unit circle in $\mathbb{C}$. For the proof, the following symmetry property of P_n is one of the essential conditions:

$$r^{2n} P_n\left(\frac{1}{r^2}\right) = P_n(r^2). \tag{5.5}$$

In fact, this property of P_n can be deduced from the covariance property (5.1) of the function S. The derivative of the inverse Birkhoff normal form is given by $\frac{1}{4}\sum_{n=0}^{\infty} P_n(r^2) Z^n = S(b, a, c, d)$.

Therefore, from $S(b,a,c,d) = S(b,c,a,d)$, we have $\sum_{n=0}^{\infty} P_n(r^2)Z^n = \sum_{n=0}^{\infty} P_n\left(\frac{1}{r^2}\right)(r^2Z)^n$, where we used the fact that r^2 is mapped to $\frac{1}{r^2}$ and Z to r^2Z, by the action of the permutation (ac). Therefore, we obtain (5.5).

6 Concluding remarks

On the basis of the expression of the derivative for the inverse of Birkhoff normal forms around the equilibria in terms of period integrals, we have considered their analytic continuation for the free rigid body dynamics, which has monodromy as we have seen. In view of the naive elliptic fibration, which naturally arises from the dynamics, we have shown that the branches of the analytic continuation give rise to a basis of the first cohomology group of the regular fibres of the fibration. Further, the monodromy of the analytic continuation of the derivative for the inverse of Birkhoff normal forms coincides with the monodromy of the naive elliptic fibration. The explicit global monodromy has been determined by the monodromy of Gauß hypergeometric differential equation and by using the techniques to compute the fundamental group of the complement of hyperplane arrangements.

In view of [31], the analysis in the present paper can be related to further possible studies on the semi-global symplectic invariants [33]. The semi-global symplectic invariants are defined for real integrable systems around their hyperbolic equilibria. However, if we deal with a real analytic integrable system, we can consider its complexification for which there is no essential difference between elliptic and hyperbolic equilibria. Then, it seems to be possible to enquire the relation between the (complexification of) semi-global symplectic invariants and the complex analytic or algebraic geometry of this complexified integrable systems, which is explained in the present paper in the free rigid body case. We hope that the results in the present paper would be helpful for such further studies.

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