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Kyoto University
A CONNECTEDNESS THEOREM OVER THE SPECTRUM OF A FORMAL POWER SERIES RING

MASAYUKI KAWAKITA

Abstract. We study the connectedness of the non-klt locus over the spectrum of a formal power series ring. In dimension 3, we prove the existence and normality of the smallest lc centre, and apply it to the ACC for minimal log discrepancies greater than 1 on smooth 3-folds.

1. Introduction

The vanishing theorem by Kodaira [18] is one of the most basic tools in algebraic geometry in characteristic zero. It is reasonable to expect a vanishing theorem on excellent schemes, but it is annoyingly unknown besides the work on surfaces by Lipman [22]. Precisely, we are interested in the relative Kodaira vanishing for a birational morphism over the spectrum of a formal power series ring $R = K[[x_1, \ldots, x_d]]$ for a field $K$ of characteristic zero. We mean by an $R$-variety an integral separated scheme of finite type over Spec $R$.

Conjecture 1.1. Let $f: Y \to X$ be a projective birational morphism of regular $R$-varieties and $L$ an $f$-ample divisor on $Y$. Then $R^i f_* \mathcal{O}_Y(K_Y/X + L) = 0$ for $i \geq 1$. Here the relative canonical divisor $K_{Y/X}$ is defined by the 0-th Fitting ideal of $\Omega_{Y/X}$.

We shall not deal with this algebraic conjecture. Instead, we study the connectedness lemma by Shokurov [29] and Kollár [19], which is an important geometric application of the vanishing theorem in birational geometry. It claims for a proper morphism $f: Y \to X$, the fibrewise connectedness of the non-klt locus of a subpair $(Y, \Delta)$ such that $\Delta$ is effective outside a locus in $X$ of codimension at least 2 and such that $-(K_Y + \Delta)$ is $f$-nef and $f$-big. We shall verify it for a germ at a regular point of $X$ in the case when $f$ is isomorphic outside the central fibre (Theorem 3.1). Investigating further in dimension 3, we obtain a desirable result on the smallest lc centre of a pair on a regular $R$-variety of dimension 3.

Theorem 1.2. Let $P \in (X, \alpha)$ be a germ of an lc but not klt pair of a regular $R$-variety $X$ of Krull dimension 3 and an $R$-ideal $\alpha$ on $X$. Then the smallest lc centre of $(X, \alpha)$ exists and it is normal.

We reduce to the case $X = \text{Spec} R$ with $K$ an algebraically closed field $k$. Theorem 3.1, the fibrewise connectedness, is proved by approximating the effective $\mathbb{R}$-divisor $f_*\Delta$ by an m-primary $\mathbb{R}$-ideal $\alpha(l)$, where $m$ is the maximal ideal sheaf, such that the non-klt locus of the subtriple coming from $\alpha(l)$ coincides with the central fibre of the original non-klt locus. The $\alpha(l)$ is descended to $\text{Spec} k[x_1, \ldots, x_d]$, on which the connectedness lemma is applied. The existence of the smallest lc centre in Theorem 1.2 is a corollary to Theorem 3.1. The hardest part of Theorem 1.2 is the normality of the smallest lc centre $C$ which is a curve. We construct an ideal
sheaf $\mathcal{I}_a$ on the normalisation $C_Y$ of $C$ with $f_C: C_Y \to C$ which satisfies $f_C^* \mathcal{I}_a \subset \mathcal{O}_{C_Y}$ and $\mathcal{O}_{C_Y} / f_C^* \mathcal{I}_a \simeq f_C^* \mathcal{O}_Y / f_C^* \mathcal{I}_a$. Then we obtain the isomorphism $\mathcal{O}_C \simeq f_C^* \mathcal{O}_{C_Y}$ meaning the normality of $C$.

Our motivation for excellent schemes stems from the notion of a generic limit of ideals due to de Fernex and Mustaţă [8]. The generic limit was used to prove the ascending chain condition (ACC) for log canonical thresholds on smooth varieties [7], the approach of which works even for the study of minimal log discrepancies [16]. We shall apply Theorem 1.2 to the ACC conjecture for minimal log discrepancies by Shokurov [28], [30] and Cascini, McKernan [24] in the case of smooth 3-folds, and settle the part of minimal log discrepancies greater than 1.

**Theorem 1.3.** Fix subsets $I \subset (0, \infty)$ and $J \subset (1, 3]$ both of which satisfy the descending chain condition. Then there exist finite subsets $I_0 \subset I$ and $J_0 \subset J$ such that if $P \in X, a = \prod_j a_j^{r_j}$ is a germ of a pair of a smooth variety $X$ of dimension 3 and an $\mathbb{R}$-ideal $a$ on $X$ with all $a_j$ non-trivial at $P$, all $r_j \in I$ and $\text{mld}_P(X, a) \in J$, then all $r_j \in I_0$ and $\text{mld}_P(X, a) \in J_0$.

The generic limit of $\mathbb{R}$-ideals $a_i$ on $P \in X = \text{Spec } \mathbb{K}[x_1, \ldots, x_d]$ is an $\mathbb{R}$-ideal on $P_K = X_K = \text{Spec } \mathbb{K}[x_1, \ldots, x_d]$ with a field extension $K$ of $k$. The ACC for minimal log discrepancies on smooth $d$-folds is reduced to the stability $\text{mld}_{P_K}(X_K, a) = \text{mld}_P(X, a)$ for general $i$. We prove it when $(X_K, a)$ is a klt pair, or even a plt pair whose lc centre has an isolated singularity, by our previous arguments [14], [15]. In dimension 3, only the case when $(X_K, a)$ has the smallest lc centre of dimension 1 remains. In this case, the estimate $\text{mld}_{P_K}(X_K, a) \leq 1$ is derived from Theorem 1.2, which is enough to prove Theorem 1.3.

The structure of the paper is as follows. After reviewing the basics of singularities in Section 2, we study the connectedness of the non-klt locus and establish Theorem 1.2 in Section 3. We discuss the ACC for minimal log discrepancies from the point of view of generic limits in Section 4. The stability of minimal log discrepancies in the klt and plt cases is shown in Section 5. Theorem 1.3 is completed in Section 6. The appendix exposing generic limits is attached.

Throughout this paper, $k$ is an algebraically closed field of characteristic zero.

**Remark 1.4.** Recently, Chatzistamatiou and Rülling proved that the higher direct images of the structure sheaf vanish for a projective birational morphism of regular excellent schemes [4].

## 2. Singularities

We review the basics of singularities in birational geometry. A good reference is [21]. A *variety* is an integral separated scheme of finite type over $\text{Spec } k$. A germ of a scheme is considered at a closed point. The *dimension* of a scheme means the Krull dimension.

An $\mathbb{R}$-ideal on a noetherian scheme $X$ is a formal product $a = \prod_j a_j^{r_j}$ of finitely many coherent ideal sheaves $a_j$ on $X$ with positive real exponents $r_j$. The $a$ to the power of $t > 0$ is $a^t := \prod_j a_j^{r_j^t}$. The co-support Cosupp $a$ of $a$ is the union of all Supp $\mathcal{O}_X / a$. The pull-back of $a$ by a morphism $Y \to X$ is $a \mathcal{O}_Y := \prod_j (a_j \mathcal{O}_Y)^{r_j}$. The $\mathbb{R}$-ideal $a$ is said to be *invertible* if all $a_j$ are invertible. In this case, if in addition $X$ is normal, then the $\mathbb{R}$-divisor $A = \sum_j r_j A_j$ with $a_j = \mathcal{O}_X(-A_j)$ is called the $\mathbb{R}$-divisor *defined by* $a$. 
Let $Z$ be an irreducible closed subset of $X$. We write $\eta_Z$ for the generic point of $Z$. The order of $a$ along $Z$ is $\text{ord}_Z a = \sum_i r_i \text{ord}_Z a_i$, where $\text{ord}_Z a_i$ is the maximal $v \in \mathbb{N} \cup \{+\infty\}$ satisfying $a_i \mathcal{O}_{X, \eta_Z} \subset \mathcal{I}_Z^v \mathcal{O}_{X, \eta_Z}$ for the ideal sheaf $\mathcal{I}_Z$ of $Z$.

We treat a triple $(X, \Delta, a)$ which consists of a normal variety $X$, an effective $\mathbb{R}$-divisor $\Delta$ on $X$ such that $K_X + \Delta$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and an $\mathbb{R}$-ideal $a = \prod_i a_i^{e_i}$ on $X$. A prime divisor $D$ on $X$ is said to be a divisor on $X$ and denoted by $\mathcal{O}_X$ the set of all divisors over $X$. The log discrepancy of $E$ with respect to $(X, \Delta, a)$ is

$$a_E(X, \Delta, a) := 1 + \text{ord}_E K_Y/(X, \Delta) - \text{ord}_E a,$$

where $K_Y/(X, \Delta) := K_Y - f^*(K_X + \Delta)$ and $\text{ord}_E a := \text{ord}_E a \mathcal{O}_Y$. Note that $c_X(E)$ and $a_E(X, \Delta, a)$ are determined by the valuation on the function field of $X$ by $\mathcal{O}_X$.

For an irreducible closed subset $Z$ of $X$, the minimal log discrepancy of $(X, \Delta, a)$ at $\eta_Z$ is

$$\text{mld}_{\eta_Z}(X, \Delta, a) := \inf \{ a_E(X, \Delta, a) \mid E \in \mathcal{D}_X, c_X(E) = Z \}.$$

It is either a non-negative real number or $-\infty$. We say that $E \in \mathcal{D}_X$ computes $\text{mld}_{\eta_Z}(X, \Delta, a)$ if $c_X(E) = Z$ and $a_E(X, \Delta, a) = \text{mld}_{\eta_Z}(X, \Delta, a)$ (or is negative when $\text{mld}_{\eta_Z}(X, \Delta, a) = -\infty$). We often reduce to the case when $Z$ is a closed point by the relation $\text{mld}_{\eta_Z}(X, \Delta, a) = \text{mld}_{\mathcal{O}}(X, \Delta, a) - \dim Z$ for a general closed point $P \in Z$ (cf. [3, Proposition 2.1]).

The triple $(X, \Delta, a)$ is said to be log canonical (lc) (resp. Kawamata log terminal (klt)) if $a_E(X, \Delta, a) \geq 0$ (resp. $> 0$) for all $E \in \mathcal{D}_X$. It is said to be purely log terminal (plt) (resp. canonical, terminal) if $a_E(X, \Delta, a) > 0$ (resp. $\geq 1$, $> 1$) for all exceptional $E \in \mathcal{D}_X$. The log canonicity of $(X, \Delta, a)$ about $P \in X$ is equivalent to $\text{mld}_{\mathcal{O}}(X, \Delta, a) \geq 0$. Let $Y$ be a normal variety with a birational morphism to $X$. A centre $c_Y(E)$ with $a_E(X, \Delta, a) \leq 0$ is called a non-klt centre on $Y$ of $(X, \Delta, a)$. The union of all non-klt centres on $Y$ is called the non-klt locus on $Y$ and denoted by $\text{Nklt}_Y(X, \Delta, a)$. When we say just a non-klt centre or the non-klt locus, we mean that it is on $X$.

A log resolution of $(X, \Delta, a)$ is a projective morphism $f: Y \to X$ from a regular variety $Y$ such that

(i) $\text{Exc } f$ is a divisor and $a \mathcal{O}_Y$ is invertible,
(ii) $\text{Exc } f \cup \text{Supp } \Delta_Y \cup \text{Cosupp } a \mathcal{O}_Y$ is a simple normal crossing (snc) divisor, where $\Delta_Y$ is the strict transform of $\Delta$, and
(iii) $f$ is isomorphic on the locus $U$ in $X$ with $U$ regular, $a|_U$ invertible and $\text{Supp } \Delta|_U \cup \text{Cosupp } a|_U$ snc.

A stratum (resp. an open stratum) of an snc divisor $\sum_{i \in I} E_i$ is an irreducible component of $\bigcap_{i \in I} E_i$ (resp. $\bigcap_{i \in I} E_i \setminus \bigcup_{i \in I} E_i$) for a subset $J$ of $I$.

By allowing a not necessarily effective $\mathbb{R}$-divisor $\Delta$, one can consider a subtriple $(X, \Delta, a = \prod_i a_i^{e_i})$. The notions of log discrepancies and lc/klt singularities are extended for subtriples. Let $f: Y \to X$ be a birational morphism from a regular variety $Y$ such that $\text{Exc } f$ is a divisor $\sum_{i \in I} E_i$. The weak transform on $Y$ of $a$ is the $\mathbb{R}$-ideal $a_Y = \prod_i a_i^{e_i}$ with $a_Y = a_i \mathcal{O}_Y (\sum_i (\text{ord}_E a_i) E_i)$.

**Definition 2.1.** Notation as above. The pull-back of $(X, \Delta, a)$ by $f$ is the subtriple $(Y, \Delta_Y, a_Y)$ where $\Delta_Y = -K_{Y/(X, \Delta)} + \sum_{i} (r_i \text{ord}_E a_i) E_i$. 

We have \( a_E(X, \Delta, a) = a_E(Y, \Delta_Y, a_Y) \) for any \( E \in \mathcal{D}_Y \). In particular when \( f \) is proper, \((X, \Delta, a)\) is lc (resp. klt) if and only if so is \((Y, \Delta_Y, a_Y)\). We use the notation \( \text{Nklt}_Y(X, \Delta, a) \) also for a subtriple \((X, \Delta, a)\).

These definitions are extended on schemes over a field \( K \) of characteristic zero and even over a formal power series ring \( R = K[[x_1, \ldots, x_d]] \) by the existence of log resolutions due to Hironaka [12] and Temkin [32], [33]. This extension is studied by de Fernex, Ein and Mustață [7], [8]. We mean by an \( R \)-variety an integral separated scheme of finite type over \( \text{Spec} \, R \).

The canonical divisor \( K_X \) on a normal \( R \)-variety \( X \) is defined by the isomorphism \( \mathcal{O}_X(K_X)|_U \cong \bigwedge^r \mathcal{O}'_{X/K}|_U \) on the regular locus \( U \) of \( X \), where \( \mathcal{O}'_{X/K} \) is the sheaf of special differentials in [7] and \( r \) is its rank. The relative canonical divisor is well understood for a birational morphism of regular \( R \)-varieties.

**Lemma 2.2** ([7, Remark A.12]). Let \( Y \to X \) be a proper birational morphism of regular \( R \)-varieties. Then \( K_{Y/X} \) is the effective divisor defined by the 0-th Fitting ideal of \( \Omega_{Y/X} \). In particular, \( K_{Y/X} \) is independent of the structure of \( X \) as an \( R \)-variety.

The log discrepancies are preserved by field extensions and completions.

**Corollary 2.3.** Let \( Y \to X \) be as in Lemma 2.2. Take an \( R' \)-variety \( X' \) as in (i), (ii) or (iii) below and set a morphism \( Y' = Y \times_X X' \to X' \) of \( R' \)-varieties.

(i) \( X' \) is a component of \( X \times_{\text{Spec} \, R} \text{Spec} \, R' \) with \( R' = R \otimes_K K' \) for a field extension \( K' \) of \( K_r \).

(ii) \( X' = \text{Spec} \, \mathcal{O}'_{X, P} \) for a germ \( P \in X \), which admits the structure of an \( R' \)-variety for a suitable \( R' = K'[x_1, \ldots, x_{d'}] \) by Cohen’s structure theorem [5].

(iii) \( X' = X \) with another structure morphism \( X \to \text{Spec} \, R' \).

Then \( K_{Y/X} \) is the pull-back of \( K_{Y/X} \). In particular, for an \( \mathbb{R} \)-ideal \( a \) on \( X \), a divisor \( E \) over \( X \) and a germ \( P \in X \), one has \( a_E(X', a \mathcal{O}'_{X'}) = a_E(X, a) \) for a component \( E' \) of \( E' \times_X X' \) and \( \text{mld}_P(X', a \mathcal{O}'_{X'}) = \text{mld}_P(X, a) \) for a point \( P' \) of \( P \times_X X' \).

This is by the regularity of the morphism \( X' \to X \). The cases (i) and (ii) for \( R = K \) are stated in [7, Lemma 2.14, Propositions 2.11, A.14] even for a normal (Q-Gorenstein) \( K \)-variety \( X \).

Suppose that \((X, \Delta, a)\) is an lc triple. Then a non-klt centre (on \( X \)) of \((X, \Delta, a)\) is often called an lc centre. An lc centre which is minimal with respect to inclusions is called a minimal lc centre. When we work over a germ \( P \in X \), the following definition makes sense.

**Definition 2.4.** Let \( P \in (X, \Delta, a) \) be a germ of an lc triple. The smallest lc centre is an lc centre of \((X, \Delta, a)\) contained in every lc centre.

If \( X \) is a variety, then the smallest lc centre exists and it is normal [9, Theorem 9.1]. It is, however, unknown for \( R \)-varieties. Theorem 1.2 states that this is the case when \( X \) is a regular \( R \)-variety of dimension 3.

3. The Smallest LC Centre on a Threefold

This section is devoted to the proof of Theorem 1.2. We work over a germ \( P \in X \) of an \( R \)-variety with \( R = K[[x_1, \ldots, x_d]] \). The maximal ideal sheaf of \( P \in X \) is denoted by \( m \). When we discuss on the spectrum of a noetherian ring, we identify an ideal in the ring with its coherent ideal sheaf.
3.A. A connectedness theorem. We start with a connectedness theorem over $X$, Theorem 3.1. Though we impose the strong condition that $f$ is isomorphic outside $P$, this theorem is sufficient in dimension 3 in order to derive the existence of the smallest lc centre, which will be seen in Subsection 3.C.

**Theorem 3.1.** Let $P \in (X, \alpha)$ be a germ of a pair on a regular $R$-variety $X$ and $f : Y \to X$ a proper birational morphism of regular $R$-varieties which is isomorphic outside $P$. Let $\Delta$ be an $\mathbb{R}$-divisor on $Y$ with $f_*\Delta \geq 0$ such that $-(K_Y + \Delta)$ is $f$-nef. Then $\text{Nklt}_Y(Y, \Delta, \alpha \mathcal{O}_Y) \cap f^{-1}(P)$ is connected.

We extract the case $\Delta = -K_Y$.

**Corollary 3.2.** Let $P \in (X, \alpha)$ be a germ of a pair on a regular $R$-variety $X$ and $f : Y \to X$ a proper birational morphism of regular $R$-varieties which is isomorphic outside $P$. Then $\text{Nklt}_Y(X, \alpha) \cap f^{-1}(P)$ is connected.

The statement for $R = k$ is a special case of the connectedness lemma by Shokurov and Kollár [19, Theorem 17.4]. Their lemma can settle Theorem 3.1 in the case when $\alpha$ is $m$-primary and $\Delta$ is $f$-exceptional. Write $\alpha = \prod_j \alpha_j^j$.

**Lemma 3.3.**

(i) In order to prove Theorem 3.1, one may assume that $X = \text{Spec} R$ with $K = k$, $f$ is projective and $\Delta$ is $f$-exceptional.

(ii) Theorem 3.1 holds in the case when $X = \text{Spec} R$ with $K = k$, $f$ is projective, $\Delta$ is $f$-exceptional and all $\alpha_j$ are $m$-primary ideals.

**Proof.** (i) Take an isomorphism $\mathcal{O}_{X,P} \simeq K'^{[x_1, \ldots, x_d]}$ with $K' = \mathcal{O}_{X,P}/m$ by Cohen’s structure theorem and set $R' = k[[x_1, \ldots, x_d]]$ for the algebraic closure of $k$ of $K'$. Because the base change $\text{Spec} R' \to X$ commutes with taking the non-klt locus by Corollary 2.3, we may assume $X = \text{Spec} R$ with $K = k$ (the $d$ may be changed).

By the flattening theorem of Raynaud and Gruson [26, Théorème 1er 5.2.2], there exists a projective morphism $f' : Y' \to X$ from a regular $R$-variety $Y'$ which is isomorphic outside $P$ and factors through $f$. Replacing $(Y, \Delta)$ with its pull-back on $Y'$, we may assume that $f$ is projective. The $\Delta' := \Delta - f^* f_\Delta$ is $f$-exceptional.

Take an invertible $\mathbb{R}$-ideal $\mathfrak{d}$ on $X$ which defines the $\mathbb{R}$-divisor $f_\mathfrak{d} \geq 0$. Then $\text{Nklt}_Y(Y, \Delta, \alpha \mathcal{O}_Y) = \text{Nklt}_Y(Y', \Delta', \alpha \mathcal{O}_{Y'})$. Replacing $\Delta$ with $\Delta'$ and $\alpha$ with $\alpha \mathfrak{d}$, we may assume that $\Delta$ is $f$-exceptional.

(ii) We use the notation $\bar{R} = k[x_1, \ldots, x_d]$ and $\bar{\mathbb{A}}^d_k = \text{Spec} \bar{R}$ with origin $\bar{P}$. By Proposition A.7, $f$ is the base change of a projective morphism $\bar{f} : \bar{Y} \to \bar{A}^d_k$ and $\bar{\alpha}$ is the pull-back of the $\mathbb{R}$-ideal $\bar{\mathfrak{a}} := \prod_j (\alpha_j \cap R)^{\bar{g}}$. Then $f^{-1}(P) \simeq \bar{f}^{-1}(\mathfrak{d})$ and $\Delta$ is the base change of an $\bar{f}$-exceptional $\mathbb{R}$-divisor $\bar{\Delta}$ such that $-(K_{\bar{Y}} + \bar{\Delta})$ is $\bar{f}$-nef. Thus $f^{-1}(P) \cap \text{Nklt}_Y(Y, \Delta, \alpha \mathcal{O}_Y) \simeq \text{Nklt}_Y(Y, \Delta, \bar{\mathfrak{a}} \mathcal{O}_{\bar{Y}})$, which is connected by [19, Theorem 17.4].

For Theorem 3.1, we take a log resolution $q : W \to Y$ of $(Y, \Delta, \alpha \mathcal{O}_Y)$ and set the composition $g = f \circ q : W \to X$ as below.

\[
\begin{array}{ccc}
W & \xrightarrow{q} & Y \\
& \searrow & \swarrow f \\
& & X
\end{array}
\]

We fix $\varepsilon > 0$ such that

(1) \[ F := \text{Nklt}_W(Y, \Delta, \alpha \mathcal{O}_Y) = \text{Nklt}_W(Y, \Delta, \alpha^{1+\varepsilon} \mathcal{O}_Y). \]
Lemma 3.5. Suppose that

\[ a \in \mathbb{R} \text{ is an } m \text{-primary ideal} \]

\[ a(l) := \prod_j (a_j + ml^{1+\varepsilon}) \]

with \( l \in \mathbb{N} \). We want to find a large \( l \) such that \( \text{Nklt}_Y(Y, \Delta, a(l) \mathcal{O}_Y) \cap f^{-1}(P) = \text{Nklt}_Y(Y, \Delta, a(l) \mathcal{O}_Y) \) in order to apply Lemma 3.3(ii).

Since \( F \) and \( g^{-1}(P) \) are divisors contained in an snc divisor, any irreducible component \( D \) of \( F \cap g^{-1}(P) \) has \( \text{codim}_W D = 1 \) or \( 2 \). Suppose \( \text{codim}_W D = 2 \). Let \( E_D \subseteq g^{-1}(P) \) and \( F_D \subseteq F \) be the unique prime divisors such that \( D \subseteq E_D \cap F_D \). We build a tower of blow-ups

\[ \cdots \to W_i \overset{E_i}{\to} W_{i-1} \to \cdots \to W_0 = W \]

as follows. Set \( W_0 := W, E_0 := E_D \) and \( F_0 := F_D \). We construct inductively the blow-up \( g_i \colon W_i \to W_{i-1} \) along \( D \) for \( i = 1 \) (resp. along \( E_{i-1} \cap F_{i-1} \) for \( i \geq 2 \)), and set \( E_i \) as the exceptional divisor of \( g_i \), and \( F_i \) as the strict transform on \( W_i \) of \( F_D \).

The composition \( g_1 \circ \cdots \circ g_i \), is denoted by \( h_i \colon W_i \to W \).

Lemma 3.4.

(i) \( aE(Y, \Delta, a^{1+\varepsilon} \mathcal{O}_Y) \leq aE_0(Y, \Delta, a^{1+\varepsilon} \mathcal{O}_Y) - i\varepsilon \text{ ord}_{F_D} a \).

(ii) \( h_i \cdot \mathcal{O}_W(-aE_i) \subseteq \mathcal{O}_W(-aE_D) \mathcal{O}_F \) for any \( a \in \mathbb{N} \).

Proof. The (i) is just a computation using \( aE_0(Y, \Delta, a\mathcal{O}_Y) \leq 0 \). The (ii) is from

\[ h_i \cdot \mathcal{O}_W(-aE_i) \cdot \mathcal{O}_F \subseteq h_i \cdot \mathcal{O}_F(-aE_i|_F) = \mathcal{O}_F(-aE_D|_F) \text{ via } F_i \simeq F_D. \]

q.e.d.

Lemma 3.5. Suppose that \((Y, \Delta)\) is klt outside \( f^{-1}(P) \). Then there exists \( l \) such that \( \text{Nklt}_W(Y, \Delta, a(l) \mathcal{O}_Y) = F \cap g^{-1}(P) \).

Proof. By (1), (2) and the assumption, \( \text{Nklt}_W(Y, \Delta, a(l) \mathcal{O}_Y) \subseteq F \cap g^{-1}(P) \) for any \( l \). Thus it suffices to prove that for every irreducible component \( D \) of \( F \cap g^{-1}(P) \), there exists \( I_D \) such that \( D \) is a non-klt centre on \( W \) of \( (Y, \Delta, a(l) \mathcal{O}_Y) \) for any \( l \geq I_D \). \( D \) has \( \text{codim}_W D = 1 \) or \( 2 \). If \( \text{codim}_W D = 1 \), then we may take any \( I_D \) such that \( I_D \text{ ord}_D m \geq \text{ ord}_D a_j \) for all \( j \). If \( \text{codim}_W D = 2 \), then we take the tower of blow-ups in (3). Note \( \text{ord}_{F_D} a_0 > 0 \). By Lemma 3.4(i), we have \( aE(Y, \Delta, a^{1+\varepsilon} \mathcal{O}_Y) \leq 0 \) whenever \( aE_0(Y, \Delta, a^{1+\varepsilon} \mathcal{O}_Y) \leq i\varepsilon \text{ ord}_{F_D} a_0 \). Fix such \( i \) and take \( I_D \) such that \( I_D \text{ ord}_E m \geq \text{ ord}_E a_j \) for all \( j \). Then for \( l \geq I_D \), \( aE(Y, \Delta, a(l) \mathcal{O}_Y) = aE(Y, \Delta, a^{1+\varepsilon} \mathcal{O}_Y) \leq 0 \), so \( D = c_w(E_i) \) is a non-klt centre on \( W \) of \((Y, \Delta, a(l) \mathcal{O}_Y)\). q.e.d.

Proof of Theorem 3.1. After the reduction in Lemma 3.3(i), we take \( l \) in Lemma 3.5. Then

\[ \text{Nklt}_Y(Y, \Delta, a(l) \mathcal{O}_Y) \cap f^{-1}(P) = q(F \cap g^{-1}(P)) \]

\[ = q(\text{Nklt}_W(Y, \Delta, a(l) \mathcal{O}_Y)) = \text{Nklt}_Y(Y, \Delta, a(l) \mathcal{O}_Y). \]

Apply Lemma 3.3(ii) to \((Y, \Delta, a(l) \mathcal{O}_Y)\).

q.e.d.

In our proof of Theorem 3.1, we do not know a relative vanishing for \( g \colon W \to X \). Instead, consider a log resolution \( f_i \colon Y_i \to X \) of \((X, a(l) m)\) which factors through \( f \), and let \( p_i \colon Y_i \to Y \) be the induced morphism as below. The \( l \) is not fixed here.

\[ \begin{array}{ccc}
Y_i & \xrightarrow{p_i} & Y \\
\downarrow{f_i} & & \downarrow{f} \\
X
\end{array} \]
The $f_l$ is isomorphic outside $P$. Let $(Y_l, \Delta_l, \mathcal{O}_{Y_l})$ be the pull-back of $(X, 0, a(l))$. Then we have a vanishing involving $\Delta_l$, which will be used in Subsection 3.C.

**Lemma 3.6.** Let $f_l = f \circ p_l : Y_l \to Y \to X$ be as above. Write $[-\Delta_l] = P_l - N_l$ by effective divisors $P_l$ and $N_l$ with no common divisors. Then

$$R^1 f_l^*(p_{\ell*} \mathcal{O}_{Y_l}(-N_l)) = 0.$$ 

**Proof.** The sheaf $R^1 f_l^*(p_{\ell*} \mathcal{O}_{Y_l}(-N_l))$ is supported in $P \cap \mathcal{O}_{X, P}$ and $\mathcal{O}' = k[[x_1, \ldots, x_d]]$ for the algebraic closure $k$ of $K'$, then $\mathcal{O}'$ is faithfully flat over $\mathcal{O}_{X, P}$. Hence taking the base change to $\text{Spec} \mathcal{O}'$, one can reduce to the case $X = \text{Spec} R$ with $K = k$ by [10, Proposition III.1.4.15] and Corollary 2.3. By Proposition A.7, $f_l$ is the base change of a projective morphism $\tilde{f}_l : \tilde{Y}_l \to \mathbb{A}^d_k$. The $a(l)$ is the pull-back of an $\mathbb{R}$-ideal $\tilde{a}(l)$ on $\mathbb{A}^d_k$, and $\Delta_l$ is the base change of the $\mathbb{R}$-divisor $\tilde{\Delta}_l$ on $\tilde{Y}_l$ such that $(\tilde{Y}_l, \tilde{\Delta}_l, \mathcal{O}_{\tilde{Y}_l})$ is the pull-back of $(\mathbb{A}^d_k, 0, \tilde{a}(l))$.

Kawamata–Viehweg vanishing theorem [17], [35] implies $R^1 f_l^* \mathcal{O}_{\tilde{Y}_l}(-\tilde{\Delta}_l) = 0$. Since $X \to \mathbb{A}^d_k$ is flat, this is base-changed to $R^1 f_{\ell*} \mathcal{O}_{Y_l}([-\Delta_l]) = 0$ by [10, Proposition III.1.4.15]. Thus, applying $f_{\ell*}$ to the exact sequence

$$0 \to \mathcal{O}_{Y_l}(P_l - N_l) \to \mathcal{O}_{Y_l}(P_l) \to \mathcal{O}_{N_l}(P_l|_{N_l}) \to 0,$$

we obtain the surjection $\mathcal{O}_X = f_{\ell*} \mathcal{O}_{Y_l}(P_l) \to f_{\ell*} \mathcal{O}_{N_l}(P_l|_{N_l})$. This homomorphism is factored as $\mathcal{O}_X \to f_{\ell*} \mathcal{O}_{N_l} \to f_{\ell*} \mathcal{O}_{N_l}(P_l|_{N_l})$, so we have the surjection $\mathcal{O}_X \to f_{\ell*} \mathcal{O}_{N_l}$. Moreover, we have the base change $R^1 f_{\ell*} \mathcal{O}_{Y_l} = 0$ of the vanishing $R^1 f_{\ell*} \mathcal{O}_{\tilde{Y}_l} = 0$ [12, p.144 (2)]. Hence applying $f_{\ell*}$ to the exact sequence

$$0 \to \mathcal{O}_{Y_l}(-N_l) \to \mathcal{O}_{Y_l} \to \mathcal{O}_{N_l} \to 0,$$

we obtain $R^1 f_{\ell*} \mathcal{O}_{Y_l}(-N_l) = 0$.

Leray spectral sequence $R^p f_{\ell*}(R^q p_{\ell*} \mathcal{O}_{Y_l}(-N_l)) \Rightarrow R^{p+q} f_{\ell*} \mathcal{O}_{Y_l}(-N_l)$ gives an injection $R^1 f_{\ell*}(p_{\ell*} \mathcal{O}_{Y_l}(-N_l)) \supseteq R^1 f_{\ell*} \mathcal{O}_{Y_l}(-N_l)$, so $R^1 f_{\ell*}(p_{\ell*} \mathcal{O}_{Y_l}(-N_l)) = 0$. q.e.d.

**3.B. Propositions in an arbitrary dimension.** We prepare two auxiliary propositions which can be stated independently of Theorem 1.2.

It is easy to see that a minimal lc centre of codimension 1 is normal.

**Proposition 3.7.** Let $(X, a)$ be a pair on a regular $R$-variety $X$, and $S$ the union of all non-klt centres of codimension 1 of $(X, a)$. Then every irreducible component of the non-normal locus of $S$ is a non-klt centre of $(X, a)$.

**Proof.** $S$ is Cohen–Macaulay since $S$ is a Cartier divisor on a regular scheme $X$. Thus any irreducible component $C$ of the non-normal locus of $S$ has codim$_S C = 2$ and mult$_S S \geq 2$. Let $E$ be the divisor over $X$ obtained at $\nu_C$ by the blow-up of $X$ along $C$. Then $a_E(X, a) = 2 - \text{ord}_E a \leq 2 - \text{mult}_E S \leq 0$, so $C = c_X(E)$ is a non-klt centre of $(X, a)$. q.e.d.

We can perturb $a$ to reduce to the case when every lc centre is minimal.

**Proposition 3.8.** Let $(X, a)$ be an lc pair on a klt $R$-variety $X$. Then there exists an $\mathbb{R}$-ideal $a'$ forming an lc pair $(X, a')$ such that a minimal lc centre of $(X, a)$ is an lc centre of $(X, a')$ and vice versa.

**Proof.** Let $\{Z_i\}$ be the set of all minimal lc centres of $(X, a = \prod a'_i)$. For each $Z_i$, fix $E_i \in \mathcal{D}_X$ computing $\text{ml}(Z_i(X, a)) = 0$. Let $\mathcal{I}_Z$ be the ideal sheaf of $Z = \bigcup Z_i$, and take an integer $l$ such that $l \text{ord}_{E_i} \mathcal{I}_Z \geq \text{ord}_{E_i} a_j$ for all $i, j$. Then $(X, a' := \prod a'_i)$. The proof is complete. q.e.d.
\[ \mathcal{X}_f(a_j + \mathcal{I}^{j'}) = \text{lc}, \text{ and } Z_i \text{ is an lc centre of } (X, a') \text{ by } \operatorname{ord}_E a' = \operatorname{ord}_E a. \] On the other hand, every lc centre of \((X, a')\) is an lc centre of \((X, a)\) contained in \(\operatorname{Cosupp} a' = Z\), so it equals some \(Z_i\).

3.C. The smallest lc centre on a threefold. We proceed to the proof of Theorem 1.2. We may assume that \(P\) is not an lc centre of \((X, a)\). By Proposition 3.8, we may assume that every lc centre of \((X, a)\) is minimal.

The existence of the smallest lc centre is a consequence of Corollary 3.2.

Proof of the existence of the smallest lc centre. Let \(\{Z_i\}\) be the set of all lc centres of \((X, a)\), which are assumed to be minimal. Proposition 3.7 implies that \(Z = \bigcup Z_i\) is regular outside \(P\). Thus we have an embedded resolution \(f : Y \to X\) of singularities of \(Z\), in which \(f\) is isomorphic outside \(P\) and induces \(f_Z : \bigcup Z_iY \to Z\) for the strict transform \(Z_iY\) of \(Z_i\). By Corollary 3.2, \(f_Z^{-1}(P) = \operatorname{Nkl}_{Y}(X, a) \cap f^{-1}(P)\) is connected, that is, there exists only one lc centre of \((X, a)\). q.e.d.

Remark 3.9. The above proof shows that if \(Z\) is the smallest lc centre of \((X, a)\), then its normalisation \(Z^\nu \to Z\) is a homeomorphism.

To complete Theorem 1.2, we must prove that the unique lc centre of \((X, a)\) is normal. (Since we have assumed that every lc centre is minimal, the existence of the smallest lc centre means the uniqueness of lc centre.) If it is of dimension 2, then it is normal by Proposition 3.7. Thus, we may assume that \((X, a)\) has the unique lc centre \(C\) which is of dimension 1.

We have an embedded resolution \(f : Y \to X\) of singularities of \(C\), in which \(f\) is isomorphic outside \(P\) and induces the normalisation \(f_C : C_Y \to C\) for the strict transform \(C_Y\) of \(C\). Note that \(f_C^{-1}(P)\) consists of one point, say \(P_y\), by Remark 3.9. We let \(n\) denote the maximal ideal sheaf of \(P_y \in Y\). Then we take a log resolution \(q : W \to Y\) of \((Y, a_{\mathcal{Y}} \cdot n)\) and set the composition \(g = f \circ q : W \to X\). We have the following diagram.

\[
\begin{array}{ccc}
W & \xrightarrow{q} & Y \\
\downarrow{g} & & \downarrow{f} \\
X & \xrightarrow{f} & C_Y \\
\downarrow{f_C} & & \downarrow{f_C} \\
C & \xrightarrow{f_C} & P
\end{array}
\]

The normality of \(C\) is equivalent to the isomorphism \(\mathcal{O}_C \simeq f_{C*}\mathcal{O}_{C_Y}\). We shall see this by constructing an ideal sheaf \(\mathcal{I}_a\) on \(C_Y\) which satisfies \(f_{C*}\mathcal{I}_a \subset \mathcal{O}_C\) and \(\mathcal{O}_C/f_{C*}\mathcal{I}_a \simeq f_{C*}\mathcal{O}_{C_Y}/f_{C*}\mathcal{I}_a\).

We fix \(\varepsilon\) in (1) for \(\Delta = -K_{Y/X}\), that is, \(F = \operatorname{Nkl}_{W}(X, a) = \operatorname{Nkl}_{W}(X, a^{1+\varepsilon})\). For the \(a(l)\) in (2), we consider a log resolution \(f_l : Y_l \to X\) of \((X, a(l))\) which factors through \(f\) as \(f_l = f \circ p_l\). We extend Lemma 3.6.

Lemma 3.10. Let \(f\) and \(f_l\) be as above. Then for an arbitrary ideal sheaf \(\mathcal{I}\) on \(Y\) containing \(p_l\mathcal{O}_{Y_l}(-N_l)\), with \(N_l\) in Lemma 3.6, one has \(R^1 f_* \mathcal{I} = 0\).

Proof. By (1) for \(\Delta = -K_{Y/X}\) and (2), we see \(p_l(\operatorname{Supp} N_l) = \operatorname{Nkl}_W(X, a(l)) \subset q(F \cap g^{-1}(P)) = C_Y \cap f^{-1}(P) = P_y\), whence the cokernel \(\mathcal{D}\) of the natural injection \(p_l\mathcal{O}_{Y_l}(-N_l) \to \mathcal{I}\) is a skyscraper sheaf. In particular, \(R^1 f_* \mathcal{I} = 0\). Apply \(f_*\) to the exact sequence

\[ 0 \to p_l \mathcal{O}_{Y_l}(-N_l) \to \mathcal{I} \to \mathcal{D} \to 0.\]

By Lemma 3.6 and \(R^1 f_* \mathcal{D} = 0\), we obtain \(R^1 f_* \mathcal{I} = 0\). q.e.d.
Since $C$ is the unique lc centre of $(X, a)$, every irreducible component of $F$ maps onto $C_Y$. Thus $F \cap q^{-1}(P_Y) \neq \emptyset$ and any irreducible component $D$ of $F \cap q^{-1}(P_Y)$ has dimension 1. We fix one such $D$, and let $E_D \subset q^{-1}(P_Y)$ and $F_D \subset F$ be the unique prime divisors such that $D \subset E_D \cap F_D$. We derive a vanishing for ideal sheaves on $Y$ close to that of $C_Y$.

**Lemma 3.11.** $R^1 f_*(q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D))) = 0$ for any $a \in \mathbb{N}$.

**Proof.** Take the tower of blow-ups in (3). For fixed $a$, choose $i \in \mathbb{N}$ such that $aE_D(X, a^{1+\varepsilon}) - i \varepsilon \text{ord}_{f_D} a \leq -a$. Then Lemma 3.4 for $\Delta = -K_Y/X$ shows

\[ h^i_u(r_W([aE(X, a^{1+\varepsilon})])E_i) \subset h^i_u(r_W(-aE_D) \subset r_W(-aE_D) + r_W(-F_D)). \]

Take $l$ such that $l \ord_{E_i} m \geq \ord_{E_D} a$ for all $j$. Then,

\[ aE(X, a^{1+\varepsilon}) = aE((X, a^{l!})) . \]

For this $l$, we take a log resolution $f_1 : Y_1 \rightarrow X$ of $(X, a^l)a$ which factors through $f_1$ such that $C_Y(E_1)$ is a divisor. Then by (4) and (5), one can apply Lemma 3.10 to $\mathcal{I} = q_*(r_W(-aE_D) + r_W(-F_D))$. q.e.d.

Now we set the ideal sheaf $\mathfrak{n}_a$ on $C_Y$ as

\[ \mathfrak{n}_a := q_*(r_W(-aE_D) + r_W(-F_D)) \cdot r_C \cdot Y. \]

**Lemma 3.12.** There exists a such that $f^* \mathfrak{n}_a \subset \mathcal{O}_C$.

**Proof.** Note that $\mathfrak{n} \mathcal{O}_C$ is an invertible ideal sheaf on $C_Y$. Set $n = \ord_{E_D} n$, then

\[ n_m \subset q_*(r_{F_D}(aE_D|F_D)) = n' \mathcal{O}_C \]

for any $l$. Take an $f$-ample divisor $A \geq 0$ on $Y$ such that $-A$ is $f$-ample and set $\mathcal{O}_C|Y \subset \mathfrak{n} \mathcal{O}_C$. By Serre vanishing theorem [10, Théorème III.2.2.1], there exists $m_0$ such that $R^1 f_* \mathfrak{I}_Y(-mA) = 0$ for any $m \geq m_0$, where $\mathfrak{I}_Y$ is the ideal sheaf of $Y$ on $Y$. Then we have the surjection $f_* \mathfrak{I}_Y(-mA) \rightarrow f_* \mathfrak{I}_Y(-mA|C_Y) = f^* \mathfrak{I}_C$, which provides

\[ f^* \mathfrak{I}_C \cdot n^m \mathcal{O}_C = f_* \mathfrak{I}_Y(-mA) \cdot \mathcal{O}_C \subset \mathcal{O}_C. \]

Combining (6) and (7), we obtain $f^* \mathfrak{I}_C \cdot n^m \mathcal{O}_C < f^* \mathcal{O}_C \cdot m \mathcal{O}_C$ for $m \geq m_0$. q.e.d.

**Proof of the normality of $C$.** Applying $f_*$ to the exact sequence

\[ 0 \rightarrow q_*(r_W(-aE_D) + r_W(-F_D)) \rightarrow r_C \rightarrow r_C/\mathfrak{n}_a \rightarrow 0 \]

and using Lemma 3.11, we obtain the surjection $r_X \rightarrow f_*(r_C/\mathfrak{n}_a)$. This homomorphism is factored as

\[ r_X \rightarrow r_C/f_C \cdot n_a \cap r_C \rightarrow f_C \cdot r_C/ f_C \cdot n_a \rightarrow f_C(r_C/ n_a). \]

so we have an isomorphism $r_C/f_C \cdot n_a \cap r_C \simeq f_C \cdot r_C/ f_C \cdot n_a$. For $a$ in Lemma 3.12, it is $r_C/f_C \cdot n_a \simeq f_C \cdot r_C/ f_C \cdot n_a$. Therefore $r_C \simeq f_C \cdot r_C$, meaning the normality of $C$. q.e.d.

Theorem 1.2 is established.

**Remark 3.13.** (i) One may prove the normality of $C$ by using Zariski’s subspace theorem [1, (10.6)]. One has an isomorphism $r_C/f_C \cdot n_a \cap r_C \simeq f_C(r_C/ n_a)$ for any $a$. By (6), the family $\{n_a\}_a$ gives the $n \mathcal{O}_C$-adic topology. Since the family $\{f_* \mathfrak{I}_Y(-mA)\}_m$ in the proof of Lemma 3.12 gives the $m$-adic topology by Zariski’s subspace theorem (cf. [13, Lemma...
Definition 4.3. Let finite was proved in [16].

\[ \text{Definition 4.1. We say that a subset I of } \mathbb{R} \text{ satisfies the ascending chain condition (ACC) (resp. the descending chain condition (DCC)) if there exist no infinite strictly increasing (resp. strictly decreasing) sequences of elements in I.} \]

Remark 4.2. I \subset \mathbb{R} is finite if and only if I satisfies both the ACC and DCC.

Definition 4.3. Let \( P \in (X, \Delta = \sum \delta \Delta_i, a = \prod a_j^{r_j}) \) be a germ of a triple. We write Coef\(_{P}(\Delta, a)\) for the set which consists of all \( \delta_i > 0 \) and all \( r_j > 0 \) with \( a_j \) non-trivial at \( P \).

**Conjecture 4.4.** (Shokurov [28], [30], Cascini, McKernan [24]). Fix \( d \in \mathbb{N} \) and subsets \( I \subset (0, \infty) \) and \( J \subset [0, \infty) \) both of which satisfy the DCC. Then there exist finite subsets \( I_0 \subset I \) and \( J_0 \subset J \) such that if \( P \in (X, \Delta, a) \) is a germ of a triple on a variety \( X \) of dimension \( d \) with Coef\(_{P}(\Delta, a) \subset I \) and mld\(_{P}(X, \Delta, a) \subset J \), then Coef\(_{P}(\Delta, a) \subset I_0 \) and mld\(_{P}(X, \Delta, a) \subset J_0 \).

Conjecture 4.4 by Cascini and McKernan is a generalisation of the original conjecture by Shokurov, which claims only the existence of \( J_0 \). When \( d = 2 \), the existence of \( J_0 \) was proved by Alexeev [2]. The motivation of this conjecture stems from the reduction by Shokurov [31] that the termination of flips follows from two conjectural properties of minimal log discrepancies: the ACC and the lower semi-continuity. For the purpose of the termination of flips, one may assume \( I \) in Conjecture 4.4 to be a finite set.

We consider Conjecture 4.4 with the assumption of the smoothness of \( X \). Then we may assume \( \Delta = 0 \) by absorbing \( \Delta \) to \( a \), since any divisor on \( X \) is a Cartier divisor.

**Conjecture 4.4’.** Fix \( d \in \mathbb{N} \) and subsets \( I \subset (0, \infty) \) and \( J \subset [0, d] \) both of which satisfy the DCC. Then there exist finite subsets \( I_0 \subset I \) and \( J_0 \subset J \) such that if \( P \in (X, a) \) is a germ of a pair on a smooth variety \( X \) of dimension \( d \) with Coef\(_{P}a \subset I \) and mld\(_{P}(X, a) \subset J \), then Coef\(_{P}a \subset I_0 \) and mld\(_{P}(X, a) \subset J_0 \).

Theorem 1.3 is Conjecture 4.4’ for \( d = 3 \) with \( J \subset (1, 3] \). Conjecture 4.4’ with \( I \) finite was proved in [16].

4. The ACC for Minimal Log Discrepancies

In this section, we discuss the ACC for minimal log discrepancies on smooth varieties from the point of view of generic limits.

4.A. Statements. We begin with the statement of the ACC conjecture.

**Definition 4.4.** We say that a subset \( I \) of \( \mathbb{R} \) satisfies the ascending chain condition (ACC) (resp. the descending chain condition (DCC)) if there exist no infinite strictly increasing (resp. strictly decreasing) sequences of elements in \( I \).

**Remark 4.2.** \( I \subset \mathbb{R} \) is finite if and only if \( I \) satisfies both the ACC and DCC.

**Definition 4.3.** Let \( P \in (X, \Delta = \sum \delta \Delta_i, a = \prod a_j^{r_j}) \) be a germ of a triple. We write Coef\(_{P}(\Delta, a)\) for the set which consists of all \( \delta_i > 0 \) and all \( r_j > 0 \) with \( a_j \) non-trivial at \( P \).

**Conjecture 4.4.** (Shokurov [28], [30], Cascini, McKernan [24]). Fix \( d \in \mathbb{N} \) and subsets \( I \subset (0, \infty) \) and \( J \subset [0, \infty) \) both of which satisfy the DCC. Then there exist finite subsets \( I_0 \subset I \) and \( J_0 \subset J \) such that if \( P \in (X, \Delta, a) \) is a germ of a triple on a variety \( X \) of dimension \( d \) with Coef\(_{P}(\Delta, a) \subset I \) and mld\(_{P}(X, \Delta, a) \subset J \), then Coef\(_{P}(\Delta, a) \subset I_0 \) and mld\(_{P}(X, \Delta, a) \subset J_0 \).

Conjecture 4.4 by Cascini and McKernan is a generalisation of the original conjecture by Shokurov, which claims only the existence of \( J_0 \). When \( d = 2 \), the existence of \( J_0 \) was proved by Alexeev [2]. The motivation of this conjecture stems from the reduction by Shokurov [31] that the termination of flips follows from two conjectural properties of minimal log discrepancies: the ACC and the lower semi-continuity. For the purpose of the termination of flips, one may assume \( I \) in Conjecture 4.4 to be a finite set.

We consider Conjecture 4.4 with the assumption of the smoothness of \( X \). Then we may assume \( \Delta = 0 \) by absorbing \( \Delta \) to \( a \), since any divisor on \( X \) is a Cartier divisor.

**Conjecture 4.4’.** Fix \( d \in \mathbb{N} \) and subsets \( I \subset (0, \infty) \) and \( J \subset [0, d] \) both of which satisfy the DCC. Then there exist finite subsets \( I_0 \subset I \) and \( J_0 \subset J \) such that if \( P \in (X, a) \) is a germ of a pair on a smooth variety \( X \) of dimension \( d \) with Coef\(_{P}a \subset I \) and mld\(_{P}(X, a) \subset J \), then Coef\(_{P}a \subset I_0 \) and mld\(_{P}(X, a) \subset J_0 \).

Theorem 1.3 is Conjecture 4.4’ for \( d = 3 \) with \( J \subset (1, 3] \). Conjecture 4.4’ with \( I \) finite was proved in [16].
4B. Reduction. We shall reduce Conjecture 4.4′ to the stability of minimal log discrepancies in taking a generic limit of \( \mathbb{R} \)-ideals. We refer to Appendix A for the definition of a generic limit and the relevant notation: \( R = k[[x_1, \ldots, x_d]] \) with maximal ideal \( m \) and \( X = \text{Spec} \, R \) with closed point \( P \), and for a field extension \( K \) of \( k \), \( R_K = K[[x_1, \ldots, x_d]] \) with maximal ideal \( m_K \) and \( X_K = \text{Spec} \, R_K \) with closed point \( P_K \).

**Conjecture 4.5** ([16, Conjecture 5.7]). Fix \( r_1, \ldots, r_e > 0 \). Let \( S = \{(a_1, \ldots, a_e_i) \}_{i \in I} \) be a collection of \( e \)-tuples of ideals in \( R = k[[x_1, \ldots, x_d]] \), and \( (a_1, \ldots, a_e) \) the generic limit of \( S \) defined in \( R_K \) with respect to a family \( \mathcal{F} = \{Z_j, (\mathcal{A}_j)(l), \mathcal{I}_l, s_l, t_{l+1} \}_{l \geq l_0} \) of approximations of \( S \). Set \( a_i = \prod_j a_i^{e_j} \) and \( a = \prod_j a_j^{e_j} \). Then after replacing \( \mathcal{F} \) with a subfamily but using the same notation,

\[
mld_{P_K}(X_K, a) = mld_{P}(X, a)
\]

for any \( i \in I_i \) with \( I \geq l_0 \).

Conjecture 4.5 is closely related to the ideal-adic semi-continuity of minimal log discrepancies.

**Conjecture 4.6** (Mustață, cf. [14, Conjecture 2.5]). Let \( P \in X = \text{Spec} \, k[[x_1, \ldots, x_d]] \) and \( m \) be as above and \( a = \prod_j a_j^{e_j} \) an \( \mathbb{R} \)-ideal on \( X \). Then there exists an integer \( l \) such that if an \( \mathbb{R} \)-ideal \( b = \prod_j b_j^{e_j} \) on \( X \) satisfies \( a_j + m^l = b_j + m^l \) for all \( j \), then \( mld(X, a) = mld(X, b) \).

**Remark 4.7.** One inequality is easy in both conjectures. One has \( mld_{P_K}(X_K, a) \geq mld_{P}(X, a_i) \) in Conjecture 4.5 by Lemma A.8, and \( mld_{P}(X, a) \geq mld_{P}(X, b) \) in Conjecture 4.6 by [14, Remark 2.5.3]. In particular, these conjectures hold in the case when \((X_K, a)\) (resp. \((X, a)\)) is not lc.

**Proposition 4.8.** Conjecture 4.5 implies Conjectures 4.4′ and 4.6.

**Proof.** Firstly, we shall see Conjecture 4.4′. It was observed by Mustață and sketched in [14, Remark 2.5.1]. Let \( \{a_i = \prod_j a_i^{e_j} \}_{i \in N} \) be an arbitrary collection of \( \mathbb{R} \)-ideals on \( X = \text{Spec} \, R \) such that \( a_j \) are non-trivial at \( P \), \( r_j \in I \) and

\[
m_j := mld(P, a_j) \in J.
\]

Then \( \sum_{j=1}^{e_j} r_j \leq \text{ord}_E a_i \leq a_E(X) = d \) for the divisor \( E \) obtained by the blow-up of \( X \) at \( P \), since \( m_j \geq 0 \). The \( I \) has the minimum, say \( t > 0 \), so \( e_j \leq t^{-1} d \). By Corollary 2.3 and Remark 4.2, it is enough to show that both the subsets \( \bigcup_{i \in N} \text{Coef}_P a_i \) of \( I \) and \( \bigcup_{i \in N} \{m_i \} \) of \( J \) satisfy the ACC. We may replace \( N \) with a countable subset \( \mathbb{N} \) on which \( e_i \) is constant, say \( e \), such that the sequences \( \{r_{ij} \} \) for \( 1 \leq j \leq e \) and \( \{m_i \} \) are non-decreasing. By \( r_{ij} \leq d \) and \( m_i \leq d \), these sequences have limits

\[
r_j := \lim_i r_{ij} \quad \text{and} \quad m := \lim_i m_i.
\]

It suffices to prove \( r_j = r_j \) and \( m_i = m \) for some \( i \).

For the collection \( S = \{(a_1, \ldots, a_e_i) \}_{i \in N} \) of \( e \)-tuples of ideals in \( R \), we take a family \( \mathcal{F} = \{(Z_j, (\mathcal{A}_j)(l)), \mathcal{I}_l, s_l, t_{l+1} \}_{l \geq l_0} \) of approximations of \( S \) and the generic limit \( (a_1, \ldots, a_e) \) of \( S \) defined in \( R_K \) with respect to \( \mathcal{F} \) as in Lemma A.8, where \( E_k \in \mathcal{D}_{X_K} \) computing

\[
M := mld_{P_K}(X_K, \prod_j a_j^{e_j})
\]
Remark 4.9

This is obvious by the above proof. Note that (8) implies

\[ l \leq a_{i(j)}(X, \prod_j a_{ij}^{(j)}) = a_{(E_i)}(X, \prod_j (a_{ij} + m^j)^{r_{ij}}) \text{ and } \text{ord}_E a_j = \text{ord}_{(E_i)} a_{ij} + l \]

for \( i \in I \) with \( z = s_j(i) \) using (iii) in Definition A.1. Hence \( \text{ord}_E a_j = \text{ord}_{(E_i)} a_{ij} \) and

\[ m_i \leq a_{(E_i)}(X, \prod_j a_{ij}^{(j)}) = a_{(E_i)}(X, \prod_j a_{ij}^{(j)}) + \sum_j (r_j - r_{ij}) \text{ord}_{(E_i)} a_{ij} \]

(8) \[ = M + \sum_j (r_j - r_{ij}) \text{ord}_{E} a_j. \]

By Conjecture 4.5, \( M = \text{mld}_P(X, \prod_j a_{ij}^{(j)}) \leq m_i \) for any \( i \in I \) after replacing \( \mathcal{F} \) with a subfamily. With (8), we obtain

\[ M \leq m_i \leq M + \sum_j (r_j - r_{ij}) \text{ord}_{E} a_j. \]

The right-hand side converges to \( M \), whence \( m_i = m = M \). Then \( \text{mld}_P(X, \prod_j a_{ij}^{(j)}) = \text{mld}_P(X, \prod_j a_{ij}^{(j)}) \), so \( r_{ij} = r_j \).

Secondly, we shall see Conjecture 4.6. Suppose the contrary. Then for every \( i \in \mathbb{N} \), there exists an \( \mathbb{R} \)-ideal \( b_i = \prod_j b_{ij}^j \) on \( X \) such that \( a_j + m^j = b_{ij} + m^j \) for all \( j \) but \( \text{mld}_P(X, a) \neq \text{mld}_P(X, b_i) \). Take a family \( \mathcal{F} = (Z_i, (\bar{b}_j(l)))_j \) of approximations of \( S = \{(b_{ij})_j \mid j \in \mathbb{N}\} \) and the generic limit \( (b_j)_j \) of \( \mathcal{F} \) defined in \( R_K \) with respect to \( \mathcal{F} \). Then for \( l > b_0 \),

\[ \bar{b}_j(l)_K = b_{ij} + m^j = a_j + m^j \]

for \( i \in I \) with \( z = s_j(i) \) satisfying \( i \geq l \), and such \( z \) form a dense subset of \( Z_l \). This implies \( \bar{b}_j(l) = ((a_j + m^j) \cap K) \otimes_k \mathcal{O}_{Z_l} \), whence

\[ \bar{b}_j(l)_K = (a_j R_K + m^j) \cap \overline{R}_K. \]

Then \( b_j = \lim \bar{b}_j(l)_K = a_j R_K \) by Remark A.3, so \( \text{mld}_P(X_K, \prod_j b_{ij}^j) = \text{mld}_P(X, a) \) by Corollary 2.3. By Conjecture 4.5, we have \( \text{mld}_P(X_K, \prod_j b_{ij}^j) = \text{mld}_P(X, b_j) \) for infinitely many \( i \), that is, \( \text{mld}_P(X, a) = \text{mld}_P(X, b_i) \), which is absurd. q.e.d.

Remark 4.9. Proposition 4.8 has the refinement that for fixed \( d \) and \( a \geq 0 \),

(i) Conjecture 4.5 for \( d \) with \( \text{mld}_P(X, a) > a \) (resp. \( \geq a \)) implies Conjecture 4.4, for \( d \) with \( J \subset (a, d) \) (resp. \( \subset [a, d] \)), and

(ii) Conjecture 4.5 for \( d \) with \( \text{mld}_P(X, a) = a \) implies Conjecture 4.6 for \( d \) with \( \text{mld}_P(X, a) = a \).

This is obvious by the above proof. Note that (8) implies \( m \leq M \).

Remark 4.10. Theorem A.9 gives Conjecture 4.6 in the case when \( \text{mld}_P(X, a) = 0 \), and then its Corollary A.10 gives Conjecture 4.5 in the case when \( \text{mld}_P(X, a) = 0 \). The order of this logic is opposite to Proposition 4.8. We expect that an effective estimate of \( l \) in Conjecture 4.6 implies Conjecture 4.5.

Theorem A.9 is reduced to the corresponding statement [6, Theorem 1.4] on a variety by the property that the log canonical threshold for an ideal in \( \mathcal{O}_{Y, Q} \) is approximated by those for ideals in \( \mathcal{O}_{Y, Q} \). This property for the minimal log discrepancy on \( X \) is a special case of Conjecture 4.5, so we do not know how to reduce Conjecture 4.6 to its variety version. The version of Conjecture 4.6 for a germ \( Q \in (Y, \Delta, a) \) of a triple on a variety \( Y \) holds when (i) \( (Y, \Delta, a) \) is klt [14, Theorem 2.6], (ii) \( Y \) is a surface [15], or (iii) \( Y \) is toric and \( Q, \Delta, a \) are torus invariant [25, Theorem 1.8].
The variety version of Theorem A.9 is globalised.

**Theorem 4.11.** Let \((Y, \Delta, a = \prod_i a_i^{r_i})\) be a triple on a variety \(Y\) and \(Z\) an irreducible closed subset of \(Y\). Suppose \(\text{mld}_{\eta_Z}(Y, \Delta, a) = 0\) and it is computed by \(E \in \mathcal{D}_Y\). Then there exists an open subset \(Y'\) of \(Y\) containing \(\eta_Z\) such that if an \(R\)-ideal \(b = \prod_j b_j^{r_j}\) on \(Y'\) satisfies \(a_j|_{Y'} + p_j = b_j + p_j\), for all \(j\), where \(p_j = \{ u \in \mathcal{O}_{Y'} \mid \text{ord}_E u > \text{ord}_E a_j \}\), then \((Y', \Delta_{|Y'}, b)\) is lc about \(Z_{|Y'}\) and \(\text{mld}_{\eta_Z}(Y', \Delta_{|Y'}, b) = 0\).

**Proof.** Take a log resolution \(f : W \to Y\) of \((Y, \Delta, a, \mathcal{F}_Z)\), where \(\mathcal{F}_Z\) is the ideal sheaf of \(Z\), such that \(E\) is realised as a divisor on \(W\). Then \(F := \text{Exc} f \cup \text{Supp} \Delta_W \cup \text{Cosupp} a \mathcal{F}_Z \mathcal{O}_Y\) is an snc divisor \(\Sigma F_i\), where \(\Delta_W\) is the strict transform of \(\Delta\). By generic smoothness [11, Corollary III.10.7], there exists an open subset \(Y'\) of \(Y\) containing \(\eta_Z\) such that if the restriction \(S' = S|_{f^{-1}(Y')}\) of a stratum \(S\) of \(\Sigma F_i\) satisfies \(S' \neq \emptyset\) and \(f(S') \subset Z' = Z_{|Y'}\), then \(S' \to Z'\) is smooth and surjective.

Set \(z = \dim Z\). We claim that for any \(Q \in Z'\),

\[(9) \quad \text{mld}_Q(Y, \Delta, a|_Q) = 0 \quad \text{and it is computed by} \quad G_Q\]

for the maximal ideal sheaf \(m_Q\) and the divisor \(G_Q\) obtained by the blow-up of \(W\) along a component of \(E \cap f^{-1}(Q)\). This can be verified from the local description at each closed point \(R \in f^{-1}(Q)\) that there exists a regular sequence of parameters \(v_1, \ldots, v_{\dim Y} \in \mathcal{O}_{Y,R}\) such that \(m_Q \mathcal{O}_{Y,R} = (v_1, \ldots, v_s, \prod_{i=1}^t v_{z+i}) \mathcal{O}_{Y,R}\) and \(F\) is given by \(\prod_{i=1}^t v_{z+i} = 0\) for some \(s, t\) with \(1 \leq s \leq t\).

Because \(\text{ord}_{G_Q} a_j = \text{ord}_E a_j\) and \(\text{ord}_{G_Q} u > \text{ord}_E u\) for \(u \in \mathcal{O}_{Y'}\), we conclude \(\text{mld}_{G_Q}(Y', \Delta_{|Y'}, b|_{G_Q}) = 0\) for \(b\) in Theorem 4.11 by [6, Theorem 1.4] (its proof works for triples). Hence \((Y', \Delta_{|Y'}, b)\) is lc about \(Z'\), and \(\text{mld}_{\eta_Z}(Y', \Delta_{|Y'}, b) = 0\) by \(a_E(Y', \Delta_{|Y'}, b) = 0\). q.e.d.

**Corollary 4.12.** Let \((Y, \Delta, a = \prod_i a_i^{r_i})\) be an lc triple on a variety \(Y\) and \(Z\) a closed subset of \(Y\) with ideal sheaf \(\mathcal{F}_Z\). Then there exists an integer \(l\) such that if an \(R\)-ideal \(b = \prod_j b_j^{r_j}\) on \(Y\) satisfies \(a_j + \mathcal{F}_Z^l = b_j + \mathcal{F}_Z^l\) for all \(j\), then \((Y, \Delta, b)\) is lc about \(Z\).

**Remark 4.13.** The author should have written the proof after [14, Theorem 2.4]. The estimate of \(l\) in [14, Remark 2.4.1] is incorrect unless \(Z\) is a closed point (so is that of \(t_1\) in [14, Lemma 3.1]).

5. The klt and plt cases

In this section, we settle Conjecture 4.5 in the klt case, and in the plt case whose lc centre has an isolated singularity. Conjecture 4.5 in these cases will be applied in the proof of Theorem 1.3 in Section 6. We keep the notation in Appendix A, so \(P \in X = \text{Spec} R\) with \(R = K[[x_1, \ldots, x_d]]\) and \(P_K \in X_K = \text{Spec} R_K\) with \(R_K = K[[x_1, \ldots, x_d]]\). In the course of the proofs, we will often replace the family \(\mathcal{F}\) with a subfamily, but we keep using the same notation \(\mathcal{F} = (Z_i, (\Delta_j(I)))_j, t_1, t_1, t_{l+1})_{l \geq b_0}\) to avoid intricacy.

5.A. The klt case.

**Theorem 5.1.** Conjecture 4.5 holds in the case when \((X_K, a)\) is klt.

**Proof.** It is shown similarly to [14, Theorem 2.6]. By Remark 4.7, it suffices to show that after replacing \(\mathcal{F}\) with a subfamily,

\[(10) \quad a_E(X, a_i) \geq \text{mld}_{P_K}(X_K, a)\]
for any \( i \in I \) (with \( l \geq l_0 \)) and any \( E \in \mathcal{G} \) with centre \( P \).

Take a subfamily in Lemma A.8 so that \( \mathrm{mld}_P(X, \prod_j (\mathcal{a}_j(l), R)^{\mathcal{r}_j}) = \mathrm{mld}_{P_k}(X_K, a) \) for \( z \in Z_i \). Then for \( i \in I \),

\[
a_E(X, \prod_j (a_{ij} + m^j)^{\mathcal{r}_j}) \geq \mathrm{mld}_{P_k}(X_K, a)
\]

by (iii) in Definition A.1. Since \((X_K, a)\) is klt, we can fix \( t > 0 \) such that \((X_K, a_1^{1+t})\) is lc. By Corollary A.10, \((X, a_i^{1+t})\) is lc for \( i \in I \) after replacing \( \mathcal{G} \) with a subfamily, whence \( a_E(X, a_i) \geq t \mathrm{ord}_E a_i = t \sum_j r_j \mathrm{ord}_E a_{ij} \). We fix \( l \geq l_0 \) such that \( l \geq (tr_j)^{-1} \mathrm{mld}_{P_k}(X_K, a) \) for all \( j \). Then,

\[
a_E(X, a_i) \geq l^{-1} \mathrm{ord}_E a_{ij} \cdot \mathrm{mld}_{P_k}(X_K, a)
\]

for any \( j \) and \( i \in I \).

If \( \mathrm{ord}_E a_{ij} < l \) for all \( j \), then \( \mathrm{ord}_E a_{ij} = \mathrm{ord}_E (a_{ij} + m^j) \), so one has \( a_E(X, a_i) = a_E(X, \prod_j (a_{ij} + m^j)^{\mathcal{r}_j}) \), and (10) follows from (11). If \( \mathrm{ord}_E a_{ij} \geq l \) for some \( j \), then (10) follows from (12).

**q.e.d.**

**Remark 5.2.** By Remark 4.7, Theorem 5.1 and Corollary A.10, Conjecture 4.5 remains open only when \((X_K, a)\) is non-klt with \( \mathrm{mld}_{P_k}(X_K, a) > 0 \).

**5.B. The plt case whose lc centre has an isolated singularity.** Suppose that \((X_K, a)\) is an lc but not klt pair every lc centre of which has codimension 1. Then by Proposition 3.7, \((X_K, a)\) has the smallest lc centre \( S_K \) and it is normal. We prove Conjecture 4.5 on the assumption that \( S_K \) has an isolated singularity. Our argument has its origin in [15], but is highly cumbersome.

**Theorem 5.3.** Conjecture 4.5 holds in the case when \((X_K, a)\) has the smallest lc centre of codimension 1 which is regular outside \( P_K \).

We let \( S_K \) denote the smallest lc centre of \((X_K, a)\). \( S_K \) is a prime divisor which is regular outside \( P_K \). We define an \( \mathbb{R} \)-ideal \( c = \prod_j c_j^{\mathcal{r}_j} \) by the expression

\[
a_j = c_j \mathcal{O}_{X_K}(-\mathrm{ord}_{S_K} a_j S_K)
\]

so that \( c_j \) does not vanish along \( S_K \). The \( a \) and \( c \mathcal{O}_{X_K}(-S_K) \) take the same order along any divisor over \( X_K \). We can fix \( t > 0 \) such that \((X_K, S_K, c^{1+t})\) is lc, since \( S_K \) is the unique lc centre of \((X_K, S_K, c)\).

We take a log resolution \( f_K : Y_K \to X_K \) of \((X_K, S_K, m_K)\), which is isomorphic outside \( P_K \). Let \( \{E_{aK}\}_a \) be the set of all \( f_K \)-exceptional prime divisors. The \( E_K = \sum_a E_{aK} \) is snc. Let \((Y_K, \Delta_K, a' = \prod_j (a'_j)^{\mathcal{r}_j})\) be the pull-back of \((X_K, 0, a)\) and \((Y_K, T_K + \Delta_K, c')\) that of \((X_K, S_K, c)\). Then \( T_K \) is the strict transform of \( S_K \). We set

\[
L_K := T_K \cap f_K^{-1}(P_K),
C_K := \text{Cosupp} c' \cap f_K^{-1}(P_K).
\]

We have the following diagram.

\[
\begin{array}{ccc}
T_K & \subset & Y_K & \supset & E_K & \supset & L_K, C_K \\
\downarrow f_K & & \downarrow & & \downarrow & & \\
S_K & \subset & X_K & \supset & P_K
\end{array}
\]
By blowing up $Y_K$ further, we may assume that $C_K$ is contained in the union of those $E_{ak}$ satisfying

$$t \operatorname{ord}_{E_{ak}} c \geq \operatorname{mld}_K(X_K, a).$$

One can see this by induction on $\max \left\{ \min_{\alpha \in J} \{ \operatorname{ord}_{E_{ak}} c \} \right\}$ in which one considers all subsets $J$ of indices satisfying $C_K \subset \bigcup_{\alpha \in J} E_{ak}$. Indeed, suppose that a subset $J$ gives the maximum of $\min_{\alpha \in J} \{ \operatorname{ord}_{E_{ak}} c \}$. Take a log resolution $p_K: Y_K \to Y_K$ of $(Y_K, T_k + E_K, \mathcal{I}_K)$ for the ideal sheaf $\mathcal{I}_K$ of $C_K$, and write the snc divisor $\operatorname{Exc} p_K$ as $\sum_{\beta \in J} E'_{\beta k}$. Then the new $C'_K$ on $Y'_K$ defined like $C_K$ is contained in $\bigcup_{\beta \in J} E'_{\beta k}$. Further for any $\beta \in J$, there exists $\alpha \in J$ such that $p_K(E'_{\beta k}) \subset E_{ak}$, for which $\operatorname{ord}_{E_{ak}} c > \operatorname{ord}_{E_{ak}} c$. Hence

$$\min_{\beta \in J'} \{ \operatorname{ord}_{E'_{\beta k}} c \} > \min_{\alpha \in J} \{ \operatorname{ord}_{E_{ak}} c \},$$

so the induction can be proceeded. Note that the order of $c$ takes value in the discrete subset $\sum_j r_j \mathbb{Z}_{>0}$ of $\mathbb{R}$.

The $f_K$ is descpicable by Proposition A.7, so replacing $\mathcal{F}$ with a subfamily, we obtain the diagram (16) in which $f_j$ is a family of log resolutions. Shrinking $Z_i$, we may assume that $E_{ak}, L_i$ and $C_K$ are the base changes of flat families $E_{al}, L_i$ and $C_i$ in $Y_i$ over $Z_i$. We may assume that $\sum_{\alpha} E_{al}$ is an snc divisor such that for every stratum $\bar{S}_j$ of $\sum_{\alpha} E_{al}$, the projections $\bar{S}_j \to Z_i$ and $\bar{S}_j \cap L_i \to Z_i$ are smooth and surjective. We may also assume that $\operatorname{ord}_{E_{al}} \delta_j(l)_z$ is constant on $z \in Z_i$ for each $\alpha$ and $j$. Their base changes in $Y_i$ are denoted by $E_{al}, L_i$ and $C_i$. We write $E_j = \sum_{\alpha} E_{al}$ and $E_l = \sum_{\alpha} E_{al}$.

We fix $m$ such that $m \operatorname{ord}_{E_{ak}} m_k \geq \operatorname{ord}_{E_{ak}} c_j$ for all $\alpha$ and $j$, and set

$$d := \prod_j (c_j + m_j^n)^{r_j}.$$

Then $\operatorname{ord}_{E_{ak}} d = t \operatorname{ord}_{E_{ak}} c$, and $(X_K, ad)$ is lc. The $d$ is defined over some $k(Z_i)$, so by replacing $\mathcal{F}$ with a subfamily, we may assume that $d$ is the base change of an $R$-ideal $\delta_i = \prod_j \delta_{ij}^{r_j}$ on $h^*_k \times \operatorname{Spec} k Z_i$ with $m^n \otimes_k \delta_{ij} \subset \delta_{ij}$, and that $\operatorname{ord}_{E_{al}}(\delta_i)_z$ is constant on $Z_i$ for each $\alpha$. By Corollary A.10, after taking a subfamily, $(X, a_i(\delta_i)_z)$ is lc for any $i \in I_i$ with $z = s_i(i)$, where $\delta_i$ is the pull-back on $X \times \operatorname{Spec} k Z_i$ of $\delta_i$.

We fix $l \geq l_0$ such that

$$l \operatorname{ord}_{E_{ak}} m_k > \operatorname{ord}_{E_{ak}} a_j + \operatorname{ord}_{X_{ak}} a_j$$

for all $\alpha$ and $j$. By Remark 4.7, for Theorem 5.3 it suffices to prove that after shrinking $Z_i$ (and taking a subfamily accordingly),

$$a_E(X, a_i) \geq \operatorname{mld}_K(X, a)$$

for any $i \in I_i$ and $E \in \mathcal{D}_K$ with centre $P$. Setting $z = s_i(i)$, we shall prove (15) by treating the three cases according to the position of $c(Y_i)_i(E)$:

(a) $c(Y_i)_i(E) \subset (L_i \cup C_i)_z$.
(b) $c(Y_i)_j(E) \subset (C_i)_z$.
(c) $c(Y_i)_i(E) \subset (L_i)_z$ and $c(Y_i)_j(E) \subset (C_i)_z$.

We let $\delta_i' = \prod_j \delta_{ij}(l)^{r_j}$ be the weak transform on $\bar{Y}_l$ of $\prod_j \delta_{ij}(l)^{r_j}$, and $\delta_i = \prod_j (\delta_{ij}')^{r_j}$ the weak transform on $\bar{Y}_l$ of $\delta_i$. 
Lemma 5.4.  
(i) $\tilde{\alpha}'(l)_{K} \mathcal{O}_{Y_{k}} = a_{i}' + l_{ij}$ with an ideal sheaf $l_{ij}$ which is contained in $\mathcal{O}_{Y_{k}}(-E_{K})^{ord_{Y_{k}}a_{j}+1}$.
(ii) $\tilde{\alpha}'(l)_{z} \mathcal{O}_{(Y_{k})_{z}} = a_{i}' + \mathcal{J}_{ij}$ with an ideal sheaf $\mathcal{J}_{ij}$ which is contained in $\mathcal{O}_{(Y_{k})_{z}}(-E_{K})^{ord_{Y_{k}}a_{j}+1}$.
(iii) $\text{Cosupp} \tilde{\alpha}'(l) = L_{k} \cup C_{i}$ after shrinking $Z_{i}$.

Proof. Write $m_{K} \mathcal{O}_{Y_{k}} = \mathcal{O}_{Y_{k}}(-M_{K})$ and $a_{j} \mathcal{O}_{Y_{k}} = a_{j}' \mathcal{O}_{Y_{k}}(-A_{jK})$. The inequality (14) means that $l_{ij} = \mathcal{O}_{Y_{k}}(A_{jK} - lM_{K})$ is an ideal sheaf contained in $\mathcal{O}_{Y_{k}}(-E_{K})^{ord_{Y_{k}}a_{j}+1}$. Then $m_{K} \mathcal{O}_{Y_{k}} = l_{ij} \mathcal{O}_{Y_{k}}(-A_{jK})$. By $\bar{a}_{j}(l)_{K}R_{K} = a_{j} + m_{K}$, we have $\bar{a}_{j}(l)_{K} \mathcal{O}_{Y_{k}} = a_{j}' \mathcal{O}_{Y_{k}}(-A_{jK}) + l_{ij} \mathcal{O}_{Y_{k}}(-A_{jK})$, which induces (i). From (i), $\text{Cosupp} \tilde{\alpha}'(l)_{K} \mathcal{O}_{Y_{k}} = \text{Cosupp} a'_{i} \cap f_{K}^{-1}(P_{K}) = L_{k} \cup C_{k}$, which is extended to $\text{Cosupp} \tilde{\alpha}'(l) = L_{k} \cup C_{i}$ in (iii). On the other hand, $\text{ord}_{E_{al}} m_{K} = \text{ord}_{E_{al}} m_{K} a_{j} = \text{ord}_{E_{al}} \bar{a}_{j}(l)_{K}R_{K} = \text{ord}_{E_{al}} \bar{a}_{j}(l)_{K} = \text{ord}_{E_{al}} a_{ij}$ by (14) and Definitions A.1, A.2. Then, (ii) is induced similarly to (i). q.e.d.

The cases (a) and (b) are not difficult.

Proof of (15) in the case (a). Set $\Delta_{i} = \sum a(1 - a_{E_{al}}(X_{K}, a))E_{al}$, base-changed to $\Delta_{K}$. Then $((Y_{i})_{z}, (\Delta_{i})_{z}, a_{i}')$ is the pull-back of $(X, 0, a_{i})$. For a divisor $(E_{al})_{z}$ containing $c_{(Y_{i})_{z}}(E)$, we have

$$a_{E}(Y_{i})_{z}, (\Delta_{i})_{z} \geq \text{ord}_{E_{al}}(E_{l} - \Delta_{i})_{z} \geq \text{ord}_{E_{al}}(E_{l} - \Delta_{i})_{z} = a_{E_{al}}(X_{K}, a) \geq \text{mld}_{P_{k}}(X_{K}, a),$$

where the first inequality follows from the log canonicity of $((Y_{i})_{z}, (E_{l})_{z})$. By Lemma 5.4(ii) and (iii), $\text{Cosupp} a'_{i} \cap (f_{l})_{z}^{-1}(P_{l}) = \text{Cosupp} \tilde{\alpha}'(l)_{z} \mathcal{O}_{(Y_{i})_{z}} = (L_{l} \cup C_{l})_{z}$, so $\text{ord}_{E_{al}} a'_{i} = 0$. Thus

$$a_{E}(X, a_{i}) = a_{E}(Y_{i})_{z}, (\Delta_{i})_{z}, a'_{i}) = a_{E}(Y_{i})_{z}, (\Delta_{i})_{z} \geq \text{mld}_{P_{k}}(X_{K}, a).$$

q.e.d.

Proof of (15) in the case (b). The $c_{(Y_{l})_{z}}(E)$ lies on some $(E_{al})_{z}$ such that $E_{al}$ satisfies (13). Then

$$a_{E}(X, a_{i}) \geq \text{ord}_{E}(d_{i})_{z} \geq \text{ord}_{E_{al}}(d_{i})_{z} = \text{ord}_{E_{al}} d = \text{ord}_{E_{al}} c \geq \text{mld}_{P_{k}}(X_{K}, a),$$

whose first inequality follows from the log canonicity of $(X, a_{i}(d_{i}))_{z}$. q.e.d.

The case (c) is reduced to the following log canonicity.

Lemma 5.5. After shrinking $Z_{i}$, the triple $((Y_{i})_{z}, (E_{l})_{z}, a'_{i})$ is lc about $(L_{l})_{z} \setminus (C_{l})_{z}$ for any $i$ in $I_{l}$ with $z = s_{l}(i) \in Z_{l}$.

Proof of (15) in the case (c) from Lemma 5.5. For $\Delta_{i} = \sum a(1 - a_{E_{al}}(X_{K}, a))E_{al}$, we have $a_{E}(X, a_{i}) = a_{E}(Y_{i})_{z}, (\Delta_{i})_{z}, a'_{i}) \geq \text{ord}_{E}(E_{l} - \Delta_{i})_{z}$ by Lemma 5.5, and have seen $\text{ord}_{E}(E_{l} - \Delta_{i})_{z} \geq \text{mld}_{P_{k}}(X_{K}, a)$ in the proof in the case (a). q.e.d.

Proof of Lemma 5.5. Pick any open stratum $F_{k}$ of the snc divisor $E_{K}$, which is extended to an open stratum $F_{l}$ of $E_{l}$. We prove Lemma 5.5 by noetherian induction. Recall that $l$ has been fixed. Let $\tilde{Q}_{l}$ be an irreducible locally closed subset of $F_{l} \cap L_{l} \setminus C_{l}$ which dominates $Z_{l}$. It suffices to show the existence of a dense open subset $\tilde{Q}_{l}'$ of $\tilde{Q}_{l}$ such that the triple $((Y_{i})_{z}, (E_{l})_{z}, a'_{i})$ is lc about $(\tilde{Q}_{l})_{z}$ for $i \in I_{l}$ with

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$z = s_i(i)$, where $Q_i \overset{\pi}{\rightarrow} \bar{Q}_i \times \bar{f}_i Y_i$. Indeed, start with a component $\bar{Q}_i$ of $\bar{F}_i \cap L_i \setminus \bar{C}_i$ and find $\bar{Q}_i'$. Take a dense open subset $Z'_i$ of $Z_i$ such that each irreducible component $\bar{Q}_i'$ of $\bar{Q}_i \setminus Z'_i|_{Z'_i}$ dominates $Z'_i$. Replace $Z_i$ with $Z'_i$ and continue the argument for each $\bar{Q}_i'$. Eventually we attain $Z_i$ such that $(Y_i)_z, (E_i)_z, a_i)$ is lc about $(F_i \cap L_i \setminus C_i)$ for $i \in I$ with $z = s_i(i) \in Z_i$. Applying this to all open strata of $E_K$, we obtain a shrunk $Z_i$ in Lemma 5.5.

We shall construct $\bar{Q}_i'$, By shrinking $\bar{Q}_i$ and $Z_i$, we may assume that $\bar{Q}_i \rightarrow Z_i$ is smooth and surjective. Let $\bar{f}_i : \bar{Y}_i \rightarrow \mathbb{A}^d_k \times \text{Spec} \mathbb{k} \bar{Q}_i$ be the base change of $\bar{f}_i$ by $\bar{Q}_i \rightarrow Z_i$. Then $\text{pr}_{\bar{Q}_i} \circ \bar{f}_i^+$ has the natural section $\bar{g}_i : \bar{Q}_i \rightarrow \bar{Y}_i^+ = \bar{Y}_i \times Z_i \bar{Q}_i$ by the immersion $\bar{Q}_i \hookrightarrow \bar{Y}_i$. We construct $f_K^+$ and $g_K$ similarly for $Q_K = \bar{Q}_i \times Y_i K$ as below.

$$
\begin{array}{c}
\xymatrix{
Y^+_i \ar[r]^-{Y^+_{i+}} & \bar{Y}^+_i \ar[r]^-{\bar{f}_i^+} \ar[d]^-{\bar{g}_i} & \bar{Y}_i \ar[d]^-{\bar{f}_i} \ar[r]^-{g_K} & Y_i \ar[d]^-{g_K} \ar[r]^-{g_K} & Y_i \ar[d]^-{f_K^+} \ar[r]^-{g_K} & Y_i \ar[d]^-{g_K} \ar[r]^-{g_K} & Y_i
}
\end{array}
$$

The $Y^+_i, Y^+_K$ are the base changes of $Y_i, Y_K$ by smooth morphisms. For a scheme or a sheaf $\square$ on $Y_i$ or $Y_K$, we mean by $\square^+$ the base change of $\square$ on $\bar{Y}_i^+$ or $Y_K^+$. For example, $a^+ = \prod_j(a_j^+)^r_j = a^o Y^+_K$. Let $\bar{g}_i$ be the ideal sheaf of $\bar{g}_i(\bar{Q}_i)$ on $\bar{Y}_i^+$ and $\bar{g}_i \in \mathcal{D}_{\bar{Y}_i^+}$ the divisor obtained by the blow-up of $\bar{Y}_i^+$ along $\bar{g}_i$. They are base-changed to $q$ on $Y_K^+$ and $G_K \in \mathcal{D}_{Y_K^+}$.

Set $\pi = \dim F_K - 1$. Similarly to (9), we see that $\text{mld}_{n_k, q_k} (Y_K^+, E_K^+, q^n a^+) = 0$ and it is computed by $G_K$. We have $\bar{a}^+(l)^{+}_K \mathcal{O}_{Y_K^+} = \prod_j (a_j^+ + l_{ij})^r_j$ and $\text{ord}_{G_K} a_j^+ = \text{ord}_{S_K} a_j < \text{ord}_{G_K} l_{ij}$ from Lemma 5.4(i), so

$$\text{mld}_{n_k, q_k} (Y_K^+, E_K^+, q^n a(l)^+ \mathcal{O}_{Y_K^+}) = 0$$
and it is computed by $G_K$.

Then $\text{mld}_{n_k, q_k} (\bar{Y}_i^+, \bar{E}_i^+, \bar{g}_i a(l)^+) = 0$ and it is computed by $G_i$. We regard $\bar{Y}_i^+$ as a family over $\bar{Q}_i$. There exists a dense open subset $\bar{Q}_i$ of $\bar{Q}_i$ such that for any closed point $q \in Q_i = \bar{Q}_i \times Y_i Y_i$ with its image $z \in Z_i$.

$$\text{mld}_q ((Y)_z, (E)_z, m_q^{\bar{a}^+}(l) \mathcal{O}_{(Y)_z}) = 0$$
and it is computed by $(G_i)_q$.

and $\text{ord}_{(G_i)_q} a^+(l) \mathcal{O}_{(Y)_z} = \text{ord}_{S_K} a_j$, where $m_q$ is the maximal ideal sheaf of $q \in (Y)_z$ and $G_i \cap G_i = G_i \times Y_i$. The $(G_i)_q$ is obtained by the blow-up of $(Y)_z$ at $q$. If $z = s_i(i)$ with $i \in I$, then $\bar{a}^+(l) \mathcal{O}_{(Y)_z} = \prod_j (a_j^+ + \mathcal{J}_{ij})^{r_j}$ and $\text{ord}_{(G_i)_q} a_j^+(l) \mathcal{O}_{(Y)_z} < \text{ord}_{(G_i)_q} \mathcal{J}_{ij}$ by Lemma 5.4(ii). Applying Theorem A.9, we have $\text{mld}_q ((Y)_z, (E)_z, m_q^a a_j^+)$ = 0, and the log canonicity of $((Y)_z, (E)_z, a_j^+)$ about $(Q_i)_z$ is concluded.

q.e.d.

Theorem 5.3 is completed.

6. THE THREEFOLD CASE

We shall prove Theorem 1.3. By Remark 4.9, the theorem follows from Conjecture 4.5 for $d = 3$ with $\text{mld}_{n_k} (X_K, a) > 1$. In Remark 5.2, Conjecture 4.5 is reduced.
to the case when $(X_k, a)$ is an lc pair which has a minimal lc centre $Z$ of positive dimension. If $d = 3$, then by Theorem 1.2, $Z$ is the smallest lc centre and it is normal. If $\dim Z = 2$, then one can apply Theorem 5.3. If $\dim Z = 1$, then $\mld_{X_k}(X_k, a) \leq 1$ by Proposition 6.1. Therefore, we obtain Theorem 1.3.

**Proposition 6.1.** Let $P \in (X, a)$ be a germ of an lc pair on a regular $R$-variety $X$ of dimension 3 with $R = K[[x_1, \ldots, x_d]]$ whose smallest lc centre is of dimension 1. Then $\mld_P(X, a) \leq 1$.

**Proof.** The smallest lc centre $C$ of $(X, a)$ is regular by Theorem 1.2. Setting $(X_0, a_0) := (X, 0, a)$ and $C_0 := C$, we build a tower of finitely many blow-ups

$$X_n \to \cdots \to X_i \xrightarrow{f_i} X_{i+1} \to \cdots \to X_0 = X$$

such that

(i) $f_i: X_i \to X_{i+1}$ is the blow-up along $C_{i+1}$,
(ii) $E_i$ is the exceptional divisor of $f_i$,
(iii) $(X_i, a_i)$ is the pull-back of $(X_{i+1}, a_{i+1})$,
(iv) $C_i$ is a regular non-klt centre on $X_i$ of $(X, a)$ mapped onto $C_{i+1}$, and
(v) $a_{E_i}(X, a) > 0$ for $i < n$ and $a_{E_n}(X, a) = 0$.

Here one can prove the effectiveness $\Delta_i \geq 0$ and the regularity of $C_i$ by induction. Indeed, if they hold for $i - 1$, then $\ord_{E_i} \Delta_i = \ord_{C_{i+1}} \Delta_{i+1} + \ord_{C_i} a_i - 1 > 0$ by Lemma 6.2. Unless $a_{E_i}(X, a) = 0$, an arbitrary lc centre $C_i$ of $(X_i, a_i)$ mapped onto $C_{i+1}$ is of dimension 1 and is minimal. The regularity of $C_i$ follows from Theorem 1.2.

Let $F$ be the divisor obtained by the blow-up of $X_n$ along an irreducible component of $E_n \cap (f_1 \circ \cdots \circ f_n)^{-1}(P)$. Then $a_F(X, a) = a_F(X_n, a_n) \leq a_F(X_n, E_n) = 1$ by $\Delta_n \geq 1 - a_{E_n}(X, a)/E_n = E_n$. q.e.d.

**Lemma 6.2.** Let $(X, a)$ be a pair on a regular $R$-variety $X$ and $Z$ a non-klt centre of $(X, a)$. Then $\ord_Z a \geq 1$. If in addition $\dim_X Z \geq 2$, then $\ord_Z a > 1$.

**Proof.** The lemma is obvious if $Z$ is a divisor, so we may assume $\dim_X Z \geq 2$. Setting $X_0 := X$, $Z_0 := Z$ and $a_0 := a$, we build a tower of finitely many blow-ups

$$X_n \to \cdots \to X_i \xrightarrow{h_i} X_i' \to \cdots \to X_0 = X$$

such that

(i) $h_i$ is the composition $X_i \xrightarrow{b_i} X_i' \xrightarrow{g_{i-1}} X_{i-1}$ of the blow-up $h_i: X_i \to X_i'$ along the strict transform on $X_i'$ of $Z_i$ and an embedded resolution $g_{i-1}: X_i' \to X_{i-1}$ of singularities of $Z_{i-1}$, in which $g_{i-1}$ is isomorphic on the regular locus of $Z_i$, 
(ii) $E_i$ is the exceptional divisor of $h_i$,
(iii) $a_i$ is the weak transform on $X_i$ of $a_{i-1}$, 
(iv) $Z_i$ is a non-klt centre on $X_i$ of $(X, a)$ mapped onto $Z_{i-1}$, and 
(v) $a_{E_i}(X, a) > 0$ for $i < n$ and $a_{E_n}(X, a) \leq 0$.

Supposing $\ord_Z a \leq 1$, we shall derive by induction two inequalities

$$\ord_Z a_i \leq 1 \quad \text{and} \quad a_{E_i}(X_i, a_i) \leq 0$$

for any $i$. The claim for $i = 0$ is trivial. If they hold for $i - 1$, then $\ord_Z a_i \leq \ord_Z a_i \leq \ord_Z a_{i-1} \leq 1$ by [12, Lemmata III.7, III.8] for an irreducible closed subset $V_i$ of $Z_i$ meeting the regular locus of $Z_i$ such that $V_i \to Z_{i-1}$ is finite and
surjective. Note that the symbol $v^{(1)}$ in [12] stands for the order. The $(X_{i-1}, 0, a_{l-1})$ is pulled back to $(X_i, \Delta_i, a_i)$ with $\text{ord}_{E_i} \Delta_i = 1 + \text{ord}_{Z_{i-1}} a_{l-1} - \text{codim}_{X_{i-1}} Z_{i-1} \leq 0$, so $a_{E_i}(X_i, a_i) \leq a_{E_i}(X_{i-1}, a_{l-1}) \leq 0$.

We obtained $a_{E_i}(X_n, a_n) \leq 0$. However, it contradicts $a_{E_i}(X_n) = 1$ and $\text{ord}_{E_i} a_n = 0$.

Appendix A. Generic limits

The generic limit is a limit of ideals. It was constructed first by de Fernex and Mustaţă [8] using ultraproducts, and then by Kollár [20] using Hilbert schemes.

We set $\bar{R} = k[x_1, \ldots, x_d]$ with maximal ideal $\bar{m}$, and $A_k^d = \text{Spec } \bar{R}$ with origin $\bar{P}$. We also set $R = k[[x_1, \ldots, x_d]]$ with $m = \bar{m}k$, and $X = \text{Spec } R$ with closed point $P$. Mostly we discuss on the spectrum of a noetherian ring, where an ideal in the ring is identified with its coherent ideal sheaf.

We introduce the notion of a family of approximated ideals by which a generic limit is defined.

**Definition A.1.** Let $S = \{(a_{i1}, \ldots, a_{ie})\}_{i \in I}$ be a collection of $e$-tuples of ideals in $R$, indexed by an infinite set $I$. A *family $\mathcal{F}$ of approximations* of $S$ consists of, with $l_0$ fixed, for each $l \geq l_0$,

(a) a variety $Z_l$,
(b) an ideal sheaf $\bar{a}_j(l)$ on $A_k^d \times _{\text{Spec } \bar{R}} Z_l$ containing $\bar{m}^l \otimes_k \mathcal{O}_{Z_l}$ for $1 \leq j \leq e$,
(c) an infinite subset $I_l$ of $I$ and a map $s_l : I_l \to Z_l(k)$, where $Z_l(k)$ is the set of $k$-points on $Z_l$, and
(d) a dominant morphism $t_{l+1} : Z_{l+1} \to Z_l$,

such that

(i) $s_j(l)$ gives a flat family of closed subschemes of $A_k^d$ parametrised by $Z_l$,
(ii) the pull-back of $\bar{a}_j(l)$ by $\text{id}_{A_k^d} \times t_{l+1}$ is $\bar{a}_j(l+1) + \bar{m}^l \otimes_k \mathcal{O}_{Z_{l+1}}$,
(iii) $a_j + m^l = \bar{a}_j(l)x_i)R$ for $i \in I_l$, where $\bar{a}_j(l)z$ is the ideal in $R$ given by $\bar{a}_j(l)$ at $z \in Z_l$,
(iv) $s_l(t_l)$ is dense in $Z_l$, and
(v) $t_{l+1} \subset I_l$ and $t_{l+1} \circ s_{l+1} = s_l \mid_{l+1}$.

The construction of $\mathcal{F}$ using Hilbert schemes is exposed in [6, Section 4]. In general, there exist essentially different families of approximations.

For a field extension $K$ of $k$, we set $\bar{R}_K = \bar{R} \otimes_k K = K[x_1, \ldots, x_d]$ with $\bar{m}_K = \bar{m} \bar{R}_K$, and $A_K^d = \text{Spec } \bar{R}_K$ with origin $P_K$. We also set $R_K = R \otimes_k K = K[[x_1, \ldots, x_d]]$ with $m_K = \bar{m} \bar{R}_K$, and $X_K = \text{Spec } R_K$ with closed point $P_K$.

**Definition A.2.** Suppose that a family $\mathcal{F}$ of approximations of $S$ is given as in Definition A.1. For this $\mathcal{F}$, take the union $K = \lim_{\rightarrow} K(Z_l)$ of the function fields $K(Z_l)$ of $Z_l$ by the inclusions $t_{l+1}^* : K(Z_{l+1}) \hookrightarrow K(Z_l)$. Then the *generic limit* of $S$ with respect to $\mathcal{F}$ is the $e$-tuple $(a_{11}, \ldots, a_{ee})$ of ideals in $R_K$ such that

$$a_j + m_K^l = \bar{a}_j(l)K \bar{R}_K$$

for all $l \geq l_0$, where $\bar{a}_j(l)K$ is the ideal in $R_K$ given by $\bar{a}_j(l)$ at the natural $K$-point $\text{Spec } K \to Z_l$.

**Remark A.3.** We have $a_j = \lim_{\rightarrow} \bar{a}_j(l)K$, by $\bar{a}_j(l)K = \bar{a}_j(l+1)K + \bar{m}_K^l$ from (ii) in Definition A.1.
Definition A.4. Let $S = (Z_t, (\tilde{a}_j(l)))_{l,l_1,s_1,t_1}^{t_1+1} \geq l_0}$ and $S' = (Z'_t, (\tilde{a}'_j(l)))_{l,l'_1,s'_1}^{l'_1+1} \geq l'_0}$ be families of approximations of $S$. A morphism $S' \to S$ consists of dominant blow-ups $f_i: Z'_i \to Z_i$ for $l \geq l'_i$, with $l'_i \geq l_0$ imposed, such that

(i) $t_{i+1} \circ f_{i+1} = t_i \circ f_i$,
(ii) the pull-back of $\tilde{a}_j(l)$ by $\text{id}_{\mathbb{A}_K^d} \times f_i$ is $\tilde{a}'_j(l)$, and
(iii) $f'_i \subseteq f_i$ and $f_i \circ s'_i = s_i|_{f'_i}$.

An $S'$ is called a subfamily of $S$ if it is equipped with a morphism $S' \to S$ as above such that all $f_i$ are open immersions.

We want to compare minimal log discrepancies over $X$ and $X_K$. The comparison of those for approximated ideals is a consequence of the existence of a family of log resolutions on an open subfamily of triples and Corollary 2.3.

Lemma A.5 (cf. [16, Proposition 3.2(iii)]). Notation as above. Let $(a_1, \ldots, a_r)$ be the generic limit of $S$ with respect to $S$. Then after replacing $S$ with a subfamily but using the same notation,

$$\text{mld}_{P_K}(X_K, \prod_j (a_j + m_{K,l})^{f_j'}) = \text{mld}_{P}(\mathbb{A}_K^d, \prod_j \tilde{a}_j(l_j^{f_j})$$

for all $r_1, \ldots, r_s > 0$ and all $z \in Z_l$ with $l \geq l_0$.

We utilise a projective morphism which is descended to $\mathbb{A}_K^d$.

Definition A.6. A projective morphism $f_K: Y_K \to X_K$ is said to be descendible if there exists a projective morphism $f_K: \tilde{Y}_K \to \mathbb{A}_K^d$ whose base change to $X_K$ is $f_K$.

Proposition A.7. Let $f_K: Y_K \to X_K$ be a projective morphism of $R_K$-varieties which is isomorphic outside $P_K$. Then $f_K$ is descendible.

Proof. Assuming $d \geq 1$, $f_K$ is the blow-up along an ideal $n_K$ in $R_K$ [23, Theorem 8.1.24]. We may assume codim$_{X_K}$ Cosupp$n_K \geq 2$, then Cosupp$n_K \subseteq P_K$, that is, $n_K$ is an $m_K$-primary ideal. Thus, $n_K$ is the pull-back of the ideal $\bar{n}_K = n_K \cap \bar{R}_K$ in $\bar{R}_K$. Since blowing-up commutes with flat base change [23, Proposition 8.1.12(c)], the blow-up of $\bar{n}_K$ along $\bar{n}_K$ is base-changed to $f_K$.

Let $f_K: Y_K \to X_K$ be a descendible projective morphism, descended to $\bar{f}_K: \bar{Y}_K \to \mathbb{A}_K^d$. This $\bar{f}_K$ is defined over $k(Z_{l_0})$ for some $l_0 \geq 0$. For $l \geq l'_0$, one can construct inductively a projective morphism $\bar{f}'_i: \bar{Y}'_i \to \mathbb{A}_K^d \times_{\text{Spec} k} Z_{l'_i}$ with a smooth open subvariety $Z_{l'_i}$ of $Z_{l'_i}$ such that (i) $\bar{Y}'_i$ is flat over $Z_{l'_i}$, (ii) $Z_{l'_i+1} \subseteq t_{l'_i+1}(Z_{l'_i})$, and (iii) $\bar{f}'_{i+1}$ and $\bar{f}'_i$ are the base changes of $f_{i+1}$ and $f_i$, by generic flatness [10, Corollaire IV.11.1.5]. These $Z_{l'_i}$ with $l'_i = s_i^{-1}(Z_i(k))$ form a subfamily $\mathcal{F}'$ of $\mathcal{F}$. Replacing $\mathcal{F}$ with $\mathcal{F}'$, we obtain a commutative diagram

$$
\begin{array}{ccc}
Y_K & \xrightarrow{f_K} & X_K \\
\downarrow{f_i} & & \downarrow{f_i} \\
Y_i & \xrightarrow{f_i} & X_i \\
\end{array}
$$

(16)
for $l \geq l_0$ (the $l_0$ is replaced) such that (i) $Z_l$ is smooth, (ii) $\bar{f}_i$ is projective, (iii) $\bar{Y}_i$ is flat over $Z_l$, and (iv) $\tilde{f}_{i+1}$, $\bar{f}_k$, $f_1$ and $f_K$ are the base changes of $\bar{f}_i$. In general, $X_K \to X \times_{\Spec k} Z_l$ is not the base change of $X_k^d \to X_k^d \times_{\Spec k} Z_l$.

Whenever an algebraic object over $X_K$ descendsible to $X_k^d$ is specified, by taking a subfamily, one can construct (16) so that it comes from a flat family over $Z_f$. For example, suppose that $E_K \in \mathcal{D}_{X_K}$ with centre $P_K$ is given. It is realised as a divisor on $Y_K$ equipped with a log resolution $f_K: Y_K \to X_K$ of $(X_K, m_K)$, which is isomorphic outside $P_K$. This $f_K$ is descended to a log resolution $\bar{f}_K$ by Proposition A.7, and $\bar{f}_i$ is extended to a family $\bar{f}_l$ of log resolutions in (16) by generic smoothness. There exists a prime divisor $\bar{E}_l$ on $\bar{Y}_l$ which is base-changed to $E_K$. By this observation, Lemma A.5 is refined as follows.

**Lemma A.8** (cf. [16, Proposition 3.2(iii)]). **Notation as above.** Fix $r_1, \ldots, r_e > 0$ and $E_K \in \mathcal{D}_{X_K}$ computing $\mld_{\bar{f}_i}(X_K, \prod_j a_{ij}^r)$. Then after replacing $\mathcal{F}$ with a subfamily but using the same notation, there exists a divisor $\bar{E}_l$ over $X_k^d \times_{\Spec k} Z_l$ for any $l \geq l_0$, base-changed to $E_K$, such that

\[
\mld_{\bar{f}_l}(X_K, \prod_j a_{ij}^r) = \mld_{g}(X_k^d, \prod_j \bar{a}_j(l)^r) = a_{(E)_l}(X_k^d, \prod_j \bar{a}_j(l)^r),
\]

\[
\ord_{E_K} a_j = \ord_{E_k}(a_j + m_K^l) = \ord_{E_l}(\bar{a}_j(l))^r < l,
\]

for all $z \in Z_l$.

We apply the ideal-adic semi-continuity of log canonicity by Kollár, and de Fernex, Ein and Mustață.

**Theorem A.9** ([20], [6], [7, Proposition 2.20]). **Let $Q \in Y$ be a germ of an lc variety and set $\bar{Y} = \Spec \bar{O}_{Y,Q}$ with closed point $\bar{Q}$. Let $a = \prod_j a_{ij}^r$ be an $\mathcal{R}$-ideal on $\bar{Y}$. Suppose $\mld_{\bar{Q}}(\bar{Y}, a) = 0$ and it is computed by $\bar{E} \in \mathcal{D}_{\bar{Y}}$. If an $\mathcal{R}$-ideal $b = \prod_j \bar{b}_j^r$ on $\bar{Y}$ satisfies $\bar{a}_j + p_j = b_j + p_j$ for all $j$, where $p_j = \{u \in \bar{O}_{\bar{Y}} \mid \ord_{\bar{E}} u > \ord_{\bar{E}} a_j\}$, then $\mld_{\bar{Q}}(\bar{Y}, b) = 0$.

**Corollary A.10.** **In Lemma A.8, if $\mld_{\bar{f}_i}(X_K, \prod_j a_{ij}^r) = 0$, then $\mld_{g}(X, \prod_j a_{ij}^r) = 0$ for any $i \in I_1$ on a subfamily. In particular, if $(X_K, \prod_j a_{ij}^r)$ is lc, then so is $(X, \prod_j a_{ij}^r)$.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
E-mail address: masayuki@kurims.kyoto-u.ac.jp