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A CONNECTEDNESS THEOREM OVER THE SPECTRUM OF A FORMAL POWER SERIES RING

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ABSTRACT. We study the connectedness of the non-klt locus over the spectrum of a formal power series ring. In dimension 3, we prove the existence and normality of the smallest lc centre, and apply it to the ACC for minimal log discrepancies greater than 1 on smooth 3-folds.

1. INTRODUCTION

The vanishing theorem by Kodaira [18] is one of the most basic tools in algebraic geometry in characteristic zero. It is reasonable to expect a vanishing theorem on excellent schemes, but it is annoyingly unknown besides the work on surfaces by Lipman [22]. Precisely, we are interested in the relative Kodaira vanishing for a birational morphism over the spectrum of a formal power series ring $R = \mathbb{K}[[x_1, \ldots, x_d]]$ for a field $\mathbb{K}$ of characteristic zero. We mean by an $R$-variety an integral separated scheme of finite type over $\text{Spec } R$.

Conjecture 1.1. Let $f : Y \to X$ be a projective birational morphism of regular $R$-varieties and $L$ an $f$-ample divisor on $Y$. Then $R^i f_* \mathcal{O}_Y(K_Y/X + L) = 0$ for $i \geq 1$. Here the relative canonical divisor $K_Y/X$ is defined by the $0$-th Fitting ideal of $\Omega_{Y/X}$.

We shall not deal with this algebraic conjecture. Instead, we study the connectedness lemma by Shokurov [29] and Kollár [19], which is an important geometric application of the vanishing theorem in birational geometry. It claims for a proper morphism $f : Y \to X$, the fibrewise connectedness of the non-klt locus of a subpair $(Y, \Delta)$ such that $\Delta$ is effective outside a locus in $X$ of codimension at least 2 and such that $-(K_Y + \Delta)$ is $f$-nef and $f$-big. We shall verify it for a germ at a regular point of $X$ in the case when $f$ is isomorphic outside the central fibre (Theorem 3.1). Investigating further in dimension 3, we obtain a desirable result on the smallest lc centre of a pair on a regular $R$-variety of dimension 3.

Theorem 1.2. Let $P \in (X, \mathfrak{a})$ be a germ of an lc but not klt pair of a regular $R$-variety $X$ of Krull dimension 3 and an $R$-ideal $\mathfrak{a}$ on $X$. Then the smallest lc centre of $(X, \mathfrak{a})$ exists and it is normal.

We reduce to the case $X = \text{Spec } R$ with $R$ an algebraically closed field $k$. Theorem 3.1, the fibrewise connectedness, is proved by approximating the effective $R$-divisor $f_* \Delta$ by an $m$-primary $R$-ideal $\mathfrak{a}(l)$, where $m$ is the maximal ideal sheaf, such that the non-klt locus of the subtriple coming from $\mathfrak{a}(l)$ coincides with the central fibre of the original non-klt locus. The $\mathfrak{a}(l)$ is descended to $\mathbb{A}_k^d = \text{Spec } k[x_1, \ldots, x_d]$, on which the connectedness lemma is applied. The existence of the smallest lc centre in Theorem 1.2 is a corollary to Theorem 3.1. The hardest part of Theorem 1.2 is the normality of the smallest lc centre $C$ which is a curve. We construct an ideal
sheaf \( n_a \) on the normalisation \( C_Y \) of \( C \) with \( f_C: C_Y \to C \) which satisfies \( f_{C*}n_a \subset \mathcal{O}_C \) and \( \mathcal{O}_C / f_{C*}n_a \cong f_{C*}\mathcal{O}_{C_Y} / f_{C*}n_a \). Then we obtain the isomorphism \( \mathcal{O}_C \cong f_{C*}\mathcal{O}_{C_Y} \) meaning the normality of \( C \).

Our motivation for excellent schemes stems from the notion of a generic limit of ideals due to de Fernex and Musta˘a [8]. The generic limit was used to prove the ascending chain condition (ACC) for log canonical thresholds on smooth varieties [7], the approach of which works even for the study of minimal log discrepancies [16]. We shall apply Theorem 1.2 to the ACC conjecture for minimal log discrepancies by Shokurov [28], [30] and Cascini, McKernan [24] in the case of smooth 3-folds, and settle the part of minimal log discrepancies greater than 1.

**Theorem 1.3.** Fix subsets \( I \subset (0, \infty) \) and \( J \subset (1, 3] \) both of which satisfy the descending chain condition. Then there exist finite subsets \( I_0 \subset I \) and \( J_0 \subset J \) such that if \( P \in (X, a = \prod_j a_j^{r_j}) \) is a germ of a pair of a smooth variety \( X \) of dimension 3 and an \( \mathbb{R} \)-ideal \( a \) on \( X \) with all \( a_j \) non-trivial at \( P \), all \( r_j \in I \) and \( \text{mld}(X, a) \in J \), then all \( r_j \in I_0 \) and \( \text{mld}(X, a) \in J_0 \).

The generic limit \( a \) of \( \mathbb{R} \)-ideals \( a_i \) on \( P \in X = \text{Spec} k[[x_1, \ldots, x_d]] \) is an \( \mathbb{R} \)-ideal on \( P_k = X_k = \text{Spec} k[[x_1, \ldots, x_d]] \) with a field extension \( k \) of \( k \). The ACC for minimal log discrepancies on smooth \( d \)-folds is reduced to the stability \( \text{mld}(P_k, a) = \text{mld}(X, a) \) for general \( i \). We prove it when \( (X_k, a) \) is a klt pair, or even a plt pair whose lc centre has an isolated singularity, by our previous arguments [14], [15]. In dimension 3, only the case when \( (X_k, a) \) has the smallest lc centre of dimension 1 remains. In this case, the estimate \( \text{mld}(P_k, a) \leq 1 \) is derived from Theorem 1.2, which is enough to prove Theorem 1.3.

The structure of the paper is as follows. After reviewing the basics of singularities in Section 2, we study the connectedness of the non-klt locus and establish Theorem 1.2 in Section 3. We discuss the ACC for minimal log discrepancies from the point of view of generic limits in Section 4. The stability of minimal log discrepancies in the klt and plt cases is shown in Section 5. Theorem 1.3 is completed in Section 6. The appendix exposing generic limits is attached.

Throughout this paper, \( k \) is an algebraically closed field of characteristic zero.

**Remark 1.4.** Recently, Chatzistamatiou and Rülling proved that the higher direct images of the structure sheaf vanish for a projective birational morphism of regular excellent schemes [4].

**2. SINGULARITIES**

We review the basics of singularities in birational geometry. A good reference is [21]. A **variety** is an integral separated scheme of finite type over Spec \( k \). A **germ** of a scheme is considered at a closed point. The **dimension** of a scheme means the Krull dimension.

An **\( \mathbb{R} \)-ideal** on a noetherian scheme \( X \) is a formal product \( a = \prod_j a_j^{r_j} \) of finitely many coherent ideal sheaves \( a_j \) on \( X \) with positive real exponents \( r_j \). The \( a \) to the power of \( t > 0 \) is \( a^t := \prod_j a_j^{r_j/ t} \). The **cosupport** \( \text{Cosupp} \ a \) of \( a \) is the union of all \( \text{Supp} \ a / a_j \). The **pull-back** of \( a \) by a morphism \( Y \to X \) is \( a\mathcal{O}_Y := \prod_j (a_j\mathcal{O}_Y)^{r_j} \). The \( \mathbb{R} \)-ideal \( a \) is said to be **invertible** if all \( a_j \) are invertible. In this case, if in addition \( X \) is normal, then the **\( \mathbb{R} \)-divisor** \( A = \sum_j r_j A_j \) with \( a_j = \mathcal{O}_X(-A_j) \) is called the **\( \mathbb{R} \)-divisor defined by \( a \).**
Let $Z$ be an irreducible closed subset of $X$. We write $\eta_Z$ for the generic point of $Z$. The order of $a$ along $Z$ is $\text{ord}_Z a = \sum_j r_j \text{ord}_a J_j$, where $\text{ord}_Z a_j$ is the maximal $v \in \mathbb{N} \cup \{+\infty\}$ satisfying $a_j O_{X, \eta_Z} \subset I_Z^v O_{X, \eta_Z}$ for the ideal sheaf $I_Z$ of $Z$.

We treat a triple $(X, \Delta, a)$ which consists of a normal variety $X$, an effective $R$-divisor $\Delta$ on $X$ such that $K_X + \Delta$ is an $R$-Cartier $R$-divisor, and an $R$-ideal $a = \prod_j a_j$ on $X$. A prime divisor $E$ on a normal variety $Y$ with a birational morphism $f : Y \to X$ is called a divisor over $X$, and the closure $\overline{f(E)}$ of the image on $X$ is called the centre of $E$ on $X$ and denoted by $c_X(E)$. We denote by $\mathcal{D}_X$ the set of all divisors over $X$. The log discrepancy of $E$ with respect to $(X, \Delta, a)$ is

$$a_E(X, \Delta, a) := 1 + \text{ord}_E K_Y/(X, \Delta) - \text{ord}_E a,$$

where $K_Y/(X, \Delta) := K_Y - f^*(K_X + \Delta)$ and $\text{ord}_E a := \text{ord}_a O_Y$. Note that $c_X(E)$ and $a_E(X, \Delta, a)$ are determined by the valuation on the function field of $X$ given by $E$.

For an irreducible closed subset $Z$ of $X$, the minimal log discrepancy of $(X, \Delta, a)$ at $\eta_Z$ is

$$\text{mld} Z(X, \Delta, a) := \inf \{ a_E(X, \Delta, a) \mid E \in \mathcal{D}_X, c_X(E) = Z \}.$$

It is either a non-negative real number or $-\infty$. We say that $E \in \mathcal{D}_X$ computes $\text{mld} Z(X, \Delta, a)$ if $c_X(E) = Z$ and $a_E(X, \Delta, a) = \text{mld} Z(X, \Delta, a)$ (or is negative when $\text{mld} Z(X, \Delta, a) = -\infty$). We often reduce to the case when $Z$ is a closed point by the relation $\text{mld} Z(X, \Delta, a) = \text{mld}_P(X, \Delta, a) - \dim Z$ for a general closed point $P \in Z$ (cf. [3, Proposition 2.1]).

The triple $(X, \Delta, a)$ is said to be log canonical (lc) (resp. Kawamata log terminal (klt)) if $a_E(X, \Delta, a) \geq 0$ (resp. $> 0$) for all $E \in \mathcal{D}_X$. It is said to be purely log terminal (plt) (resp. canonical, terminal) if $a_E(X, \Delta, a) > 0$ (resp. $\geq 1, > 1$) for all exceptional $E \in \mathcal{D}_X$. The log canonicity of $(X, \Delta, a)$ about $P \in X$ is equivalent to $\text{mld}_P(X, \Delta, a) \geq 0$. Let $Y$ be a normal variety with a birational morphism to $X$. A centre $c_Y(E)$ with $a_E(X, \Delta, a) \leq 0$ is called a non-klt centre on $Y$ of $(X, \Delta, a)$. The union of all non-klt centres on $Y$ is called the non-klt locus on $Y$ and denoted by $\text{Nklt}_Y(X, \Delta, a)$. When we say just a non-klt centre or the non-klt locus, we mean that it is on $X$.

A log resolution of $(X, \Delta, a)$ is a projective morphism $f : Y \to X$ from a regular variety $Y$ such that

(i) $\text{Exc} f$ is a divisor and $aO_Y$ is invertible,

(ii) $\text{Exc} f \cup \text{Supp} \Delta_Y \cup \text{Cosupp} aO_Y$ is a simple normal crossing (snc) divisor, where $\Delta_Y$ is the strict transform of $\Delta$, and

(iii) $f$ is isomorphic on the locus $U$ in $X$ with $U$ regular, $a|_U$ invertible and $\text{Supp} \Delta|_U \cup \text{Cosupp} a|_U$ snc.

A stratum (resp. an open stratum) of an snc divisor $\sum_{i \in I} E_i$ is an irreducible component of $\bigcap_{i \in I} E_i$ (resp. $\bigcap_{i \in I} E_i \setminus \bigcup_{j \in J} E_i$) for a subset $J$ of $I$.

By allowing a not necessarily effective $R$-divisor $\Delta$, one can consider a subtriple $(X, \Delta, a = \prod_j a_j')$. The notions of log discrepancies and lc/klt singularities are extended for subtriples. Let $f : Y \to X$ be a birational morphism from a regular variety $Y$ such that $\text{Exc} f$ is a divisor $\sum_{i \in I} E_i$. The weak transform on $Y$ of $a$ is the $R$-ideal $a_Y = \prod_j a_j f'$ with $a_Y = a_j f' \sum_i (\text{ord}_E a_j') E_i$.

**Definition 2.1.** Notation as above. The pull-back of $(X, \Delta, a)$ by $f$ is the subtriple $(Y, \Delta_Y, a_Y)$ where $\Delta_Y = -K_Y/(X, \Delta) + \sum_{i,j} (r_j \text{ord}_E a_j) E_i$. 


We have $a_E(X,\Delta,a) = a_E(Y,\Delta_Y,a_Y)$ for any $E \in \mathcal{P}$. In particular when $f$ is proper, $(X,\Delta,a)$ is lc (resp. klt) if and only if so is $(Y,\Delta_Y,a_Y)$. We use the notation $\text{Nckt}(X,\Delta,a)$ also for a subtriple $(X,\Delta,a)$.

These definitions are extended on schemes over a field $K$ of characteristic zero and even over a formal power series ring $R = K[[x_1, \ldots, x_d]]$ by the existence of log resolutions due to Hironaka [12] and Temkin [32], [33]. This extension is studied by de Fernex, Ein and Mustață [7], [8]. We mean by an $R$-variety an integral separated scheme of finite type over $\text{Spec}\, R$.

The canonical divisor $K_X$ on a normal $R$-variety $X$ is defined by the isomorphism $\mathcal{O}_X(K_X)|_U \cong \bigwedge^r \Omega_X|_U$ on the regular locus $U$ of $X$, where $\Omega_X/K$ is the sheaf of special differentials in [7] and $r$ is its rank. The relative canonical divisor is well understood for a birational morphism of regular $R$-varieties.

**Lemma 2.2** ([7, Remark A.12]). Let $Y \to X$ be a proper birational morphism of regular $R$-varieties. Then $K_{Y/X}$ is the effective divisor defined by the $0$-th Fitting ideal of $\Omega_Y/K$. In particular, $K_{Y/X}$ is independent of the structure of $X$ as an $R$-variety.

The log discrepancies are preserved by field extensions and completions.

**Corollary 2.3.** Let $Y \to X$ be as in Lemma 2.2. Take an $R'$-variety $X'$ as in (i), (ii) or (iii) below and set a morphism $Y' = Y \times_X X' \to X'$ of $R'$-varieties.

(i) $X'$ is a component of $X \times_{\text{Spec}\, R} \text{Spec}\, R'$ with $R' = R \otimes_K K'$ for a field extension $K'$ of $K$.

(ii) $X' = \text{Spec}\, \mathcal{O}_{X,P}$ for a germ $P \in X$, which admits the structure of an $R'$-variety for a suitable $R' = K'[[x_1, \ldots, x_d]]$ by Cohen’s structure theorem [5].

(iii) $X' = X$ with another structure morphism $X \to \text{Spec}\, R'$.

Then $K_{Y/X'}$ is the pull-back of $K_{Y/X}$. In particular, for an $R$-ideal $a$ on $X$, a divisor $E$ over $X$ and a germ $P \in X$, one has $a_E(X',a\mathcal{O}_{X'}) = a_E(X,a)$ for a component $E'$ of $E \times_X X'$ and $\text{mld}_{P'}(X',a\mathcal{O}_{X'}) = \text{mld}_P(X,a)$ for a point $P'$ of $P \times_X X'$.

This is by the regularity of the morphism $X' \to X$. The cases (i) and (ii) for $R = K$ are stated in [7, Lemma 2.14, Propositions A.11, A.14] even for a normal (Q-Gorenstein) $K$-variety $X$.

Suppose that $(X,\Delta,a)$ is an lc triple. Then a non-klt centre (on $X$) of $(X,\Delta,a)$ is often called an lc centre. An lc centre which is minimal with respect to inclusions is called a minimal lc centre. When we work over a germ $P \in X$, the following definition makes sense.

**Definition 2.4.** Let $P \in (X,\Delta,a)$ be a germ of an lc triple. The smallest lc centre is an lc centre of $(X,\Delta,a)$ contained in every lc centre.

If $X$ is a variety, then the smallest lc centre exists and it is normal [9, Theorem 9.1]. It is, however, unknown for $R$-varieties. Theorem 1.2 states that this is the case when $X$ is a regular $R$-variety of dimension $3$.

3. THE SMALLEST LC CENTRE ON A THREEFOLD

This section is devoted to the proof of Theorem 1.2. We work over a germ $P \in X$ of an $R$-variety with $R = K[[x_1, \ldots, x_d]]$. The maximal ideal sheaf of $P \in X$ is denoted by $m$. When we discuss on the spectrum of a noetherian ring, we identify an ideal in the ring with its coherent ideal sheaf.
3.A. **A connectedness theorem.** We start with a connectedness theorem over $X$, Theorem 3.1. Though we impose the strong condition that $f$ is isomorphic outside $P$, this theorem is sufficient in dimension 3 in order to derive the existence of the smallest lc centre, which will be seen in Subsection 3.C.

**Theorem 3.1.** Let $P \in (X, a)$ be a germ of a pair on a regular $R$-variety $X$ and $f : Y \to X$ a proper birational morphism of regular $R$-varieties which is isomorphic outside $P$. Let $\Delta$ be an $\mathbb{R}$-divisor on $Y$ with $f, \Delta \geq 0$ such that $-(K_Y + \Delta)$ is $f$-nef. Then $\operatorname{Nkl}_{fY}(Y, \Delta, aO_Y) \cap f^{-1}(P)$ is connected.

We extract the case $\Delta = -K_Y/X$.

**Corollary 3.2.** Let $P \in (X, a)$ be a germ of a pair on a regular $R$-variety $X$ and $f : Y \to X$ a proper birational morphism of regular $R$-varieties which is isomorphic outside $P$. Then $\operatorname{Nkl}_Y(Y, a) \cap f^{-1}(P)$ is connected.

The statement for $R = k$ is a special case of the connectedness lemma by Shokurov and Kollár [19, Theorem 17.4]. Their lemma can settle Theorem 3.1 in the case when $a$ is $m$-primary and $\Delta$ is $f$-exceptional. Write $a = \prod_j a_j$.

**Lemma 3.3.**

(i) In order to prove Theorem 3.1, one may assume that $X = \text{Spec} R$ with $K = k$, $f$ is projective and $\Delta$ is $f$-exceptional.

(ii) Theorem 3.1 holds in the case when $X = \text{Spec} R$ with $K = k$, $f$ is projective, $\Delta$ is $f$-exceptional and all $a_j$ are $m$-primary ideals.

**Proof.** (i) Take an isomorphism $\mathcal{O}_{X, P} \cong K''[[x_1, \ldots, x_d]]$ with $K' = \mathcal{O}_{X, P}/m$ by Cohen’s structure theorem and set $R' = k[[x_1, \ldots, x_d]]$ for the algebraic closure $k$ of $K'$. Because the base change $\text{Spec} R' \to X$ commutes with taking the non-klt locus by Corollary 2.3, we may assume $X = \text{Spec} R$ with $K = k$ (the $d$ may be changed). By the flattening theorem of Raynaud and Gruson [26, Théorème 1ère 5.2.2], there exists a projective morphism $f' : Y' \to X$ from a regular $R$-variety $Y'$ which is isomorphic outside $P$ and factors through $f$. Replacing $(Y, \Delta)$ with its pull-back on $Y'$, we may assume that $f$ is projective. The $\Delta' := \Delta - f^* f_\Delta$ is $f$-exceptional. Take an invertible $\mathbb{R}$-ideal $d$ on $X$ which defines the $\mathbb{R}$-divisor $f_\Delta \geq 0$. Then $\operatorname{Nkl}_{fY}(Y, \Delta, aO_Y) = \operatorname{Nkl}_{fY}(Y, \Delta', aO_Y)$. Replacing $\Delta$ with $\Delta'$ and $a$ with $a d$, we may assume that $\Delta$ is $f$-exceptional.

(ii) We use the notation $\bar{R} = \bar{k}[x_1, \ldots, x_d]$ and $\bar{A}_m^d = \text{Spec} \bar{R}$ with origin $\bar{P}$. By Proposition A.7, $f$ is the base change of a projective morphism $\bar{f} : \bar{Y} \to \bar{A}_m^d$ and $a$ is the pull-back of the $\bar{R}$-ideal $\bar{a} = \prod_j (a_j \cap \bar{R})/\bar{R}$. Then $f^{-1}(P) \cong \bar{f}^{-1}(P)$ and $\Delta$ is the base change of an $\bar{f}$-exceptional $\mathbb{R}$-divisor $\bar{\Delta}$ such that $-(K_{\bar{Y}} + \bar{\Delta})$ is $\bar{f}$-nef. Thus $f^{-1}(P) \supset \operatorname{Nkl}_{fY}(Y, \Delta, aO_Y) \cong \operatorname{Nkl}_{fY}(Y, \bar{\Delta}, \bar{a}O_{\bar{Y}})$, which is connected by [19, Theorem 17.4].

For Theorem 3.1, we take a log resolution $q : W \to Y$ of $(Y, \Delta, aO_Y)$ and set the composition $g = f \circ q : W \to X$ as below.

\[
\begin{array}{ccc}
W & \xrightarrow{q} & Y \\
\downarrow g & & \downarrow f \\
\downarrow & & \\
X & & \\
\end{array}
\]

We fix $\epsilon > 0$ such that

\[F := \operatorname{Nkl}_W(Y, \Delta, aO_Y) = \operatorname{Nkl}_W(Y, \Delta, a^{1+\epsilon}O_Y).
\]
Lemma 3.5. That a in (3). Note ord $D$ has codim $l$. There exists

By (1), (2) and the assumption, $N_{klt} f^{-1}(P) = N_{klt} Y, a(l) O_Y$ in order to apply Lemma 3.3(ii).

Since $F$ and $g^{-1}(P)$ are divisors contained in an snc divisor, any irreducible component $D$ of $F \cap g^{-1}(P)$ has codim$_W D = 1$ or 2. Suppose codim$_W D = 2$. Let $E_D \subset g^{-1}(P)$ and $F_D \subset F$ be the unique prime divisors such that $D \subset E_D \cap F_D$. We build a tower of blow-ups

$W_i \to W_{i-1} \to \cdots \to W_0 = W$

as follows. Set $W_0 := W$, $E_0 := E_D$ and $F_0 := F_D$. We construct inductively the blow-up $g_i: W_i \to W_{i-1}$ along $D$ for $i = 1$ (resp. along $E_{i-1} \cap F_{i-1}$ for $i \geq 2$), and set $E_i$ as the exceptional divisor of $g_i$, and $F_i$ as the strict transform on $W_i$ of $F_D$. The composition $g_1 \circ \cdots \circ g_i$, is denoted by $h_i: W_i \to W$.

Lemma 3.4. (i) $a_{E_i}(Y, a(l), a^{1+\varepsilon} O_Y) \leq a_{E_0}(Y, a(l), a^{1+\varepsilon} O_Y) - i \epsilon$ for all $a \in \mathbb{N}$.

Proof. (i) is just a computation using $a_{E_0}(Y, a(l), a^{1+\varepsilon} O_Y) \leq 0$. The (ii) is from $h_i, O_W(-aE_i) \subset O_W(-aE_D) \cap (-F_D)$ for any $a \in \mathbb{N}$.

Lemma 3.5. Suppose that $(Y, \Delta)$ is klt outside $f^{-1}(P)$. Then there exists $l$ such that $N_{klt} (Y, a(l) O_Y) = F \cap g^{-1}(P)$.

Proof. By (1), (2) and the assumption, $N_{klt} Y, a(l) O_Y \subset F \cap g^{-1}(P)$ for any $l$. Thus it suffices to prove that for every irreducible component $D$ of $F \cap g^{-1}(P)$, there exists $l_D$ such that $D$ is a non-klt centre on $W$ of $(Y, a(l) O_Y)$ for any $l \geq l_D$. $D$ has codim$_W D = 1$ or 2. If codim$_W D = 1$, then we may take any $l_D$ such that $l_D \text{ord}_D m \geq \text{ord}_D a_j$ for all $j$. If codim$_W D = 2$, then we take the tower of blow-ups in (3). Note $\text{ord}_D a > 0$. By Lemma 3.4(i), we have $a_{E_i}(Y, a(l), a^{1+\varepsilon} O_Y) \leq 0$ whenever $a_{E_i}(Y, a(l), a^{1+\varepsilon} O_Y) \leq 0$ whenever $a_{E_i}(Y, a(l), a^{1+\varepsilon} O_Y) \leq 0$ for all $j$. Then for $l \geq l_D$, $a_{E_i}(Y, a(l) O_Y) = a_{E_i}(Y, a(l) O_Y) \leq 0$, so $D = c_{W}(E_i)$ is a non-klt centre on $W$ of $(Y, a(l) O_Y)$. q.e.d.

Proof of Theorem 3.1. After the reduction in Lemma 3.3(i), we take $l$ in Lemma 3.5. Then

$N_{klt} (Y, a(l) O_Y) \cap f^{-1}(P) = q(F \cap g^{-1}(P))$

$= q(N_{klt} Y, a(l) O_Y) = N_{klt} (Y, a(l) O_Y)$. q.e.d.

In our proof of Theorem 3.1, we do not know a relative vanishing for $g: W \to X$. Instead, consider a log resolution $f_j: Y_j \to X$ of $(X, a(l) m)$ which factors through $f$, and let $p_j: Y_j \to Y$ be the induced morphism as below. The $l$ is not fixed here.
The $f_l$ is isomorphic outside $P$. Let $(Y_l, \Delta_l, \cO_{Y_l})$ be the pull-back of $(X, 0, a(l))$. Then we have a vanishing involving $\Delta_l$, which will be used in Subsection 3.C.

**Lemma 3.6.** Let $f_l = f \circ p_l : Y_l \to Y \to X$ be as above. Write $[-\Delta_l] = P_l - N_l$ by effective divisors $P_l$ and $N_l$ with no common divisors. Then

$$R^1 f_! (p_* \cO_{Y_l}(-N_l)) = 0.$$ 

**Proof.** The sheaf $R^0 f_! (p_* \cO_{Y_l}(-N_l))$ is supported in $P$. Set $\cO_{X, P} \simeq K[[x_1, \ldots, x_d]]$ and $R' = k[[x_1, \ldots, x_d]]$ for the algebraic closure $k$ of $K'$, then $R'$ is faithfully flat over $\cO_{X, P}$. Hence taking the base change to $\text{Spec} R'$, one can reduce to the case $X = \text{Spec} R$ with $K = k$ by [10, Proposition III.1.4.15] and Corollary 2.3. By Proposition A.7, $f_1$ is the base change of a projective morphism $\tilde{f}_1 : \tilde{Y}_1 \to \mathbb{A}^d_k$. The $a(l)$ is the pull-back of an $\mathbb{R}$-ideal $\tilde{a}(l)$ on $\mathbb{A}^d_k$ and $\Delta_l$ is the base change of the $\mathbb{R}$-divisor $\Delta_\tilde{l}$ on $\tilde{Y}_1$ such that $(\tilde{Y}_1, \Delta_\tilde{l}, \tilde{\cO}_{\tilde{Y}_1})$ is the pull-back of $(\mathbb{A}^d_k, 0, \tilde{a}(l))$.

Kawamata–Viehweg vanishing theorem [17], [35] implies $R^1 \tilde{f}_! \tilde{\cO}_{\tilde{Y}_1}([\Delta_\tilde{l}]) = 0$. Since $X \to \mathbb{A}^d_k$ is flat, this is base-changed to $R^1 f_! \cO_{Y_l}([\Delta_l]) = 0$ by [10, Proposition III.1.4.15]. Thus, applying $f_!$, to the exact sequence

$$0 \to \cO_{Y_l}(P_l - N_l) \to \cO_{Y_l}(P_l) \to \cO_{N_l}(P|_{N_l}) \to 0,$$

we obtain the surjection $\cO_X = f_! \cO_{Y_l}(P_l) \twoheadrightarrow f_! \cO_{N_l}(P|_{N_l})$. This homomorphism is factored as $\cO_X \to f_! \cO_{Y_l} \twoheadrightarrow f_! \cO_{N_l}(P|_{N_l})$, so we have the surjection $\cO_X \to f_! \cO_{N_l}$. Moreover, we have the base change $R^1 f_! \cO_{Y_l} = 0$ of the vanishing $R^1 f_! \cO_{\tilde{Y}_1} = 0$ [12, p.144 (2)]. Hence applying $f_!$ to the exact sequence

$$0 \to \cO_{Y_l}(-N_l) \to \cO_{Y_l} \to \cO_{N_l} \to 0,$$

we obtain $R^1 f_! \cO_{Y_l}(-N_l) = 0$.

Leray spectral sequence $R^p f_! (R^q p_* \cO_{Y_l}(-N_l)) \Rightarrow R^{p+q} f_! \cO_{Y_l}(-N_l)$ gives an injection $R^1 f_! (p_* \cO_{Y_l}(-N_l)) \hookrightarrow R^1 f_! \cO_{Y_l}(-N_l)$, so $R^1 f_! (p_* \cO_{Y_l}(-N_l)) = 0$. q.e.d.

**3.B. Propositions in an arbitrary dimension.** We prepare two auxiliary propositions which can be stated independently of Theorem 1.2.

It is easy to see that a minimal lc centre of codimension 1 is normal.

**Proposition 3.7.** Let $(X, a)$ be a pair on a regular $R$-variety $X$, and $S$ the union of all non-klt centres of codimension 1 of $(X, a)$. Then every irreducible component of the non-normal locus of $S$ is a non-klt centre of $(X, a)$.

**Proof.** $S$ is Cohen–Macaulay since $S$ is a Cartier divisor on a regular scheme $X$. Thus any irreducible component $C$ of the non-normal locus of $S$ has codim$_S C = 2$ and $\text{mult}_S S \leq 2$. Let $E$ be the divisor over $X$ obtained at $a_C$ by the blow-up of $X$ along $C$. Then $a_E(X, a) = 2 - \text{ord}_E a \leq 2 - \text{mult}_S S \leq 0$, so $C = c_X(E)$ is a non-klt centre of $(X, a)$.

We can perturb $a$ to reduce to the case when every lc centre is minimal.

**Proposition 3.8.** Let $(X, a)$ be an lc pair on a klt $R$-variety $X$. Then there exists an $\mathbb{R}$-ideal $a'$ forming an lc pair $(X, a')$ such that a minimal lc centre of $(X, a)$ is an lc centre of $(X, a')$ and vice versa.

**Proof.** Let $\{Z_i\}$ be the set of all minimal lc centres of $(X, a = \prod_i a_i')$. For each $Z_i$, fix $E_i \in \cD_X$ computing $\text{ml}_{\cO_{Z_i}}(X, a) = 0$. Let $\cI_Z$ be the ideal sheaf of $Z = \bigcup_j Z_i$, and take an integer $l$ such that $\text{ord}_{E_i} \cI_Z \geq \text{ord}_{E_j} a_j$ for all $i, j$. Then $(X, a' := \prod_i a_i'^{|l_i|})$
\[ \prod_j (a_j + \mathcal{I}_j^{(j)}) \text{ is lc, and } Z_i \text{ is an lc centre of } (X, a') \text{ by ord}_{E_i} a' = \text{ord}_{E_i} a. \] On the other hand, every lc centre of \((X, a')\) is an lc centre of \((X, a)\) contained in Cosupp\(a = Z_i\), so it equals some \(Z_i\).

The existence of the smallest lc centre is a consequence of Corollary 3.2.

Proof of the existence of the smallest lc centre. Let \(\{Z_i\}\) be the set of all lc centres of \((X, a)\), which are assumed to be minimal. Proposition 3.7 implies that \(Z = \bigcup_i Z_i\) is regular outside \(P\). Thus we have an embedded resolution \(f: Y \rightarrow X\) of singularities of \(Z\), in which \(f\) is isomorphic outside \(P\) and induces \(f_Z: \bigcup_i Z_iY \rightarrow Z\) for the strict transform \(Z_iY\) of \(Z_i\). By Corollary 3.2, \(f_Z^{-1}(P) = \text{Nkl}_{Y}(X, a) \cap f^{-1}(P)\) is connected, that is, there exists only one lc centre of \((X, a)\).

Remark 3.9. The above proof shows that if \(Z\) is the smallest lc centre of \((X, a)\), then its normalisation \(Z' \rightarrow Z\) is a homeomorphism.

To complete Theorem 1.2, we must prove that the unique lc centre of \((X, a)\) is normal. (Since we have assumed that every lc centre is minimal, the existence of the smallest lc centre means the uniqueness of lc centre.) If it is of dimension 2, then it is normal by Proposition 3.7. Thus, we may assume that \((X, a)\) has the unique lc centre \(C\) which is of dimension 1.

We have an embedded resolution \(f: Y \rightarrow X\) of singularities of \(C\), in which \(f\) is isomorphic outside \(P\) and induces the normalisation \(f_C: C_Y \rightarrow C\) for the strict transform \(C_Y\) of \(C\). Note that \(f_C^{-1}(P)\) consists of one point, say \(P_Y\), by Remark 3.9.

We let \(n\) denote the maximal ideal sheaf of \(P_Y \in Y\). Then we take a log resolution \(q: W \rightarrow Y\) of \((Y, a\mathcal{O}_Y \cdot n)\) and set the composition \(g = f \circ q: W \rightarrow X\). We have the following diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{q} & Y \\
\downarrow g & & \downarrow f \\
X & \xrightarrow{f} & C \\
\end{array}
\]

The normality of \(C\) is equivalent to the isomorphism \(\mathcal{O}_C \cong f_{C*}\mathcal{O}_{C_Y}\). We shall see this by constructing an ideal sheaf \(n_a\) on \(C_Y\) which satisfies \(f_{C*}n_a \subset \mathcal{O}_C\) and \(\mathcal{O}_C/f_{C*}n_a \cong f_{C*}\mathcal{O}_{C_Y}/f_{C*}n_{a_Y}\).

We fix \(\epsilon\) in (1) for \(\Delta = -K_{Y/X}\), that is, \(F = \text{Nkl}_{Y}(X, a) = \text{Nkl}_{Y}(X, a^{1+\epsilon})\). For the \(a(l)\) in (2), we consider a log resolution \(f_l: Y_l \rightarrow X\) of \((X, a(l)m)\) which factors through \(f\) as \(f_l = f \circ p_l\). We extend Lemma 3.6.

Lemma 3.10. Let \(f\) and \(f_l = f \circ p_l\) be as above. Then for an arbitrary ideal sheaf \(\mathcal{I}\) on \(Y\) containing \(p_{l*}\mathcal{O}_{Y_l}(-N_l)\), with \(N_l\) in Lemma 3.6, one has \(R^1 f_{*}\mathcal{I} = 0\).

Proof. By (1) for \(\Delta = -K_{Y/X}\) and (2), we see \(p_{l*}(\text{Supp}\ N_l) = \text{Nkl}_{Y}(X, a(l)) \subset q(F \cap g^{-1}(P)) = C_Y \cap f^{-1}(P) = P_Y\), whence the cokernel \(\mathcal{I}\) of the natural injection \(p_{l*}\mathcal{O}_{Y_l}(-N_l) \rightarrow \mathcal{I}\) is a skyscraper sheaf. In particular, \(R^1 f_{*}\mathcal{I} = 0\). Apply \(f_l\) to the exact sequence

\[
0 \rightarrow p_{l*}\mathcal{O}_{Y_l}(-N_l) \rightarrow \mathcal{I} \rightarrow \mathcal{I}/\mathcal{I}f \rightarrow 0.
\]

By Lemma 3.6 and \(R^1 f_{*}\mathcal{I} = 0\), we obtain \(R^1 f_{*}\mathcal{I} = 0\).
Since $C$ is the unique lc centre of $(X, a)$, every irreducible component of $F$ maps onto $C_Y$. Thus $F \cap q^{-1}(P_Y) \neq \emptyset$ and any irreducible component $D$ of $F \cap q^{-1}(P_Y)$ has dimension 1. We fix one such $D$, and let $E_D \subset q^{-1}(P_Y)$ and $F_D \subset F$ be the unique prime divisors such that $D \subset E_D \cap F_D$. We derive a vanishing for ideal sheaves on $Y$ close to that of $C_Y$.

**Lemma 3.11.** $R^1f_*(q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D))) = 0$ for any $a \in \mathbb{N}$.

**Proof.** Take the tower of blow-ups in (3). For fixed $a$, choose $i \in \mathbb{N}$ such that $aE_D(X, a^{i+\varepsilon}) \equiv 0 \bmod F_D$ and $a \leq -a$. Then Lemma 3.4 for $\Delta = -K_Y/X$ shows

$$h_\ast \mathcal{O}_W([aE(X, a^{i+\varepsilon})]_{E_i}) \subset h_\ast \mathcal{O}_W(-aE_i) \subset \mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D).$$

Take $l$ such that $l \bmod E_i m \geq \bmod E_i a_j$ for all $j$. Then,

$$aE(X, a^{i+\varepsilon}) = aE(X, a/l).$$

For this $l$, we take a log resolution $f_I: Y_I \to X$ of $(X, a/l)m$ which factors through $f$, such that $C_Y(E_i)$ is a divisor. Then by (4) and (5), one can apply Lemma 3.10 to $\mathcal{F} = q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D))$. q.e.d.

Now we set the ideal sheaf $n_a$ on $C_Y$ as

$$n_a := q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D)) \cdot \mathcal{O}_{C_Y}.$$

**Lemma 3.12.** There exists a such that $f_C, n_a \subset \mathcal{O}_C$.

**Proof.** Note that $n\mathcal{O}_{C_Y}$ is an invertible ideal sheaf on $C_Y$. Set $n = \bmod E_D n$, then

$$n_m \subset q_*(\mathcal{O}_D(-nE_D|F_D)) = n_m \mathcal{O}_{C_Y}$$

for any $l$. Take an $f$-exceptional divisor $A \geq 0$ on $Y$ such that $-A$ is $f$-ample and set $\mathcal{O}_{C_Y}(-A|_{C_Y}) = n_m \mathcal{O}_{C_Y}$. By Serre vanishing theorem [10, Théorème III.2.2.1], there exists $m_0$ such that $R^1f_*\mathcal{F}_{C_Y}(-mA) = 0$ for any $m \geq m_0$, where $\mathcal{F}_{C_Y}$ is the ideal sheaf of $C_Y$ on $Y$. Then we have the surjection $f_*\mathcal{F}_{C_Y}(-mA) \to f_C, \mathcal{O}_{C_Y}(-mA|_{C_Y}) = f_C, n_{m_0} \mathcal{O}_{C_Y}$, which provides

$$f_C, n_{m_0} \mathcal{O}_{C_Y} = f_*\mathcal{F}_{C_Y}(-mA) \cdot \mathcal{O}_C \subset \mathcal{O}_C.$$

Combining (6) and (7), we obtain $f_C, n_{m_0} \subset f_C, n_{m_0} \mathcal{O}_{C_Y} \subset \mathcal{O}_C$ for $m \geq m_0$. q.e.d.

**Proof of the normality of $C$.** Applying $f_*$ to the exact sequence

$$0 \to q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D)) \to \mathcal{O}_Y \to \mathcal{O}_{C_Y}/n_a \to 0$$

and using Lemma 3.11, we obtain the surjection $\mathcal{O}_X \to f_C, (\mathcal{O}_{C_Y}/n_a)$. This homomorphism is factored as

$$\mathcal{O}_X \to \mathcal{O}_C/f_C, n_a \cap \mathcal{O}_C \to f_C, \mathcal{O}_{C_Y}/f_C, n_a \to f_C, (\mathcal{O}_{C_Y}/n_a),$$

so we have an isomorphism $\mathcal{O}_C/f_C, n_a \cap \mathcal{O}_C \simeq f_C, \mathcal{O}_{C_Y}/f_C, n_a$. For $a$ in Lemma 3.12, it is $\mathcal{O}_C/f_C, n_a \simeq f_C, \mathcal{O}_{C_Y}/f_C, n_a$. Therefore $\mathcal{O}_C \simeq f_C, \mathcal{O}_{C_Y}$, meaning the normality of $C$.

Theorem 1.2 is established.

**Remark 3.13.** For $a$ one may prove the normality of $C$ by using Zariski’s subspace theorem [1, (10.6)]. One has an isomorphism $\mathcal{O}_C/f_C, n_a \cap \mathcal{O}_C \simeq f_C, (\mathcal{O}_{C_Y}/n_a)$ for any $a$. By (6), the family $\{n_a\}_a$ gives the $n\mathcal{O}_{C_Y}$-adic topology. Since the family $\{f_*, \mathcal{F}_{C_Y}(-mA)\}_m$ in the proof of Lemma 3.12 gives the $m$-adic topology by Zariski’s subspace theorem (cf. [13, Lemma...
Definition 4.3. Let finite was proved in [16].

Coef

Remark We say that a subset \( I \) of the point of view of generic limits.

Conjecture 4.4 (Shokurov [28], [30], Cascini, McKernan [24])

\[ \text{for the set which consists of all } \delta_i > 0 \text{ and all } r_j > 0 \text{ with } a_j \text{ non-trivial at } P. \]

Conjecture 4.4 by Cascini and McKernan is a generalisation of the original conjecture by Shokurov, which claims only the existence of \( J_0 \). When \( d = 2 \), the existence of \( J_0 \) was proved by Alexeev [2]. The motivation of this conjecture stems from the reduction by Shokurov [31] that the termination of flips follows from two conjectural properties of minimal log discrepancies: the ACC and the lower semi-continuity. For the purpose of the termination of flips, one may assume \( I \) in Conjecture 4.4 to be a finite set.

We consider Conjecture 4.4 with the assumption of the smoothness of \( X \). Then we may assume \( \Delta = 0 \) by absorbing \( \Delta \) to \( a \), since any divisor on \( X \) is a Cartier divisor.

Conjecture 4.4'. Fix \( d \in \mathbb{N} \) and subsets \( I \subset (0, \infty) \) and \( J \subset [0, \infty) \) both of which satisfy the DCC. Then there exist finite subsets \( I_0 \subset I \) and \( J_0 \subset J \) such that if \( P \in (X, a) \) is a germ of a pair on a smooth variety \( X \) of dimension \( d \) with \( \text{Coef}_P a \subset I \) and \( \text{mld}_P(X, a) \in J \), then \( \text{Coef}_P a \subset I_0 \) and \( \text{mld}_P(X, a) \in J_0 \).

Theorem 1.3 is Conjecture 4.4' for \( d = 3 \) with \( J \subset (1, 3] \). Conjecture 4.4' with \( I \) finite was proved in [16].
4.B. Reduction. We shall reduce Conjecture 4.4' to the stability of minimal log discrepancies in taking a generic limit of \( \mathbb{R} \)-ideals. We refer to Appendix A for the definition of a generic limit and the relevant notation: \( R = k[[x_1, \ldots, x_d]] \) with maximal ideal \( m \) and \( X = \text{Spec} R \) with closed point \( P \), and for a field extension \( K \) of \( k, R_K = K[[x_1, \ldots, x_d]] \) with maximal ideal \( m_K \) and \( X_K = \text{Spec} R_K \) with closed point \( P_K \).

**Conjecture 4.5** ([16, Conjecture 5.7]). Fix \( r_1, \ldots, r_e > 0 \). Let \( S = \{(a_1, \ldots, a_e)\}_{i \in I} \) be a collection of \( e \)-tuples of ideals in \( R = k[[x_1, \ldots, x_d]] \), and \( (a_1, \ldots, a_e) \) the generic limit of \( S \) defined in \( R_K \) with respect to a family \( \mathcal{F} = (Z_i, (\alpha_j(l))_{j}, l_i, s_i, t_i + 1)_{1 \leq l_0} \) of approximations of \( S \). Set \( a_i = \prod_j a_{ij}^j \) and \( a = \prod_j a_j^j \). Then after replacing \( \mathcal{F} \) with a subfamily but using the same notation,

\[ \text{mld}_{P_K}(X_K, a) = \text{mld}_{P}(X, a) \]

for any \( i \in I \), with \( l \geq l_0 \).

Conjecture 4.5 is closely related to the ideal-adic semi-continuity of minimal log discrepancies.

**Conjecture 4.6** (Mustată, cf. [14, Conjecture 2.5]). Let \( P \in X = \text{Spec} k[[x_1, \ldots, x_d]] \) and \( m \) be as above and \( a = \prod_j a_j^j \) an \( \mathbb{R} \)-ideal on \( X \). Then there exists an integer \( l \) such that if an \( \mathbb{R} \)-ideal \( b = \prod_j b_j^j \) on \( X \) satisfies \( a_j + m_j^l = b_j + m_j^l \) for all \( j \), then \( \text{mld}_{P}(X, a) = \text{mld}_{P}(X, b) \).

**Remark 4.7.** One inequality is easy in both conjectures. One has \( \text{mld}_{P_K}(X_K, a) \geq \text{mld}_{P}(X, a) \) in Conjecture 4.5 by Lemma A.8, and \( \text{mld}_{P}(X, a) \geq \text{mld}_{P}(X, b) \) in Conjecture 4.6 by [14, Remark 2.5.3]. In particular, these conjectures hold in the case when \( (X_K, a) \) (resp. \( (X, a) \)) is not lc.

**Proposition 4.8.** Conjecture 4.5 implies Conjectures 4.4' and 4.6.

*Proof.* Firstly, we shall see Conjecture 4.4'. It was observed by Mustată and sketched in [14, Remark 2.5.1]. Let \( \{a_i = \prod_{j=1}^{e} a_{ij}^j\}_{i \in N} \) be an arbitrary collection of \( \mathbb{R} \)-ideals on \( X = \text{Spec} R \) such that \( a_{ij} \) are non-trivial at \( P \), \( r_{ij} \in I \) and

\[ m_i := \text{mld}_{P}(X, a_i) \in J. \]

Then \( \sum_{i=1}^{e} r_{ij} \leq \text{ord}_{E} a_i \leq a_k(X) = d \) for the divisor \( E \) obtained by the blow-up of \( X \) at \( P \), since \( m_i \geq 0 \). The \( I \) has the minimum, say \( t > 0 \), so \( e_j \leq t^{-1} d \). By Corollary 2.3 and Remark 4.2, it is enough to show that both the subsets \( \bigcup_{i \in N} \text{Coeff}_{a_i} I \) and \( \bigcup_{i \in N} \{m_i\} \) of \( J \) satisfy the ACC. We may replace \( N \) with a countable subset \( N \) on which \( e_i \) is constant, say \( e \), such that the sequences \( \{r_{ij}\}_{i \in N} \) for \( 1 \leq j \leq e \) and \( \{m_i\}_{i \in N} \) are non-decreasing. By \( r_{ij} \leq d \) and \( m_i \leq d \), these sequences have limits

\[ r_j := \lim_{i \to j} r_{ij} \quad \text{and} \quad m := \lim_{i \to j} m_i. \]

It suffices to prove \( r_{ij} = r_j \) and \( m_i = m \) for some \( i \).

For the collection \( S = \{(a_1, \ldots, a_e)\}_{i \in N} \) of \( e \)-tuples of ideals in \( R \), we take a family \( \mathcal{F} = (Z_i, (\alpha_j(l))_{j}, l_i, s_i, t_i + 1)_{1 \leq l_0} \) of approximations of \( S \) and the generic limit \( (a_1, \ldots, a_e) \) of \( S \) defined in \( R_K \) with respect to \( \mathcal{F} \) as in Lemma A.8, where \( E_K \in \mathcal{D}_{X_K} \), computing

\[ M := \text{mld}_{P_K}(X_K, \prod_{j} a_j^j) \]
is fixed. It is extended to $E_l$ over $X \times \text{Spec} \mathbb{K}$, and we have $M = \text{mld}_P(X, \prod_j (a_{ij} + m^j)^{r_j}) = a_{(E_l),j}(X, \prod_j (a_{ij} + m^j)^{r_j})$ and $\text{ord}_{E_l} a_j = \text{ord}_{(E_l),j}(a_{ij} + m^j) < l$ for $i \in I_l$ with $z = s_j(l)$ using (iii) in Definition A.1. Hence $\text{ord}_{E_l} a_j = \text{ord}_{(E_l),j}$, $a_{ij}$ and

(8) \[ m_i \leq a_{(E_l),j}(X, \prod_j (a_{ij} + m^j)^{r_j}) = a_{(E_l),j}(X, \prod_j (a_{ij}^{r_j}) + \sum_j (r_j - r_{ij}) \text{ord}_{(E_l),j} a_{ij} \]

\[ = M + \sum_j (r_j - r_{ij}) \text{ord}_{E_l} a_j. \]

By Conjecture 4.5, $M = \text{mld}_P(X, \prod_j (a_{ij}^{r_j}) \leq m_i$ for any $i \in I_l$ after replacing $\mathcal{F}$ with a subfamily. With (8), we obtain

$M \leq m_i \leq M + \sum_j (r_j - r_{ij}) \text{ord}_{E_l} a_j$.

The right-hand side converges to $M$, whence $m_i = m = M$. Then $\text{mld}_P(X, \prod_j (a_{ij}^{r_j}) = \text{mld}_P(X, \prod_j a_{ij}^{r_j})$, so $r_{ij} = r_j$.

Secondly, we shall see Conjecture 4.6. Suppose the contrary. Then for every $i \in \mathbb{N}$, there exists an $\mathbb{R}$-ideal $b_i = \prod_j b_{ij}^{r_j}$ on $X$ such that $a_j + m^j = b_{ij} + m^j$ for all $j$ but $\text{mld}_P(X, a) \neq \text{mld}_P(X, b_i)$. Take a family $\mathcal{F} = (Z_i, (b_{ij}(l)))_{l \in \mathbb{N}, i \in I}$ of approximations of $S = \{(b_{ij})_{j \in \mathbb{N}}$ and the generic limit $(b_{ij})_j$ of $S$ defined in $R_K$ with respect to $\mathcal{F}$. Then for $l \geq l_0$,

$b_{ij}(l)_R = b_{ij} + m^j = a_j + m^j$

for $i \in I_l$ with $z = s_j(l)$ satisfying $i \geq l$, and such $z$ form a dense subset of $Z_l$. This implies $b_{ij}(l) = ((a_j + m^j) \cap \mathcal{R}) \otimes_k \mathcal{O}_{Z_l}$, whence

$b_{ij}(l)_K = (a_j R_K + m^j) \cap R_K$.

Then $b_j = \lim_{l \to \infty} b_{ij}(l)_K = a_j R_K$ by Remark A.3, so $\text{mld}_P(X_K, \prod_j b_{ij}^{r_j}) = \text{mld}_P(X, a)$ by Corollary 2.3. By Conjecture 4.5, we have $\text{mld}_P(X_K, \prod_j b_{ij}^{r_j}) = \text{mld}_P(X, b_i)$ for infinitely many $i$, that is, $\text{mld}_P(X, a) = \text{mld}_P(X, b_i)$, which is absurd. q.e.d.

Remark 4.9. Proposition 4.8 has the refinement that for fixed $d$ and $a \geq 0$,

(i) Conjecture 4.5 for $d$ with $\text{mld}_P(X_K, a) > a$ (resp. $\geq a$) implies Conjecture 4.4 for $d$ with $J \subset (a, d)$ (resp. $\subset [a, d]$), and

(ii) Conjecture 4.5 for $d$ with $\text{mld}_P(X_K, a) = a$ implies Conjecture 4.6 for $d$ with $\text{mld}_P(X, a) = a$.

This is obvious by the above proof. Note that (8) implies $m \leq M$.

Remark 4.10. Theorem A.9 gives Conjecture 4.6 in the case when $\text{mld}_P(X, a) = 0$, and then its Corollary A.10 gives Conjecture 4.5 in the case when $\text{mld}_P(X_K, a) = 0$. The order of this logic is opposite to Proposition 4.8. We expect that an effective estimate of $l$ in Conjecture 4.6 implies Conjecture 4.5.

Theorem A.9 is reduced to the corresponding statement [6, Theorem 1.4] on a variety by the property that the log canonical threshold for an ideal in $\mathcal{O}_{Y,Q}$ is approximated by those for ideals in $\mathcal{O}_{Y,Q}$. This property for the minimal log discrepancy on $X$ is a special case of Conjecture 4.5, so we do not know how to reduce Conjecture 4.6 to its variety version. The version of Conjecture 4.6 for a germ $Q \in (Y, \Delta, a)$ of a triple on a variety $Y$ holds when (i) $(Y, \Delta, a)$ is klt [14, Theorem 2.6], (ii) $Y$ is a surface [15], or (iii) $Y$ is toric and $Q, \Delta, a$ are torus invariant [25, Theorem 1.8].
The variety version of Theorem A.9 is globalised.

**Theorem 4.11.** Let \( (Y, \Delta, a = \prod I_i a_i^{l_i}) \) be a triple on a variety \( Y \) and \( Z \) an irreducible closed subset of \( Y \). Suppose \( \text{mld}_{R_Y}(Y, \Delta, a) = 0 \) and it is computed by \( E \in \mathcal{D}_Y \). Then there exists an open subset \( Y' \) of \( Y \) containing \( \eta_Z \) such that if an \( \mathbb{R} \)-ideal \( b = \prod j b_j' \) on \( Y' \) satisfies \( a_j |_{Y'} + p_j = b_j + p_j \) for all \( j \), where \( p_j = \{ u \in \mathcal{O}_{Y'} : \text{ord}_{E} u > \text{ord}_{E} a_j \} \), then \( (Y', \Delta|_{Y'}, b) \) is lc about \( Z|_{Y'} \) and \( \text{mld}_{R_{Y'}}(Y', \Delta|_{Y'}, b) = 0 \).

**Proof.** Take a log resolution \( f : W \to Y \) of \( (Y, \Delta, a, \mathcal{F}_Z) \), where \( \mathcal{F}_Z \) is the ideal sheaf of \( Z \), such that \( E \) is realised as a divisor on \( W \). Then \( F := \text{Exc} f \cup \text{Supp} \Delta_W \cup \text{Cosupp} \mathcal{F}_Z \mathcal{O}_W \) is an snc divisor \( \sum F_i \), where \( \Delta_W \) is the strict transform of \( \Delta \). By generic smoothness [11, Corollary III.10.7], there exists an open subset \( Y' \) of \( Y \) containing \( \eta_Z \) such that if the restriction \( S' = S|_{f^{-1}(Y')} \) of a stratum \( S \) of \( \sum F_i \) satisfies \( S' \neq \emptyset \) and \( f(S') \subset Z' = Z|_{Y'} \), then \( S' \to Z' \) is smooth and surjective.

Set \( \varepsilon = \dim Z \). We claim that for any \( Q \in Z' \),

\[
\text{mld}_{\mathcal{O}}(Y, \Delta, a m^\varepsilon_{\mathcal{O}}) = 0 \quad \text{and it is computed by } G_{\mathcal{O}}
\]

for the maximal ideal sheaf \( m_Q \) and the divisor \( G_{\mathcal{O}} \) obtained by the blow-up of \( W \) along a component of \( E \cap f^{-1}(Q) \). This can be verified from the local description at each closed point \( R \in f^{-1}(Q) \) that there exists a regular sequence of parameters \( v_1, \ldots, v_{\dim Y} \in \mathcal{O}_{Y,R} \) such that \( m_Q \mathcal{O}_{Y,R} = (v_1, \ldots, v_{\varepsilon}, \prod_{t=1}^{\varepsilon} v_{t+i}) \mathcal{O}_{Y,R} \) and \( F \) is given by \( \prod_{t=1}^{\varepsilon} v_{t+i} = 0 \) for some \( s \), \( t \) with \( 1 \leq s \leq t \).

Because \( \text{ord}_{G_{\mathcal{O}}} a_j = \text{ord}_{E} a_j \) and \( \text{ord}_{G_{\mathcal{O}}} u \geq \text{ord}_{E} u \) for \( u \in \mathcal{O}_{Y'} \), we conclude \( \text{mld}_{\mathcal{O}}(Y', \Delta|_{Y'}, a m^\varepsilon_{\mathcal{O}}) = 0 \) for \( b \) in Theorem 4.11 by [6, Theorem 1.4] (its proof works for triples). Hence \( (Y', \Delta|_{Y'}, b) \) is lc about \( Z' \), and \( \text{mld}_{R_{Y'}}(Y', \Delta|_{Y'}, b) = 0 \) by \( a_E(Y', \Delta|_{Y'}, b) = 0 \).

**Corollary 4.12.** Let \( (Y, \Delta, a = \prod I_i a_i^{l_i}) \) be an lc triple on a variety \( Y \) and \( Z \) a closed subset of \( Y \) with ideal sheaf \( \mathcal{F}_Z \). Then there exists an integer \( l \) such that if an \( \mathbb{R} \)-ideal \( b = \prod j b_j' \) on \( Y \) satisfies \( a_j + \mathcal{F}_Z = b_j + \mathcal{F}_Z \) for all \( j \), then \( (Y, \Delta, b) \) is lc about \( Z \).

**Remark 4.13.** The author should have written the proof after [14, Theorem 2.4]. The estimate of \( l \) in [14, Remark 2.4.1] is incorrect unless \( Z \) is a closed point (so is that of \( l_1 \) in [14, Lemma 3.1]).

### 5. The klt and plt cases

In this section, we settle Conjecture 4.5 in the klt case, and in the plt case whose lc centre has an isolated singularity. Conjecture 4.5 in these cases will be applied in the proof of Theorem 1.3 in Section 6. We keep the notation in Appendix A, so \( P \in X = \text{Spec} R \) with \( R = k[[x_1, \ldots, x_d]] \) and \( P_K \in X_K = \text{Spec} R_K \) with \( R_K = K[[x_1, \ldots, x_d]] \). In the course of the proofs, we will often replace the family \( \mathcal{F} \) with a subfamily, but we keep using the same notation \( \mathcal{F} = (Z_i, (a_j(l)|_j), l_1, s_1, t_1+1)_{l \geq l_0} \) to avoid intricacy.

#### 5.A. The klt case

**Theorem 5.1.** Conjecture 4.5 holds in the case when \( (X_K, a) \) is klt.

**Proof.** It is shown similarly to [14, Theorem 2.6]. By Remark 4.7, it suffices to show that after replacing \( \mathcal{F} \) with a subfamily,

\[
\text{mld}_{P_K}(X_K, a) \geq a_E(X, a_i) \quad \text{for all } a_i \leq m_{l_i}(X_K, a) \quad (10)
\]
for any \( i \in I \) (with \( l \geq l_0 \)) and any \( E \in \mathcal{D}_X \) with centre \( P \).

Take a subfamily in Lemma A.8 so that \( \mld_P(X, \prod_j (\tilde{a}_j(l), R)^{r_j}) = \mld_P(X_K, a) \) for \( \tilde{z} \in Z_i \). Then for \( i \in I \),

\[
    a_E(X, \prod_j (a_{ij} + m^j)^{r_j}) \geq \mld_P(X_K, a)
\]

by (iii) in Definition A.1. Since \((X_K, a)\) is klt, we can fix \( t > 0 \) such that \((X_K, a^{1+t})\) is lc. By Corollary A.10, \((X, a_i^{1+t})\) is lc for \( i \in I \) after replacing \( \mathcal{F} \) with a subfamily, whence \( a_E(X, a_i) \geq t \ord_E a_i = t \sum_j r_j \ord_E a_{ij} \). We fix \( l \geq l_0 \) such that \( l \geq (tr)^{-1} \mld_P(X_K, a) \) for all \( j \). Then,

\[
    a_E(X, a_i) \geq l^{-1} \ord_E a_{ij} \cdot \mld_P(X_K, a)
\]

for any \( j \) and \( i \in I \).

If \( \ord_E a_{ij} < l \) for all \( j \), then \( \ord_E a_{ij} = \ord_E (a_{ij} + m^j) \), so one has \( a_E(X, a_i) = a_E(X, \prod_j (a_{ij} + m^j)^{r_j}) \), and (10) follows from (11). If \( \ord_E a_{ij} \geq l \) for some \( j \), then (10) follows from (12).

**Remark 5.2.** By Remark 4.7, Theorem 5.1 and Corollary A.10, Conjecture 4.5 remains open only when \((X_K, a)\) is non-klt with \( \mld_P(X_K, a) > 0 \).

### 5.B. The plt case whose lc centre has an isolated singularity

Suppose that \((X_K, a)\) is an lc but not klt pair every lc centre of which has codimension 1. Then by Proposition 3.7, \((X_K, a)\) has the smallest lc centre \( S_K \) and it is normal. We prove Conjecture 4.5 on the assumption that \( S_K \) has an isolated singularity. Our argument has its origin in [15], but is highly cumbersome.

**Theorem 5.3.** Conjecture 4.5 holds in the case when \((X_K, a)\) has the smallest lc centre of codimension 1 which is regular outside \( P_K \).

We let \( S_K \) denote the smallest lc centre of \((X_K, a)\). \( S_K \) is a prime divisor which is regular outside \( P_K \). We define an \( \mathbb{R} \)-ideal \( c = \prod_j c_j^\alpha_j \) by the expression

\[
    a_j = c_j c_{X_K}(-\ord_{S_K} a_j) S_K
\]

so that \( c_j \) does not vanish along \( S_K \). The \( a \) and \( c c_{X_K}(-S_K) \) take the same order along any divisor over \( X_K \). We can fix \( t > 0 \) such that \((X_K, S_K, c^{1+t})\) is lc, since \( S_K \) is the unique lc centre of \((X_K, S_K, c)\).

We take a log resolution \( f_K : Y_K \to X_K \) of \((X_K, S_K, m_K)\), which is isomorphic outside \( P_K \). Let \( \{E_{ak}\}_a \) be the set of all \( f_K \)-exceptional prime divisors. The \( E_K = \sum_a E_{ak} \) is snc. Let \((Y_K, K_{X_K}, a' = \prod_j (a'_j)^{r_j})\) be the pull-back of \((X_K, 0, a)\) and \((Y_K, T_K + \Delta_K, c')\) that of \((X_K, S_K, c)\). Then \( T_K \) is the strict transform of \( S_K \). We set

\[
    L_K := T_K \cap f_K^{-1}(P_K),
\]

\[
    C_K := \text{Cosupp} c' \cap f_K^{-1}(P_K).
\]

We have the following diagram.

\[
    \begin{array}{cccc}
    T_K & \subset & Y_K & \supset & E_K & \supset & L_K, C_K \\
    \downarrow f_K & & & & & \\
    S_K & \subset & X_K & \supset & P_K
    \end{array}
\]
By blowing up $Y_K$ further, we may assume that $C_K$ is contained in the union of those $E_{\alpha K}$ satisfying

$$
tord_{E_{\alpha K}} c \geq \text{mld}_{p_c}(X_K, a).
$$

(13)

One can see this by induction on $\max_j \{ \min_{a \in J} \{ \text{ord}_{E_{\alpha K}} c \} \}$ in which one considers all subsets $J$ of indices satisfying $C_K \subset \bigcup_{a \in J} E_{\alpha K}$. Indeed, suppose that a subset $J$ gives the maximum of $\min_{a \in J} \{ \text{ord}_{E_{\alpha K}} c \}$. Take a log resolution $p_K : Y'_K \to Y_K$ of $(Y_K, T_K + E_K, \mathcal{I}_c)$. Then the new $C_K'$ on $Y'_K$ defined like $C_K$ is contained in $\bigcup_{a \in J} E'_{\beta K}$. Further for any $\beta \in J'$, there exists $\alpha \in J$ such that $p_K(E'_{\beta K}) \subset E_{\alpha K}$, for which $\text{ord}_{E'_{\beta K}} c > \text{ord}_{E_{\alpha K}} c$. Hence

$$
\min_{\beta \in J'} \{ \text{ord}_{E'_{\beta K}} c \} > \min_{a \in J} \{ \text{ord}_{E_{\alpha K}} c \},
$$

so the induction can be proceeded. Note that the order of $c$ takes value in the discrete subset $\sum_{j} r_j \mathbb{Z}_{\geq 0}$ of $\mathbb{R}$.

The $f_K$ is descendible by Proposition A.7, so replacing $\mathcal{F}$ with a subfamily, we obtain the diagram (16) in which $f_j$ is a family of log resolutions. Shrinking $Z_l$, we may assume that $E_{\alpha K}, L_K$ and $C_K$ are the base changes of flat families $E_{\alpha l}, L_l$ and $C_l$ in $Y_l$ over $Z_l$. We may assume that $\sum_{a \in J} E_{al}$ is an snc divisor such that for every stratum $S_l$ of $\sum_{a \in J} E_{\alpha l}$, the projections $S_l \to Z_l$ and $S_l \cap L_l \to Z_l$ are smooth and surjective. We may also assume that $\text{ord}_{E_{\alpha l}} \bar{a}_j(l)\cdot z$ is constant on $z \in Z_l$ for each $\alpha$ and $j$. Their base changes in $Y_l$ are denoted by $E_{al}, L_l$ and $C_l$. We write $E_l = \sum_{a \in J} E_{al}$ and $E_l = \sum_{a \in J} E_{al}$.

We fix $m$ such that $m \text{ord}_{E_{\alpha K}} m_K \geq \text{ord}_{E_{\alpha K}} c_j$ for all $\alpha$ and $j$, and set

$$d := \prod_j (c_j + m_j^{\nu_j} r_j).$$

Then $\text{ord}_{E_{\alpha K}} d = t \text{ord}_{E_{\alpha K}} c$, and $(X_K, \text{ad})$ is lc. The $d$ is defined over some $k(Z_l)$, so by replacing $\mathcal{F}$ with a subfamily, we may assume that $d$ is the base change of an $\mathbb{R}$-ideal $\mathfrak{d}_l = \prod_i \mathfrak{d}_l^{i_j}$ on $\mathbb{A}_{k}^d \times \text{Spec} k Z_l$ with $\mathfrak{m}_l \otimes_k \mathfrak{d}_l \subset \mathfrak{d}_l^{i_j}$, and that $\text{ord}_{E_{\alpha l}}(\mathfrak{d}_l)^{i_j}$ is constant on $Z_l$ for each $\alpha$. By Corollary A.10, after taking a subfamily, $(X, a_0(\mathfrak{d}_l)^{i_j})$ is lc for any $i \in I_l$ with $z = s_l(i)$, where $\mathfrak{d}_l$ is the pull-back on $X \times \text{Spec} k Z_l$ of $\mathfrak{d}_l$.

We fix $l \geq l_0$ such that

$$l \text{ord}_{E_{\alpha K}} m_K > \text{ord}_{E_{\alpha K}} a_j + \text{ord}_X a_j$$

for all $\alpha$ and $j$. By Remark 4.7, for Theorem 5.3 it suffices to prove that after shrinking $Z_l$ (and taking a subfamily accordingly),

$$a_E(X, a_l) \geq \text{mld}_{p_c}(X_K, a)$$

(15)

for any $i \in I_l$ and $E \in \mathcal{D}_K$ with centre $P$. Setting $z = s_l(i)$, we shall prove (15) by treating the three cases according to the position of $c(l)_{\alpha l}(E)$:

(a) $c(l)_{\alpha l}(E) \notin (L_l \cup C_l)_z$.

(b) $c(l)_{\alpha l}(E) \subset (C_l)_z$.

(c) $c(l)_{\alpha l}(E) \subset (L_l)_z$ and $c(l)_{\alpha l}(E) \notin (C_l)_z$.

We let $\bar{a}'(l) = \prod_j \bar{a}'_j(l)^{r_j}$ be the weak transform on $\tilde{Y}_l$ of $\prod_j \bar{a}_j(l)^{r_j}$, and $a'_j = \prod_j (a'_j)^{r_j}$ the weak transform on $(Y_l)_z$ of $a_l$. 

Lemma 5.4.  
(i) \( \delta_j'(l)K \mathcal{O}_{Y_K} = a'_j + l_{ij} \) with an ideal sheaf \( l_{ij} \) which is contained in \( \mathcal{O}_{Y_K}(E_K)_{a_j}^{ord_K a_j + 1} \).
(ii) \( \delta_j'(l)K \mathcal{O}_{(Y)} = a'_j + \mathcal{L}_{ij} \) with an ideal sheaf \( \mathcal{L}_{ij} \) which is contained in \( \mathcal{O}_{(Y)}(E_Y)_{a_j}^{ord_Y a_j + 1} \).
(iii) Cosupp \( \delta_j'(l)K \mathcal{O}_{Y_K} \) is induced to Cosupp \( \delta_j'(l)K \mathcal{O}_{Z} \) after shrinking \( Z_l \).

Proof. Write \( m_K \mathcal{O}_{Y_K} = \mathcal{O}_{Y_K}(-M_K) \) and \( a_j \mathcal{O}_{Y_K} = a'_j \mathcal{O}_{Y_K}(-A_{jk}) \). The inequality (14) means that \( l_{ij} = \mathcal{O}_{Y_K}(A_{jk} - lM_K) \) is an ideal sheaf contained in \( \mathcal{O}_{Y_K}(E_K)_{a_j}^{ord_K a_j + 1} \). Then \( m_K \mathcal{O}_{Y_K} = l_{ij} \mathcal{O}_{Y_K}(-A_{jk}) \). By \( \delta_j'(l)K \mathcal{O}_{Y_K} = a'_j \mathcal{O}_{Y_K}(-A_{jk}) + l_{ij} \mathcal{O}_{Y_K}(-A_{jk}) \), which induces (i). From (i),
\[
\text{Cosupp}(\delta_j'(l)K \mathcal{O}_{Y_K}) = \text{Cosupp} a'_j \cap f_K^{-1}(P_K) = L_K \cup C_K,
\]
which is extended to Cosupp \( \delta_j'(l)K \mathcal{O}_{Z} \) in (iii). On the other hand, \( ord_E \alpha \mathcal{O}_{E_K} = ord_{E_K} a_j \mathcal{O}_{E_K} = \mathcal{O}_{(Y)}(E_Y)_{a_j}^{ord_Y a_j + 1} \).

The cases (a) and (b) are not difficult.

Proof of (15) in the case (a). Set \( D_l = \sum a(1 - a_E(X_K, a))E_{al} \), base-changed to \( D_K \). Then \( \mathcal{O}_{(Y)}(\Delta_l)_{z} a'_j \) is the pull-back of \( (X, 0, a_i) \). For a divisor \( E_{al} \) containing \( c_{(Y)}(E) \), we have
\[
a_E((Y_l)_{z}, (\Delta_l)_{z}) \geq ord_E(E_l - \Delta_l)_{z} \geq ord_{E_{al}}(E_l - \Delta_l)_{z} = a_E(X_K, a) \geq \text{mld}_{P_{l}}(X_K, a),
\]
where the first inequality follows from the log canonicity of \( ((Y_l)_{z}, (E_l)_{z}) \). By Lemma 5.4(ii) and (iii), Cosupp \( a'_j \cap (f_i)^{-1}(P) = \text{Cosupp} \delta_j'(l)K \mathcal{O}_{(Y)} = (L_l \cup C_l)_{z} \), so \( ord_E a'_j = 0 \). Thus
\[
a_E(X, a_i) = a_E((Y_l)_{z}, (\Delta_l)_{z}, a'_j) = a_E((Y_l)_{z}, (\Delta_l)_{z}) \geq \text{mld}_{P_{l}}(X_K, a).
\]
q.e.d.

Proof of (15) in the case (b). The \( c_{(Y)}(E) \) lies on some \( (E_{al})_z \) such that \( E_{al} \) satisfies (13). Then
\[
a_E(X, a_i) \geq ord_{E}((\delta_i)_{z}) \geq ord_{E_{al}}((\delta_i)_{z}) = ord_{E_{al}} d = t ord_{E_{al}} c \geq \text{mld}_{P_{l}}(X_K, a),
\]
whose first inequality follows from the log canonicity of \( (X, a_i(\delta_i))_{z} \).
q.e.d.

The case (c) is reduced to the following log canonicity.

Lemma 5.5. After shrinking \( Z_l \), the triple \( ((Y_l)_{z}, (E_l)_{z}, a'_j) \) is lc about \( (L_l)_{z} \setminus C_l \) for any \( i \in I_l \) with \( z = s_i(l) \in Z_l \).

Proof of (15) in the case (c) from Lemma 5.5. For \( D_l = \sum a(1 - a_E(X_K, a))E_{al} \), we have \( a_E(X, a_i) = a_E((Y_l)_{z}, (\Delta_l)_{z}, a'_j) \geq ord_E(E_l - \Delta_l)_{z} \) by Lemma 5.5, and have seen \( ord_E(E_l - \Delta_l)_{z} \geq \text{mld}_{P_{l}}(X_K, a) \) in the proof in the case (a).
q.e.d.

Proof of Lemma 5.5. Pick any open stratum \( F_l \) of the snc divisor \( E_K \), which is extended to an open stratum \( F_l \) of \( E_l \). We prove Lemma 5.5 by noetherian induction. Recall that \( l \) has been fixed. Let \( Q_l \) be an irreducible locally closed subset of \( \tilde{F_l} \setminus L_l \). Then \( D_l \), which dominates \( Z_l \). It suffices to show the existence of a dense open subset \( Q_l \) of \( Q_l \) such that the triple \( ((Y_l)_{z}, (E_l)_{z}, a'_j) \) is lc about \( (Q_l')_{z} \) for \( i \in I_l \) with
\[ z = s_i(i), \text{ where } O'_q = \overline{O}_q \times_{\overline{F}_q} Y. \] Indeed, start with a component \( \bar{Q}_i \) of \( \overline{F}_q \) and find \( \bar{Q}_i' \). Take a dense open subset \( Z'_j \) of \( Z_j \) such that each irreducible component \( \bar{Q}_i' \) of \( \overline{Q}_i \cap Z'_j \) dominates \( Z_j' \). Replace \( Z_i \) with \( Z_i' \) and continue the argument for each \( \bar{Q}_i' \). Eventually we attain \( Z_i \) such that \((Y_j)_z, (E_l)_z, a_i') \) is lc about \((F_j \cap L_j \setminus \mathcal{C}_l)\) for \( i \in I_l \) with \( z = s_i(i) \in Z_l \). Applying this to all open strata of \( E_K \), we obtain a shrunk \( Z_i \) in Lemma 5.5.

We shall construct \( \bar{Q}_i' \). By shrinking \( \bar{Q}_i \) and \( Z_i \), we may assume that \( \bar{Q}_i \to Z_i \) is smooth and surjective. Let \( \bar{f}_i^+: \bar{Y}_i^+ \to A^d_k \times \text{Spec} \bar{k} \bar{Q}_i \) be the base change of \( \bar{f}_i \) by \( \bar{Q}_i \to Z_i \). Then \( \text{pr}_{\bar{Q}_i} \circ \bar{f}_i^+ \) has the natural section \( \bar{g}_i: \bar{Q}_i \to \bar{Y}_i^+ = \bar{Y}_i \times_{\bar{Q}_i} \bar{Q}_i \) by the immersion \( \bar{Q}_i \to \bar{Y}_i \). We construct \( f^+_K \) and \( g_K \) similarly for \( Q_K = \bar{Q}_i \times_{\bar{Y}_i} Y_K \) as below.

The \( Y_i^+ \), \( Y_K^+ \) are the base changes of \( Y_i \), \( Y_K \) by smooth morphisms. For a scheme or a sheaf \( \square \) on \( Y_i \) or \( Y_K \), we mean by \( \square^+ \) the base change of \( \square \) on \( Y_i^+ \) or \( Y_K^+ \). For example, \( a^+ = \prod_j (a_{ij})^{r_j} = a^+ \mathcal{O}_{Y_i^+}^+ \). Let \( \mathfrak{q}_i \) be the ideal sheaf of \( \bar{g}_i(\bar{Q}_i) \) on \( \bar{Y}_i^+ \) and \( \bar{G}_i \subset \mathcal{O}_{\bar{Y}_i^+} \) the divisor obtained by the blow-up of \( Y_i^+ \) along \( \mathfrak{q}_i \). They are base-changed to \( q \) on \( Y_K^+ \) and \( G_K \in \mathcal{D}_{Y_K^+} \).

Set \( n = \dim F_K - 1 \). Similarly to (9), we see that \( \text{mld}_{\text{g}_{K,(q)}(Y_i^+, E_i, q^n a^+)} = 0 \) and it is computed by \( G_K \). We have \( \mathfrak{a}^+(l)_K \mathcal{O}_{Y_K^+} = \prod_j (a_{ij}^+ + 1)_{ij} \), \( r_j \) and \( \text{ord}_{G_K} a_{ij}^+ = \text{ord}_{E_K} a_{ij} < \text{ord}_{G_K} l_{ij} \) from Lemma 5.4(i), so

\[
\text{mld}_{\text{g}_{K,(q)}(Y_i^+, E_i, q^n a^+(l)_K \mathcal{O}_{Y_K^+})} = 0 \quad \text{and it is computed by } G_K.
\]

Then \( \text{mld}_{\text{g}_{K,(q)}(Y_i^+, E_i, q^n a^+(l)_K \mathcal{O}_{Y_K^+})} = 0 \) and it is computed by \( G_i \). We regard \( Y_i^+ \) as a family over \( Q_i \). There exists a dense open subset \( \bar{Q}_i' \) of \( \bar{Q}_i \) such that for any closed point \( q \in \bar{Q}_i' = \bar{Q}_i \times_{\bar{Q}_i} Y_i' \) with its image \( z \in Z_i \), \( \text{mld}_{(Y_i)_z, (E_i)_z, m_q^a(\mathfrak{a}^+(l)_K \mathcal{O}_{(Y_i)_z})} = 0 \) and it is computed by \( (G_i)_q \).

Theorem 5.3 is completed.

6. The threefold case

We shall prove Theorem 1.3. By Remark 4.9, the theorem follows from Conjecture 4.5 for \( d = 3 \) with \( \text{mld}_{K,(a)}(X_K, a) > 1 \). In Remark 5.2, Conjecture 4.5 is reduced
to the case when \((X, a)\) is an lc pair which has a minimal lc centre \(Z\) of positive dimension. If \(d = 3\), then by Theorem 1.2, \(Z\) is the smallest lc centre and it is normal. If \(\dim Z = 2\), then one can apply Theorem 5.3. If \(\dim Z = 1\), then \(\text{mld}_X(X, a) \leq 1\) by Proposition 6.1. Therefore, we obtain Theorem 1.3.

**Proposition 6.1.** Let \(P \in (X, a)\) be a germ of an lc pair on a regular \(R\)-variety \(X\) of dimension 3 with \(R = K[[x_1, \ldots, x_d]]\) whose smallest lc centre is of dimension 1. Then \(\text{mld}_P(X, a) \leq 1\).

**Proof.** The smallest lc centre \(C\) of \((X, a)\) is regular by Theorem 1.2. Setting \((X_0, A_0, a_0) := (X, 0, a)\) and \(C_0 := C\), we build a tower of finitely many blow-ups

\[X_n \to \cdots \to X_i \overset{f_i}{\to} X_{i-1} \to \cdots \to X_0 = X\]

such that

(i) \(f_i : X_i \to X_{i-1}\) is the blow-up along \(C_{i-1}\).
(ii) \(E_i\) is the exceptional divisor of \(f_i\).
(iii) \((X_i, \Delta_i, a_i)\) is the pull-back of \((X_{i-1}, \Delta_{i-1}, a_{i-1})\).
(iv) \(C_i\) is a regular non-klt centre on \(X_i\) of \((X, a)\) mapped onto \(C_{i-1}\), and
(v) \(a_{E_i}(X, a) > 0\) for \(i < n\) and \(a_{E_i}(X, a) = 0\).

Here one can prove the effectiveness \(\Delta_i \geq 0\) and the regularity of \(C_i\) by induction. Indeed, if they hold for \(i - 1\), then \(\text{ord}_{E_i} \Delta_i = \text{ord}_{C_{i-1}} \Delta_{i-1} + \text{ord}_{C_{i-1}} a_{i-1} - 1 > 0\) by Lemma 6.2. Unless \(a_{E_i}(X, a) = 0\), an arbitrary lc centre \(C_i\) of \((X_i, \Delta_i, a_i)\) mapped onto \(C_{i-1}\) is of dimension 1 and is minimal. The regularity of \(C_i\) follows from Theorem 1.2.

Let \(F\) be the divisor obtained by the blow-up of \(X_n\) along an irreducible component of \(E_n \cap (f_1 \circ \cdots \circ f_n)^{-1}(P)\). Then \(a_F(X, a) = a_F(X_n, a_n) \leq a_F(X_n, E_n) = 1\) by \(\Delta_n \geq (1 - a_{E_n}(X, a))E_n = E_n\).

**q.e.d.**

**Lemma 6.2.** Let \((X, a)\) be a pair on a regular \(R\)-variety \(X\) and \(Z\) a non-klt centre of \((X, a)\). Then \(\text{ord}_Z a \geq 1\). If in addition \(\text{codim}_X Z \geq 2\), then \(\text{ord}_Z a > 1\).

**Proof.** The lemma is obvious if \(Z\) is a divisor, so we may assume \(\text{codim}_X Z \geq 2\). Setting \(X_0 := X, Z_0 := Z\) and \(a_0 := a\), we build a tower of finitely many blow-ups

\[X_n \to \cdots \to X_i \overset{h_i}{\to} X_{i-1} \to \cdots \to X_0 = X\]

such that

(i) \(f_i\) is the composition \(X_i \overset{h_i}{\to} X_i' \to X_{i-1}' \overset{g_{i-1}}{\to} X_{i-1}\) of the blow-up \(h_i : X_i \to X_i'\) along the strict transform on \(X_i'\) of \(Z_{i-1}\) and an embedded resolution \(g_{i-1} : X_i' \to X_{i-1}\) of singularities of \(Z_{i-1}\), in which \(g_{i-1}\) is isomorphic on the regular locus of \(Z_{i-1}\),
(ii) \(E_i\) is the exceptional divisor of \(h_i\),
(iii) \(a_i\) is the weak transform on \(X_i\) of \(a_{i-1}\),
(iv) \(Z_i\) is a non-klt centre on \(X_i\) of \((X, a)\) mapped onto \(Z_{i-1}\), and
(v) \(a_{E_i}(X, a) > 0\) for \(i < n\) and \(a_{E_i}(X, a) \leq 0\).

Supposing \(\text{ord}_Z a \leq 1\), we shall derive by induction two inequalities

\[\text{ord}_Z a_i \leq 1\quad\text{and}\quad a_{E_i}(X, a_i) \leq 0\]

for any \(i\). The claim for \(i = 0\) is trivial. If they hold for \(i - 1\), then \(\text{ord}_Z a_i \leq \text{ord}_Z a_{i-1} \leq \text{ord}_Z a_{i-1} \leq 1\) by [12, Lemmata III.7, III.8] for an irreducible closed subset \(V_i\) of \(Z_i\) meeting the regular locus of \(Z_i\) such that \(V_i \to Z_{i-1}\) is finite and
surjective. Note that the symbol \( v^{(1)} \) in [12] stands for the order. The \((X_{i-1},0,a_{i-1})\) is pulled back to \((X_i,\Delta_i,a_i)\) with \(\text{ord}_{\mathcal{E}_i} \Delta_i = 1 + \text{ord}_{\mathcal{Z}_{i-1}} a_{i-1} - \text{codim}_{X_i} Z_{i-1} \leq 0\), so \(\text{ord}_{\mathcal{E}_i} (X_i,\Delta_i,a_i) \leq \text{ord}_{\mathcal{E}_i} (X_i,\Delta_i,a_i) = \text{ord}_{\mathcal{E}_i} (X_{i-1},a_{i-1}) \leq 0\).

We obtained \(\text{ord}_{\mathcal{E}_i} (X_n,a_n) \leq 0\). However, it contradicts \(\text{ord}_{\mathcal{E}_i} a_n = 0\).

**APPENDIX A. GENERIC LIMITS**

The generic limit is a limit of ideals. It was constructed first by de Fernex and Mustaţă [8] using ultraproducts, and then by Kollár [20] using Hilbert schemes. We set \( \mathcal{R} = k[x_1,\ldots,x_d] \) with maximal ideal \( \mathfrak{m} \), and \( \mathbb{A}^d_k = \text{Spec} \mathcal{R} \) with origin \( \mathfrak{P} \). We also set \( R = k[[x_1,\ldots,x_d]] \) with \( \mathfrak{m} = \mathfrak{m}R \), and \( X = \text{Spec} R \) with closed point \( P \). Mostly we discuss on the spectrum of a noetherian ring, where an ideal in the ring is identified with its coherent ideal sheaf.

We introduce the notion of a family of approximated ideals by which a generic limit is defined.

**Definition A.1.** Let \( S = \{(a_1,\ldots,a_e)\}_{i \in I} \) be a collection of \( e \)-tuples of ideals in \( R \), indexed by an infinite set \( I \). A family \( \mathcal{F} \) of approximations of \( S \) consists of, for each \( I \geq I_0 \),

\[
\begin{align*}
\text{(a)} & \text{ a variety } Z_i, \\
\text{(b)} & \text{ an ideal sheaf } \mathfrak{a}_j(l) \text{ on } \mathbb{A}^d_k \times_{\text{Spec} k} Z_i \text{ containing } \mathfrak{m}^{\ell} \otimes_k \mathcal{O}_{Z_i} \text{ for } 1 \leq j \leq e, \\
\text{(c)} & \text{ an infinite subset } I_j \text{ of } I \text{ and a map } s_j : I_j \to Z_i(k), \text{ where } Z_i(k) \text{ is the set of } k\text{-points on } Z_i, \text{ and} \\
\text{(d)} & \text{ a dominant morphism } t_{i+1} : Z_{i+1} \to Z_i,
\end{align*}
\]

such that

\[
\begin{align*}
\text{(i)} & \text{ } \mathfrak{a}_j(l) \text{ gives a flat family of closed subschemes of } \mathbb{A}^d_k \text{ parametrised by } Z_i, \\
\text{(ii)} & \text{ the pull-back of } \mathfrak{a}_j(l) \text{ by } \text{id}_{\mathbb{A}^d_k} \times t_{i+1} \text{ is } \mathfrak{a}_j(l+1) + \mathfrak{m}^{l+1} \otimes_k \mathcal{O}_{Z_{i+1}}, \\
\text{(iii)} & \text{ } a_j + \mathfrak{m}^{l} = \mathfrak{a}_j(l)(a_j)_{i}(R) \text{ for } i \in I_j, \text{ where } \mathfrak{a}_j(l)_{i} \text{ is the ideal in } \mathcal{R} \text{ given by } \mathfrak{a}_j(l) \text{ at } z \in Z_i, \\
\text{(iv)} & \text{ } s_j(I_l) \text{ is dense in } Z_{l}, \text{ and} \\
\text{(v)} & \text{ } I_{i+1} \subseteq I_j \text{ and } t_{i+1} \circ s_{j+1} = s_j |_{l_{i+1}}.
\end{align*}
\]

The construction of \( \mathcal{F} \) using Hilbert schemes is exposed in [6, Section 4]. In general, there exist essentially different families of approximations.

For a field extension \( K \) of \( k \), we set \( \mathbb{A}^d_K = K \otimes_k \mathbb{A}^d_k \) with \( \mathfrak{m}_K = \mathfrak{m}K \), and \( \mathbb{A}^d_K = \text{Spec} \mathcal{R}_K \) with origin \( \mathfrak{P}_K \). We also set \( R_K = K \otimes_k R = K[[x_1,\ldots,x_d]] \) with \( \mathfrak{m}_K = \mathfrak{m}R_K \), and \( X_K = \text{Spec} R_K \) with closed point \( P_K \).

**Definition A.2.** Suppose that a family \( \mathcal{F} \) of approximations of \( S \) is given as in Definition A.1. For this \( \mathcal{F} \), take the union \( K = \lim_{i \to \infty} K(Z_i) \) of the function fields \( K(Z_i) \) of \( Z_i \) by the inclusions \( t_{i+1}^* : K(Z_i) \hookrightarrow K(Z_{i+1}) \). Then the generic limit of \( S \) with respect to \( \mathcal{F} \) is the \( e \)-tuple \((a_1,\ldots,a_e)\) of ideals in \( R_K \) such that

\[
a_j + \mathfrak{m}_K^l = \mathfrak{a}_j(l)_{K} \mathcal{R}_K
\]

for all \( l \geq l_0 \), where \( \mathfrak{a}_j(l)_{K} \) is the ideal in \( \mathcal{R}_K \) given by \( \mathfrak{a}_j(l) \) at the natural \( K \)-point \( \text{Spec} K \to Z_i \).

**Remark A.3.** We have \( a_j = \lim_{l \to \infty} \mathfrak{a}_j(l)K \), by \( \mathfrak{a}_j(l)_{K} = \mathfrak{a}_j(l+1)K + \mathfrak{m}_K^l \) from (ii) in Definition A.1.
Theorem A.6. Let $\mathcal{F} = (Z_i, (\bar{a}_i(l)))_{l \geq 0}$ and $\mathcal{F}' = (Z'_i, (\bar{a}'_i(l)))_{l \geq l'_0}$ be families of approximations of $S$. A morphism $\mathcal{F}' \to \mathcal{F}$ consists of dominant morphisms $f_i: Z'_i \to Z_i$ for $l \geq l'_0$, with $l'_0 \geq l_0$ imposed, such that

(i) $t_{i+1} \circ f_{i+1} = f_{i} \circ t'_{i+1}$,
(ii) the pull-back of $\bar{a}_i(l)$ by $f_i$ is $\bar{a}'_i(l)$, and
(iii) $l'_r \subset l_i$ and $f_i \circ s'_r = s_i|_{l'_r}$.

An $\mathcal{F}'$ is called a subfamily of $\mathcal{F}$ if it is equipped with a morphism $\mathcal{F}' \to \mathcal{F}$ as above such that all $f_i$ are open immersions.

We want to compare minimal log discrepancies over $X$ and $X_K$. The comparison of those for approximated ideals is a consequence of the existence of a family of log resolutions on an open subfamily of triples and Corollary 2.3.

Lemma A.5 (cf. [16, Proposition 3.2(ii)]). Notation as above. Let $(a_1, \ldots, a_\nu)$ be the generic limit of $S$ with respect to $\mathcal{F}$. Then after replacing $\mathcal{F}$ with a subfamily but using the same notation,

$$\text{mld}(X_K, \prod_j (a_j + m_K)^{r_j}) = \text{mld}(\bar{a}_K, \prod_j \bar{a}_j(l)_{l})$$

for all $r_1, \ldots, r_\nu > 0$ and all $l \in Z_0$ with $l \geq l_0$.

We utilise a projective morphism which is descended to $\mathcal{A}_K$.

Definition A.6. A projective morphism $f_K: Y_K \to X_K$ is said to be descendible if there exists a projective morphism $\bar{f}_K: \bar{Y}_K \to \mathcal{A}_K$ whose base change to $X_K$ is $f_K$.

Proposition A.7. Let $f_K: Y_K \to X_K$ be a projective morphism of $R_K$-varieties which is isomorphic outside $P_K$. Then $f_K$ is descendible.

Proof. Assuming $\text{codim}_{X_K} \text{Cosupp}_{X_K} \geq 2$, we may assume codim_{X_K} \text{Cosupp}_{X_K} \subset P_K$, that is, $n_K$ is an $m_K$-primary ideal. Thus, $n_K$ is the pull-back of the ideal $n_K = n_K \cap R_K$ in $R_K$.

Since blowing-up commutes with flat base change [23, Proposition 8.1.12(c)], the blow-up of $\mathcal{A}_K$ along $\bar{n}_K$ is base-changed to $f_K$.

We utilise a projective morphism which is descended to $\mathcal{A}_K$.

$$\text{mld}(X_K, \prod_j (a_j + m_K)^{r_j}) = \text{mld}(\bar{a}_K, \prod_j \bar{a}_j(l)_{l})$$

for all $r_1, \ldots, r_\nu > 0$ and all $l \in Z_0$ with $l \geq l_0$.

Let $f_K: Y_K \to X_K$ be a descendible projective morphism, descended to $\bar{f}_K: \bar{Y}_K \to \mathcal{A}_K$. This $\bar{f}_K$ is defined over $k(Z_0)$ for some $l'_0 \geq l_0$. For $l \geq l'_0$, one can construct inductively a projective morphism $\bar{f}_K: \bar{Y}'_l \to \mathcal{A}_K \times \text{Spec} K_l$ with a smooth open subvariety $Z'_l$ of $Z_l$ such that (i) $\bar{Y}'_l$ is flat over $Z'_l$, (ii) $Z'_l \subset l^{-1}_l(Z'_0)$, and (iii) $\bar{f}'_{l+1}$ and $\bar{f}'_{l}$ are the base changes of $\bar{f}'_l$, by generic flatness [10, Corollaire IV.11.1.5].

These $Z'_l$ with $l'_r = s^{-1}_r(Z'_l(k))$ form a subfamily $\mathcal{F}'$ of $\mathcal{F}$. Replacing $\mathcal{F}$ with $\mathcal{F}'$, we obtain a commutative diagram

$$\begin{array}{cccc}
Y_K & \xrightarrow{f_K} & Y_l & \xrightarrow{f_l} Y'_l \\
| & | & | & | \\
X_K & \xrightarrow{f_K} & X \times \text{Spec} K_l & \xrightarrow{f_l} \mathcal{A}_K \times \text{Spec} K_l
\end{array}$$
for \( l \geq l_0 \) (the \( l_0 \) is replaced) such that (i) \( Z_l \) is smooth, (ii) \( \tilde{f}_i \) is projective, (iii) \( \tilde{Y}_l \) is flat over \( Y_l \), and (iv) \( \tilde{f}_{i+1}, \tilde{f}_k, f_l \) and \( f_K \) are the base changes of \( \tilde{f}_i \). In general, \( X_K \to X \times \text{Spec} k Z_l \) is not the base change of \( \mathbb{k}_K \to \mathbb{k}_K \times \text{Spec} k Z_l \).

Whenever an algebraic object over \( X_K \) descendible to \( \mathbb{k}_K \) is specified, by taking a subfamily, one can construct \( (16) \) so that it comes from a flat family over \( Z_l \). For example, suppose that \( E_K \in \mathcal{D}_{X_K} \) with centre \( P_K \) is given. It is realised as a divisor on \( Y_K \) equipped with a log resolution \( f_K : Y_K \to X_K \) of \( (X_K, m_K) \), which is isomorphic outside \( P_K \). This \( f_K \) is descended to a log resolution \( \bar{f}_K \) by Proposition A.7, and \( \bar{f}_K \) is extended to a family \( \tilde{f}_i \) of log resolutions in \( (16) \) by generic smoothness. There exists a prime divisor \( \bar{E}_l \) on \( \bar{Y}_l \) which is base-changed to \( E_K \). By this observation, Lemma A.5 is refined as follows.

**Lemma A.8** (cf. [16, Proposition 3.2(iii)]). **Notation as above.** Fix \( r_1, \ldots, r_e > 0 \) and \( E_K \in \mathcal{D}_{X_K} \) computing \( \text{mld}_P(K, \prod_j a_{i,j}^{r_j}) \). Then after replacing \( \mathcal{F} \) with a subfamily but using the same notation, there exists a divisor \( \tilde{E}_l \) over \( \mathbb{k}_K \times \text{Spec} k Z_l \) for any \( l \geq l_0 \), base-changed to \( E_K \), such that

\[
\text{mld}_P(K, \prod_j a_{i,j}^{r_j}) = \text{mld}_P(\mathbb{k}_K, \prod_j \bar{a}_j(l)_{z'}^{r_j}) = a_{(\bar{E}_l),r}(\mathbb{k}_K, \prod_j \bar{a}_j(l)_{z'}^{r_j}),
\]

\[
\text{ord}_{E_K} a_j = \text{ord}_{E_K}(a_j + m_K) = \text{ord}_{E} a_j < l,
\]

for all \( z \in Z_l \).

We apply the ideal-adic semi-continuity of log canonicity by Kollár, and de Fernex, Ein and Mustaţă.

**Theorem A.9** ([20], [6], [7, Proposition 2.20]). Let \( Q \in Y \) be a germ of an lc variety and set \( \bar{Y} = \text{Spec} \mathcal{O}_{Y,Q} \) with closed point \( Q \). Let \( a = \prod_j a_{i,j}^{r_j} \) be an \( \mathbb{R} \)-ideal on \( \bar{Y} \). Suppose \( \text{mld}_Q(\bar{Y}, a) = 0 \) and it is computed by \( \bar{E} \in \mathcal{D}_{\bar{Y}} \). If an \( \mathbb{R} \)-ideal \( b = \prod_j b_{i,j}^{r_j} \) on \( \bar{Y} \) satisfies \( a_j + p_j = b_j + p_j \) for all \( j \), where \( p_j = \{ u \in \mathcal{O}_{\bar{Y}} \mid \text{ord}_u > \text{ord}_\bar{E} a_j \} \), then \( \text{mld}_Q(\bar{Y}, b) = 0 \).

**Corollary A.10.** In Lemma A.8, if \( \text{mld}_P(K, \prod_j a_{i,j}^{r_j}) = 0 \), then \( \text{mld}_P(X, \prod_j a_{i,j}^{r_j}) = 0 \) for any \( i \in I_l \) on a subfamily. In particular, if \( (X, \prod_j a_{i,j}^{r_j}) \) is lc, then so is \( (X, \prod_j a_{i,j}^{r_j}) \).

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