TOWARD A GEOMETRIC ANALOGUE OF DIRICHLET’S UNIT THEOREM

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ABSTRACT. In this article, we propose a geometric analogue of Dirichlet’s unit theorem on arithmetic varieties [18], that is, if \( X \) is a normal projective variety over a finite field and \( D \) is a pseudo-effective \( \mathbb{Q} \)-Cartier divisor on \( X \), does it follow that \( D \) is \( \mathbb{Q} \)-effective? We also give affirmative answers on an abelian variety and a projective bundle over a curve.

INTRODUCTION

Let \( K \) be a number field and \( O_K \) the ring of integers in \( K \). Let \( K(C) \) be the set of all embeddings \( K \rightarrow C \). For \( \sigma \in K(C) \), the complex conjugation of \( \sigma \) is denoted by \( \overline{\sigma} \), that is, \( \overline{\sigma}(x) = \overline{\sigma(x)} \) \( (x \in K) \). Here we define \( \Xi_K \) and \( \Xi^0_K \) to be

\[
\Xi_K := \left\{ \xi \in \mathbb{R}^{K(C)} | \xi(\sigma) = \xi(\overline{\sigma}) \ (\forall \sigma) \right\},
\]

\[
\Xi^0_K := \left\{ \xi \in \Xi_K | \sum_{\sigma \in K(C)} \xi(\sigma) = 0 \right\}.
\]

The Dirichlet unit theorem asserts that the group \( O_K^{\times} \) consisting of units in \( O_K \) is a finitely generated abelian group of rank \( s := \dim_{\mathbb{R}} \Xi^0_K \).

Let us consider the homomorphism \( L : K^{\times} \rightarrow \mathbb{R}^{K(C)} \) given by

\[
L(x)(\sigma) := \log |\sigma(x)| \quad (x \in K^{\times}, \sigma \in K(C)).
\]

It is easy to see the following:

(i) For a compact set \( B \) in \( \mathbb{R}^{K(C)} \), the set \( \{ x \in O_K^{\times} | L(x) \in B \} \) is finite.
(ii) \( L : K^{\times} \rightarrow \mathbb{R}^{K(C)} \) extends to \( L_{\mathbb{R}} : K^{\times} \otimes \mathbb{R} \rightarrow \mathbb{R}^{K(C)} \).
(iii) \( L_{\mathbb{R}} : O_K^{\times} \otimes \mathbb{R} \rightarrow \mathbb{R}^{K(C)} \) is injective.
(iv) \( L_{\mathbb{R}}(O_K^{\times} \otimes \mathbb{R}) \subseteq \Xi^0_K \).

By using (i) and (iii), we can see that \( O_K^{\times} \) is a finitely generated abelian group. The most essential part of the Dirichlet unit theorem is to show that \( O_K^{\times} \) is of rank \( s \), which is equivalent to see that, for any \( \xi \in \Xi^0_K \), there is \( x \in O_K^{\times} \otimes \mathbb{R} \) with \( L_{\mathbb{R}}(x) = \xi \).

In order to understand the equality \( L_{\mathbb{R}}(x) = \xi \) in terms of Arakelov geometry, let us introduce several notations for arithmetic divisors on the arithmetic curve \( \text{Spec}(O_K) \). An arithmetic \( \mathbb{R} \)-divisor on \( \text{Spec}(O_K) \) is a pair \((D, \xi)\) consisting of an \( \mathbb{R} \)-divisor \( D \) on \( \text{Spec}(O_K) \) and \( \xi \in \Xi_K \). We often denote the pair \((D, \xi)\) by \( D_\xi \). The

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arithmetic principal $\mathcal{R}$-divisor $(x)_\mathcal{R}$ of $x \in K^\times \otimes \mathcal{R}$ is the arithmetic $\mathcal{R}$-divisor given by
\[
(x)_\mathcal{R} := \left( \sum P \text{ord}_P(x)[P], -2L_\mathcal{R}(x) \right),
\]
where $P$ runs over the set of all maximal ideals of $O_K$ and
\[
\text{ord}_P(x) := a_1 \text{ord}_P(x_1) + \cdots + a_r \text{ord}_P(x_r)
\]
for $x = x_1^{a_1} \cdots x_r^{a_r}$ ($x_1, \ldots, x_r \in K^\times$ and $a_1, \ldots, a_r \in \mathcal{R}$). The arithmetic degree $\deg(\overline{D})$ of an arithmetic $\mathcal{R}$-divisor $\overline{D} = (\sum P a_P[P], \xi)$ is defined to be
\[
\overline{\deg}(\overline{D}) := \sum P a_P \log \#(O_K/P) + \frac{1}{2} \sum_{\sigma \in K(C)} \xi(\sigma).
\]
Note that
\[
\overline{\deg}(x)_\mathcal{R} = 0 \quad (x \in K^\times \otimes \mathcal{R})
\]
by virtue of the product formula. Further, $\overline{D} = (\sum P a_P[P], \xi)$ is said to be effective if $a_P \geq 0$ for all $P$ and $\xi(\sigma) \geq 0$ for all $\sigma$.

In [18, SubSection 3.4], we proved the following:

(0.1) “If $\overline{\deg}(\overline{D}) \geq 0$, then $\overline{D} + (x)_\mathcal{R}$ is effective for some $x \in K^\times \otimes \mathcal{R}$.”

This implies the essential part of the Dirichlet unit theorem. Indeed, we set $\overline{D} = (0, \xi)$ for $\xi \in \mathbb{Z}_K^0$. As $\overline{\deg}(\overline{D}) = 0$, by the assertion (0.1), $\overline{D} + (y)_\mathcal{R}$ is effective for some $y \in K^\times \otimes \mathcal{R}$, and hence $\overline{D} + (y)_\mathcal{R} = (0, 0)$ because $\deg(\overline{D} + (y)_\mathcal{R}) = 0$. Here we set $y = u_1^{a_1} \cdots u_r^{a_r}$ such that $u_1, \ldots, u_r \in K^\times$, $a_1, \ldots, a_r \in \mathcal{R}$ and $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. By using the linear independency of $a_1, \ldots, a_r$ over $\mathbb{Q}$, $\text{ord}_P(y) = 0$ implies $\text{ord}_P(u_i) = 0$ for all maximal ideals $P$ of $O_K$ and $i = 1, \ldots, r$, that is, $u_i \in O_K^\times$ for $i = 1, \ldots, r$. Therefore, $\xi = L_\mathcal{R}(y^2)$ and $y \in O_K^\times \otimes \mathcal{R}$, as required. In this sense, the above property (0.1) is an Arakelov theoretic interpretation of the classical Dirichlet unit theorem.

In [18] and [19], we considered a higher dimensional analogue of (0.1). In the higher dimensional case, the condition “$\overline{\deg}(\overline{D}) \geq 0$” should be replaced by the pseudo-effectivity of $\overline{D}$. Of course, this analogue is not true in general (cf. [5]). It is however a very interesting problem to find a sufficient condition for the existence of an arithmetic small $\mathcal{R}$-section, that is, an element $x$ such that
\[
x = x_1^{a_1} \cdots x_r^{a_r} \quad (x_1, \ldots, x_r \text{ are rational functions and } a_1, \ldots, a_r \in \mathcal{R})
\]
and $\overline{D} + (x)_\mathcal{R}$ is effective. For example, in [18] and [19], we proved that if $D$ is numerically trivial and $\overline{D}$ is pseudo-effective, then $\overline{D}$ has an arithmetic small $\mathcal{R}$-section. In this paper, we would like to consider a geometric analogue of the Dirichlet unit theorem in the above sense.

Let $X$ be a normal projective variety over an algebraically closed field $k$. Let $\text{Div}(X)$ denote the group of Cartier divisors on $X$. Let $\mathbb{K}$ be either the field $\mathbb{Q}$ of rational numbers or the field $\mathbb{R}$ of real numbers. We define $\text{Div}(X)_{\mathbb{K}}$ to be $\text{Div}(X)_{\mathbb{K}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K}$, whose element is called a $\mathbb{K}$-Cartier divisor on $X$. For $\mathbb{K}$-Cartier divisors $D_1$ and $D_2$, we say that $D_1$ is $\mathbb{K}$-linearly equivalent to $D_2$, which
is denoted by $D_1 \sim_K D_2$, if there are non-zero rational functions $\phi_1, \ldots, \phi_r$ on $X$ and $a_1, \ldots, a_r \in K$ such that
\[ D_1 - D_2 = a_1(\phi_1) + \cdots + a_r(\phi_r). \]

Let $D$ be a $K$-Cartier divisor on $X$. We say that $D$ is big if there is an ample $Q$-Cartier divisor $A$ on $X$ such that $D - A$ is $K$-linearly equivalent to an effective $K$-Cartier divisor. Further, $D$ is said to be pseudo-effective if $D + B$ is big for any big $K$-Cartier divisor $B$ on $X$. Note that if $D$ is $K$-effective (i.e. $D$ is $K$-linearly equivalent to an effective $K$-Cartier divisor), then $D$ is pseudo-effective. The converse of the above statement holds on toric varieties (for example, [4, Proposition 4.9]). However, it is not true in general. In the case where $k$ is uncountable (for example, $k = \mathbb{C}$), several examples are known such as non-torsion numerically trivial invertible sheaves and Mumford’s example on a minimal ruled surface (cf. [8, Chapter 1, Example 10.6] and [14]). Nevertheless, we would like to propose the following question:

**Question 0.2** ($K$-version). We assume that $k$ is an algebraic closure of a finite field. If a $K$-Cartier divisor $D$ on $X$ is pseudo-effective, does it follow that $D$ is $K$-effective?

This question is a geometric analogue of the fundamental question introduced in [18]. In this sense, it turns out to be a geometric Dirichlet’s unit theorem if it is true, so that we often say that a $K$-Cartier divisor $D$ has the Dirichlet property if $D$ is $K$-effective. Note that the $\mathbb{R}$-version implies the $Q$-version (cf. Proposition 1.5). Moreover, the $\mathbb{R}$-version does not hold in general. In Example 3.2, we give an example, so that, for the $\mathbb{R}$-version, the question should be

“Under what conditions does it follow that $D$ is $K$-effective?”.

Further, the $Q$-version implies the following question due to Keel (cf. [10, Question 0.9] and Remark 2.4). The similar arguments on an algebraic surface are discussed in the recent article by Langer [12, Conjecture 1.7~1.9 and Lemma 1.10].

**Question 0.3** (S. Keel). We assume that $k$ is an algebraic closure of a finite field and $X$ is an algebraic surface over $k$. Let $D$ be a Cartier divisor on $X$. If $(D \cdot C) > 0$ for all irreducible curves $C$ on $X$, is $D$ ample?

By virtue of the Zariski decomposition, Question 0.2 on an algebraic surface is equivalent to ask the following:

“If $D$ is nef, then is $D$ $K$-effective?”.

One might expect that $D$ is semiample (cf. [10, Question 0.8.2]). However, Totaro [24, Theorem 6.1] found a Cartier divisor $D$ on an algebraic surface over a finite field such that $D$ is nef but not semiample. Totaro’s example does not give a counter example of our question because we assert only the $Q$-effectivity in Question 0.2.

Inspired by the paper [3] due to Biswas and Subramanian, we have the following partial answer to the above question.
**Theorem 0.4.** We assume that $k$ is an algebraic closure of a finite field. Let $C$ be a smooth projective curve over $k$ and let $E$ be a locally free sheaf of rank $r$ on $C$. Let $\mathbb{P}(E)$ be the projective bundle of $E$, that is, $\mathbb{P}(E) := \text{Proj} \left( \bigoplus_{m=0}^{\infty} \text{Sym}^m(E) \right)$. If $D$ is a pseudo-effective $K$-Cartier divisor on $\mathbb{P}(E)$, then $D$ is $K$-effective.

In addition to the above result, we can also give an affirmative answer for the $Q$-version of Question 0.2 on abelian varieties.

**Proposition 0.5.** We assume that $k$ is an algebraic closure of a finite field. Let $A$ be an abelian variety over $k$. If $D$ is a pseudo-effective $Q$-Cartier divisor on $A$, then $D$ is $Q$-effective.

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1. Preliminaries

Let $k$ be an algebraic closed field. Let $C$ be a smooth projective curve over $k$ and let $E$ be a locally free sheaf of rank $r$ on $C$. The projective bundle $\mathbb{P}(E)$ of $E$ is given by

\[
\mathbb{P}(E) := \text{Proj} \left( \bigoplus_{m=0}^{\infty} \text{Sym}^m(E) \right).
\]

The canonical morphism $\mathbb{P}(E) \to C$ is denoted by $f_E$. A tautological divisor $\Theta_E$ on $\mathbb{P}(E)$ is a Cartier divisor on $\mathbb{P}(E)$ such that $\mathcal{O}_{\mathbb{P}(E)}(\Theta_E)$ is isomorphic to the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$. We say that $E$ is **strongly semistable** if, for any surjective morphism $\pi : C' \to C$ of smooth projective curves, $\pi^*(E)$ is semistable. By definition, if $E$ is strongly semistable and $\pi : C' \to C$ is a surjective morphism of smooth projective curves over $k$, then $\pi^*(E)$ is also strongly semistable. A filtration

\[
0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E
\]

of $E$ is called the **strong Harder-Narasimhan filtration** if

\[
\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_{s-1}/E_s) > \mu(E_s/E_{s-1})
\]

and $E_i/E_{i-1}$ is a strongly semistable locally free sheaf on $C$ for each $i = 1, \ldots, s$.

Recall the following well-known facts (F1)–(F5) on strong semistability.

(F1) A locally free sheaf $E$ on $C$ is strongly semistable if and only if $\Theta_E - f_E^*(\xi_E/r)$ is nef, where $\xi_E$ is a Cartier divisor on $C$ with $\mathcal{O}_C(\xi_E) \simeq \text{det}(E)$ (for example, see [16, Proposition 7.1, (3)]).

(F2) Let $\pi : C' \to C$ be a surjective morphism of smooth projective curves over $k$ such that the function field of $C'$ is a separable extension field over the function field of $C$. If $E$ is semistable, then $\pi^*(E)$ is also semistable (for example, see [16, Proposition 7.1, (1)])]. In particular, if $\text{char}(k) = 0$, then $E$ is strongly semistable if and only if $E$ is semistable. Moreover, in the case where $\text{char}(k) > 0$, $E$ is strongly semistable if and only if $(E^m)^*(E)$
is semistable for all \( m \geq 0 \), where \( F : C \to C \) is the absolute Frobenius map and

\[
F^m = \frac{m}{F \circ \cdots \circ F}.
\]

(F3) If \( E \) and \( G \) are strongly semistable locally free sheaves on \( C \), then \( \text{Sym}^m(E) \) and \( E \otimes G \) are also strongly semistable for all \( m \geq 1 \) (for example, see \([16, \text{Theorem 7.2 and Corollary 7.3}]\)).

(F4) There is a surjective morphism \( \pi : C' \to C \) of smooth projective curves over \( k \) such that \( \pi^*(E) \) has the strong Harder-Narasimham filtration (cf. \([11, \text{Theorem 7.2}]\)).

(F5) We assume that \( k \) is an algebraic closure of a finite field. If \( E \) is a strongly semistable locally free sheaf on \( C \) with \( \det(E) \cong \mathcal{O}_C \), then there is a surjective morphism \( \pi : C' \to C \) of smooth projective curves over \( k \) such that \( \pi^*(E) \cong \mathcal{O}_{C'}^{\oplus \text{rk} E} \) (cf. \([1, \text{p. 557}], [23, \text{Theorem 3.2}] \) and \([3]\)).

The purpose of this section is to prove the following characterizations of pseudo-effective \( \mathbb{R} \)-Cartier divisors and nef \( \mathbb{R} \)-Cartier divisors on \( \mathbb{P}(E) \). This is essentially due to Nakayama \([22, \text{Lemma 3.7}]\) in which he works over the complex number field.

**Proposition 1.1.** We assume that \( E \) has the strong Harder-Narasimham filtration:

\[
0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E.
\]

Then, for an \( \mathbb{R} \)-divisor \( A \) on \( C \), we have the following:

1. \( \Theta_E - f^*(A) \) is pseudo-effective if and only if \( \deg(A) \leq \mu(E_1) \).
2. \( \Theta_E - f^*(A) \) is nef if and only if \( \deg(A) \leq \mu(E/E_{s-1}) \).

Let us begin with the following lemma.

**Lemma 1.2.** We assume that \( E \) has a filtration

\[
0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E
\]

such that \( E_i/E_{i-1} \) is a strongly semistable locally free sheaf on \( C \) and \( \deg(E_i/E_{i-1}) < 0 \) for all \( i = 1, \ldots, s \). Then, \( H^0(C, \text{Sym}^m(E) \otimes G) = 0 \) for \( m \geq 1 \) and a strongly semistable locally free sheaf \( G \) on \( C \) with \( \deg(G) \leq 0 \).

**Proof.** We prove it by induction on \( s \). In the case where \( s = 1 \), \( E \) is strongly semistable and \( \deg(E) < 0 \), so that \( \text{Sym}^m(E) \otimes G \) is also strongly semistable by (F3) and

\[
\deg(\text{Sym}^m(E) \otimes G) < 0.
\]

Therefore, \( H^0(C, \text{Sym}^m(E) \otimes G) = 0 \).

Here we assume that \( s > 1 \). Let us consider an exact sequence

\[
0 \to E_{s-1} \to E \to E/E_{s-1} \to 0.
\]

By \([9, \text{Chapter II, Exercise 5.16, (c)}]\), there is a filtration

\[
\text{Sym}^m(E) = F^0 \supsetneq F^1 \supsetneq \cdots \supsetneq F^m \supsetneq F^{m+1} = 0
\]
such that
\[ F^j / F^{j+1} \simeq \text{Sym}^j(E_{s-1}) \otimes \text{Sym}^{m-j}(E/E_{s-1}) \]
for each \( j = 0, \ldots, m \). By using the hypothesis of induction,
\[ H^0(C, (F^j / F^{j+1}) \otimes G) = 0 \]
for \( j = 1, \ldots, m \) because \( \text{Sym}^{m-j}(E/E_{s-1}) \otimes G \) is strongly semistable by (F3) and
\[ \deg(\text{Sym}^{m-j}(E/E_{s-1}) \otimes G) \leq 0. \]
Moreover, since \( \text{Sym}^m(E/E_{s-1}) \otimes G \) is strongly semistable by (F3) and
\[ \deg(\text{Sym}^m(E/E_{s-1}) \otimes G) < 0, \]
we have
\[ H^0(C, (F^0 / F^1) \otimes G) = H^0(C, \text{Sym}^m(E/E_{s-1}) \otimes G) = 0. \]
Therefore, by using an exact sequence
\[ 0 \to F^{j+1} \otimes G \to F^j \otimes G \to (F^j / F^{j+1}) \otimes G \to 0, \]
we have
\[ H^0(C, F^{j+1} \otimes G) \to H^0(C, F^j \otimes G) \]
for \( j = 0, \ldots, m \), which implies that \( H^0(C, \text{Sym}^m(E) \otimes G) = 0 \), as required. \( \square \)

**Proof of Proposition 1.1.** It is sufficient to show the following:

(a) If \( A \) is a \( \mathbb{Q} \)-Cartier divisor and \( \deg(A) < \mu(E_1) \), then \( \Theta_E - f^*(A) \) is \( \mathbb{Q} \)-effective.

(b) If \( A \) is a \( \mathbb{Q} \)-Cartier divisor and \( \deg(A) > \mu(E_1) \), then \( \Theta_E - f^*(A) \) is not pseudo-effective.

(c) If \( \Theta_E - f^*(A) \) is nef, then \( \deg(A) \leq \mu(E/E_{s-1}) \).

(d) If \( \Theta_E - f^*(A) \) is not nef, then \( \deg(A) > \mu(E/E_{s-1}) \).

(a) Let \( \theta \) be a divisor on \( C \) with \( \deg(\theta) = 1 \). As \( E_1 \) is strongly semistable, by (F1), \( \Theta_{E_1} - \mu(E_1) f^{*}_{E_1}(\theta) \) is nef, so that we can see that \( \Theta_{E_1} - f^{*}_{E_1}(A) \) is nef and big because
\[ \Theta_{E_1} - \deg(A) f^{*}_{E_1}(\theta) = \Theta_{E_1} - \mu(E_1) f^{*}_{E_1}(\theta) + (\mu(E_1) - \deg(A)) f^{*}_{E_1}(\theta). \]
Therefore, there is a positive integer \( m_1 \) such that \( m_1 A \) is a divisor on \( C \) and
\[ H^0 \left( \mathbb{P}(E_1), \mathcal{O}_{\mathbb{P}(E_1)}(m_1 \Theta_{E_1} - f^{*}_{E_1}(m_1 A)) \right) \neq 0. \]
In addition,
\[ H^0 \left( \mathbb{P}(E_1), \mathcal{O}_{\mathbb{P}(E_1)}(m_1 \Theta_{E_1} - f^{*}_{E_1}(m_1 A)) \right) = H^0(C, \text{Sym}^{m_1}(E_1) \otimes \mathcal{O}_C(-m_1 A)) \]
\[ \subseteq H^0(C, \text{Sym}^{m_1}(E) \otimes \mathcal{O}_C(-m_1 A)) \]
\[ = H^0 \left( \mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m_1 \Theta_E - f^{*}_{E}(m_1 A)) \right), \]
so that \( \Theta_E - f^*_{E}(A) \) is \( \mathbb{Q} \)-effective.
(b) Let $B$ be an ample $\mathbb{Q}$-divisor on $C$ with $\deg(B) < \deg(A) - \mu(E_1)$. Let $\pi : C' \to C$ be a surjective morphism of smooth projective curves over $k$ such that $\pi^*(-A + B)$ is a Cartier divisor on $C'$. Note that

$$\mu((\pi^*(E_i/E_{i-1}) \otimes \mathcal{O}_{C'}(\pi^*(-A + B)))) < 0$$

for $i = 1, \ldots, s$, and hence, by Lemma 1.2,

$$H^0(C', \text{Sym}^m(\pi^*(E)) \otimes \mathcal{O}_{C'}(m\pi^*(-A + B)))) = 0$$

for all $m \geq 1$. In particular, if $b$ is a positive integer such that $b(-A + B)$ is a Cartier divisor, then

$$H^0\left(C, \text{Sym}^{mb}(E) \otimes \mathcal{O}_C(mb(-A + B))\right) = 0$$

for $m \geq 1$. Here we assume that $\Theta_E - f^*_E(A)$ is pseudo-effective. Let $a$ be a positive integer such that $\Theta_E - f^*_E(A) + af^*_E(B)$ is ample. Then

$$(a - 1)(\Theta_E - f^*_E(A)) + \Theta_E - f^*_E(A) + af^*_E(B) = a(\Theta_E + f^*_E(-A + B))$$

is big, so that we can find a positive integer $m_1$ such that

$$H^0\left(C, \text{Sym}^{m_1ab}(E) \otimes \mathcal{O}_C(m_1ab(-A + B))\right) = 0$$

which is a contradiction.

(c) Note that

$$\mathbb{P}(E/E_{s-1}) \subseteq \mathbb{P}(E), \Theta_{E/E_{s-1}} \sim \Theta_E|_{\mathbb{P}(E/E_{s-1})} \quad \text{and} \quad f_{E/E_{s-1}} = f_E|_{\mathbb{P}(E/E_{s-1})},$$

so that $\Theta_{E/E_{s-1}} - f^*_E/E_{s-1}(A)$ is nef on $\mathbb{P}(E/E_{s-1})$. Let $\xi_{E/E_{s-1}}$ be a Cartier divisor on $C$ with $\mathcal{O}_C(\xi_{E/E_{s-1}}) \simeq \det(E/E_{s-1})$. If we set $e = \text{rk}E/E_{s-1}$ and $G = \xi_{E/E_{s-1}}/e - A$, then

$$\Theta_{E/E_{s-1}} - f^*_E/E_{s-1}(A) = \Theta_{E/E_{s-1}} - f^*_E/E_{s-1}(\xi_{E/E_{s-1}}/e) + f^*_E/E_{s-1}(G).$$

Since $\Theta_{E/E_{s-1}} - f^*_E/E_{s-1}(\xi_{E/E_{s-1}}/e)$ is nef by (F1) and

$$\left(\Theta_{E/E_{s-1}} - f^*_E/E_{s-1}(\xi_{E/E_{s-1}}/e)\right)^e = 0,$$

we have

$$0 \leq \left(\Theta_{E/E_{s-1}} - f^*_E/E_{s-1}(A)\right)^e = e \deg(G).$$

Therefore, $\deg(G) \geq 0$, and hence $\deg(A) \leq \mu(E/E_{s-1})$.

(d) We can find an irreducible curve $C_0$ of $X$ such that $(\Theta_E - f^*_E(A) \cdot C_0) < 0$. Clearly $C_0$ is flat over $C$. Let $C_1$ be the normalization of $C_0$ and $h : C_1 \to C$ the induced morphism. Let us consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{P}(E) & \xrightarrow{\mathbb{P}(h)} & \mathbb{P}(h^*(E)) \\
\downarrow f_E & & \downarrow f_{h^*(E)} \\
C & \leftarrow & C_1
\end{array}$$
Note that \( \mathbb{P}(h^*) (\Theta_E - f_E^*(A)) \sim_\mathbb{R} \Theta_{h^*(E)} - f_{h^*(E)}^*(h^*(A)) \). Further, there is a section \( S \) of \( f_{h^*(E)} \) such that \( \mathbb{P}(h^*) (S) = C_0 \). Let \( Q \) be the quotient line bundle of \( h^*(E) \) corresponding to the section \( S \). As
\[
0 = h^*(E_0) \subset h^*(E_1) \subset h^*(E_2) \subset \cdots \subset h^*(E_{s-1}) \subset h^*(E_s) = h^*(E)
\]
is the Harder-Narasimham filtration of \( h^*(E) \), we can easily see
\[
\deg(Q) \geq \mu(h^*(E/E_{s-1})) = \deg(h) \mu(E/E_{s-1}).
\]
On the other hand,
\[
\deg(Q) - \deg(h) \deg(A) = (\Theta_{h^*(E)} - f_{h^*(E)}^*(h^*(A)) \cdot S) = (\Theta_E - f_E^*(A) \cdot C_0) < 0,
\]
and hence \( \mu(E/E_{s-1}) < \deg(A) \). \( \square \)

Finally let us consider the following three results.

**Lemma 1.3.** Let \( K \) be either \( Q \) or \( R \). Let \( \mu : X' \to X \) be a generically finite morphism of normal projective varieties over \( k \). For a \( K \)-Cartier divisor \( D \) on \( X \), \( D \) is \( K \)-effective if and only if \( \mu^*(D) \) is \( K \)-effective.

**Proof.** Clearly, if \( D \) is \( K \)-effective, then \( \mu^*(D) \) is \( K \)-effective. Let \( K \) and \( K' \) be the function fields of \( X \) and \( X' \), respectively. Here we assume that \( \mu^*(D) \) is \( K' \)-effective, that is, there are \( \phi'_1, \ldots, \phi'_r \in K'^* \) and \( a_1, \ldots, a_r \in K \) such that \( \mu^*(D) + a_1(\phi'_1) + \cdots + a_r(\phi'_r) \) is effective, so that
\[
\mu_*(\mu^*(D) + a_1(\phi'_1) + \cdots + a_r(\phi'_r)) = \deg(\mu)D + a_1\mu_*(\phi'_1) + \cdots + a_r\mu_*(\phi'_r)
\]
is effective. Note that \( \mu_*(\phi'_i) = (N_{K'/K}(\phi'_i)) \) (cf. [7, Proposition 1.4]), where \( N_{K'/K} \) is the norm map of \( K' \) over \( K \), and hence
\[
D + (a_1/ \deg(\mu))(N_{K'/K}(\phi'_1)) + \cdots + (a_r/ \deg(\mu))(N_{K'/K}(\phi'_r))
\]
is effective. Therefore, \( D \) is \( K \)-effective. \( \square \)

**Lemma 1.4.** Let \( K \) be either \( Q \) or \( R \). We assume that \( k \) is an algebraic closure of a finite field. Let \( X \) be a normal projective variety over \( k \) and \( D \) a \( K \)-Cartier divisor on \( X \). If \( D \) is numerically trivial, then \( D \) is \( K \)-linearly equivalent to the zero divisor.

**Proof.** If \( K = Q \), then the assertion is well-known, so that we assume that \( K = R \). We set \( D = a_1D_1 + \cdots + a_rD_r \), where \( D_1, \ldots, D_r \) are Cartier divisors on \( X \) and \( a_1, \ldots, a_r \in R \). Considering a \( Q \)-basis of \( Qa_1 + \cdots + Qa_r \) in \( R \), we may assume that \( a_1, \ldots, a_r \) are linearly independent over \( Q \). Let \( C \) be an irreducible curve on \( X \). Note that
\[
0 = (D \cdot C) = a_1(D_1 \cdot C) + \cdots + a_r(D_r \cdot C)
\]
and \( (D_1 \cdot C), \ldots, (D_r \cdot C) \in \mathbb{Z} \), and hence \( (D_1 \cdot C) = \cdots = (D_r \cdot C) = 0 \) because \( a_1, \ldots, a_r \) are linearly independent over \( Q \). Thus \( D_1, \ldots, D_r \) are numerically equivalent to zero, so that \( D_1, \ldots, D_r \) are \( Q \)-linearly equivalent to the zero divisor. Therefore, the assertion follows. \( \square \)

**Proposition 1.5.** Let \( X \) be a normal projective variety over \( k \) and let \( D \) be a \( Q \)-Cartier divisor on \( X \). If \( D \) is \( R \)-effective, then \( D \) is \( Q \)-effective.
Lemma 2.1. Let $D$ be a Cartier divisor on $X$, and $D$ is $\mathbb{R}$-effective. Let $b_1, \ldots, b_l \in \mathbb{R}$ such that $D + b_1\psi_1 + \cdots + b_l\psi_l$ is effective. We set $V = \mathbb{Q}b_1 + \cdots + \mathbb{Q}b_l \subseteq \mathbb{R}$. If $V \subseteq \mathbb{Q}$, then $b_1, \ldots, b_l \in \mathbb{Q}$, so that we may assume that $V \nsubseteq \mathbb{Q}$.

Proof. As $D$ is $\mathbb{R}$-effective, there are non-zero rational functions $\psi_1, \ldots, \psi_l$ on $X$ and $b_1, \ldots, b_l \in \mathbb{R}$ such that $D + b_1\psi_1 + \cdots + b_l\psi_l$ is effective. We set $V = \mathbb{Q}b_1 + \cdots + \mathbb{Q}b_l \subseteq \mathbb{R}$. If $V \subseteq \mathbb{Q}$, then $b_1, \ldots, b_l \in \mathbb{Q}$, so that we may assume that $V \nsubseteq \mathbb{Q}$.

Claim 1.5.1. There are non-zero rational functions $\phi_1, \ldots, \phi_r$ on $X$, $a_1, \ldots, a_r \in \mathbb{R}$ and a $\mathbb{Q}$-Cartier divisor $D'$ on $X$ such that $D \sim_{\mathbb{Q}} D'$, $D' + a_1\phi_1 + \cdots + a_r\phi_r$ is effective and $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$.

Proof. We can find a basis $a_1, \ldots, a_r$ of $V$ over $\mathbb{Q}$ with the following properties:

(i) If we set $b_i = \sum_{j=1}^r c_{ij}a_j$, then $c_{ij} \in \mathbb{Z}$ for all $i, j$.

(ii) If $V \cap \mathbb{Q} \neq \{0\}$, then $a_1 \in \mathbb{Q}^\times$.

We put $\phi_j = \prod_{i=1}^r \psi_i^{c_{ij}}$. Note that $\sum_{i=1}^r b_i(\psi_i) = \sum_{j=1}^r a_j(\phi_j)$. Therefore, in the case where $V \cap \mathbb{Q} = \{0\}$, $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$ and $D + \sum_{j=1}^r a_j(\phi_j)$ is effective. Otherwise, $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$ and $(D + a_1(\phi_1)) + \sum_{j=2}^r a_j(\phi_j)$ is effective. □

We set $L = D' + a_1(\phi_1) + \cdots + a_r(\phi_r)$. Let $\Gamma$ be a prime divisor with $\Gamma \nsubseteq \text{Supp}(L)$. Then

$$0 = \text{mult}_\Gamma(L) = \text{mult}_\Gamma(D') + a_1 \text{ord}_\Gamma(\phi_1) + \cdots + a_r \text{ord}_\Gamma(\phi_r),$$

so that $\text{mult}_\Gamma(D') = -a_1 \text{ord}_\Gamma(\phi_1) = \cdots = -a_r \text{ord}_\Gamma(\phi_r) = 0$ because $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Thus,

$$\text{Supp}(D'), \text{Supp}((\phi_1)), \ldots, \text{Supp}((\phi_r)) \subseteq \text{Supp}(L).$$

Therefore, we can find $a'_1, \ldots, a'_r \in \mathbb{Q}$ such that $D' + a'_1(\phi_1) + \cdots + a'_r(\phi_r)$ is effective, and hence $D$ is $\mathbb{Q}$-effective. □

2. Proof of Theorem 0.4

Let $k$ be an algebraic closure of a finite field. Let $C$ be a smooth projective curve over $k$. Let us begin with the following lemma.

Lemma 2.1. Let $K$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let $A$ be a $K$-Cartier divisor on $C$. If $\text{deg}(A) \geq 0$, then $A$ is $K$-effective.

Proof. If $K = \mathbb{Q}$, then the assertion is obvious. We assume that $K = \mathbb{R}$. If $\text{deg}(A) = 0$, the assertion follows from Lemma 1.4. Next we consider the case where $\text{deg}(A) > 0$. We can find a $\mathbb{Q}$-Cartier divisor $A'$ such that $A' \leq A$ and $\text{deg}(A') > 0$. Thus the previous observation implies the assertion. □

As a consequence of (F3), (F4) and (F5), we have the following splitting theorem, which was obtained by Biswas and Parameswaran [2, Proposition 2.1].

Theorem 2.2. For a locally free sheaf $E$ on $C$, there are a surjective morphism $\pi : C' \to C$ of smooth projective curves over $k$ and invertible sheaves $L_1, \ldots, L_r$ on $C'$ such that $\pi^*(E) \cong L_1 \oplus \cdots \oplus L_r$. 
Proof. For reader’s convenience, we give a sketch of the proof. First we assume that $E$ is strongly semistable. Let $\xi_E$ be a Cartier divisor on $C$ with $O_C(\xi_E) \simeq \det(E)$. Let $h : B \to C$ be a surjective morphism of smooth projective curves over $k$ such that $h^*(\xi_E)$ is divisible by $\mathrm{rk}(E)$. We set $E' = h^*(E) \otimes O_B(-h^*(\xi_E)/\mathrm{rk}(E))$. As $\det(E') \simeq O_B$, the assertion follows from (F5).

By the above observation, it is sufficient to find a surjective morphism $\pi : C' \to C$ of smooth projective curves over $k$ and strongly semistable locally free sheaves $Q_1, \ldots, Q_n$ on $C'$ such that

$$\pi^*(E) = Q_1 \oplus \cdots \oplus Q_n.$$  

Moreover, by (F4), we may assume that $E$ has the strong Harder-Narasimham filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = E.$$  

Clearly we may further assume that $n \geq 2$. For a non-negative integer $m$, we set

$$C_m := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k),$$

where the morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is given by $x \mapsto x^{1/p^m}$. Let $F_k^m : C_m \to C$ be the relative $m$-th Frobenius morphism over $k$. Put

$$G_{i,j}^m := (F_k^m)^*( (E_j/E_i) \otimes (E_i/E_{i-1})^\vee ) \otimes \omega_{C_m}$$

for $i = 1, \ldots, n-1$ and $j = i, \ldots, n$. We can find a positive integer $m$ such that

$$\mu(G_{i,j+1}^m) = p^m (\mu(E_{i+1}/E_i) - \mu(E_i/E_{i-1})) + \deg(\omega_C) < 0$$

for all $i = 1, \ldots, n-1$. By using (F3), we can see that

$$0 = G_{i,i}^m \subsetneq G_{i,i+1}^m \subsetneq G_{i,i+2}^m \subsetneq \cdots \subsetneq G_{i,n-1}^m \subsetneq G_{i,n}^m$$

is the strong Harder-Narasimham filtration of $G_{i,n}^m$, so that $H^0\left(C_m, G_{i,n}^m\right) = \{0\}$, which yields

$$\mathrm{Ext}^1((F_k^m)^*(E/E_i), (F_k^m)^*(E_i/E_{i-1})) = 0$$

because of Serre’s duality theorem. Therefore, an exact sequence

$$0 \to (F_k^m)^*(E_i/E_{i-1}) \to (F_k^m)^*(E/E_{i-1}) \to (F_k^m)^*(E/E_i) \to 0$$

splits, that is, $(F_k^m)^*(E_i/E_{i-1}) \simeq (F_k^m)^*(E_i/E_{i-1}) \oplus (F_k^m)^*(E/E_i)$ for $i = 1, \ldots, n - 1$, and hence

$$(F_k^m)^*(E) \simeq \bigoplus_{i=1}^n (F_k^m)^*(E_i/E_{i-1}),$$

as required. \hfill \square

Proof of Theorem 0.4. By virtue of Theorem 2.2 and Lemma 1.3, we may assume that

$$E \simeq L_1 \oplus \cdots \oplus L_r$$

for some invertible sheaves $L_1, \ldots, L_r$ on $C$. We set

$$d = \max\{\deg(L_1), \ldots, \deg(L_r)\} \quad \text{and} \quad I = \{i \mid \deg(L_i) = d\}.$$
There is a $\mathcal{K}$-Cartier divisor $A$ on $C$ such that $D \sim_{\mathcal{K}} \lambda \Theta_E - f^*_E(A)$ for some $\lambda \in \mathcal{K}$. Let $M$ be an ample divisor on $C$ such that $T := \Theta_E + f^*_E(M)$ is ample. As $D$ is pseudo-effective, we have

$$0 \leq (D \cdot T^{r-2} \cdot f^*_E(M)) = ((\lambda T - f^*_E(A + \lambda M)) \cdot T^{r-2} \cdot f^*_E(M)) = \lambda \deg(M),$$

and hence $\lambda \geq 0$. If $\lambda = 0$, then $0 \leq (D \cdot T^{r-1}) = \deg(-A)$. Thus, by Lemma 2.1, $-A$ is $\mathcal{K}$-effective, so that the assertion follows.

We assume that $\lambda > 0$. Replacing $D$ by $D/\lambda$, we may assume that $\lambda = 1$. Let $\xi$ be a Cartier divisor on $C$ such that $\mathcal{O}_C(\xi) \simeq L_{i_0}$ for some $i_0 \in I$. Note that the first part $E_1$ of the strong Harder-Narasimham filtration of $E$ is $\bigoplus_{i \in I} L_i$, so that, by Proposition 1.1, $\deg(A) \leq \deg(\xi)$. If we set $B = \xi - A$, then, by Lemma 2.1, $B$ is $\mathcal{K}$-effective because $\deg(B) \geq 0$. Moreover, as

$$\Theta_E - f^*_E(A) = \Theta_E - f^*_E(\xi) + f^*_E(B),$$

it is sufficient to consider the case where $D = \Theta_E - f^*_E(\xi)$. In this case, the assertion is obvious because

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(D)) = H^0(C, E \otimes \mathcal{O}_C(-\xi)) = H^0\left(C, \bigoplus_{i=1}^r L_i \otimes \mathcal{O}_C(-\xi)\right) \neq \{0\}.$$

As a consequence of Theorem 0.4, we can recover a result due to [3].

**Corollary 2.3.** Let $k$, $C$ and $E$ be same as in Theorem 0.4. We assume that $r = 2$. Let $D$ be a Cartier divisor on $\mathbb{P}(E)$ such that $(D \cdot Y) > 0$ for all irreducible curves $Y$ on $\mathbb{P}(E)$. Then $D$ is ample.

**Proof.** As $D$ is nef, $D$ is pseudo-effective, so that, by Theorem 0.4, there is an effective $\mathbb{Q}$-Cartier divisor $E$ on $X$ such that $D \sim_{\mathbb{Q}} E$. As $E \neq 0$, we have $(D \cdot D) = (D \cdot E) > 0$. Therefore, $D$ is ample by Nakai-Moishezon criterion.

**Remark 2.4.** The argument in the proof of Corollary 2.3 actually shows that the $\mathbb{Q}$-version of Question 0.2 on algebraic surfaces implies Question 0.3.

### 3. Numerically effectivity on abelian varieties

The purpose of this section is to give an affirmative answer for the $\mathbb{Q}$-version of Question 0.2 on abelian varieties. Let $A$ be an abelian variety over an algebraically closed field $k$. A key observation is the following proposition.

**Proposition 3.1.** If a $\mathbb{Q}$-Cartier divisor $D$ on $A$ is nef, then $D$ is numerically equivalent to a $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier divisor.

**Proof.** We prove it by induction on $\dim A$. If $\dim A \leq 1$, then the assertion is obvious. Clearly we may assume that $D$ is a Cartier divisor, so that we set $L = \mathcal{O}_A(D)$. As $L \otimes [-1]^*L$ is numerically equivalent to $L^{\otimes 2}$ (cf. [21, p.75, (iv)]), we may assume that $L$ is symmetric, that is, $L \simeq [-1]^*L$. Let $K(L)$ be the closed subgroup of $A$ given by $K(L) = \{x \in A \mid T^*_x(L) \simeq L\}$ (cf. [21, p.60, Definition]). If $K(L)$ is finite, then $L$ is nef and big by virtue of [21, p.150,
The Riemann-Roch theorem], so that $D$ is $Q$-effective. Otherwise, let $B$ be the connected component of $K(L)$ containing 0.

**Claim 3.1.1.** \[1\] \( T^*_x(L)|_B \simeq L|_B \) for all \( x \in A \).

\( 2 \) \( L^{\otimes 2}|_{B+x} \simeq \mathcal{O}_{B+x} \) for \( x \in A \).

**Proof.** (1) Let $N$ be an invertible sheaf on $A \times A$ given by

$$N = m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1}) ,$$

where $p_i : A \times A \to A$ is the projection to the $i$-th factor ($i = 1, 2$) and $m$ is the addition morphism. Note that $N|_{B \times A} \simeq \mathcal{O}_{B \times A}$ (cf. [21, p.123, §13]). Fixing $x \in A$, let us consider a morphism $\alpha : B \to B \times A$ given by $\alpha(y) = (y, x)$. Then

\[
\mathcal{O}_B \simeq \alpha^* \left( m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1}) \right) \mid_{B \times A} \simeq T^*_x(L)|_B \otimes L^{-1}|_B ,
\]

as required.

(2) First we consider the case where $x = 0$. As $N|_{B \times A} \simeq \mathcal{O}_{B \times A}$, we have $N|_{B \times B} \simeq \mathcal{O}_{B \times B}$. Using a morphism $\beta : B \to B \times B$ given by $\beta(y) = (y, -y)$, we have

\[
\mathcal{O}_B \simeq \beta^* (N|_{B \times B}) = L^{-1}\mid_B \otimes [-1]^*(L^{-1})\mid_B \simeq L^{\otimes -2}|_B ,
\]

as required.

In general, for $x \in A$, by (1) and the previous observation together with the following commutative diagram

\[
\begin{array}{ccc}
B + x & \longrightarrow & A \\
\downarrow T_{-x} & & \downarrow T_{-x} \\
B & \longrightarrow & A,
\end{array}
\]

we can see

\[
\mathcal{O}_{B+x} = T^*_x(\mathcal{O}_B) \simeq T^*_x \left( L^{\otimes 2}\mid_B \right) \simeq T^*_x \left( T^*_x(L)^{\otimes 2}\mid_B \right) = T^*_x \left( T^*_x(L^{\otimes 2})\mid_B \right) = T^*_x \left( T^*_x(L^{\otimes 2})\right)\mid_{B+x} = L^{\otimes 2}|_{B+x} .
\]

\[\square\]

Let $\pi : A \to A/B$ be the canonical homomorphism. By (2) in the above claim,

\[
\dim_{k(y)} H^0 \left( \pi^{-1}(y), L^{\otimes 2} \right) = 1
\]

for all $y \in A/B$, so that, by [21, p.51, Corollary 2], $\pi_* (L^{\otimes 2})$ is an invertible sheaf on $A/B$ and $\pi_* (L^{\otimes 2}) \otimes k(y) \twoheadrightarrow H^0 (\pi^{-1}(y), L^{\otimes 2})$. Therefore, the natural homomorphism $\pi^* (\pi_*(L^{\otimes 2})) \to L^{\otimes 2}$ is an isomorphism, that is, there is a $Q$-Cartier divisor $D'$ on $A/B$ such that $\pi^* (D') \sim_Q D$. Note that $D'$ is also nef, so that, by the hypothesis of induction, $D'$ is numerically equivalent to a $Q$-effective $Q$-Cartier divisor, and hence the assertion follows. \[\square\]

**Proof of Proposition 0.5.** Proposition 0.5 is a consequence of Lemma 1.4 and Proposition 3.1 because a pseudo-effective $Q$-Cartier divisor on an abelian variety is nef. \[\square\]
Example 3.2. Here we show that the \( \mathbb{R} \)-version of Question 0.2 does not hold in general. Let \( k \) be an algebraically closed field (\( k \) is not necessarily an algebraic closure of a finite field). Let \( C \) be an elliptic curve over \( k \) and \( A := C \times C \). Let \( \text{NS}(A) \) be the Néron-Severi group of \( A \). Note that \( \rho := \text{rk}\, \text{NS}(A) \geq 3 \). By using the Hodge index theorem, we can find a basis \( e_1, \ldots, e_\rho \) of \( \text{NS}(A)_\mathbb{Q} := \text{NS}(A) \otimes_\mathbb{Z} \mathbb{Q} \) with the following properties:

1. \( e_1 \) is the class of the divisor \( \{0\} \times C + C \times \{0\} \). In particular, \( (e_1 \cdot e_1) = 2 \).
2. \( (e_i \cdot e_i) < 0 \) for all \( i = 2, \ldots, \rho \).
3. \( (e_i \cdot e_j) = 0 \) for all \( 1 \leq i \neq j \leq \rho \).

We set \( \lambda_i := -(e_i \cdot e_i) \) for \( i = 2, \ldots, \rho \). Let \( \overline{\text{Amp}}(A) \) be the closed cone in \( \text{NS}(A)_\mathbb{R} := \text{NS}(A) \otimes_\mathbb{Z} \mathbb{R} \) generated by ample \( \mathbb{Q} \)-Cartier divisors on \( A \). It is well known that

\[
\text{Amp}(A) = \left\{ \xi \in \text{NS}(A)_\mathbb{R} \mid (\xi^2) \geq 0, \ (\xi \cdot e_1) \geq 0 \right\}
\]

\[
= \left\{ x_1 e_1 + x_2 e_2 + \cdots + x_\rho e_\rho \mid \lambda_2 x_2^2 + \cdots + \lambda_\rho x_\rho^2 \leq 2x_1^2, \ x_1 \geq 0 \right\}.
\]

We choose \( (a_2, \ldots, a_\rho) \in \mathbb{R}^{\rho-1} \) such that

\[
(a_2, \ldots, a_\rho) \notin \mathbb{Q}^{\rho-1} \quad \text{and} \quad \lambda_2 a_2^2 + \cdots + \lambda_\rho a_\rho^2 = 2.
\]

Let \( E_i \) be a \( \mathbb{Q} \)-Cartier divisor on \( A \) such that the class of \( E_i \) in \( \text{NS}(A)_\mathbb{Q} \) is equal to \( e_i \) for \( i = 1, \ldots, \rho \). If we set \( D := E_1 + a_2 E_2 + \cdots + a_\rho E_\rho \), then we have the following claim, which is sufficient for our purpose.

Claim 3.2.1. \( D \) is nef and \( D \) is not numerically equivalent to an effective \( \mathbb{R} \)-Cartier divisor.

Proof. Clearly \( D \) is nef. If we set \( e_i' = e_i/\sqrt{2} \) and \( e_i' = e_i/\sqrt{\lambda_i} \) for \( i = 2, \ldots, \rho \), then

\[
\overline{\text{Amp}}(A) = \left\{ y_1 e_1' + y_2 e_2' + \cdots + y_\rho e_\rho' \mid y_2^2 + \cdots + y_\rho^2 \leq y_1^2, \ y_1 \geq 0 \right\}.
\]

Therefore, as \([D] \in \partial(\overline{\text{Amp}}(A)_\mathbb{R})\), we can choose

\[
H \in \text{Hom}_\mathbb{R}(\text{NS}(A)_\mathbb{R}, \mathbb{R})
\]

such that

\[
H \geq 0 \text{ on } \overline{\text{Amp}}(A) \quad \text{and} \quad \{H = 0\} \cap \overline{\text{Amp}}(A) = \mathbb{R}_{\geq 0}[D],
\]

where \([D]\) is the class of \( D \) in \( \text{NS}(A)_\mathbb{R} \). We assume that \( D \) is numerically equivalent to an effective \( \mathbb{R} \)-Cartier divisor \( c_1 \Gamma_1 + \cdots + c_r \Gamma_r \), where \( c_1, \ldots, c_r \in \mathbb{R}_{>0} \) and \( \Gamma_1, \ldots, \Gamma_r \) are prime divisors on \( A \). As \([D]\) \neq 0, we may assume that \( c_1, \ldots, c_r \in \mathbb{R}_{>0} \). Note that \([\Gamma_1], \ldots, [\Gamma_r] \in \overline{\text{Amp}}(A)\) and

\[
0 = H([D]) = c_1 H([\Gamma_1]) + \cdots + c_r H([\Gamma_r]),
\]

so that \( H([\Gamma_1]) = \cdots = H([\Gamma_r]) = 0 \), and hence \([\Gamma_1], \ldots, [\Gamma_r] \in \mathbb{R}_{\geq 0}[D]\). In particular, there is \( t \in \mathbb{R}_{\geq 0} \) with \([\Gamma_1] = t[D] \). Here we can set

\[
[\Gamma_1] = b_1 e_1 + \cdots + b_\rho e_\rho \quad (b_1, \ldots, b_\rho \in \mathbb{Q}).
\]
Thus $b_1 = t$, $b_2 = ta_2, \ldots, b_\rho = ta_\rho$. As $[\Gamma_1] \neq 0$, $t \in \mathbb{Q}^\times$, and hence $(a_2, \ldots, a_\rho) = t^{-1}(b_2, \ldots, b_\rho) \in \mathbb{Q}^{\rho-1}$. This is a contradiction. 

**Remark 3.3.** Let $k$ be an algebraic closure of a finite field and let $X$ be a normal projective variety over $k$. Let $\text{NS}(X)$ be the Néron-Severi group of $X$ and $\text{NS}(X)_R := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\overline{\text{Eff}}(X)$ be the closed cone in $\text{NS}(X)_R$ generated by pseudo-effective $\mathbb{R}$-Cartier divisors on $X$. We assume that $\overline{\text{Eff}}(X)$ is a rational polyhedral cone, that is, there are pseudo-effective $\mathbb{Q}$-Cartier divisors $D_1, \ldots, D_n$ on $X$ such that $\overline{\text{Eff}}(X)$ is generated by the classes of $D_1, \ldots, D_n$. Then the $\mathbb{Q}$-version of Question 0.2 implies the $\mathbb{R}$-version of Question 0.2.

**Example 3.4.** This is an example due to Yuan [25]. Let us fix an algebraically closed field $k$ and an integer $g \geq 2$. Let $C$ be a smooth projective curve over $k$ and $f : X \to C$ an abelian scheme over $C$ of relative dimension $g$. Let $L$ be an $f$-ample invertible sheaf on $X$ such that $[-1]^*(L) \simeq L$ and $L$ is trivial along the zero section of $f : X \to C$.

**Claim 3.4.1.**
2. $L$ is nef.

**Proof.** (1) As $[2]^*(L)|_{f^{-1}(x)} \simeq L^\otimes 4|_{f^{-1}(x)}$ for all $x \in C$, there is an invertible sheaf $M$ on $C$ such that $[2]^*(L) \simeq L^\otimes 4 \otimes f^*(M)$. Let $Z_0$ be the zero section of $f : X \to C$. Then
\[
\mathcal{O}_{Z_0} \simeq [2]^*(L|_{Z_0}) = [2]^*(L)|_{Z_0} \simeq L^\otimes 4 \otimes f^*(M)|_{Z_0} \simeq M,
\]
so that we have the assertion.

(2) Let $A$ be an ample invertible sheaf on $C$ such that $L \otimes f^*(A)$ is ample. Let $\Delta$ be a horizontal curve on $X$. As $f \circ [2^n] = f$ and $[2^n]^*(L) \simeq L^\otimes 4^n$ by using (1),
\[
0 \leq (L \otimes f^*(A) \cdot [2^n]^*(\Delta)) = ([2^n]^*(L \otimes f^*(A)) \cdot \Delta) = (L^\otimes 4^n \otimes f^*(A) \cdot \Delta),
\]
so that $(L \cdot \Delta) \geq -4^{-n}(f^*(A) \cdot \Delta)$ for all $n > 0$. Thus $(L \cdot \Delta) \geq 0$. 

**Claim 3.4.2.** If the characteristic of $k$ is zero and $f$ is non-isotrivial, then $L$ does not have the Dirichlet property (i.e. $L$ is not $\mathbb{Q}$-effective).

**Proof.** The following proof is due to Yuan [25]. An alternative proof can be found in [6, Theorem 4.3]. We need to see that $H^0(X, L^\otimes n) = 0$ for all $n > 0$. We set $d_n = \text{rk} f_*(L^\otimes n)$. By changing the base $C$ if necessarily, we may assume that all $(d_n)^2$-torsion points on the generic fiber $X_\eta$ of $f : X \to C$ are defined over the function field of $C$. By using the algebraic theta theory due to Mumford (especially [20, the last line in page 81]), there is an invertible sheaf $M$ on $C$ such that $f_*(L^\otimes n) = M^\otimes d_n$. On the other hand, by [13],
\[
\deg(\det(f_*(L^\otimes n)) \otimes 2 \otimes f_*(\omega_{X/C})^\otimes d_n) = 0,
\]
that is, $2 \deg(M) + \deg(f_*(\omega_{X/C})) = 0$. As $f$ is non-isotrivial, we can see that $\deg(f_*(\omega_{X/C})) > 0$, so that $\deg(M) < 0$, and hence the assertion follows. 

If the characteristic of $k$ is positive, we do not know the $\mathbb{Q}$-effectivity of $L$ in general. In [15], there is an example with the following properties:
(1) \( g = 2 \) and \( C = \mathbb{P}^1_k \).

(2) There are an abelian surface \( A \) over \( k \) and an isogeny \( h : A \times \mathbb{P}^1_k \to X \) over \( \mathbb{P}^1_k \).

**Claim 3.4.3.** In the above example, \( L \) has the Dirichlet property.

**Proof.** Replacing \( L \) by \( L^\otimes n \), we may assume that \( d := \text{rk} \ f_*(L) > 0 \). Let

\[
p_1 : A \times \mathbb{P}^1_k \to A \quad \text{and} \quad p_2 : A \times \mathbb{P}^1_k \to \mathbb{P}^1_k
\]

be the projections to \( A \) and \( \mathbb{P}^1_k \), respectively. Note that \( h^*(L) \) is symmetric and \( h^*(L) \) is trivial along the zero section of \( p_2 \). Since \( \omega_{A \times \mathbb{P}^1_k/\mathbb{P}^1_k} \cong p_1^*(\omega_A) \), we have

\[
(p_2)_*(\omega_{A \times \mathbb{P}^1_k/\mathbb{P}^1_k}) \cong \mathcal{O}_{\mathbb{P}^1_k},
\]

so that, by [13], \( \deg(\det((p_2)_*(h^*(L)))) = 0 \), that is, if we set

\[
(p_2)_*(h^*(L)) = \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_d),
\]

then \( a_1 + \cdots + a_d = 0 \). Thus \( a_i \geq 0 \) for some \( i \), and hence

\[
H^0(A \times \mathbb{P}^1_k, h^*(L)) \neq 0.
\]

Therefore, \( L \) is \( \mathbb{Q} \)-effective by Lemma 1.3. \( \square \)

The above claim suggests that the set of preperiodic points of the map \([2] : X \to X\) is not dense in the analytification \( X_v^{\text{an}} \) at any place \( v \) of \( \mathbb{P}^1_k \) with respect to the analytic topology (cf. [5]).

**References**


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