TOWARD A GEOMETRIC ANALOGUE OF DIRICHLET'S UNIT THEOREM

ATSUSHI MORIWAKI

ABSTRACT. In this article, we propose a geometric analogue of Dirichlet’s unit theorem on arithmetic varieties [18], that is, if \( X \) is a normal projective variety over a finite field and \( D \) is a pseudo-effective \( \mathbb{Q} \)-Cartier divisor on \( X \), does it follow that \( D \) is \( \mathbb{Q} \)-effective? We also give affirmative answers on an abelian variety and a projective bundle over a curve.

INTRODUCTION

Let \( K \) be a number field and \( O_K \) the ring of integers in \( K \). Let \( K(\mathbb{C}) \) be the set of all embeddings \( K \hookrightarrow \mathbb{C} \). For \( \sigma \in K(\mathbb{C}) \), the complex conjugation of \( \sigma \) is denoted by \( \overline{\sigma} \), that is, \( \overline{\sigma}(x) = \overline{\sigma(x)} \) (\( x \in K \)). Here we define \( X_K \) and \( X_0^K \) to be

\[
X_K := \{ x \in O_K^{\times} | \overline{\sigma}(x) = \sigma(\overline{x}) \ (\forall \sigma) \},
\]

\[
X_0^K := \{ x \in X_K | \sum_{\sigma \in K(\mathbb{C})} \overline{\sigma}(x) = 0 \}.
\]

The Dirichlet unit theorem asserts that the group \( O_K^{\times} \) consisting of units in \( O_K \) is a finitely generated abelian group of rank \( s := \dim_{\mathbb{R}} X_0^K \).

Let us consider the homomorphism \( L : K^{\times} \rightarrow R^{K(\mathbb{C})} \) given by

\[
L(x)(\sigma) := \log |\sigma(x)| \quad (x \in K^{\times}, \sigma \in K(\mathbb{C})).
\]

It is easy to see the following:

(i) For a compact set \( B \) in \( R^{K(\mathbb{C})} \), the set \( \{ x \in O_K^{\times} | L(x) \in B \} \) is finite.
(ii) \( L : K^{\times} \rightarrow R^{K(\mathbb{C})} \) extends to \( L_{\mathbb{R}} : K^{\times} \otimes \mathbb{R} \rightarrow R^{K(\mathbb{C})} \).
(iii) \( L_{\mathbb{R}} : O_K^{\times} \otimes \mathbb{R} \rightarrow R^{K(\mathbb{C})} \) is injective.
(iv) \( L_{\mathbb{R}}(O_K^{\times} \otimes \mathbb{R}) \subseteq X_0^K \).

By using (i) and (iii), we can see that \( O_K^{\times} \) is a finitely generated abelian group. The most essential part of the Dirichlet unit theorem is to show that \( O_K^{\times} \) is of rank \( s \), which is equivalent to see that, for any \( \xi \in X_0^K \), there is \( x \in O_K^{\times} \otimes \mathbb{R} \) with \( L_{\mathbb{R}}(x) = \xi \).

In order to understand the equality \( L_{\mathbb{R}}(x) = \xi \) in terms of Arakelov geometry, let us introduce several notations for arithmetic divisors on the arithmetic curve \( \text{Spec}(O_K) \). An arithmetic \( \mathbb{R} \)-divisor on \( \text{Spec}(O_K) \) is a pair \((D, \xi)\) consisting of an \( \mathbb{R} \)-divisor \( D \) on \( \text{Spec}(O_K) \) and \( \xi \in X_K \). We often denote the pair \((D, \xi)\) by \( \mathbf{D} \). The
arithmetic principal $\mathbb{R}$-divisor $(x)_{\mathbb{R}}$ of $x \in K^\times \otimes \mathbb{R}$ is the arithmetic $\mathbb{R}$-divisor given by
\[
(x)_{\mathbb{R}} := \left( \sum_p \text{ord}_p(x)[P], -2d_{\mathbb{R}}(x) \right),
\]
where $P$ runs over the set of all maximal ideals of $O_K$ and
\[
\text{ord}_p(x) := a_1 \text{ord}_p(x_1) + \cdots + a_r \text{ord}_p(x_r)
\]
for $x = x_1^{a_1} \cdots x_r^{a_r}$ ($x_1, \ldots, x_r \in K^\times$ and $a_1, \ldots, a_r \in \mathbb{R}$). The arithmetic degree $\deg(D)$ of an arithmetic $\mathbb{R}$-divisor $D = (\sum_p a_p[P], \xi)$ is defined to be
\[
\deg(D) := \sum_p a_p \log \#(O_K/P) + \frac{1}{2} \sum_{\sigma \in \mathbb{K}(C)} \xi(\sigma).
\]
Note that
\[
\deg((x)_{\mathbb{R}}) = 0 \quad (x \in K^\times \otimes \mathbb{R})
\]
by virtue of the product formula. Further, $D = (\sum_p a_p[P], \xi)$ is said to be effective if $a_p \geq 0$ for all $P$ and $\xi(\sigma) \geq 0$ for all $\sigma$.

In [18, Subsection 3.4], we proved the following:

(0.1) “If $\deg(D) \geq 0$, then $D + (x)_{\mathbb{R}}$ is effective for some $x \in K^\times \otimes \mathbb{R}$.”

This implies the essential part of the Dirichlet unit theorem. Indeed, we set $D = (0, \xi)$ for $\xi \in \mathbb{Z}_K$. As $\deg(D) = 0$, by the assertion (0.1), $D + (y)_{\mathbb{R}}$ is effective for some $y \in K^\times \otimes \mathbb{R}$, and hence $D + (y)_{\mathbb{R}} = (0, 0)$ because $\deg(D + (y)_{\mathbb{R}}) = 0$. Here we set $y = u_1^{a_1} \cdots u_r^{a_r}$ such that $u_1, \ldots, u_r \in K^\times$, $a_1, \ldots, a_r \in \mathbb{R}$ and $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. By using the linear independency of $a_1, \ldots, a_r$ over $\mathbb{Q}$, $\text{ord}_p(y) = 0$ implies $\text{ord}_p(u_i) = 0$ for all maximal ideals $P$ of $O_K$ and $i = 1, \ldots, r$, that is, $u_i \in O_K^\times$ for $i = 1, \ldots, r$. Therefore, $\xi = L_{\mathbb{R}}(y^2)$ and $y \in O_K^\times \otimes \mathbb{R}$, as required. In this sense, the above property (0.1) is an Arakelov theoretic interpretation of the classical Dirichlet unit theorem.

In [18] and [19], we considered a higher dimensional analogue of (0.1). In the higher dimensional case, the condition “$\deg(D) \geq 0$” should be replaced by the pseudo-effectivity of $D$. Of course, this analogue is not true in general (cf. [5]). It is however a very interesting problem to find a sufficient condition for the existence of an arithmetic small $\mathbb{R}$-section, that is, an element $x$ such that
\[
x = x_1^{a_1} \cdots x_r^{a_r} \quad (x_1, \ldots, x_r \text{ are rational functions and } a_1, \ldots, a_r \in \mathbb{R})
\]
and $D + (x)_{\mathbb{R}}$ is effective. For example, in [18] and [19], we proved that if $D$ is numerically trivial and $D$ is pseudo-effective, then $D$ has an arithmetic small $\mathbb{R}$-section. In this paper, we would like to consider a geometric analogue of the Dirichlet unit theorem in the above sense.

Let $X$ be a normal projective variety over an algebraically closed field $k$. Let $\text{Div}(X)$ denote the group of Cartier divisors on $X$. Let $\mathbb{K}$ be either the field $\mathbb{Q}$ of rational numbers or the field $\mathbb{R}$ of real numbers. We define $\text{Div}(X)_{\mathbb{K}}$ to be $\text{Div}(X)_{\mathbb{K}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K}$, whose element is called a $\mathbb{K}$-Cartier divisor on $X$. For $\mathbb{K}$-Cartier divisors $D_1$ and $D_2$, we say that $D_1$ is $\mathbb{K}$-linearly equivalent to $D_2$, which
is denoted by $D_1 \sim_\mathbb{K} D_2$, if there are non-zero rational functions $\phi_1, \ldots, \phi_r$ on $X$ and $a_1, \ldots, a_r \in \mathbb{K}$ such that

$$D_1 - D_2 = a_1(\phi_1) + \cdots + a_r(\phi_r).$$

Let $D$ be a $\mathbb{K}$-Cartier divisor on $X$. We say that $D$ is big if there is an ample $\mathbb{Q}$-Cartier divisor $A$ on $X$ such that $D - A$ is $\mathbb{K}$-linearly equivalent to an effective $\mathbb{K}$-Cartier divisor. Further, $D$ is said to be pseudo-effective if $D + B$ is big for any big $\mathbb{K}$-Cartier divisor $B$ on $X$. Note that if $D$ is $\mathbb{K}$-effective (i.e. $D$ is $\mathbb{K}$-linearly equivalent to an effective $\mathbb{K}$-Cartier divisor), then $D$ is pseudo-effective. The converse of the above statement holds on toric varieties (for example, [4, Proposition 4.9]). However, it is not true in general. In the case where $k$ is uncountable (for example, $k = \mathbb{C}$), several examples are known such as non-torsion numerically trivial invertible sheaves and Mumford’s example on a minimal ruled surface (cf. [8, Chapter 1, Example 10.6] and [14]). Nevertheless, we would like to propose the following question:

**Question 0.2** ($\mathbb{K}$-version). We assume that $k$ is an algebraic closure of a finite field. If a $\mathbb{K}$-Cartier divisor $D$ on $X$ is pseudo-effective, does it follow that $D$ is $\mathbb{K}$-effective?

This question is a geometric analogue of the fundamental question introduced in [18]. In this sense, it turns out to be a geometric Dirichlet’s unit theorem if it is true, so that we often say that a $\mathbb{K}$-Cartier divisor $D$ has the Dirichlet property if $D$ is $\mathbb{K}$-effective. Note that the $\mathbb{R}$-version implies the $\mathbb{Q}$-version (cf. Proposition 1.5). Moreover, the $\mathbb{R}$-version does not hold in general. In Example 3.2, we give an example, so that, for the $\mathbb{R}$-version, the question should be

“Under what conditions does it follow that $D$ is $\mathbb{K}$-effective?”. 

Further, the $\mathbb{Q}$-version implies the following question due to Keel (cf. [10, Question 0.9] and Remark 2.4). The similar arguments on an algebraic surface are discussed in the recent article by Langer [12, Conjecture 1.7–1.9 and Lemma 1.10].

**Question 0.3** (S. Keel). We assume that $k$ is an algebraic closure of a finite field and $X$ is an algebraic surface over $k$. Let $D$ be a Cartier divisor on $X$. If $(D \cdot C) > 0$ for all irreducible curves $C$ on $X$, is $D$ ample?

By virtue of the Zariski decomposition, Question 0.2 on an algebraic surface is equivalent to ask the following:

“If $D$ is nef, then is $D$ $\mathbb{K}$-effective?”.

One might expect that $D$ is semiample (cf. [10, Question 0.8.2]). However, Totaro [24, Theorem 6.1] found a Cartier divisor $D$ on an algebraic surface over a finite field such that $D$ is nef but not semiample. Totaro’s example does not give a counter example of our question because we assert only the $\mathbb{Q}$-effectivity in Question 0.2.

Inspired by the paper [3] due to Biswas and Subramanian, we have the following partial answer to the above question.
**Theorem 0.4.** We assume that $k$ is an algebraic closure of a finite field. Let $C$ be a smooth projective curve over $k$ and let $E$ be a locally free sheaf of rank $r$ on $C$. Let $\mathbb{P}(E)$ be the projective bundle of $E$, that is, $\mathbb{P}(E) := \text{Proj}\left(\bigoplus_{m=0}^{\infty} \text{Sym}^m(E)\right)$. If $D$ is a pseudo-effective $\mathbb{K}$-Cartier divisor on $\mathbb{P}(E)$, then $D$ is $\mathbb{K}$-effective.

In addition to the above result, we can also give an affirmative answer for the $\mathbb{Q}$-version of Question 0.2 on abelian varieties.

**Proposition 0.5.** We assume that $k$ is an algebraic closure of a finite field. Let $A$ be an abelian variety over $k$. If $D$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor on $A$, then $D$ is $\mathbb{Q}$-effective.

Finally I would like to thank Prof. Biswas, Prof. Keel, Prof. Langer, Prof. Tanaka and Prof. Totaro for their helpful comments. Especially I would like to thank the referee for the suggestions.

1. Preliminaries

Let $k$ be an algebraic closed field. Let $C$ be a smooth projective curve over $k$ and let $E$ be a locally free sheaf of rank $r$ on $C$. The projective bundle $\mathbb{P}(E)$ of $E$ is given by

$$\mathbb{P}(E) := \text{Proj}\left(\bigoplus_{m=0}^{\infty} \text{Sym}^m(E)\right).$$

The canonical morphism $\mathbb{P}(E) \to C$ is denoted by $f_E$. A tautological divisor $\Theta_E$ on $\mathbb{P}(E)$ is a Cartier divisor on $\mathbb{P}(E)$ such that $\mathcal{O}_{\mathbb{P}(E)}(\Theta_E)$ is isomorphic to the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$. We say that $E$ is strongly semistable if, for any surjective morphism $\pi : C' \to C$ of smooth projective curves, $\pi^*(E)$ is semistable. By definition, if $E$ is strongly semistable and $\pi : C' \to C$ is a surjective morphism of smooth projective curves over $k$, then $\pi^*(E)$ is also strongly semistable. A filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E$$

of $E$ is called the strong Harder-Narasimhan filtration if

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_{s-1}/E_{s-2}) > \mu(E_s/E_{s-1})$$

and $E_i/E_{i-1}$ is a strongly semistable locally free sheaf on $C$ for each $i = 1, \ldots, s$. Recall the following well-known facts (F1)–(F5) on strong semistability.

(F1) A locally free sheaf $E$ on $C$ is strong semistable if and only if $\Theta_E - f_E^*(\zeta_E/r)$ is nef, where $\zeta_E$ is a Cartier divisor on $C$ with $\mathcal{O}_C(\zeta_E) \simeq \text{det}(E)$ (for example, see [16, Proposition 7.1, (3)]).

(F2) Let $\pi : C' \to C$ be a surjective morphism of smooth projective curves over $k$ such that the function field of $C'$ is a separable extension field over the function field of $C$. If $E$ is semistable, then $\pi^*(E)$ is also semistable (for example, see [16, Proposition 7.1, (1)]). In particular, if $\text{char}(k) = 0$, then $E$ is strongly semistable if and only if $E$ is semistable. Moreover, in the case where $\text{char}(k) > 0$, $E$ is strongly semistable if and only if $(E^m)^*(E)$
is semistable for all $m \geq 0$, where $F : C \to C$ is the absolute Frobenius map and

$$F^m = \underbrace{F \circ \cdots \circ F}_{m}.$$

(F3) If $E$ and $G$ are strongly semistable locally free sheaves on $C$, then $\text{Sym}^m(E)$ and $E \otimes G$ are also strongly semistable for all $m \geq 1$ (for example, see [16, Theorem 7.2 and Corollary 7.3]).

(F4) There is a surjective morphism $\pi : C' \to C$ of smooth projective curves over $k$ such that $\pi^*(E)$ has the strong Harder-Narasimham filtration (cf. [11, Theorem 7.2]).

(F5) We assume that $k$ is an algebraic closure of a finite field. If $E$ is a strongly semistable locally free sheaf on $C$ with $\det(E) \simeq O_C$, then there is a surjective morphism $\pi : C' \to C$ of smooth projective curves over $k$ such that $\pi^*(E) \simeq O^\oplus_{C'}^{\text{rk}E}$ (cf. [1, p. 557], [23, Theorem 3.2] and [3]).

The purpose of this section is to prove the following characterizations of pseudo-effective $R$-Cartier divisors and nef $R$-Cartier divisors on $\mathbb{P}(E)$. This is essentially due to Nakayama [22, Lemma 3.7] in which he works over the complex number field.

**Proposition 1.1.** We assume that $E$ has the strong Harder-Narasimham filtration:

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E.$$

Then, for an $R$-divisor $A$ on $C$, we have the following:

1. $\Theta_E - f^*(A)$ is pseudo-effective if and only if $\deg(A) \leq \mu(E_1)$.
2. $\Theta_E - f^*(A)$ is nef if and only if $\deg(A) \leq \mu(E/E_{s-1})$.

Let us begin with the following lemma.

**Lemma 1.2.** We assume that $E$ has a filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E$$

such that $E_i/E_{i-1}$ is a strongly semistable locally free sheaf on $C$ and $\deg(E_i/E_{i-1}) < 0$ for all $i = 1, \ldots, s$. Then, $H^0(C, \text{Sym}^m(E) \otimes G) = 0$ for $m \geq 1$ and a strongly semistable locally free sheaf $G$ on $C$ with $\deg(G) \leq 0$.

**Proof.** We prove it by induction on $s$. In the case where $s = 1$, $E$ is strongly semistable and $\deg(E) < 0$, so that $\text{Sym}^m(E) \otimes G$ is also strongly semistable by (F3) and

$$\deg(\text{Sym}^m(E) \otimes G) < 0.$$

Therefore, $H^0(C, \text{Sym}^m(E) \otimes G) = 0$.

Here we assume that $s > 1$. Let us consider an exact sequence

$$0 \to E_{s-1} \to E \to E/E_{s-1} \to 0.$$

By [9, Chapter II, Exercise 5.16, (c)], there is a filtration

$$\text{Sym}^m(E) = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} = 0$$
such that
\[ F^j / F^{j+1} \simeq \text{Sym}^j(E_{s-1}) \otimes \text{Sym}^{m-j}(E / E_{s-1}) \]
for each \( j = 0, \ldots, m \). By using the hypothesis of induction,
\[ H^0(C, (F^j / F^{j+1}) \otimes G) = 0 \]
for \( j = 1, \ldots, m \) because \( \text{Sym}^{m-j}(E / E_{s-1}) \otimes G \) is strongly semistable by (F3) and
\[ \text{deg}(\text{Sym}^{m-j}(E / E_{s-1}) \otimes G) \leq 0. \]
Moreover, since \( \text{Sym}^m(E / E_{s-1}) \otimes G \) is strongly semistable by (F3) and
\[ \text{deg}(\text{Sym}^m(E / E_{s-1}) \otimes G) < 0, \]
we have
\[ H^0(C, (F^0 / F^1) \otimes G) = H^0(C, \text{Sym}^m(E / E_{s-1}) \otimes G) = 0. \]
Therefore, by using an exact sequence
\[ 0 \to F^{j+1} \otimes G \to F^j \otimes G \to (F^j / F^{j+1}) \otimes G \to 0, \]
we have
\[ H^0(C, F^{j+1} \otimes G) \to H^0(C, F^j \otimes G) \]
for \( j = 0, \ldots, m \), which implies that \( H^0(C, \text{Sym}^m(E) \otimes G) = 0 \), as required. \( \square \)

**Proof of Proposition 1.1.** It is sufficient to show the following:

(a) If \( A \) is a \( \mathbb{Q} \)-Cartier divisor and \( \text{deg}(A) < \mu(E_1) \), then \( \Theta_E - f^*(A) \) is \( \mathbb{Q} \)-effective.

(b) If \( A \) is a \( \mathbb{Q} \)-Cartier divisor and \( \text{deg}(A) > \mu(E_1) \), then \( \Theta_E - f^*(A) \) is not pseudo-effective.

(c) If \( \Theta_E - f^*(A) \) is nef, then \( \text{deg}(A) \leq \mu(E / E_{s-1}) \).

(d) If \( \Theta_E - f^*(A) \) is not nef, then \( \text{deg}(A) > \mu(E / E_{s-1}) \).

(a) Let \( \theta \) be a divisor on \( C \) with \( \text{deg}(\theta) = 1 \). As \( E_1 \) is strongly semistable, by (F1), \( \Theta_{E_1} - \mu(E_1)f_{E_1}^*(\theta) \) is nef, so that we can see that \( \Theta_{E_1} - f_{E_1}^*(A) \) is nef and big because
\[ \Theta_{E_1} - \text{deg}(A)f_{E_1}^*(\theta) = \Theta_{E_1} - \mu(E_1)f_{E_1}^*(\theta) + (\mu(E_1) - \text{deg}(A))f_{E_1}^*(\theta). \]
Therefore, there is a positive integer \( m_1 \) such that \( m_1A \) is a divisor on \( C \) and
\[ H^0\left( \mathbb{P}(E_1), \mathcal{O}_{\mathbb{P}(E_1)}(m_1\Theta_{E_1} - f_{E_1}^*(m_1A)) \right) \neq 0. \]
In addition,
\[ H^0\left( \mathbb{P}(E_1), \mathcal{O}_{\mathbb{P}(E_1)}(m_1\Theta_{E_1} - f_{E_1}^*(m_1A)) \right) = H^0(C, \text{Sym}^{m_1}(E_1) \otimes \mathcal{O}_C(-m_1A)) \]
\[ \subseteq H^0(C, \text{Sym}^{m_1}(E) \otimes \mathcal{O}_C(-m_1A)) \]
\[ = H^0\left( \mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m_1\Theta_{E} - f_{E}^*(m_1A)) \right), \]
so that \( \Theta_E - f_{E}^*(A) \) is \( \mathbb{Q} \)-effective.
Therefore, $\deg CQ$ is a positive integer such that $Q = \deg CQ$ is big, so that we can find a positive integer $m$ such that $CQ$ is a Cartier divisor, then
\[
H^0 \left( CQ, \text{Sym}^m(\pi^*(E)) \otimes CQ(m\pi^*(-A + B)) \right) = 0
\]
for all $m \geq 1$. In particular, if $b$ is a positive integer such that $b(-A + B)$ is a Cartier divisor, then
\[
H^0 \left( CQ, \text{Sym}^m b(E) \otimes CQ(mb(-A + B)) \right) = 0
\]
for $m \geq 1$. Here we assume that $\Theta E - f_E^*(A)$ is pseudo-effective. Let $a$ be a positive integer such that $\Theta E - f_E^*(A) + af_E^*(B)$ is ample. Then
\[
(a - 1)(\Theta E - f_E^*(A)) + \Theta E - f_E^*(A) + af_E^*(B) = a(\Theta E + f_E^*(-A + B))
\]
is big, so that we can find a positive integer $m_1$ such that
\[
H^0 \left( CQ, \text{Sym}^{m_1 ab}(E) \otimes CQ(m_1 ab(-A + B)) \right)
\]
\[
= H^0 \left( \mathcal{P}(E), \mathcal{O}_{\mathcal{P}(E)}(m_1 ab(\Theta E + f_E^*(-A + B))) \right) \neq 0,
\]
which is a contradiction.

(c) Note that
\[
\mathcal{P}(E/E_{s-1}) \subseteq \mathcal{P}(E), \quad \Theta E/E_{s-1} \sim \Theta E|_{\mathcal{P}(E/E_{s-1})} \quad \text{and} \quad f_{E/E_{s-1}} = f_E|_{\mathcal{P}(E/E_{s-1})},
\]
so that $\Theta E/E_{s-1} - f_{E/E_{s-1}}^*(A)$ is nef on $\mathcal{P}(E/E_{s-1})$. Let $\xi_{E/E_{s-1}}$ be a Cartier divisor on $C$ with $\mathcal{O}_C(\xi_{E/E_{s-1}}) \simeq \det(E/E_{s-1})$. If we set $e = \text{rk} E/E_{s-1}$ and $G = \xi_{E/E_{s-1}}/e - A$, then
\[
\Theta E/E_{s-1} - f_{E/E_{s-1}}^*(A) = \Theta E/E_{s-1} - f_{E/E_{s-1}}^*(\xi_{E/E_{s-1}}/e) + f_{E/E_{s-1}}^*(G).
\]
Since $\Theta E/E_{s-1} - f_{E/E_{s-1}}^*(\xi_{E/E_{s-1}}/e)$ is nef by (F1) and
\[
\left( \Theta E/E_{s-1} - f_{E/E_{s-1}}^*(\xi_{E/E_{s-1}}/e) \right)^e = 0,
\]
we have
\[
0 \leq \left( \Theta E/E_{s-1} - f_{E/E_{s-1}}^*(A) \right)^e = e \deg(G).
\]
Therefore, $\deg(G) \geq 0$, and hence $\deg(A) \leq \mu(E/E_{s-1})$.

(d) We can find an irreducible curve $C_0$ of $X$ such that $(\Theta E - f_E^*(A) \cdot C_0) < 0$. Clearly $C_0$ is flat over $C$. Let $C_1$ be the normalization of $C_0$ and $h : C_1 \to C$ the induced morphism. Let us consider the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{P}(E) & \xrightarrow{\mathcal{P}(h)} & \mathcal{P}(h^*(E)) \\
\downarrow f_E & & \downarrow f_{h^*(E)} \\
C & \xleftarrow{h} & C_1
\end{array}
\]
Note that $\mathcal{P}(h)^*(\Theta_E - f_E^*(A)) \sim_{\mathbb{R}} \Theta_{h^*(E)} - f_{h^*(E)}^*(h^*(A))$. Further, there is a section $S$ of $f_{h^*(E)}$ such that $\mathcal{P}(h)_*(S) = C_0$. Let $Q$ be the quotient line bundle of $h^*(E)$ corresponding to the section $S$. As

$$0 = h^*(E_0) \subseteq h^*(E_1) \subseteq h^*(E_2) \subseteq \cdots \subseteq h^*(E_{s-1}) \subseteq h^*(E_s) = h^*(E)$$

is the Harder-Narasimhan filtration of $h^*(E)$, we can easily see

$$\deg(Q) \geq \mu(h^*(E/E_{s-1})) = \deg(h)\mu(E/E_{s-1}).$$

On the other hand,

$$\deg(Q) - \deg(h)\deg(A) = (\Theta_{h^*(E)} - f_{h^*(E)}^*(h^*(A)) \cdot S) = (\Theta_E - f_E^*(A) \cdot C_0) < 0,$$

and hence $\mu(E/E_{s-1}) < \deg(A)$.

Finally let us consider the following three results.

**Lemma 1.3.** Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let $\mu : X' \to X$ be a generically finite morphism of normal projective varieties over $k$. For a $\mathbb{K}$-Cartier divisor $D$ on $X$, $D$ is $\mathbb{K}$-effective if and only if $\mu^*(D)$ is $\mathbb{K}$-effective.

**Proof.** Clearly, if $D$ is $\mathbb{K}$-effective, then $\mu^*(D)$ is $\mathbb{K}$-effective. Let $K$ and $K'$ be the function fields of $X$ and $X'$, respectively. Here we assume that $\mu^*(D)$ is $\mathbb{K}$-effective, that is, there are $\phi_1', \ldots, \phi_r' \in K'^\times$ and $a_1, \ldots, a_r \in \mathbb{K}$ such that $\mu^*(D) + a_1(\phi_1') + \cdots + a_r(\phi_r')$ is effective, so that

$$\mu_*(\mu^*(D) + a_1(\phi_1') + \cdots + a_r(\phi_r')) = \deg(\mu)D + a_1\mu_*(\phi_1') + \cdots + a_r\mu_*(\phi_r')$$

effective. Note that $\mu_*(\phi_i') = (N_{K'/K}(\phi_i'))$ (cf. [7, Proposition 1.4]), where $N_{K'/K}$ is the norm map of $K'$ over $K$, and hence

$$D + (a_1/\deg(\mu))(N_{K'/K}(\phi_1')) + \cdots + (a_r/\deg(\mu))(N_{K'/K}(\phi_r'))$$

is effective. Therefore, $D$ is $\mathbb{K}$-effective.

**Lemma 1.4.** Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. We assume that $k$ is an algebraic closure of a finite field. Let $X$ be a normal projective variety over $k$ and $D$ a $\mathbb{K}$-Cartier divisor on $X$. If $D$ is numerically trivial, then $D$ is $\mathbb{K}$-linearly equivalent to the zero divisor.

**Proof.** If $\mathbb{K} = \mathbb{Q}$, then the assertion is well-known, so that we assume that $\mathbb{K} = \mathbb{R}$. We set $D = a_1D_1 + \cdots + a_rD_r$, where $D_1, \ldots, D_r$ are Cartier divisors on $X$ and $a_1, \ldots, a_r \in \mathbb{R}$. Considering a $\mathbb{Q}$-basis of $qa_1 + \cdots + qa_r$ in $\mathbb{R}$, we may assume that $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Let $C$ be an irreducible curve on $X$. Note that

$$0 = (D \cdot C) = a_1(D_1 \cdot C) + \cdots + a_r(D_r \cdot C)$$

and $(D_1 \cdot C), \ldots, (D_r \cdot C) \in \mathbb{Z}$, and hence $(D_1 \cdot C) = \cdots = (D_r \cdot C) = 0$ because $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Thus $D_1, \ldots, D_r$ are numerically equivalent to zero, so that $D_1, \ldots, D_r$ are $\mathbb{Q}$-linearly equivalent to the zero divisor. Therefore, the assertion follows.

**Proposition 1.5.** Let $X$ be a normal projective variety over $k$ and let $D$ be a $\mathbb{Q}$-Cartier divisor on $X$. If $D$ is $\mathbb{R}$-effective, then $D$ is $\mathbb{Q}$-effective.
Lemma 2.1. Let \( \kappa \) be an algebraic closure of a finite field. Let \( C \) be a smooth projective curve over \( \kappa \). Let us begin with the following lemma.

**Lemma 2.1.** Let \( \mathbb{K} \) be either \( \mathbb{Q} \) or \( \mathbb{R} \). Let \( A \) be a \( \mathbb{K} \)-Cartier divisor on \( C \). If \( \deg(A) \geq 0 \), then \( A \) is \( \mathbb{K} \)-effective.

**Proof.** If \( \mathbb{K} = \mathbb{Q} \), then the assertion is obvious. We assume that \( \mathbb{K} = \mathbb{R} \). If \( \deg(A) = 0 \), the assertion follows from Lemma 1.4. Next we consider the case where \( \deg(A) > 0 \). We can find a \( \mathbb{Q} \)-Cartier divisor \( A' \) such that \( A' \leq A \) and \( \deg(A') > 0 \). Thus the previous observation implies the assertion.

As a consequence of (F3), (F4) and (F5), we have the following splitting theorem, which was obtained by Biswas and Parameswaran [2, Proposition 2.1].

**Theorem 2.2.** For a locally free sheaf \( E \) on \( C \), there are a surjective morphism \( \pi : C' \to C \) of smooth projective curves over \( \kappa \) and invertible sheaves \( L_1, \ldots, L_r \) on \( C' \) such that \( \pi^*(E) \simeq L_1 \oplus \cdots \oplus L_r \).
Proof. For reader’s convenience, we give a sketch of the proof. First we assume that $E$ is strongly semistable. Let $\xi_E$ be a Cartier divisor on $C$ with $\mathcal{O}_C(\xi_E) \simeq \det(E)$. Let $h : B \to C$ be a surjective morphism of smooth projective curves over $k$ such that $h^*(\xi_E)$ is divisible by $\text{rk}(E)$. We set $E' = h^*(E) \otimes \mathcal{O}_B(-h^*(\xi_E)/\text{rk}(E))$. As $\det(E') \simeq \mathcal{O}_B$, the assertion follows from (F5).

By the above observation, it is sufficient to find a surjective morphism $\pi : C' \to C$ of smooth projective curves over $k$ and strongly semistable locally free sheaves $Q_1, \ldots, Q_n$ on $C'$ such that

$$
\pi^*(E) = Q_1 \oplus \cdots \oplus Q_n.
$$

Moreover, by (F4), we may assume that $E$ has the strong Harder-Narasimham filtration

$$
0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = E.
$$

Clearly we may further assume that $n \geq 2$. For a non-negative integer $m$, we set

$$
C_m := X \times_{\text{Spec}(k)} \text{Spec}(k),
$$

where the morphism $\text{Spec}(k) \to \text{Spec}(k)$ is given by $x \mapsto x^{1/p^m}$. Let $F_m^i : C_m \to C$ be the relative $m$-th Frobenius morphism over $k$. Put

$$
G^m_{i,j} := (F_m^i)^* ( (E_j/E_i) \otimes (E_i/E_{i-1}) ) \otimes \omega_{C_m}
$$

for $i = 1, \ldots, n-1$ and $j = i, \ldots, n$. We can find a positive integer $m$ such that

$$
\mu (G^m_{i+1,j+1}) = p^m (\mu(E_{i+1}/E_i) - \mu(E_i/E_{i-1})) + \text{deg}(\omega_C) < 0
$$

for all $i = 1, \ldots, n-1$. By using (F3), we can see that

$$
0 = G^m_{i,i} \subsetneq G^m_{i,i+1} \subsetneq G^m_{i,i+2} \subsetneq \cdots \subsetneq G^m_{i,n-1} \subsetneq G^m_{i,n}
$$

is the strong Harder-Narasimham filtration of $G^m_{i,n}$, so that $H^0 \left( C_m, G^m_{i,n} \right) = \{0\}$, which yields

$$
\text{Ext}^1 \left( (F_m^i)^*(E/E_i), (F_m^i)^*(E_i/E_{i-1}) \right) = 0
$$

because of Serre’s duality theorem. Therefore, an exact sequence

$$
0 \to (F_m^i)^*(E/E_{i-1}) \to (F_m^i)^*(E/E_i) \to (F_m^i)^*(E/E_i) \to 0
$$

splits, that is, $(F_m^i)^*(E/E_i) \simeq (F_m^i)^*(E_i/E_{i-1}) \oplus (F_m^i)^*(E/E_i)$ for $i = 1, \ldots, n-1$, and hence

$$
(F_m^i)^*(E) \simeq \bigoplus_{i=1}^n (F_m^i)^*(E_i/E_{i-1}),
$$

as required. \qed

Proof of Theorem 0.4. By virtue of Theorem 2.2 and Lemma 1.3, we may assume that

$$
E \simeq L_1 \oplus \cdots \oplus L_r
$$

for some invertible sheaves $L_1, \ldots, L_r$ on $C$. We set

$$
d = \max\{\deg(L_1), \ldots, \deg(L_r)\} \quad \text{and} \quad I = \{i \mid \deg(L_i) = d\}.
$$
There is a $K$-Cartier divisor $A$ on $C$ such that $D \sim_K \lambda \Theta_E - f^*_E(A)$ for some $\lambda \in K$. Let $M$ be an ample divisor on $C$ such that $T := \Theta_E + f^*_E(M)$ is ample. As $D$ is pseudo-effective, we have

$$0 \leq (D \cdot T^{r-2} \cdot f^*_E(M)) = ((\lambda T - f^*_E(A + \lambda M)) \cdot T^{r-2} : f^*_E(M)) = \lambda \deg(M),$$

and hence $\lambda \geq 0$. If $\lambda = 0$, then $0 \leq (D \cdot T^{r-1}) = \deg(-A)$. Thus, by Lemma 2.1, $-A$ is $K$-effective, so that the assertion follows.

We assume that $\lambda > 0$. Replacing $D$ by $D/\lambda$, we may assume that $\lambda = 1$. Let $\xi$ be a Cartier divisor on $C$ such that $\mathcal{O}_C(\xi) \cong L_{i_0}$ for some $i_0 \in I$. Note that the first part $E_1$ of the strong Harder-Narasimham filtration of $E$ is $\bigoplus_{i \in I} L_i$, so that, by Proposition 1.1, $\deg(A) \leq \deg(\xi)$. If we set $B = \xi - A$, then, by Lemma 2.1, $B$ is $K$-effective because $\deg(B) \geq 0$. Moreover, as

$$\Theta_E - f^*_E(A) = \Theta_E - f^*_E(\xi) + f^*_E(B),$$

it is sufficient to consider the case where $D = \Theta_E - f^*_E(\xi)$. In this case, the assertion is obvious because

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(D)) = H^0(C, E \otimes \mathcal{O}_C(-\xi)) = H^0 \left(C, \bigoplus_{i=1}^r L_i \otimes \mathcal{O}_C(-\xi) \right) \neq \{0\}.$$

As a consequence of Theorem 0.4, we can recover a result due to [3].

**Corollary 2.3.** Let $k, C$ and $E$ be same as in Theorem 0.4. We assume that $r = 2$. Let $D$ be a Cartier divisor on $\mathbb{P}(E)$ such that $(D \cdot Y) > 0$ for all irreducible curves $Y$ on $\mathbb{P}(E)$. Then $D$ is ample.

**Proof.** As $D$ is nef, $D$ is pseudo-effective, so that, by Theorem 0.4, there is an effective $Q$-Cartier divisor $E$ on $X$ such that $D \sim_Q E$. As $E \neq 0$, we have $(D \cdot D) = (D \cdot E) > 0$. Therefore, $D$ is ample by Nakai-Moishezon criterion.

**Remark 2.4.** The argument in the proof of Corollary 2.3 actually shows that the $Q$-version of Question 0.2 on algebraic surfaces implies Question 0.3.

3. Numerically effectiveness on abelian varieties

The purpose of this section is to give an affirmative answer for the $Q$-version of Question 0.2 on abelian varieties. Let $A$ be an abelian variety over an algebraically closed field $k$. A key observation is the following proposition.

**Proposition 3.1.** If a $Q$-Cartier divisor $D$ on $A$ is nef, then $D$ is numerically equivalent to a $Q$-effective $Q$-Cartier divisor.

**Proof.** We prove it by induction on $\dim A$. If $\dim A \leq 1$, then the assertion is obvious. Clearly we may assume that $D$ is a Cartier divisor, so that we set $L = \mathcal{O}_A(D)$. As $L \otimes [-1]^*(L)$ is numerically equivalent to $L^{\otimes 2}$ (cf. [21, p.75, (iv)]), we may assume that $L$ is symmetric, that is, $L \simeq [-1]^*(L)$. Let $K(L)$ be the closed subgroup of $A$ given by $K(L) = \{x \in A \mid T^*_x(L) \simeq L \}$ (cf. [21, p.60, Definition]). If $K(L)$ is finite, then $L$ is nef and big by virtue of [21, p.150,
The Riemann-Roch theorem, so that $D$ is $\mathbb{Q}$-effective. Otherwise, let $B$ be the connected component of $K(L)$ containing 0. 

**Claim 3.1.1.** 
(1) $T^*_x(L)|_B \simeq L|_B$ for all $x \in A$.

(2) $L^{\otimes 2}|_{B+x} \simeq \mathcal{O}_{B+x}$ for $x \in A$.

**Proof.** (1) Let $N$ be an invertible sheaf on $A \times A$ given by

$$N = m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1}),$$

where $p_i : A \times A \to A$ is the projection to the $i$-th factor ($i = 1, 2$) and $m$ is the addition morphism. Note that $N|_{B \times A} \simeq \mathcal{O}_{B \times A}$ (cf. [21, p.123, §13]). Fixing $x \in A$, let us consider a morphism $\alpha : B \to B \times A$ given by $\alpha(y) = (y, x)$. Then

$$\mathcal{O}_B \simeq \alpha^* \left( m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1}) \right) \simeq T^*_x(L)|_B \otimes L^{-1}|_B,$$

as required.

(2) First we consider the case where $x = 0$. As $N|_{B \times A} \simeq \mathcal{O}_{B \times A}$, we have $N|_{B \times B} \simeq \mathcal{O}_{B \times B}$. Using a morphism $\beta : B \to B \times B$ given by $\beta(y) = (y, -y)$, we have

$$\mathcal{O}_B \simeq \beta^*(N|_{B \times B}) = L^{-1}|_B \otimes [-1]^*(L^{-1})|_B \simeq L^{\otimes -2}|_B,$$

as required.

In general, for $x \in A$, by (1) and the previous observation together with the following commutative diagram

$$
\begin{array}{ccc}
B + x & \longrightarrow & A  \\
\downarrow T_{-x} & & \downarrow T_{-x}  \\
B & \longrightarrow & A,
\end{array}
$$

we can see

$$\mathcal{O}_{B+x} = T^*_{-x} (\mathcal{O}_B) \simeq T^*_{-x} \left( L^{\otimes 2}|_B \right) \simeq T^*_{-x} \left( T^*_x(L^{\otimes 2})|_B \right) \simeq T^*_{-x} \left( T^*_x(L^{\otimes 2}) \right)|_{B+x} = L^{\otimes 2}|_{B+x}. \square$$

Let $\pi : A \to A/B$ be the canonical homomorphism. By (2) in the above claim,

$$\dim_k(y) H^0\left( \pi^{-1}(y), L^{\otimes 2} \right) = 1$$

for all $y \in A/B$, so that, by [21, p.51, Corollary 2], $\pi_*(L^{\otimes 2})$ is an invertible sheaf on $A/B$ and $\pi_*(L^{\otimes 2}) \otimes k(y) \xrightarrow{\sim} H^0(\pi^{-1}(y), L^{\otimes 2})$. Therefore, the natural homomorphism $\pi^*(\pi_*(L^{\otimes 2})) \to L^{\otimes 2}$ is an isomorphism, that is, there is a $\mathbb{Q}$-Cartier divisor $D'$ on $A/B$ such that $\pi^*(D') \sim_\mathbb{Q} D$. Note that $D'$ is also nef, so that, by the hypothesis of induction, $D'$ is numerically equivalent to a $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier divisor, and hence the assertion follows. \square

**Proof of Proposition 0.5.** Proposition 0.5 is a consequence of Lemma 1.4 and Proposition 3.1 because a pseudo-effective $\mathbb{Q}$-Cartier divisor on an abelian variety is nef. \square
Example 3.2. Here we show that the $\mathbb{R}$-version of Question 0.2 does not hold in general. Let $k$ be an algebraically closed field ($k$ is not necessarily an algebraic closure of a finite field). Let $C$ be an elliptic curve over $k$ and $A := C \times C$. Let $\text{NS}(A)$ be the Néron-Severi group of $A$. Note that $\rho := \text{rk} \text{NS}(A) \geq 3$. By using the Hodge index theorem, we can find a basis $e_1, \ldots, e_\rho$ of $\text{NS}(A)_{\mathbb{Q}} := \text{NS}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the following properties:

1. $e_1$ is the class of the divisor $\{0\} \times C + C \times \{0\}$. In particular, $(e_1 \cdot e_1) = 2$.
2. $(e_i \cdot e_i) < 0$ for all $i = 2, \ldots, \rho$.
3. $(e_i \cdot e_j) = 0$ for all $1 \leq i \neq j \leq \rho$.

We set $\lambda_i := -(e_i \cdot e_i)$ for $i = 2, \ldots, \rho$. Let $\overline{\text{Amp}}(A)$ be the closed cone in $\text{NS}(A)_{\mathbb{R}} := \text{NS}(A) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by ample $\mathbb{Q}$-Cartier divisors on $A$. It is well known that

$$\overline{\text{Amp}}(A) = \left\{ \xi \in \text{NS}(A)_{\mathbb{R}} \mid (\xi^2) \geq 0, (\xi \cdot e_1) \geq 0 \right\} = \left\{ x_1 e_1 + x_2 e_2 + \cdots + x_\rho e_\rho \mid \lambda_2 x_2^2 + \cdots + \lambda_\rho x_\rho^2 \leq 2x_1^2, x_1 \geq 0 \right\}.$$ 

We choose $(a_2, \ldots, a_\rho) \in \mathbb{R}^{\rho-1}$ such that

$$(a_2, \ldots, a_\rho) \notin \mathbb{Q}^{\rho-1} \quad \text{and} \quad \lambda_2 a_2^2 + \cdots + \lambda_\rho a_\rho^2 = 2.$$ 

Let $E_i$ be a $\mathbb{Q}$-Cartier divisor on $A$ such that the class of $E_i$ in $\text{NS}(A)_{\mathbb{Q}}$ is equal to $e_i$ for $i = 1, \ldots, \rho$. If we set $D := E_1 + a_2 E_2 + \cdots + a_\rho E_\rho$, then we have the following claim, which is sufficient for our purpose.

Claim 3.2.1. $D$ is nef and $D$ is not numerically equivalent to an effective $\mathbb{R}$-Cartier divisor.

Proof. Clearly $D$ is nef. If we set $e'_i = e_i/\sqrt{2}$ and $e'_i = e_i/\sqrt{-2}$ for $i = 2, \ldots, \rho$, then

$$\overline{\text{Amp}}(A) = \left\{ y_1 e'_1 + y_2 e'_2 + \cdots + y_\rho e'_\rho \mid y_2^2 + \cdots + y_\rho^2 \leq y_1^2, y_1 \geq 0 \right\}.$$ 

Therefore, as $[D] \in \partial(\overline{\text{Amp}}(A)_{\mathbb{R}})$, we can choose

$$H \in \text{Hom}_{\mathbb{R}}(\text{NS}(A)_{\mathbb{R}}, \mathbb{R})$$

such that

$$H \geq 0 \text{ on } \overline{\text{Amp}}(A) \quad \text{and} \quad \{ H = 0 \} \cap \overline{\text{Amp}}(A) = \mathbb{R}_{\geq 0}[D],$$

where $[D]$ is the class of $D$ in $\text{NS}(A)_{\mathbb{R}}$. We assume that $D$ is numerically equivalent to an effective $\mathbb{R}$-Cartier divisor $c_1 \Gamma_1 + \cdots + c_r \Gamma_r$, where $c_1, \ldots, c_r \in \mathbb{R}_{>0}$ and $\Gamma_1, \ldots, \Gamma_r$ are prime divisors on $A$. As $[D] \neq 0$, we may assume that $c_1, \ldots, c_r \in \mathbb{R}_{>0}$. Note that $[\Gamma_1], \ldots, [\Gamma_r] \in \overline{\text{Amp}}(A)$ and

$$0 = H([D]) = c_1 H([\Gamma_1]) + \cdots + c_r H([\Gamma_r]),$$

so that $H([\Gamma_1]) = \cdots = H([\Gamma_r]) = 0$, and hence $[\Gamma_1], \ldots, [\Gamma_r] \in \mathbb{R}_{\geq 0}[D]$. In particular, there is $t \in \mathbb{R}_{\geq 0}$ with $[\Gamma_1] = t[D]$. Here we can set

$$[\Gamma_1] = b_1 e_1 + \cdots + b_\rho e_\rho \quad (b_1, \ldots, b_\rho \in \mathbb{Q}).$$
Thus $b_1 = t$, $b_2 = ta_2, \ldots, b_\rho = ta_\rho$. As $[\Gamma_1] \neq 0$, $t \in \mathbb{Q}^\times$, and hence $(a_2, \ldots, a_\rho) = t^{-1}(b_2, \ldots, b_\rho) \in \mathbb{Q}^{\rho-1}$. This is a contradiction. \qed

**Remark 3.3.** Let $k$ be an algebraic closure of a finite field and let $X$ be a normal projective variety over $k$. Let $\text{NS}(X)$ be the Néron-Severi group of $X$ and $\text{NS}(X)_\mathbb{R} := \text{NS}(X) \otimes \mathbb{Z} \otimes \mathbb{R}$. Let $\overline{\text{Eff}}(X)$ be the closed cone in $\text{NS}(X)_\mathbb{R}$ generated by pseudo-effective $\mathbb{R}$-Cartier divisors on $X$. We assume that $\overline{\text{Eff}}(X)$ is a rational polyhedral cone, that is, there are pseudo-effective $\mathbb{Q}$-Cartier divisors $D_1, \ldots, D_n$ on $X$ such that $\overline{\text{Eff}}(X)$ is generated by the classes of $D_1, \ldots, D_n$. Then the $\mathbb{Q}$-version of Question 0.2 implies the $\mathbb{R}$-version of Question 0.2.

**Example 3.4.** This is an example due to Yuan [25]. Let us fix an algebraically closed field $k$ and an integer $g \geq 2$. Let $C$ be a smooth projective curve over $k$ and $f : X \to C$ an abelian scheme over $C$ of relative dimension $g$. Let $L$ be an $f$-ample invertible sheaf on $X$ such that $[-1]^* (L) \simeq L$ and $L$ is trivial along the zero section of $f : X \to C$.

**Claim 3.4.1.**

2. $L$ is nef.

**Proof.**

(1) As $[2]^* (L)|_{f^{-1}(x)} \simeq L \otimes^4|_{f^{-1}(x)}$ for all $x \in C$, there is an invertible sheaf $M$ on $C$ such that $[2]^* (L) \simeq L \otimes^4 \circ f^* (M)$. Let $Z_0$ be the zero section of $f : X \to C$. Then

$$\mathcal{O}_{Z_0} \simeq [2]^* (L|_{Z_0}) = [2]^* (L)|_{Z_0} \simeq L \otimes^4 \circ f^* (M)|_{Z_0} \simeq M,$$

so that we have the assertion.

(2) Let $A$ be an ample invertible sheaf on $C$ such that $L \otimes f^* (A)$ is ample. Let $\Delta$ be a horizontal curve on $X$. As $f \circ [2^n] = f$ and $[2^n]^* (L) \simeq L \otimes^4$ by using (1),

$$0 \leq (L \otimes f^* (A) \cdot [2^n]^* (\Delta)) = ([2^n]^* (L \otimes f^* (A)) \cdot \Delta) = (L \otimes^4 \circ f^* (A) \cdot \Delta),$$

so that $(L \cdot \Delta) \geq -4^{-n}(f^* (A) \cdot \Delta)$ for all $n > 0$. Thus $(L \cdot \Delta) \geq 0$. \qed

**Claim 3.4.2.** If the characteristic of $k$ is zero and $f$ is non-isotrivial, then $L$ does not have the Dirichlet property (i.e. $L$ is not $\mathbb{Q}$-effective).

**Proof.** The following proof is due to Yuan [25]. An alternative proof can be found in [6, Theorem 4.3]. We need to see that $H^0(X, L^{\otimes n}) = 0$ for all $n > 0$. We set $d_n = \text{rk} f_* (L^{\otimes n})$. By changing the base $C$ if necessarily, we may assume that all $(d_n)^2$-torsion points on the generic fiber $X_\eta$ of $f : X \to C$ are defined over the function field of $C$. By using the algebraic theta theory due to Mumford (especially [20, the last line in page 81]), there is an invertible sheaf $M$ on $C$ such that $f_* (L^{\otimes n}) = M^{\otimes d_n}$. On the other hand, by [13],

$$\text{deg} (\det (f_* (L^{\otimes n})) \otimes 2 \otimes f_* (\omega_{X/C})^{\otimes d_n}) = 0,$$

that is, $2 \text{deg}(M) + \text{deg}(f_* (\omega_{X/C})) = 0$. As $f$ is non-isotrivial, we can see that $\text{deg}(f_* (\omega_{X/C})) > 0$, so that $\text{deg}(M) < 0$, and hence the assertion follows. \qed

If the characteristic of $k$ is positive, we do not know the $\mathbb{Q}$-effectivity of $L$ in general. In [15], there is an example with the following properties:
(1) $g = 2$ and $C = \mathbb{P}^1_k$.

(2) There are an abelian surface $A$ over $k$ and an isogeny $h : A \times \mathbb{P}^1_k \to X$ over $\mathbb{P}^1_k$.

**Claim 3.4.3.** In the above example, $L$ has the Dirichlet property.

**Proof.** Replacing $L$ by $L^{\otimes n}$, we may assume that $d := \text{rk } f_*(L) > 0$. Let

$$p_1 : A \times \mathbb{P}^1_k \to A \quad \text{and} \quad p_2 : A \times \mathbb{P}^1_k \to \mathbb{P}^1_k$$

be the projections to $A$ and $\mathbb{P}^1_k$, respectively. Note that $h^*(L)$ is symmetric and $h^*(L)$ is trivial along the zero section of $p_2$. Since $\omega_{A \times \mathbb{P}^1_k/\mathbb{P}^1_k} \simeq p_1^*(\omega_A)$, we have

$$(p_2)_*(\omega_{A \times \mathbb{P}^1_k/\mathbb{P}^1_k}) \simeq \mathcal{O}_{\mathbb{P}^1_k},$$

so that, by [13], $\deg(\det((p_2)_*(h^*(L)))) = 0$, that is, if we set

$$(p_2)_*(h^*(L)) = \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_d),$$

then $a_1 + \cdots + a_d = 0$. Thus $a_i \geq 0$ for some $i$, and hence

$$H^0(A \times \mathbb{P}^1_k, h^*(L)) \neq 0.$$ 

Therefore, $L$ is $\mathbb{Q}$-effective by Lemma 1.3. \qed

The above claim suggests that the set of preperiodic points of the map $[2] : X \to X$ is not dense in the analytification $X^an$ at any place $v$ of $\mathbb{P}^1_k$ with respect to the analytic topology (cf. [5]).

**References**


Department of Mathematics, Faculty of Science, Kyoto University, Kyoto, 606-8502, Japan

E-mail address: moriwaki@math.kyoto-u.ac.jp