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Lax pairs on Yang-Baxter deformed backgrounds

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Abstract: We explicitly derive Lax pairs for string theories on Yang-Baxter deformed backgrounds, 1) gravity duals for noncommutative gauge theories, 2) γ-deformations of S^5, 3) Schrödinger spacetimes and 4) abelian twists of the global AdS_5. Then we can find out a concise derivation of Lax pairs based on simple replacement rules. Furthermore, each of the above deformations can be reinterpreted as twisted boundary conditions with the undeformed background by using the rules. As another derivation, the Lax pair for gravity duals for noncommutative gauge theories is reproduced from the one for a q-deformed $AdS_5 \times S^5$ by taking a scaling limit.

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Contents

1 Introduction

The AdS/CFT correspondence is a fascinating subject in the study of string theory. The most famous one among a lot of variations is a duality between 10D type IIB string theory on the AdS$_5$×S^5 background and the 4D $\mathcal{N} = 4$ super Yang-Mills theory at large N limit [1]. A great progress is that an integrable structure has been discovered behind this duality [2]. On the string-theory side, the Green-Schwarz string action on AdS$_5$×S^5 is constructed as a 2D coset sigma model [3] and the $\mathbb{Z}_4$-grading of the supercoset ensures the classical integrability [4] (For a big review of the AdS$_5$×S^5 superstring, see [5]). Although the essential mechanism of the duality has not been fully understood yet, the integrability has played a crucial role in checking conjectured relations in the AdS/CFT.

It would be significant to consider integrable deformations of the AdS/CFT. It may shed light on a deeper structure behind gauge/gravity dualities beyond the conformal invariance. On the string-theory side, an influential way is to employ the Yang-Baxter sigma model description [6, 7]. This is a systematic way to study integrable deformations of 2D non-linear sigma models. By following this approach, an integrable deformation is specified by picking up a skew-symmetric classical r-matrix which satisfies the modified classical

Yang-Baxter equation (mCYBE). The original argument [6, 7] was restricted to principal chiral models. It is generalized to symmetric cosets by Delduc-Magro-Vicedo [8].[1] Then they succeeded in constructing a q-deformed action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring [19, 20]. This formulation is also based on the mCYBE.

After that, it has been reformulated in [21] based on the (non-modified) classical Yang-Baxter equation (CYBE), where the Lax pair and the kappa transformation should be reconstructed and this generalization is not so trivial. An advantage in comparison to the mCYBE case is that partial deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$ can be considered. This is because the zero map $R = 0$ is allowed for the CYBE while not for the mCYBE. Furthermore, one can find many solutions of the CYBE. In fact, in a series of papers [22–29], many examples of (skew-symmetric) classical r-matrices have been identified with the well-known backgrounds such as γ-deformations of S^5 [30, 31], gravity duals for noncommutative (NC) gauge theories [32, 33] and Schrödinger spacetimes [34–36], in addition to new backgrounds [22]. This identification may be called the gravity/CYBE correspondence [23] (For a short summary, see [37]) and indicate that the moduli space of (a certain class of) solutions of type IIB supergravity can be described by the CYBE.[2]

In the recent, this correspondence has been generalized to integrable deformations of 4D Minkowski spacetime [39]. In particular, (T-duals of) 4D (A)dS spaces are reproduced as Yang-Baxter deformations of the Minkowski spacetimes. Furthermore, this development has an intimate connection with kappa-Minkowski spacetime [40–42] via preceding works e.g., [43]. For a recent argument with a gravity dual, see [28, 29].

It is also remarkable that the gravity/CYBE correspondence seems to be valid beyond the integrability. There are many examples of non-integrable AdS/CFT correspondences. An example is the case of $\mathrm{AdS}_5 \times T^{1,1}$ [44], for which the non-integrability has been shown by the existence of chaotic string solutions on $R\times T^{1,1}$ [45–47]. Thus TsT transformations of $T^{1,1}$ [30, 48] are regarded as non-integrable deformations. However, these deformations can be described as Yang-Baxter deformations [49]. Hence the gravity/CYBE correspondence would be applicable to a much wider class of solutions of type IIB supergravity.

We will proceed to study Yang-Baxter deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring by focusing upon Lax pairs. The universal expression of Lax pair, without taking specific classical r-matrices and concrete coordinate systems, has already been presented in [21]. Then classical r-matrices are identified and hence the associated Lax pairs are already obtained in an implicit way. However, explicit expressions of the Lax pairs have not been evaluated yet, while those are quite useful in studying classical solution by the use of the classical inverse scattering method. Thus, in this paper, we will derive explicit expressions of Lax pairs for string theories on popular examples of deformed backgrounds: 1) gravity duals for NC gauge theories [32, 33], 2) γ-deformations of S^5 [30, 31], 3) Schrödinger spacetimes [34–36] and 4) abelian twists of the global AdS_5 [50, 51]. Then we can find out a concise derivation of Lax pairs based on simple replacement rules. Furthermore, each of the above deformations can be reinterpreted as twisted boundary conditions with the undeformed

[1] For earlier arguments related to this generalization, see [9–18].

[2] This is quite analogous to the bubbling scenario proposed by Lin-Lunin-Maldacena [38]. Here the moduli space is described by droplet configurations in a free fermion system.

background by using the rules. As another derivation, the Lax pair for gravity duals for NC gauge theories is reproduced from the one for a q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ [19, 20, 52] by taking a scaling limit introduced in [53].

It would be helpful for readers to summarize the novelties of this paper below:

1. Explicit forms of Lax pairs are computed, while the associated classical r-matrices have already been obtained in [23, 24, 27].

2. Twisted boundary conditions are determined explicitly, while general arguments for TsT transformations have been provided in [31, 58].

3. A scaling limit of the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ [53] is confirmed at the level of Lax pairs.

This paper is organized as follows. Section 2 gives a brief review of Yang-Baxter deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring. In section 3, we explicitly derive Lax pairs for gravity duals of NC gauge theories with two methods, (i) Yang-Baxter deformation (argued in [24]) and (ii) a scaling limit of q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ (introduced in [53]). The resulting Lax pairs are identical under a unitary transformation. In section 4, we compute Lax pairs for γ-deformations of S^5 by evaluating the abstract expression given in [23]. The resulting Lax pairs agree with Frolov's results [31] up to gauge transformations. Section 5 argues Lax pairs for Schrödinger spacetimes by evaluating the abstract forms given in [27]. In section 6, we derive Lax pairs for abelian twists of the global AdS_5 from the results of [23]. Section 7 is devoted to conclusion and discussion. In appendix A, our convention and notation are summarized. In appendix B, we explicitly derive a Lax pair for a q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ by evaluating the universal Lax pair [19, 20] with a coordinate system [52].

2 Yang-Baxter deformations of string on $\mathrm{AdS}_5\times\mathrm{S}^5$

We shall give a brief review of Yang-Baxter deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action based on the CYBE case [21].[3]

The deformed classical action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring is given by

$$S=-\frac{\sqrt{\lambda_{\mathrm{c}}}}{4}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta})\mathrm{STr}\left[A_\alpha\, d\circ\frac{1}{1-\eta R_g\circ d}(A_\beta)\right], \tag{2.1}$$

where the left-invariant one-form A_α is defined as

$$A_\alpha\equiv g^{-1}\partial_\alpha g\,,\qquad g\in\mathrm{SU}(2,2|4) \tag{2.2}$$

with the world-sheet index $\alpha=(\tau,\sigma)$. Here the conformal gauge is supposed and the world-sheet metric is taken as $\gamma^{\alpha\beta}=\mathrm{diag}(-1,+1)$. Hence there is no coupling of the dilaton to the world-sheet scalar curvature. The anti-symmetric tensor $\epsilon^{\alpha\beta}$ is normalized as $\epsilon^{\tau\sigma}=+1$. The constant λ_{c} is the 't Hooft coupling. Note that η is a deformation parameter and hence the undeformed action [3] is reproduced when $\eta=0$.

[3]For the mCYBE case [19, 20], see appendix B.

A key ingredient in our analysis is the operator R_g defined as

$$R_g(X) \equiv g^{-1}R(gXg^{-1})g\,, \qquad X \in \mathfrak{su}(2,2|4)\,, \tag{2.3}$$

where a linear R-operator $R : \mathfrak{su}(2,2|4) \to \mathfrak{su}(2,2|4)$ is a solution of the classical Yang-Baxter equation (CYBE),[4]

$$[R(X),R(Y)] - R([R(X),Y] + [X,R(Y)]) = 0\,. \tag{2.4}$$

This R-operator is related to a skew-symmetric classical r-matrix in the tensorial notation through the following supertrace operation on the second site:

$$R(X) = \mathrm{STr}_2[r(1\otimes X)] = \sum_i (a_i\,\mathrm{STr}[b_iX] - b_i\,\mathrm{STr}[a_iX])\,, \tag{2.5}$$

where the classical r-matrix is represented by

$$r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i\otimes b_i - b_i \otimes a_i) \qquad \text{with} \qquad a_i, b_i \in \mathfrak{su}(2,2|4)\,. \tag{2.6}$$

The projection operator d is defined as

$$d \equiv P_1 + 2P_2 - P_3\,, \tag{2.7}$$

where $P_i\,(i = 0,1,2,3)$ are projections to the $\mathbb{Z}_4$-graded components of $\mathfrak{su}(2,2|4)$. In particular, $P_0(\mathfrak{su}(2,2|4))$ is a local symmetry of the classical action, $\mathfrak{so}(1,4)\oplus\mathfrak{so}(5)$. Note that the numerical coefficients are fixed by requiring the kappa-symmetry [21].

It is convenient to introduce the light-cone expression of A_α like

$$A_\pm \equiv A_\tau \pm A_\sigma\,, \tag{2.8}$$

when we will study Lax pair in the following sections.

The bosonic part of the Lagrangian. Our aim here is to explicitly derive Lax pairs for the bosonic part of deformed actions. Hence it is convenient to rewrite the bosonic part of the deformed Lagrangian (2.1) as

$$L = \frac{\sqrt{\lambda_c}}{2}\,\mathrm{STr}(A_-\,P_2(J_+))\,, \tag{2.9}$$

where $J_\pm$ is a deformed current defined as

$$J_\pm \equiv \frac{1}{1 \mp 2\eta R_g \circ P_2}\,A_\pm\,. \tag{2.10}$$

[4]In the original work [21], a wider class of R-operators whose image is given by $\mathfrak{gl}(4|4)$ has been proposed. The $\mathfrak{gl}(4|4)$ image is restricted on $\mathfrak{su}(2,2|4)$ in essential under the coset projection d as pointed out in [28, 29]. We will concentrate here on a restricted class in which the image is $\mathfrak{su}(2,2|4)$ from the beginning, so as to deal with pre-projected quantities like the deformed current J itself, without introducing extra generators. For general cases argued in [22, 25], a more detailed study would be necessary.

Note here that the factor 2 in front of η comes from the projection operator d given in (2.7). By solving the following equation,

$$(1 \mp 2\eta R_g \circ P_2)\, J_\pm = A_\pm\,, \tag{2.11}$$

the deformed current $J_\pm$ is determined.[5] Then the metric and NS-NS two-form are evaluated from the symmetric and skew-symmetric parts regarding the world-sheet coordinates in (2.9), respectively.

Taking a variation of the Lagrangian (2.9), the equation of motion is obtained as follows

$$\mathcal{E} \equiv \partial_+ P_2(J_-) + \partial_- P_2(J_+) + [J_+, P_2(J_-)] + [J_-, P_2(J_+)] = 0\,. \tag{2.12}$$

By definition, the undeformed current $A_\pm$ satisfies the flatness condition,

$$\mathcal{Z} \equiv \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0\,. \tag{2.13}$$

Then, in terms of the deformed current $J_\pm$, this condition can be rewritten as follows

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] + 2\eta\, R_g(\mathcal{E}) + 4\eta^2\, \mathrm{CYBE}_{Rg}(P_2(J_+), P_2(J_-)) = 0\,, \tag{2.14}$$

where we have introduced a new quantity defined as

$$\mathrm{CYBE}_{Rg}(X, Y) \equiv [R_g(X), R_g(Y)] - R_g([R_g(X), Y] + [X, R_g(Y)])\,. \tag{2.15}$$

Note that $\mathrm{CYBE}_{Rg}(X, Y)$ vanishes if the R-operator satisfies the CYBE in (2.4). The relation (2.14) means that $J_\pm$ also satisfies the flatness condition with the equation of motion $\mathcal{E} = 0$. That is, $J_\pm$ satisfies the flatness condition only on the on-shell, while $A_\pm$ do even on the off-shell.

It is helpful to decompose $J_\pm$ with the projection operators P_0 and P_2 like

$$J_\pm = P_0(J_\pm) + P_2(J_\pm) \equiv J_\pm^{(0)} + J_\pm^{(2)}\,, \tag{2.16}$$

where we have used the completeness condition $P_0 + P_2 = 1$. For the concrete expressions of the projection operators, see appendix A. Then the equation of motion (2.12) can be rewritten into the following form:

$$\mathcal{E} = \partial_+ J_-^{(2)} + \partial_- J_+^{(2)} + [J_+^{(0)}, J_-^{(2)}] + [J_-^{(0)}, J_+^{(2)}] = 0\,. \tag{2.17}$$

The flatness condition (2.13) can also be rewritten in a similar way:

$$\mathcal{Z} = P_0(\mathcal{Z}) + P_2(\mathcal{Z}) = 0\,. \tag{2.18}$$

[5] In order to derive the metric and NS-NS two-form, it is enough to determine $P_2(J_\pm)$ by solving the projected conditions

$$(1 \mp 2\eta P_2 \circ R_g)\, P_2(J_\pm) = P_2(A_\pm)\,,$$

as done in a series of the previous papers [22–29]. However, it is necessary here to determine $J_\pm$ itself so as to evaluate the form of Lax pair.

With the help of the linear independence of the grade 0 and grade 2 parts, one can obtain the following two conditions:

$$\begin{aligned} P_0(\mathcal{Z}) &= \partial_+ J_-^{(0)} - \partial_- J_+^{(0)} + [J_+^{(0)}, J_-^{(0)}] + [J_+^{(2)}, J_-^{(2)}] + 2\eta\, P_0(R_g(\mathcal{E})) = 0\,, \\ P_2(\mathcal{Z}) &= \partial_+ J_-^{(2)} - \partial_- J_+^{(2)} + [J_+^{(0)}, J_-^{(2)}] + [J_+^{(2)}, J_-^{(0)}] + 2\eta\, P_2(R_g(\mathcal{E})) = 0\,. \end{aligned} \tag{2.19}$$

Note here that the terms proportional to η vanish on the on-shell, i.e., $\mathcal{E} = 0$.

Then the three conditions in (2.17) and (2.19) can be recast into the following set of the equations $\mathcal{C}_i = 0$ $(i = 1, 2, 3)$:

$$\begin{aligned} \mathcal{C}_1 &\equiv \partial_- J_+^{(2)} - [J_+^{(2)}, J_-^{(0)}]\,, \\ \mathcal{C}_2 &\equiv \partial_+ J_-^{(2)} + [J_+^{(0)}, J_-^{(2)}]\,, \\ \mathcal{C}_3 &\equiv \partial_+ J_-^{(0)} - \partial_- J_+^{(0)} + [J_+^{(0)}, J_-^{(0)}] + [J_+^{(2)}, J_-^{(2)}]\,. \end{aligned} \tag{2.20}$$

Namely, $\mathcal{C}_i = 0$ $(i = 1, 2, 3)$ are satisfied on the on-shell and are equivalent to the equation of motion (2.12) and the flatness condition (2.13).

Lax pair. Finally, a Lax pair for the deformed action is given by

$$\mathcal{L}_\pm = J_\pm^{(0)} + \lambda^{\pm 1} J_\pm^{(2)} \tag{2.21}$$

with a spectral parameter $\lambda \in \mathbb{C}$.[6] Note that the existence of the Lax pair (2.21) is based on the $\mathbb{Z}_2$-grading of $\mathrm{AdS}_5 \times \mathrm{S}^5$.

As a matter of course, the flatness condition of $\mathcal{L}_\pm$

$$0 = \partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] \tag{2.22}$$

is equivalent to the equation of motion $\mathcal{E} = 0$ [in (2.12)] and the flatness condition $\mathcal{Z} = 0$ [in (2.13)] . In order to confirm the equivalence, it is helpful to notice that the right-hand side of (2.22) can be rewritten in terms of $\mathcal{C}_i$ as follows:

$$\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = -\lambda\, \mathcal{C}_1 + \frac{1}{\lambda}\, \mathcal{C}_2 + \mathcal{C}_3\,. \tag{2.23}$$

Thus we have shown the equivalence.

In the following sections, we will evaluate explicit forms of the Lax pair (2.21) for some examples of classical r-matrices.

3 Lax pairs for gravity duals of NC gauge theories

In this section, let us study Lax pairs for gravity duals of noncommutative (NC) gauge theories from the viewpoint of Yang-Baxter deformations. The integrability of this background was recently shown in [24] in the sense of the kinematical integrability. The Lax pair was implicitly derived in [24], but the explicit expression has not been computed yet.

First of all, we evaluate explicit forms of Lax pairs with classical r-matrices in [24]. The resulting Lax pairs depend on two deformation parameters. Next, one may consider another derivation for a special one-parameter case. Then the associated Lax pair can also be reproduced by taking a scaling limit [53] of the one for a q-deformed AdS_5 [19, 20, 52].

[6] Please do not confuse the spectral parameter λ with the 't Hooft coupling λ_c!

3.1 Lax pairs from Yang-Baxter deformations

In the context of Yang-Baxter deformations, abelian Jordanian classical r-matrices [24]

$$r = c_1\, p_2 \wedge p_3 + c_2\, p_0 \wedge p_1 \tag{3.1}$$

are associated with gravity duals of NC gauge theories [32, 33]. The classical r-matrices in (3.1) consist of the translation generators p_μ in $\mathfrak{su}(2,2)$ (For our convention, see appendix A). Then the deformation parameters c_1 and c_2 are related to magnetic and electric NS-NS two-forms in the gravity solutions [32, 33], respectively.

Because the square of the associated R-operator vanishes due to the properties of p_μ's, the classical r-matrices (3.1) are called Jordanian type. The classical r-matrices do not contain any generators in $\mathfrak{su}(4)$, and hence only the AdS$_5$ part is deformed. Therefore we will concentrate on only the AdS$_5$ part below.

The deformed metric and NS-NS two-form. To derive the metric and NS-NS two-form from the Lagrangian (2.9), let us introduce a coordinate system through a parametrization of an SU(2,2) element as follows:

$$g_a(\tau,\sigma) = \exp\Big[p_0\, x^0 + p_1\, x^1 + p_2\, x^2 + p_3\, x^3\Big]\ \exp\left[\gamma_5^a\, \frac{1}{2}\log z\right] \quad \in \mathrm{SU}(2,2)\,. \tag{3.2}$$

By solving the relation in (2.11), the deformed current J_α is explicitly determined as

$$\begin{aligned} J_\pm =& \frac{z}{z^4 - 4{c_2}^2\eta^2}\Big[(z^2\partial_\pm x^0 \pm 2c_2\eta\partial_\pm x^1)\, p_0 + (z^2\partial_\pm x^1 \pm 2c_2\eta\partial_\pm x^0)\, p_1\Big] \\ &+ \frac{z}{z^4 + 4{c_1}^2\eta^2}\Big[(z^2\partial_\pm x^2 \pm 2c_1\eta\partial_\pm x^3)\, p_2 + (z^2\partial_\pm x^3 \mp 2c_1\eta\partial_\pm x^2)\, p_3\Big] \\ &+ \frac{1}{2z}\partial_\pm z\, \gamma_5^a\,. \end{aligned} \tag{3.3}$$

Then the resulting metric and NS-NS two-form are given by

$$\begin{aligned} ds^2 &= \frac{z^2[-(dx^0)^2 + (dx^1)^2]}{z^4 - 4c_2^2\eta^2} + \frac{z^2[(dx^2)^2 + (dx^3)^2]}{z^4 + 4c_1^2\eta^2} + \frac{dz^2}{z^2}\,, \\ B &= -\frac{2c_2\eta}{z^4 - 4c_2^2\eta^2}dx^0 \wedge dx^1 + \frac{2c_1\eta}{z^4 + 4c_1^2\eta^2}dx^2 \wedge dx^3\,. \end{aligned} \tag{3.4}$$

This result exactly agrees with the gravity duals of NC gauge theories [32, 33]. When $c_1 = c_2 = 0$, the Poincaré AdS$_5$ is reproduced.

Lax pair. Let us derive the associated Lax pairs. Now $\mathcal{L}_\pm^{\rm NC}$ are explicitly evaluated as

$$\begin{aligned} \mathcal{L}_\pm^{\rm NC} =& \frac{z}{z^4 - 4c_2^2\eta^2}\Bigg[(z^2\partial_\pm x^0 \pm 2c_2\eta\partial_\pm x^1)\left(\frac{\lambda^{\pm1}}{2}\gamma_0^a - n_{05}^a\right) \\ &\qquad + (z^2\partial_\pm x^1 \pm 2c_2\eta\partial_\pm x^0)\left(\frac{\lambda^{\pm1}}{2}\gamma_1^a - n_{15}^a\right)\Bigg] \\ &+ \frac{z}{z^4 + 4c_1^2\eta^2}\Bigg[(z^2\partial_\pm x^2 \pm 2c_1\eta\partial_\pm x^3)\left(\frac{\lambda^{\pm1}}{2}\gamma_2^a - n_{25}^a\right) \\ &\qquad + (z^2\partial_\pm x^3 \mp 2c_1\eta\partial_\pm x^2)\left(\frac{\lambda^{\pm1}}{2}\gamma_3^a - n_{35}^a\right)\Bigg] + \frac{\lambda^{\pm1}\partial_\pm z}{2z}\gamma_5^a\,. \end{aligned} \tag{3.5}$$

In the undeformed limit c_1, $c_2 \to 0$, the above expressions are reduced to

$$\mathcal{L}_{\pm}^{\mathrm{PAdS}_5} = \frac{\partial_\pm x^\mu}{z}\left(\frac{\lambda^{\pm 1}}{2}\gamma_\mu^a - n_{\mu 5}^a\right) + \frac{\lambda^{\pm 1}\partial_\pm z}{2z}\gamma_5^a\,. \tag{3.6}$$

This is nothing but a Lax pair for the Poincaré AdS_5.

Another derivation of Lax pair. It would be of good significance to describe another derivation of Lax pair (3.5).

The undeformed current is now given by

$$A_\pm = \frac{1}{z}\partial_\pm x^\mu p_\mu + \frac{1}{2z}\partial_\pm z\,\gamma_5^a\,. \tag{3.7}$$

Then, by comparing the deformed current (3.3) with the undeformed one (3.7), the deformation under our consideration can be reinterpreted as the following replacement rules:

$$\begin{aligned}
\frac{1}{z^2}\,\partial_\pm x^0 &\longrightarrow \frac{z^2}{z^4 - 4{c_2}^2\eta^2}\left[\partial_\pm x^0 \pm \frac{2c_2\eta}{z^2}\partial_\pm x^1\right],\\
\frac{1}{z^2}\,\partial_\pm x^1 &\longrightarrow \frac{z^2}{z^4 - 4{c_2}^2\eta^2}\left[\partial_\pm x^1 \pm \frac{2c_2\eta}{z^2}\partial_\pm x^0\right],\\
\frac{1}{z^2}\,\partial_\pm x^2 &\longrightarrow \frac{z^2}{z^4 + 4{c_1}^2\eta^2}\left[\partial_\pm x^2 \pm \frac{2c_1\eta}{z^2}\partial_\pm x^3\right],\\
\frac{1}{z^2}\,\partial_\pm x^3 &\longrightarrow \frac{z^2}{z^4 + 4{c_1}^2\eta^2}\left[\partial_\pm x^3 \mp \frac{2c_1\eta}{z^2}\partial_\pm x^2\right].
\end{aligned} \tag{3.8}$$

The above concise rules give rise to another simple derivation of the Lax pair (3.5). By applying the rules to the undeformed Lax pair (3.6), the desired one (3.5) can be reproduced. This derivation is quite similar to Frolov's construction of Lax pair for string on the γ-deformed S^5 [31].

Twisted boundary condition. In fact, due to the rule (3.8), the deformation can be regarded as a twisted boundary condition with the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$, as argued in [31].

For simplicity, suppose $c_1 \neq 0$ and $c_2 = 0$. The analysis for the case with $c_2 \neq 0$ is quite similar, though there is a subtlety for the signature of the metric (For the detail, see [32, 33]).

After performing the Yang-Baxter deformation (equivalently the associated TsT transformation), the original coordinates $\tilde{x}^2$ and $\tilde{x}^3$ for the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ are mapped to x^2 and x^3. Then the relations are given by

$$\begin{aligned}
\frac{1}{z^2}\,\partial_\pm \tilde{x}^2 &= \frac{z^2}{z^4 + 4{c_1}^2\eta^2}\left[\partial_\pm x^2 \pm \frac{2c_1\eta}{z^2}\partial_\pm x^3\right],\\
\frac{1}{z^2}\,\partial_\pm \tilde{x}^3 &= \frac{z^2}{z^4 + 4{c_1}^2\eta^2}\left[\partial_\pm x^3 \mp \frac{2c_1\eta}{z^2}\partial_\pm x^2\right].
\end{aligned} \tag{3.9}$$

These relations indicate the following equivalence of Noether currents

$$\tilde{P}_2^\alpha = P_2^\alpha\,, \qquad \tilde{P}_3^\alpha = P_3^\alpha\,, \tag{3.10}$$

where P_i^α and $\tilde{P}_i^\alpha$ $(i = 2,3)$ are conserved currents associated with translation invariance for x^i and $\tilde{x}^i$ directions, respectively. The τ-component of the relations means that the momentum $p_i \equiv P_i^\tau$ is identical to $\tilde{p}_i \equiv \tilde{P}_i^\tau$, namely $p_i = \tilde{p}_i$. Then, evaluating the σ-component of (3.10) leads to the relations:

$$\partial_\sigma \tilde{x}^2 = \partial_\sigma x^2 + \frac{2c_1\eta}{\sqrt{\lambda_c}}\, p_3\,, \qquad \partial_\sigma \tilde{x}^3 = \partial_\sigma x^3 - \frac{2c_1\eta}{\sqrt{\lambda_c}}\, p_2\,. \tag{3.11}$$

Finally, by integrating these expressions, one can figure out that the deformed background with the usual periodic boundary condition is equivalent to the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ with a twisted boundary condition:

$$\tilde{x}^2(\sigma = 2\pi) = \tilde{x}^2(\sigma = 0) + \frac{2c_1\eta}{\sqrt{\lambda_c}}\, P_3\,, \qquad \tilde{x}^3(\sigma = 2\pi) = \tilde{x}^3(\sigma = 0) - \frac{2c_1\eta}{\sqrt{\lambda_c}}\, P_2\,. \tag{3.12}$$

Here P_i are Noether charges for translation invariance in the x^i directions.

Thus the Yang-Baxter deformation with the classical r-matrix (3.1) can be reinterpreted as a twisted boundary condition with the usual $\mathrm{AdS}_5\times\mathrm{S}^5$.

3.2 A scaling limit of a Lax pair for a q-deformed AdS_5

In section 3.1, we have derived the Lax pair (3.5) as Yang-Baxter deformations of AdS_5. Here we shall reproduce it as a scaling limit of a Lax pair for a q-deformed AdS_5.

A scaling limit of a q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$. We first give a short review of a scaling limit of a q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ [53]. In this limit, the metric and NS-NS two-form in (3.4) can be reproduced.

The starting point is the q-deformed metric and NS-NS two-form,

$$\begin{aligned}
ds^2_{\mathrm{AdS}_5} &= \sqrt{1+\varkappa^2}\left[\frac{1}{1-\varkappa^2\sinh^2\rho}\left(-\cosh^2\rho\, dt^2 + d\rho^2\right)\right.\\
&\qquad\left. + \frac{\sinh^2\rho}{1+\varkappa^2\sin^2\zeta\sinh^4\rho}\left[d\zeta^2 + \cos^2\zeta\,(d\psi_1)^2\right] + \sinh^2\rho\sin^2\zeta\,(d\psi_2)^2\right],\\
B_{\mathrm{AdS}_5} &= \varkappa\sqrt{1+\varkappa^2}\,\frac{\sinh^4\rho\sin 2\zeta}{1+\varkappa^2\sin^2\zeta\sinh^4\rho}\, d\psi_1\wedge d\zeta\,.
\end{aligned} \tag{3.13}$$

Let us next rescale the coordinates as follows:

$$\begin{aligned}
&t = \sqrt{\varkappa}\, x^0\,, \quad \psi_1 = \frac{\sqrt{\varkappa}}{\cos\zeta_0}\, x^2\,, \quad \psi_2 = \frac{\sqrt{\varkappa}}{\sin\zeta_0}\, x^1\,,\\
&\zeta = \zeta_0 + \sqrt{\varkappa}\, x^3\,, \qquad \rho = \operatorname{arcsinh}\left[\frac{1}{\sqrt{\varkappa}\, z}\right].
\end{aligned} \tag{3.14}$$

Here new coordinates $x^0\,, x^1\,, x^2\,, x^3\,, z$ and a real constant ζ_0 have been introduced.

After taking the $\varkappa \to 0$ limit, the resulting metric and NS-NS two-form are given by

$$\begin{aligned}
ds^2 &= \frac{-(dx^0)^2 + (dx^1)^2}{z^2} + \frac{z^2[(dx^2)^2 + (dx^3)^2]}{z^4 + \sin\zeta_0^2} + \frac{dz^2}{z^2}\,,\\
B &= \frac{\sin\zeta_0}{z^4 + \sin\zeta_0^2}\, dx^2\wedge dx^3\,.
\end{aligned} \tag{3.15}$$

This result exactly agrees with a one-parameter case of (3.4) through the identification

$$2c_1\,\eta = \sin\zeta_0\,, \qquad 2c_2\,\eta = 0\,. \tag{3.16}$$

For the S^5 part, this limit is nothing but the undeformed limit.

Lax pair — the third derivation. Next, we will derive the Lax pair (3.5) by taking the scaling limit of a Lax pair for the q-deformed AdS_5. This is the third derivation.

The Lax pair for the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ was originally constructed in [19, 20]. With a coordinate system [52], the Lax pair can be evaluated explicitly, as shown in appendix B. The remaining task is to take the scaling limit of the Lax pair (B.22).

The first is to rewrite the Lax pair (B.22) in terms of the coordinates (3.14) with (3.16). For later convenience, the spectral parameter should be flipped as $\lambda\to-\lambda$. Then, taking the $\varkappa\to 0$ limit leads to the following expression:

$$\begin{aligned}\widetilde{\mathcal{L}}_\pm = {}& \frac{\partial_\pm x^0}{z}\left[-\frac{\lambda^{\pm1}}{2}i\gamma_5^a + in_{15}^a\right] + \frac{\partial_\pm x^1}{z}\left[-i\frac{\lambda^{\pm1}}{2}\gamma_0^a - in_{01}^a\right] \\ &+ \frac{z(z^2\partial_\pm x^2+\eta\partial_\pm x^3)}{z^4+4c_1^2\eta^2}\left[\frac{\lambda^{\pm1}}{2}\gamma_2^a - n_{12}^a\right] \\ &+ \frac{z(z^2\partial_\pm x^3-\eta\partial_\pm x^2)}{z^4+4c_1^2\eta^2}\left[\frac{\lambda^{\pm1}}{2}\gamma_3^a - n_{13}^a\right] - \frac{\lambda^{\pm1}\partial_\pm z}{2z}\gamma_1^a\,. \end{aligned} \tag{3.17}$$

In order to see that the result (3.17) is identical to the Lax pair (3.5), it is necessary to perform a unitary transformation like

$$\widetilde{\mathcal{L}}_\pm \;\longrightarrow\; \mathcal{U}\,\widetilde{\mathcal{L}}_\pm\mathcal{U}^{-1}\,,\qquad \mathcal{U}\equiv\begin{pmatrix} U & 0\\ 0 & 1\end{pmatrix}\,,\qquad U\equiv\begin{pmatrix} 1 & i & -i & 1\\ i & 1 & -1 & i\\ -i & 1 & 1 & i\\ -1 & i & i & 1\end{pmatrix}\,. \tag{3.18}$$

After that, the transformed Lax pair agrees with the one (3.5), namely,

$$\mathcal{L}_\pm^{\mathrm{NC}} = \mathcal{U}\,\widetilde{\mathcal{L}}_\pm\mathcal{U}^{-1} \qquad \text{when}\;\; c_2=0\,. \tag{3.19}$$

Thus the scaling limit works well at the level of Lax pair.

It would be nice to consider this relation at the level of classical r-matrix. One may interpret the scaling limit as a rescaling of Drinfeld-Jimbo type classical r-matrix [54–56].

4 Lax pairs for γ-deformations of S^5

In this section, we shall study Yang-Baxter deformations with classical r-matrices corresponding to γ-deformations of S^5. Concretely speaking, the associated Lax pairs are computed explicitly. The resulting expressions nicely agree with the Lax pairs obtained via TsT transformations of S^5 [31]. We will omit the AdS_5 part in the following.

Let us consider abelian classical r-matrices, which have been found in [23],

$$r = \mu_3\,h_4\wedge h_5 + \mu_1\,h_5\wedge h_6 + \mu_2\,h_6\wedge h_4\,. \tag{4.1}$$

Here h_4, h_5 and h_6 are the three Cartan generators in $\mathfrak{su}(4)$, and μ_i ($i=1,2,3$) are deformation parameters. For our convention of the generators, see appendix A. The r-matrices (4.1) deform only S^5 and correspond to γ-deformations of S^5.

The deformed metric and NS-NS two-form. It is helpful to use the following representative of a group element of SU(4),

$$g_s(\tau,\sigma) = \exp\left[\frac{i}{2}(\phi_1\, h_4 + \phi_2\, h_5 + \phi_3\, h_6)\right] \exp\left[-\zeta n^s_{13}\right] \exp\left[-\frac{i}{2}\, r\, \gamma^s_1\right] . \tag{4.2}$$

By solving the equations in (2.11), the deformed current $J_\alpha^{\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3}$ is determined as

$$\begin{aligned}
J_\pm^{\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3} = &-i\partial_\pm r\, \frac{\gamma^s_1}{2} - \partial_\pm\zeta \left[i \sin r\, \frac{\gamma^s_3}{2} + \cos r\, n^s_{13}\right] \\
&- G(\hat{\gamma}_i)\Bigg[\partial_\pm\phi_1 \pm (\hat{\gamma}_3 \sin^2 r \sin^2\zeta\partial_\pm\phi_2 - \hat{\gamma}_2\cos^2 r\partial_\pm\phi_3) \\
&\qquad + \hat{\gamma}_1 \sin^2 r\cos^2 r \sin^2\zeta \sum_{i=1}^{3}\hat{\gamma}_i\partial_\pm\phi_i\Bigg] \\
&\times \left[\cos\zeta\left(i\sin r\,\frac{\gamma^s_2}{2} + \cos r\, n^s_{12}\right) + \sin\zeta\, n^s_{23}\right] \\
&- G(\hat{\gamma}_i)\Bigg[\partial_\pm\phi_2 \pm (\hat{\gamma}_1 \cos^2 r\partial_\pm\phi_3 - \hat{\gamma}_3\sin^2 r\cos^2\zeta\partial_\pm\phi_1) \\
&\qquad + \hat{\gamma}_2 \sin^2 r\cos^2 r \cos^2\zeta \sum_{i=1}^{3}\hat{\gamma}_i\partial_\pm\phi_i\Bigg] \\
&\times \left[\sin\zeta\left(i\,\sin r\,\frac{1}{2}\gamma^s_4 + \cos r\, n^s_{14}\right) + \cos\zeta\, n^s_{34}\right] \\
&+ G(\hat{\gamma}_i)\Bigg[\partial_\pm\phi_3 \pm (\hat{\gamma}_2 \sin^2 r\cos^2\zeta\partial_\pm\phi_1 - \hat{\gamma}_1\sin^2 r\sin^2\zeta\partial_\pm\phi_2) \\
&\qquad + \hat{\gamma}_3 \sin^4 r\sin^2\zeta \cos^2\zeta \sum_{i=1}^{3}\hat{\gamma}_i\partial_\pm\phi_i\Bigg] \\
&\times \left[i\cos r\,\frac{\gamma^s_5}{2} - \sin r\, n^s_{15}\right] .
\end{aligned} \tag{4.3}$$

Here the parameters $\hat{\gamma}_i$ are defined as

$$\hat{\gamma}_i \equiv 8\,\eta\mu_i\,, \tag{4.4}$$

and the scalar function $G(\hat{\gamma}_i)$ is

$$G^{-1}(\hat{\gamma}_i) \equiv 1 + \sin^2 r(\hat{\gamma}_1^2\cos^2 r\sin^2\zeta + \hat{\gamma}_2^2\cos^2 r\cos^2\zeta + \hat{\gamma}_3^2\sin^2 r\sin^2\zeta\cos^2\zeta)\,. \tag{4.5}$$

This deformed current (4.3) will play an important role in the following analysis.

Substituting the deformed current (4.3) into the Lagrangian (2.9) leads to the background

$$\begin{aligned}
ds^2 &= \sum_{i=1}^{3}\left(d{\rho_i}^2 + G(\hat{\gamma}_i){\rho_i}^2 d{\phi_i}^2\right) + G(\hat{\gamma}_i){\rho_1}^2{\rho_2}^2{\rho_3}^2\left(\sum_{i=1}^{3}\hat{\gamma}_i d\phi_i\right)^2 , \\
B_2 &= G(\hat{\gamma}_i)\left(\hat{\gamma}_3{\rho_1}^2{\rho_2}^2 d\phi_1\wedge d\phi_2 + \hat{\gamma}_1{\rho_2}^2{\rho_3}^2 d\phi_2\wedge d\phi_3 + \hat{\gamma}_2{\rho_3}^2{\rho_1}^2 d\phi_3\wedge d\phi_1\right).
\end{aligned} \tag{4.6}$$

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Here new coordinates ρ_i $(i = 1, 2, 3)$ are defined as

$$\rho_1 \equiv \sin r \cos\zeta \,, \qquad \rho_2 \equiv \sin r \sin\zeta \,, \qquad \rho_3 \equiv \cos r \,. \tag{4.7}$$

The metric and NS-NS two-form in (4.6) agree with 3-parameter γ-deformations of S^5 [31].

A particular one-parameter case with

$$\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3 \equiv \hat{\gamma} \tag{4.8}$$

corresponds to the Lunin-Maldacena solution [30] described by

$$\begin{aligned} ds^2 &= \sum_{i=1}^{3}\left(d\rho_i{}^2 + G\rho_i{}^2 d\phi_i{}^2\right) + G\hat{\gamma}^2 \rho_1{}^2\rho_2{}^2\rho_3{}^2 \left(\sum_{i=1}^{3} d\phi_i\right)^2 \,, \\ B_2 &= G\,\hat{\gamma}(\rho_1{}^2\rho_2{}^2 d\phi_1\wedge d\phi_2 + \rho_2{}^2\rho_3{}^2 d\phi_2\wedge d\phi_3 + \rho_3{}^2\rho_1{}^2 d\phi_3\wedge d\phi_1)\,, \end{aligned} \tag{4.9}$$

where the scalar function G is defined as

$$G^{-1} \equiv 1 + \frac{\hat{\gamma}^2}{4}(\sin^2 2r + \sin^4 r \sin^2 2\zeta)\,. \tag{4.10}$$

This background is a holographic dual of the β-deformation of the $\mathcal{N} = 4$ super Yang-Mills theory [57].

Lax pair. The next task is to evaluate the Lax pair in (2.21) with the classical r-matrix (4.1). The components $\mathcal{L}_\pm^{\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3}$ are given by

$$\begin{aligned} \mathcal{L}_\pm^{\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3} &= -i\frac{\lambda^{\pm1}}{2}\partial_\pm r\,\gamma_1^s - \partial_\pm\zeta\left[i\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_3^s + \cos r\,n_{13}^s\right] \\ &\quad - G(\hat{\gamma}_i)\Bigg[\partial_\pm\phi_1 \pm (\hat{\gamma}_3\sin^2 r\sin^2\zeta\partial_\pm\phi_2 - \hat{\gamma}_2\cos^2 r\partial_\pm\phi_3) \\ &\qquad\qquad + \hat{\gamma}_1\sin^2 r\cos^2 r\sin^2\zeta\sum_{i=1}^{3}\hat{\gamma}_i\partial_\pm\phi_i\Bigg] \\ &\quad \times\left[\cos\zeta\left(i\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_2^s + \cos r\,n_{12}^s\right) + \sin\zeta\,n_{23}^s\right] \\ &\quad - G(\hat{\gamma}_i)\Bigg[\partial_\pm\phi_2 \pm (\hat{\gamma}_1\cos^2 r\partial_\pm\phi_3 - \hat{\gamma}_3\sin^2 r\cos^2\zeta\partial_\pm\phi_1) \\ &\qquad\qquad + \hat{\gamma}_2\sin^2 r\cos^2 r\cos^2\zeta\sum_{i=1}^{3}\hat{\gamma}_i\partial_\pm\phi_i\Bigg] \\ &\quad \times\left[\sin\zeta\left(i\,\sin r\,\frac{\lambda^{\pm1}}{2}\gamma_4^s + \cos r\,n_{14}^s\right) + \cos\zeta\,n_{34}^s\right] \\ &\quad + G(\hat{\gamma}_i)\Bigg[\partial_\pm\phi_3 \pm (\hat{\gamma}_2\sin^2 r\cos^2\zeta\partial_\pm\phi_1 - \hat{\gamma}_1\sin^2 r\sin^2\zeta\partial_\pm\phi_2) \\ &\qquad\qquad + \hat{\gamma}_3\sin^4 r\sin^2\zeta\cos^2\zeta\sum_{i=1}^{3}\hat{\gamma}_i\partial_\pm\phi_i\Bigg] \\ &\quad \times\left[i\cos r\,\frac{\lambda^{\pm1}}{2}\gamma_5^s - \sin r\,n_{15}^s\right]. \end{aligned} \tag{4.11}$$

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Note here that, in the undeformed limit $\hat{\gamma}_i \to 0$, $\mathcal{L}_{\pm}^{\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3}$ becomes

$$\begin{aligned}
\mathcal{L}_{\pm}^{S} = &-i\,\partial_{\pm}r\frac{\lambda^{\pm 1}}{2}\gamma_1^s - \partial_{\pm}\zeta\left[i\sin r\,\frac{\lambda^{\pm 1}}{2}\gamma_3^s + \cos r\,n_{13}^s\right] \\
&- \partial_{\pm}\phi_1\left[\cos\zeta\left(i\sin r\,\frac{\lambda^{\pm 1}}{2}\gamma_2^s + \cos r\,n_{12}^s\right) + \sin\zeta\,n_{23}^s\right] \\
&- \partial_{\pm}\phi_2\left[\sin\zeta\left(i\,\sin r\,\frac{\lambda^{\pm 1}}{2}\gamma_4^s + \cos r\,n_{14}^s\right) + \cos\zeta\,n_{34}^s\right] \\
&+ \partial_{\pm}\phi_3\left[i\cos r\,\frac{\lambda^{\pm 1}}{2}\gamma_5^s - \sin r\,n_{15}^s\right].
\end{aligned} \tag{4.12}$$

This is just a Lax pair for the undeformed S^5.

Another derivation of Lax pair. Then, let us consider a simple derivation of the Lax pair (4.11), as in the previous section. The undeformed current is given by

$$\begin{aligned}
A_{\pm} = &-i\partial_{\pm}r\,\frac{\gamma_1^s}{2} - \partial_{\pm}\zeta\left[i\sin r\,\frac{\gamma_3^s}{2} + \cos r\,n_{13}^s\right] \\
&- \partial_{\pm}\phi_1\left[\cos\zeta\left(i\sin r\,\frac{\gamma_2^s}{2} + \cos r\,n_{12}^s\right) + \sin\zeta\,n_{23}^s\right] \\
&- \partial_{\pm}\phi_2\left[\sin\zeta\left(i\,\sin r\,\frac{\gamma_4^s}{2} + \cos r\,n_{14}^s\right) + \cos\zeta\,n_{34}^s\right] \\
&+ \partial_{\pm}\phi_3\left[i\cos r\,\frac{\gamma_5^s}{2} - \sin r\,n_{15}^s\right].
\end{aligned} \tag{4.13}$$

By comparing the deformed current (4.3) with the undeformed one (4.13), we can identify the following replacement rules:

$$\begin{aligned}
\partial_{\pm}\phi_1 \longrightarrow G(\hat{\gamma}_i)\Bigg[&\partial_{\pm}\phi_1 \pm (\hat{\gamma}_3\sin^2 r\sin^2\zeta\partial_{\pm}\phi_2 - \hat{\gamma}_2\cos^2 r\partial_{\pm}\phi_3) \\
&+ \hat{\gamma}_1\sin^2 r\cos^2 r\sin^2\zeta\sum_{i=1}^{3}\hat{\gamma}_i\partial_{\pm}\phi_i\Bigg], \\
\partial_{\pm}\phi_2 \longrightarrow G(\hat{\gamma}_i)\Bigg[&\partial_{\pm}\phi_2 \pm (\hat{\gamma}_1\cos^2 r\partial_{\pm}\phi_3 - \hat{\gamma}_3\sin^2 r\cos^2\zeta\partial_{\pm}\phi_1), \\
&+ \hat{\gamma}_2\sin^2 r\cos^2 r\cos^2\zeta\sum_{i=1}^{3}\hat{\gamma}_i\partial_{\pm}\phi_i\Bigg], \\
\partial_{\pm}\phi_3 \longrightarrow G(\hat{\gamma}_i)\Bigg[&\partial_{\pm}\phi_3 \pm (\hat{\gamma}_2\sin^2 r\cos^2\zeta\partial_{\pm}\phi_1 - \hat{\gamma}_1\sin^2 r\sin^2\zeta\partial_{\pm}\phi_2) \\
&+ \hat{\gamma}_3\sin^4 r\sin^2\zeta\cos^2\zeta\sum_{i=1}^{3}\hat{\gamma}_i\partial_{\pm}\phi_i\Bigg].
\end{aligned} \tag{4.14}$$

Due to these replacement rules (4.14), the deformed Lax pair (4.11) can be reconstructed from the undeformed one (4.12). In fact, the replacement rules (4.14) are identical to a TsT-transformation [31],[7] and hence the Lax pair (4.11) is equivalent to the one in [31].[8] This result further confirms the correspondence between a Yang-Baxter deformation with (4.1) and a TsT transformation [23].

Finally, it is worth mentioning the reinterpretation of the deformation as a twisted boundary condition. This fact was originally shown in [31]. The twisted boundary condition with the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ is given by

$$\begin{aligned}
\tilde{\phi}_1(\sigma=2\pi) &= \tilde{\phi}_1(\sigma=0) + \gamma_3 J_2 - \gamma_2 J_3 + 2\pi n_1\,, \\
\tilde{\phi}_2(\sigma=2\pi) &= \tilde{\phi}_2(\sigma=0) + \gamma_1 J_3 - \gamma_3 J_1 + 2\pi n_2\,, \\
\tilde{\phi}_3(\sigma=2\pi) &= \tilde{\phi}_3(\sigma=0) + \gamma_2 J_1 - \gamma_1 J_2 + 2\pi n_3\,,
\end{aligned} \tag{4.15}$$

with $\gamma_i \equiv \hat{\gamma}/\sqrt{\lambda_{\mathrm{c}}}$. Here J_i are Noether charges for rotation invariance in the ϕ_i directions. Integers n_i are winding numbers along the ϕ_i directions.

5 Lax pairs for Schrödinger spacetimes

Let us consider a classical r-matrix which deforms both AdS_5 and S^5. Such an r-matrix contains generators of both $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$. A simple example is the following [27]:

$$r = \frac{i}{4\sqrt{2}}\,(p_0 - p_3)\wedge(h_4 + h_5 + h_6)\,. \tag{5.1}$$

For convention of the generators, see appendix A. This r-matrix (5.1) is associated with Schrödinger spacetimes realized in type IIB supergravity [34–36], as shown in [27].

The deformed metric and NS-NS two-form. The bosonic group elements of SU(2,2) and SU(4) are parameterized as follows

$$\begin{aligned}
g_a(\tau,\sigma) &= \exp\left[x^0 p_0 + x^1 p_1 + x^2 p_2 + x^3 p_3\right] \exp\left[\gamma_5^a\,\frac{1}{2}\,\log z\right] &&\in\ \mathrm{SU}(2,2)\,, \\
g_s(\tau,\sigma) &= \exp\left[\frac{i}{2}(\psi_1 h_4 + \psi_2 h_5 + \psi_3 h_6)\right] \exp\left[-\zeta n_{13}^s\right] \exp\left[-\frac{i}{2} r\,\gamma_1^s\right] &&\in\ \mathrm{SU}(4)\,.
\end{aligned} \tag{5.2}$$

[7] In our argument, the rules are identified on the off-shell level, but the one in [31] is done on the on-shell.

[8] To see this equivalence (up to small differences of convention), we have to perform a gauge transformation

$$h = \exp[-i\zeta n_{02}]\exp\left[\frac{i}{2}r\gamma_2\right]\exp\left[\frac{\pi}{2}(n_{12} + i n_{03})\right]$$

and a Möbius transformation for the spectral parameter $\lambda \to \frac{\lambda+1}{\lambda-1}$.

The deformed current $J_\pm$ can be expanded in terms of the generators of $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$. Then, by solving the equation in (2.11), $J_\pm$ is determined as follows

$$
\begin{aligned}
J^a_\pm &= \frac{1}{z}\partial_\pm x^1 p_1 + \frac{1}{z}\partial_\pm x^2 p_2 + \frac{1}{2z}\partial_\pm z\,\gamma^a_5 \\
&\quad + \frac{1}{\sqrt{2}\,z}\partial_\pm x^+\,(p_0 + p_3) \\
&\quad + \frac{1}{\sqrt{2}\,z}\left[\partial_\pm x^- \pm \eta\partial_\pm\chi \pm \frac{\eta}{2}\sin^2\mu(\partial_\pm\psi + \cos\theta\partial_\pm\phi) + \frac{\eta^2}{z^2}\partial_\pm x^+\right](p_0 - p_3)\,, \\
J^s_\pm &= -\frac{i}{2}\partial_\pm\mu\,\gamma^s_1 - \frac{1}{2}\partial_\pm\theta\left[\frac{i}{2}\sin\mu\,\gamma^s_3 + \cos\mu\,n^s_{13}\right] \\
&\quad - \left[\partial_\pm\chi \pm \frac{\eta\,\partial_\pm x^+}{z^2}\right]\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_4 + \cos\mu\,n^s_{14} + n^s_{23}\right)\right. \\
&\quad \left. + \cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_2 + \cos\mu\,n^s_{12} + n^s_{34}\right) - \frac{i}{2}\cos\mu\,\gamma^s_5 + \sin\mu\,n^s_{15}\right] \\
&\quad + \frac{1}{2}\partial_\pm\phi\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_4 + \cos\mu\,n^s_{14} - n^s_{23}\right)\right. \\
&\quad \left. - \cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_2 + \cos\mu\,n^s_{12} - n^s_{34}\right)\right] \\
&\quad - \frac{1}{2}\partial_\pm\psi\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_4 + \cos\mu\,n^s_{14} + n^s_{23}\right)\right. \\
&\quad \left. + \cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_2 + \cos\mu\,n^s_{12} + n^s_{34}\right)\right].
\end{aligned}
\tag{5.3}
$$

Here we have performed a coordinate transformation,

$$
\begin{aligned}
&x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}\,, \\
&r = \mu\,, \quad \zeta = \frac{1}{2}\theta\,, \quad \psi_1 = \chi + \frac{1}{2}(\psi+\phi)\,, \quad \psi_2 = \chi + \frac{1}{2}(\psi - \phi)\,, \quad \psi_3 = \chi\,.
\end{aligned}
$$

With the deformed current (5.3), the resulting background is given by

$$
\begin{aligned}
ds^2 &= \frac{-2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \eta^2\frac{(dx^+)^2}{z^4} + ds^2_{\mathrm{S}^5}\,, \\
B_2 &= \frac{\eta}{z^2}\,dx^+ \wedge (d\chi + \omega)\,.
\end{aligned}
\tag{5.4}
$$

Here the S^5 metric is written as an S^1-fibration over $\mathbb{C}\mathrm{P}^2$,

$$
\begin{aligned}
ds^2_{\mathrm{S}^5} &= (d\chi + \omega)^2 + ds^2_{\mathbb{C}\mathrm{P}^2}\,, \\
ds^2_{\mathbb{C}\mathrm{P}^2} &= d\mu^2 + \sin^2\mu\left(\Sigma_1^2 + \Sigma_2^2 + \cos^2\mu\,\Sigma_3^2\right).
\end{aligned}
\tag{5.5}
$$

Now χ is the fiber coordinate and ω is a one-form potential of the Kähler form on $\mathbb{C}\mathrm{P}^2$. The symbols Σ_i $(i = 1,2,3)$ and ω are defined as

$$
\begin{aligned}
\Sigma_1 &\equiv \frac{1}{2}(\cos\psi\,d\theta + \sin\psi\sin\theta\,d\phi)\,, \\
\Sigma_2 &\equiv \frac{1}{2}(\sin\psi\,d\theta - \cos\psi\sin\theta\,d\phi)\,, \\
\Sigma_3 &\equiv \frac{1}{2}(d\psi + \cos\theta\,d\phi)\,, \qquad \omega \equiv \sin^2\mu\,\Sigma_3\,.
\end{aligned}
\tag{5.6}
$$

It is remarkable that only the AdS_5 metric is deformed while the S^5 part is not, in spite of the expression of the classical r-matrix (5.1). On the other hand, the NS-NS two-form carries two indices, one of which is from AdS_5 and the other is S^5.

Lax pair. In a similar way, one can evaluate the associated Lax pair. The resulting expression is a bit messy but given by

$$\begin{aligned}
\mathcal{L}_\pm^{\rm Sch} &= \frac{1}{z}\partial_\pm x^1\left[\frac{\lambda^{\pm1}}{2}\gamma_1^a - n_{15}^a\right] + \frac{1}{z}\partial_\pm x^2\left[\frac{\lambda^{\pm1}}{2}\gamma_2^a - n_{25}^a\right] + \frac{\lambda^{\pm1}}{2z}\partial_\pm z\,\gamma_5^a \\
&\quad + \frac{1}{\sqrt{2}\,z}\partial_\pm x^+\left[\frac{\lambda^{\pm1}}{2}\gamma_0^a + \frac{\lambda^{\pm1}}{2}\gamma_3^a - n_{05}^a - n_{35}^a\right] \\
&\quad + \frac{1}{\sqrt{2}\,z}\left[\partial_\pm x^- \pm \eta\partial_\pm\chi \pm \frac{\eta}{2}\sin^2\mu(\partial_\pm\psi + \cos\theta\partial_\pm\phi) + \frac{\eta^2}{z^2}\partial_\pm x^+\right] \\
&\quad \times\left[\frac{\lambda^{\pm1}}{2}\gamma_0^a - \frac{\lambda^{\pm1}}{2}\gamma_3^a - n_{05}^a + n_{35}^a\right] \\
&\quad - \frac{i\lambda^{\pm1}}{2}\partial_\pm\mu\,\gamma_1^s - \frac{1}{2}\partial_\pm\theta\left[\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_3^s + \cos\mu\,n_{13}^s\right] \\
&\quad - \left[\partial_\pm\chi \pm \frac{\eta\,\partial_\pm x^+}{z^2}\right]\left[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_4^s + \cos\mu\,n_{14}^s + n_{23}^s\right)\right. \\
&\quad \left. + \cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_2^s + \cos\mu\,n_{12}^s + n_{34}^s\right) - \frac{i\lambda^{\pm1}}{2}\cos\mu\,\gamma_5^s + \sin\mu\,n_{15}^s\right] \\
&\quad + \frac{1}{2}\partial_\pm\phi\left[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_4^s + \cos\mu\,n_{14}^s - n_{23}^s\right)\right. \\
&\quad \left. - \cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_2^s + \cos\mu\,n_{12}^s - n_{34}^s\right)\right] \\
&\quad - \frac{1}{2}\partial_\pm\psi\left[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_4^s + \cos\mu\,n_{14}^s + n_{23}^s\right)\right. \\
&\quad \left. + \cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_2^s + \cos\mu\,n_{12}^s + n_{34}^s\right)\right].
\end{aligned} \tag{5.7}$$

It would be helpful to check the undeformed limit. As $\eta \to 0$, the above Lax pair $\mathcal{L}_\pm^{\rm Sch}$ is reduced to the following:

$$\begin{aligned}
\mathcal{L}_\pm &= \frac{1}{z}\partial_\pm x^1\left[\frac{\lambda^{\pm1}}{2}\gamma_1^a - n_{15}^a\right] + \frac{1}{z}\partial_\pm x^2\left[\frac{\lambda^{\pm1}}{2}\gamma_2^a - n_{25}^a\right] + \frac{\lambda^{\pm1}}{2z}\partial_\pm z\,\gamma_5^a \\
&\quad + \frac{1}{\sqrt{2}\,z}\partial_\pm x^+\left[\frac{\lambda^{\pm1}}{2}\gamma_0^a + \frac{\lambda^{\pm1}}{2}\gamma_3^a - n_{05}^a - n_{35}^a\right] \\
&\quad + \frac{1}{\sqrt{2}\,z}\partial_\pm x^-\left[\frac{\lambda^{\pm1}}{2}\gamma_0^a - \frac{\lambda^{\pm1}}{2}\gamma_3^a - n_{05}^a + n_{35}^a\right] \\
&\quad - \frac{i\lambda^{\pm1}}{2}\partial_\pm\mu\,\gamma_1^s - \frac{1}{2}\partial_\pm\theta\left[\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_3^s + \cos\mu\,n_{13}^s\right] \\
&\quad - \partial_\pm\chi\left[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_4^s + \cos\mu\,n_{14}^s + n_{23}^s\right)\right. \\
&\qquad \left. + \cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma_2^s + \cos\mu\,n_{12}^s + n_{34}^s\right) - \frac{i\lambda^{\pm1}}{2}\cos\mu\,\gamma_5^s + \sin\mu\,n_{15}^s\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\partial_\pm\phi\left[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma^s_4+\cos\mu\,n^s_{14}-n^s_{23}\right)\right.\\
&\qquad\qquad\left.-\cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma^s_2+\cos\mu\,n^s_{12}-n^s_{34}\right)\right]\\
&-\frac{1}{2}\partial_\pm\psi\left[\sin\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma^s_4+\cos\mu\,n^s_{14}+n^s_{23}\right)\right.\\
&\qquad\qquad\left.+\cos\frac{\theta}{2}\left(\frac{i\lambda^{\pm1}}{2}\sin\mu\,\gamma^s_2+\cos\mu\,n^s_{12}+n^s_{34}\right)\right].
\end{aligned}
\tag{5.8}
$$

Another derivation of Lax pair. Let us consider another derivation of the Lax pair again. One can see the replacement rules by comparing the deformed current with the undeformed one, as before.

The undeformed current is decomposed into the AdS_5 and S^5 components like

$$
A_\pm=A^a_\pm+A^s_\pm\,,
\tag{5.9}
$$

where $A^a_\pm$ and $A^s_\pm$ are given by

$$
\begin{aligned}
A^a_\pm&=\frac{1}{z}\partial_\pm x^1p_1+\frac{1}{z}\partial_\pm x^2p_2+\frac{1}{2z}\partial_\pm z\,\gamma^a_5+\frac{1}{\sqrt{2}\,z}\partial_\pm x^+(p_0+p_3)+\frac{1}{\sqrt{2}\,z}\partial_\pm x^-(p_0-p_3)\,,\\
A^s_\pm&=-\frac{i}{2}\partial_\pm\mu\,\gamma^s_1-\frac{1}{2}\partial_\pm\theta\left[\frac{i}{2}\sin\mu\,\gamma^s_3+\cos\mu\,n^s_{13}\right]\\
&\quad-\partial_\pm\chi\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_4+\cos\mu\,n^s_{14}+n^s_{23}\right)\right.\\
&\qquad\qquad\left.+\cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_2+\cos\mu\,n^s_{12}+n^s_{34}\right)-\frac{i}{2}\cos\mu\,\gamma^s_5+\sin\mu\,n^s_{15}\right]\\
&\quad+\frac{1}{2}\partial_\pm\phi\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_4+\cos\mu\,n^s_{14}-n^s_{23}\right)\right.\\
&\qquad\qquad\left.-\cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_2+\cos\mu\,n^s_{12}-n^s_{34}\right)\right]\\
&\quad-\frac{1}{2}\partial_\pm\psi\left[\sin\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_4+\cos\mu\,n^s_{14}+n^s_{23}\right)\right.\\
&\qquad\qquad\left.+\cos\frac{\theta}{2}\left(\frac{i}{2}\sin\mu\,\gamma^s_2+\cos\mu\,n^s_{12}+n^s_{34}\right)\right].
\end{aligned}
\tag{5.10}
$$

Thus, by comparing the deformed current (5.3) with the undeformed one (5.10), one can see the following replacement rules:

$$
\begin{aligned}
\frac{1}{z^2}\,\partial_\pm x^-&\longrightarrow\frac{1}{z^2}\,\partial_\pm x^-\pm\frac{\eta}{z^2}\left[\partial_\pm\chi+\frac{1}{2}\sin^2\mu\,(\partial_\pm\psi+\cos\theta\partial_\pm\phi)\right]+\frac{\eta^2}{z^4}\,\partial_\pm x^+\,,\\
\partial_\pm\chi&\longrightarrow\partial_\pm\chi\pm\frac{\eta}{z^2}\,\partial_\pm x^+\,.
\end{aligned}
\tag{5.11}
$$

Then the Lax pair (5.7) can be reproduced by applying the replacement rules (5.11) to the undeformed Lax pair (5.8).

Finally, let us mention about the reinterpretation of the deformation as a twisted boundary condition. Similarly, one can see that the following boundary condition

$$
\begin{aligned}
\tilde{x}^{-}(\sigma=2\pi) &= \tilde{x}^{-}(\sigma=0) + \frac{\eta}{\sqrt{\lambda_{\rm c}}} J_{\chi}, \\
\tilde{\chi}(\sigma=2\pi) &= \tilde{\chi}(\sigma=0) - \frac{\eta}{\sqrt{\lambda_{\rm c}}} P_{-} + 2\pi n_{\chi}\,,
\end{aligned} \tag{5.12}
$$

with the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ is equivalent to the deformed geometry with the usual periodic boundary condition. Here P_- and J_χ are Noether charges for translation and rotation invariance for the x^- and χ directions, respectively. An integer n_χ is a winding number for the χ direction. It may be interesting to consider a relation between the above argument and the symmetric two-form studied in [59].

6 Lax pairs for abelian twists of the global AdS_5

Finally, we will consider abelian twists of the global AdS_5 as Yang-Baxter deformations. The twists are associated with the classical r-matrix,

$$
r = -\frac{i}{2} n^a_{12} \wedge n^a_{03}\,. \tag{6.1}
$$

This is composed of two Cartan generators of $\mathfrak{su}(2,2)$ (For our convention, see appendix A). This r-matrix deforms only the AdS_5 part, hence we will omit the S^5 part hereafter.

The deformed metric and NS-NS two-form. We will work with the following parameterization of a group element of $\mathrm{SU}(2,2)$:

$$
g_a(\tau,\sigma) = \exp\left[\frac{i}{2}(\phi_1\, h_1 + \phi_2\, h_2 + \tau\, h_3)\right] \exp\left[-\theta\, n^a_{13}\right] \exp\left[-\rho\, \frac{\gamma^a_1}{2}\right] \quad \in \mathrm{SU}(2,2)\,. \tag{6.2}
$$

The deformed current $J_\pm$ is expanded in terms of the basis of $\mathfrak{su}(2,2)$. Then, by solving the equation in (2.11), $J_\pm$ can be determined as follows

$$
\begin{aligned}
J_\pm = &-\partial_\pm\rho \frac{1}{2}\gamma^a_1 - \partial_\pm\theta\left[\frac{1}{2}\sinh\rho\,\gamma^a_3 + \cosh\rho\, n^a_{13}\right] \\
&+ i\partial_\pm\tau\left[\frac{1}{2}\cosh\rho\,\gamma^a_5 + \sinh\rho\, n^a_{15}\right] \\
&- \hat{G}\left(\partial_\pm\phi_1 \mp \eta\sin^2\theta\sinh^2\rho\,\partial_\pm\phi_2\right) \\
&\times\left[\cos\theta\left(\frac{1}{2}\sinh\rho\,\gamma^a_2 + \cosh\rho\, n^a_{12}\right) + \sin\theta\, n^a_{23}\right] \\
&+ i\hat{G}\left(\partial_\pm\phi_2 \pm \eta\cos^2\theta\sinh^2\rho\,\partial_\pm\phi_1\right) \\
&\times\left[\sin\theta\left(\frac{1}{2}\sinh\rho\,\gamma^a_0 - \cosh\rho\, n^a_{01}\right) - \cos\theta\, n^a_{03}\right],
\end{aligned} \tag{6.3}
$$

where $\hat{G}$ is a scalar function defined as follows

$$
\hat{G}^{-1} \equiv 1 + \eta^2\sin^2\theta\cos^2\theta\sinh^2\rho\,. \tag{6.4}
$$

By using the current (6.3), the deformed metric and NS-NS two-form are given by

$$\begin{aligned} ds^2 &= -\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \frac{\cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2}{1+\eta^2\sin^2\theta\cos^2\theta\sinh^4\rho}\right), \\ B_2 &= -\frac{\eta\sin^2\theta\cos^2\theta\sinh^4\rho}{1+\eta^2\sin^2\theta\cos^2\theta\sinh^4\rho}\, d\phi_1\wedge d\phi_2\,. \end{aligned} \tag{6.5}$$

This result precisely agrees with abelian twists of the global AdS_5 [50, 51].

Lax pair. The next is to determine the associated Lax pair. By using the deformed current (6.3), the Lax pair can be explicitly evaluated as follows

$$\begin{aligned} \mathcal{L}^{\mathrm{AT}}_{\pm} &= -\partial_\pm\rho\frac{\lambda^{\pm 1}}{2}\gamma_1^a - \partial_\pm\theta\left[\frac{\lambda^{\pm1}}{2}\sinh\rho\,\gamma_3^a + \cosh\rho\, n_{13}^a\right] \\ &\quad + i\partial_\pm\tau\left[\frac{\lambda^{\pm1}}{2}\cosh\rho\,\gamma_5^a + \sinh\rho\, n_{15}^a\right] \\ &\quad - \hat{G}\left(\partial_\pm\phi_1 \mp \eta\sin^2\theta\sinh^2\rho\partial_\pm\,\phi_2\right) \\ &\quad \times\left[\cos\theta\left(\frac{\lambda^{\pm1}}{2}\sinh\rho\,\gamma_2^a + \cosh\rho\, n_{12}^a\right) + \sin\theta\, n_{23}^a\right] \\ &\quad + i\hat{G}\left(\partial_\pm\phi_2 \pm \eta\cos^2\theta\sinh^2\rho\partial_\pm\,\phi_1\right) \\ &\quad \times\left[\sin\theta\left(\frac{\lambda^{\pm1}}{2}\sinh\rho\,\gamma_0^a - \cosh\rho\, n_{01}^a\right) - \cos\theta\, n_{03}^a\right]. \end{aligned} \tag{6.6}$$

In the $\eta\to 0$ limit, $\mathcal{L}^{\mathrm{AT}}_{\pm}$ is reduced to the following form:

$$\begin{aligned} \mathcal{L}^{\mathrm{GAdS}_5}_{\pm} &= -\partial_\pm\rho\frac{\lambda^{\pm 1}}{2}\gamma_1^a - \partial_\pm\theta\left[\frac{\lambda^{\pm1}}{2}\sinh\rho\,\gamma_3^a + \cosh\rho\, n_{13}^a\right] \\ &\quad + i\partial_\pm\tau\left[\frac{\lambda^{\pm1}}{2}\cosh\rho\,\gamma_5^a + \sinh\rho\, n_{15}^a\right] \\ &\quad - \partial_\pm\phi_1\left[\cos\theta\left(\frac{\lambda^{\pm1}}{2}\sinh\rho\,\gamma_2^a + \cosh\rho\, n_{12}^a\right) + \sin\theta\, n_{23}^a\right] \\ &\quad + i\partial_\pm\phi_2\left[\sin\theta\left(\frac{\lambda^{\pm1}}{2}\sinh\rho\,\gamma_0^a - \cosh\rho\, n_{01}^a\right) - \cos\theta\, n_{03}^a\right]. \end{aligned} \tag{6.7}$$

This is nothing but a Lax pair for the global AdS_5.

Another derivation of Lax pair. Even in this case, one can read off the replacement rules as well.

The undeformed current is

$$\begin{aligned} A_\pm &= -\partial_\pm\rho\frac{1}{2}\gamma_1^a - \partial_\pm\theta\left[\frac{1}{2}\sinh\rho\,\gamma_3^a + \cosh\rho\, n_{13}^a\right] \\ &\quad + i\partial_\pm\tau\left[\frac{1}{2}\cosh\rho\,\gamma_5^a + \sinh\rho\, n_{15}^a\right] \\ &\quad - \partial_\pm\phi_1\left[\cos\theta\left(\frac{1}{2}\sinh\rho\,\gamma_2^a + \cosh\rho\, n_{12}^a\right) + \sin\theta\, n_{23}^a\right] \\ &\quad + i\partial_\pm\phi_2\left[\sin\theta\left(\frac{1}{2}\sinh\rho\,\gamma_0^a - \cosh\rho\, n_{01}^a\right) - \cos\theta\, n_{03}^a\right]. \end{aligned} \tag{6.8}$$

Then, by comparing the deformed current (6.3) with the undeformed one (6.8), the replacement rules are identified as follows

$$\begin{aligned}\partial_\pm\phi_1 &\longrightarrow \hat{G}\left(\partial_\pm\phi_1 \mp \eta\sin^2\theta\sinh^2\rho\,\partial_\pm\phi_2\right),\\ \partial_\pm\phi_2 &\longrightarrow \hat{G}\left(\partial_\pm\phi_2 \pm \eta\cos^2\theta\sinh^2\rho\,\partial_\pm\phi_1\right).\end{aligned}\tag{6.9}$$

By applying the replacement rules to the undeformed Lax pair (6.7), one can reproduce the Lax pair (6.6) as well.

Again, one can reinterpret the deformation as a twisted boundary condition. After performing a similar analysis, the twisted boundary condition

$$\begin{aligned}\tilde{\phi}_1(\sigma=2\pi) &= \tilde{\phi}_1(\sigma=0) + \frac{\eta}{\sqrt{\lambda_c}}J_2 + 2\pi n_1\,,\\ \tilde{\phi}_2(\sigma=2\pi) &= \tilde{\phi}_2(\sigma=0) - \frac{\eta}{\sqrt{\lambda_c}}J_1 + 2\pi n_2\end{aligned}\tag{6.10}$$

with the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ is equivalent to the deformed background with a usual periodic boundary condition. Here J_i are Noether charges for rotation invariance in the ϕ_i directions. Integers n_i are winding numbers along the ϕ_i directions.

7 Conclusion and discussion

We have explicitly derived Lax pairs for string theories on Yang-Baxter deformed backgrounds, 1) gravity duals for NC gauge theories, 2) γ-deformations of S^5, 3) Schrödinger spacetimes and 4) abelian twists of the global AdS_5. As another derivation, the Lax pair for gravity duals for NC gauge theories has been reproduced from the one for a q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ by taking a scaling limit.

As a byproduct, we have found a simple derivation of Lax pairs at least for all of the examples we have discussed here. After choosing a classical r-matrix and introducing a coordinate system, the replacement rules have been found out by comparing the deformed current J with the undeformed current A. Then, by applying the rules to a Lax pair for the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$, one can construct the resulting Lax pair associated with the deformation. In addition, we have shown that each of the deformations considered here can be reinterpreted as a twisted boundary condition with the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$, as in the work of [31]. It would be interesting to study the fermionic sector by following the work [58].

This simple derivation really helps us to check the direct computation of Lax pairs based on Yang-Baxter deformations. In addition, it enables us to derive Lax pairs for Yang-Baxter deformations of Minkowski spacetime [39], for which the universal expression of Lax pair has not been obtained yet. Our procedure can play a significant role in studying along this direction. The result would be reported in another place [60].

A more general question is what is the class of classical r-matrix for which one can deduce the replacement rule. Probably, it would be possible for some restricted r-matrices. This is also concerned with another question, what is the class of classical r-matrices for which the insertion of the operator can be eliminated by changing a boundary condition on the string world-sheet. It would be quite important to answer these questions.

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We believe that our concise prescription to construct Lax pairs would be helpful for further understanding of the gravity/CYBE correspondence.

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A Notation and convention

We summarize here our notation and convention of the $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ generators.

The gamma matrices. Let us first introduce the following gamma matrices:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma_0 = i\gamma_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

To embed $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ into $\mathfrak{su}(2,2|4)$, we follow an 8×8 matrix representation as

$$\gamma^a_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma^a_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{with} \qquad \mu = 0,1,2,3\,,$$
$$\gamma^s_i = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_i \end{pmatrix}, \quad \gamma^s_5 = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_5 \end{pmatrix} \qquad \text{with} \qquad i = 1,2,3,4\,. \tag{A.2}$$

Note that each block of the matrices is a 4×4 matrix.

The $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ generators. The Lie algebras $\mathfrak{su}(2,2) \sim \mathfrak{so}(2,4)$ and $\mathfrak{su}(4) \sim \mathfrak{so}(6)$ are spanned as follows:

$$\mathfrak{su}(2,2) = \mathrm{span}_{\mathbb{R}} \left\{ \gamma^a_\mu\,, \gamma^a_5\,, n^a_{\mu\nu} = \frac{1}{4}[\gamma^a_\mu\,, \gamma^a_\nu]\,, n^a_{\mu 5} = \frac{1}{4}[\gamma^a_\mu\,, \gamma^a_5] \mid \mu\,, \nu = 0,1,2,3 \right\},$$
$$\mathfrak{su}(4) = \mathrm{span}_{\mathbb{R}} \left\{ \gamma^s_i\,, \gamma^s_5\,, n^s_{ij} = \frac{1}{4}[\gamma^s_i\,, \gamma^s_j]\,, n^s_{i5} = \frac{1}{4}[\gamma^s_i\,, \gamma^s_5] \mid i,j = 1,2,3,4 \right\}. \tag{A.3}$$

The subalgebras $\mathfrak{so}(1,4)$ and $\mathfrak{so}(5)$ in the spinor representation are formed as follows

$$\begin{aligned} \mathfrak{so}(1,4) &= \mathrm{span}_{\mathbb{R}}\{\, n^a_{\mu\nu}\,,\, n^a_{\mu 5} \mid \mu\,,\nu = 0,1,2,3\,\}\,, \\ \mathfrak{so}(5) &= \mathrm{span}_{\mathbb{R}}\{\, n^s_{ij}\,,\, n^s_{i5} \mid i,j = 1,2,3,4\,\}\,. \end{aligned} \tag{A.4}$$

For a coset construction of Poincaré AdS_5, it is useful to employ the following basis:

$$\mathfrak{su}(2,2) = \mathrm{span}_{\mathbb{R}}\{\, p_\mu\,, k_\mu\,, h_1\,, h_2\,, h_3\,, n^a_{13}\,, n^a_{10}\,, n^a_{23}\,, n^a_{20} \mid \mu = 0,1,2,3\,\}\,. \tag{A.5}$$

Here the generators p_μ, k_μ and the Cartan generators $h_1\,, h_2\,, h_3$ are defined as follows

$$\begin{aligned} &p_\mu \equiv \frac{1}{2}\gamma^a_\mu - n^a_{\mu 5}\,, \qquad k_\mu \equiv \frac{1}{2}\gamma^a_\mu + n^a_{\mu 5}\,, \\ &h_1 \equiv 2i\, n^a_{12} = \mathrm{diag}(-1,1,-1,1,0,0,0,0)\,, \\ &h_2 \equiv 2n^a_{30} = \mathrm{diag}(-1,1,1,-1,0,0,0,0)\,, \\ &h_3 \equiv \gamma^a_5 = \mathrm{diag}(1,1,-1,-1,0,0,0,0)\,. \end{aligned}$$

Note that the generators p_μ and k_μ commute each other,

$$[p_\mu\,, p_\nu] = [k_\mu\,, k_\nu] = [p_\mu\,, k_\nu] = 0 \qquad \text{for} \qquad \mu\,,\nu = 0,1,2,3\,. \tag{A.6}$$

For the S^5 part, the Cartan generators $h_4\,, h_5\,, h_6$ of $\mathfrak{su}(4)$ are given by

$$\begin{aligned} &h_4 \equiv 2i\, n^s_{12} = \mathrm{diag}(0,0,0,0,-1,1,-1,1)\,, \\ &h_5 \equiv 2i\, n^s_{34} = \mathrm{diag}(0,0,0,0,-1,1,1,-1)\,, \\ &h_6 \equiv \gamma^s_5 = \mathrm{diag}(0,0,0,0,1,1,-1,-1)\,. \end{aligned} \tag{A.7}$$

Since non-Cartan generators of $\mathfrak{su}(4)$ are not used in our analysis here, we will not write them down explicitly.

The bosonic coset projectors. In deriving the bosonic part of Lax pairs, it is necessary to employ the coset projectors P_0 and P_2 regarding the $\mathbb{Z}_2$-grading property. The projectors P_0 and P_2 are decomposed into the AdS_5 part and the S^5 part like

$$P_0(x) = P^a_0(x) + P^s_0(x)\,, \qquad P_2(x) = P^a_2(x) + P^s_2(x)\,, \tag{A.8}$$

where $P^{a,s}_0$ and $P^{a,s}_2$ are the following coset projectors for $\mathfrak{so}(2,4)$ and $\mathfrak{su}(4)$,

$$\begin{aligned} &P^a_0: \quad \mathfrak{su}(2,2) \longrightarrow \mathfrak{so}(1,4)\,, \qquad P^a_2: \quad \mathfrak{su}(2,2) \longrightarrow \frac{\mathfrak{su}(2,2)}{\mathfrak{so}(1,4)}\,, \\ &P^s_0: \quad \mathfrak{su}(4) \longrightarrow \mathfrak{so}(5)\,, \qquad P^s_2: \quad \mathfrak{su}(4) \longrightarrow \frac{\mathfrak{su}(4)}{\mathfrak{so}(5)}\,. \end{aligned} \tag{A.9}$$

These coset projectors can be represented by the $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ generators as follows:

$$
\begin{aligned}
P_0^a(x) &= \frac{1}{2}\sum_{\mu,\nu=0}^{3}\frac{\mathrm{Tr}[n^a_{\mu\nu}x]}{\mathrm{Tr}[n^a_{\mu\nu}n^a_{\mu\nu}]}n^a_{\mu\nu} + \sum_{\mu=0}^{3}\frac{\mathrm{Tr}[n^a_{\mu5}x]}{\mathrm{Tr}[n^a_{\mu5}n^a_{\mu5}]}n^a_{\mu5}\,,\\
P_2^a(x) &= \sum_{\mu=0}^{3}\frac{\mathrm{Tr}[\gamma^a_{\mu}x]}{\mathrm{Tr}[\gamma^a_{\mu}\gamma^a_{\mu}]}\gamma^a_{\mu} + \frac{\mathrm{Tr}[\gamma^a_{5}x]}{\mathrm{Tr}[\gamma^a_{5}\gamma^a_{5}]}\gamma^a_{5}\,,\\
P_0^s(x) &= \frac{1}{2}\sum_{\mu,\nu=1}^{4}\frac{\mathrm{Tr}[n^s_{\mu\nu}x]}{\mathrm{Tr}[n^s_{\mu\nu}n_{\mu\nu}]}n^s_{\mu\nu} + \sum_{\mu=1}^{4}\frac{\mathrm{Tr}[n^s_{\mu5}x]}{\mathrm{Tr}[n^s_{\mu5}n^s_{\mu5}]}n^s_{\mu5}\,,\\
P_2^s(x) &= \sum_{\mu=1}^{4}\frac{\mathrm{Tr}[\gamma^s_{\mu}x]}{\mathrm{Tr}[\gamma^s_{\mu}\gamma^s_{\mu}]}\gamma^s_{\mu} + \frac{\mathrm{Tr}[\gamma^s_{5}x]}{\mathrm{Tr}[\gamma^s_{5}\gamma^s_{5}]}\gamma^s_{5}\,.
\end{aligned}
\tag{A.10}
$$

The projectors are utilized in evaluating the deformed metric, NS-NS two-form and Lax pair.

B A Lax pair for a q-deformed AdS$_5\times$S^5

In this appendix, let us consider a q-deformed AdS$_5\times$S^5 by employing the Yang-Baxter sigma model based on the mCYBE. Then we explicitly present a Lax pair for a string theory on this background.

A typical skew-symmetric solution of the mCYBE is Drinfeld-Jimbo type [54–56]. The classical action of the deformed AdS$_5\times$S^5 superstring associated with this r-matrix was constructed by Delduc-Magro-Vicedo [19, 20]. The metric (in the string frame) and NS-NS two-form have been computed in [52]. The deformed background is often called the η-deformed AdS$_5\times$S^5. Some specific limits [61] and a mirror description [62–64] have been studied. For various classical solutions, see [65–76]. Two-parameter generalizations have also been studied in [61, 77]. For some arguments towards the complete supergravity solution, see [53, 78, 79]. More recently, another integrable deformation (called the λ-deformation) has been argued in [80–87]. This deformation is closely related to the Yang-Baxter deformation by a Poisson-Lie duality [80–83, 88–90].

B.1 Yang-Baxter deformations from the mCYBE

Let us first give a short review on the Yang-Baxter deformations of the AdS$_5\times$S^5 superstring based on the mCYBE case [19, 20].

A q-deformed classical action of the AdS$_5\times$S^5 superstring [19, 20] is given by

$$
S = -\frac{\sqrt{\lambda_c}}{4}(1+\eta^2)\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta})\mathrm{STr}\left[A_\alpha\, d\circ\frac{1}{1-\eta R_g\circ d}(A_\beta)\right], \tag{B.1}
$$

The definition of A_α and R_g is the same as in section 2. A main difference is that the linear R-operator should satisfy the mCYBE

$$
[R(X),R(Y)] - R([R(X),Y]+[X,R(Y)]) = [X,Y]\,. \tag{B.2}
$$

The projection operators d is slightly different from the CYBE case like

$$d \equiv P_1 + \frac{2}{1-\eta^2} P_2 - P_3 \,. \tag{B.3}$$

Namely, the coefficient in front of P_2 depends on η. This comes from the difference of the kappa transformation.

The bosonic part of the Lagrangian. We consider here the bosonic part of the deformed action (B.1). The Lagrangian can be rewritten into a simple form,

$$L = \frac{\sqrt{\lambda_c}}{2} \frac{1+\eta^2}{1-\eta^2} \, \mathrm{STr}(A_- \, P_2(J_+)) \,, \tag{B.4}$$

with the deformed current $J_\pm$ defined as

$$J_\pm \equiv \frac{1}{1 \mp \varkappa R_g \circ P_2} A_\pm \,, \qquad \varkappa \equiv \frac{2\eta}{1-\eta^2} \,. \tag{B.5}$$

The expression of $J_\pm$ is determined by solving the following equations:

$$(1 \mp \varkappa R_g \circ P_2) \, J_\pm = A_\pm \,. \tag{B.6}$$

By taking a variation of (B.4), the equation of motion is given by

$$\mathcal{E} \equiv \partial_+ P_2(J_-) + \partial_- P_2(J_+) + [J_+, P_2(J_-)] + [J_-, P_2(J_+)] = 0 \,. \tag{B.7}$$

The undeformed current $A_\pm$ automatically satisfies the flatness condition

$$\mathcal{Z} \equiv \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \,, \tag{B.8}$$

which can be rewritten in terms of $J_\pm$ as follows

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] + \varkappa \, R_g(\mathcal{E}) + \varkappa^2 \, \mathrm{CYBE}_{Rg}(P_2(J_+), P_2(J_-)) = 0 \,. \tag{B.9}$$

Note that the quantity

$$\mathrm{CYBE}_{Rg}(X, Y) \equiv [R_g(X), R_g(Y)] - R_g([R_g(X), Y] + [X, R_g(Y)]) \tag{B.10}$$

results in $[X, Y]$, if the R-operator we are dealing with satisfies the mCYBE (B.2). Thus, due to the mCYBE, the condition (B.9) is reduced to

$$\mathcal{Z} = \partial_+ J_- - \partial_- J_+ + [J_+, J_-] + \varkappa \, R_g(\mathcal{E}) + \varkappa^2 \, [P_2(J_+), P_2(J_-)] = 0 \,. \tag{B.11}$$

In comparison to the CYBE case, the deformed current $J_\pm$ no longer satisfies the flatness condition, even if the equation of motion (B.7) is imposed.

Finally, a Lax pair [19, 20] is given by

$$\mathcal{L}_\pm = P_0(J_\pm) + \lambda^{\pm 1} \sqrt{1+\varkappa^2} \, P_2(J_\pm) \tag{B.12}$$

with a spectral parameter $\lambda \in \mathbb{C}$. The flatness condition of $\mathcal{L}_\pm$

$$\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_- + [\mathcal{L}_+, \mathcal{L}_-] = 0 \tag{B.13}$$

leads to the equation of motion (B.7) and the flatness condition (B.9).

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B.2 A Lax pair for a q-deformed AdS$_5\times$S^5

We shall study the bosonic part with a classical r-matrix of Drinfeld-Jimbo type [54–56],

$$r_{\rm DJ} = -i\sum_{a<b} E_{ab}\wedge E_{ba} - i\sum_{c<d} E_{cd}\wedge E_{dc}\,, \tag{B.14}$$

where E_{ab} $(a,b=1,\dots,4)$ and E_{cd} $(c,d=5,\dots,8)$ are the fundamental representations of $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$, respectively. This is a solution of the mCYBE (B.2).

To construct the bosonic part of the deformed AdS$_5\times$S^5 with the global coordinates, we will work with a bosonic element represented by

$$g = g_a\cdot g_s \quad \in\ \mathrm{SU}(2,2)\times\mathrm{SU}(4)\,, \tag{B.15}$$

where the group elements of SU(2,2) and SU(4) are parameterized as follows, respectively,

$$\begin{aligned}
g_a(\tau,\sigma) &= \exp\left[\frac{i}{2}\sum_{i=1}^{3}\psi_i h_i\right]\exp[-\zeta\, n^a_{13}]\exp\left[-\frac{1}{2}\,\rho\,\gamma^a_1\right],\\
g_s(\tau,\sigma) &= \exp\left[\frac{i}{2}\sum_{i=1}^{3}\phi_i h_{i+3}\right]\exp[-\xi\, n^s_{13}]\exp\left[-\frac{i}{2}\,r\,\gamma^s_1\right].
\end{aligned} \tag{B.16}$$

The AdS$_5$ part is described by the coordinates $\psi_3\,(\equiv\, t)\,,\psi_1\,,\psi_2\,,\zeta\,,\rho$. The S^5 part is parameterized by the angle variables $\phi_1\,,\phi_2\,,\phi_3\,,\xi\,,r$.

In the present case, the deformed current $J_\pm$ is decomposed into two pieces: $J_\pm = J^a_\pm + J^s_\pm$. Then, by solving the equation in (B.6), $J^a_\pm$ and $J^s_\pm$ are determined as follows

$$\begin{aligned}
J^a_\pm = &-f_a(\rho)\partial_\pm\rho\left[\frac{1}{2}(\gamma^a_1 \pm i\varkappa\sinh\rho\,\gamma^a_5)\pm i\varkappa\cosh\rho\, n^a_{15}\right]\\
&+f_a(\rho)\partial_\pm t\left[\frac{1}{2}\cosh\rho\,(i\gamma^a_5\pm\varkappa\sinh\rho\,\gamma^a_1)+i(1+\varkappa^2)\sinh\rho\, n^a_{15}\right]\\
&-g_a(\rho\,,\zeta)\partial_\pm\zeta\Big[\frac{1}{2}\sinh\rho\,\left(\gamma^a_3\pm\varkappa\sin\zeta\sinh^2\rho\gamma^a_2\right)\\
&\pm i\varkappa\sinh\rho\cosh\rho\left(n^a_{35}\pm\varkappa\sin\zeta\sinh^2\rho\, n^a_{25}\right)\mp\varkappa\cos\zeta\sinh^2\rho\, n^a_{23}\\
&+\cosh\rho\left(n^a_{13}\pm\varkappa\sin\zeta\sinh^2\rho\, n^a_{12}\right)\Big]\\
&-g_a(\rho\,,\zeta)\partial_\pm\psi_1\Big[\frac{1}{2}\cos\zeta\sinh\rho(\gamma^a_2\mp\varkappa\sin\zeta\sinh^2\rho\,\gamma^a_3)\\
&+\sin\zeta(1+\varkappa^2\sinh^4\rho)\, n^a_{23}+\cos\zeta\cosh\rho\,(n^a_{12}\mp\varkappa\sin\zeta\sinh^2\rho\, n^a_{13})\\
&\pm i\varkappa\cos\zeta\sinh\rho\cosh\rho(n^a_{25}\mp\varkappa\sin\zeta\sinh^2\rho\, n^a_{35})\Big]\\
&+\partial_\pm\psi_2\Big[i\sin\zeta\sinh\rho\left(\frac{1}{2}\gamma^a_0\pm i\varkappa\cosh\rho\, n^a_{05}\right)\\
&-i\sin\zeta\cosh\rho\, n^a_{01}-i\cos\zeta(n^a_{03}\mp\varkappa\sin\zeta\sinh^2\rho\, n^a_{02})\Big]\,,
\end{aligned} \tag{B.17}$$

$$
\begin{aligned}
J^s_\pm = &-f_s(r)\partial_\pm r\left[i\frac{1}{2}(\gamma^s_1 \mp \varkappa\sin r\,\gamma^s_5) \mp \varkappa\cos r\, n^s_{15}\right]\\
&- g_s(r,\xi)\partial_\pm\xi\Big[i\frac{1}{2}\sin r\left(\gamma^s_3 \mp \varkappa\sin\xi\sin^2 r\,\gamma^s_2\right)\\
&\pm\varkappa\cos\xi\sin^2 r\, n^s_{23} \mp \varkappa\sin r\cos r\left(n^s_{35} \mp \varkappa\sin\xi\sin^2 r\, n^s_{25}\right)\\
&+\cos r\left(n^s_{13} \mp \varkappa\sin\xi\sin^2 r\, n^s_{12}\right)\Big]\\
&- g_s(r,\xi)\partial_\pm\phi_1\Big[i\frac{1}{2}\sin r\cos\xi\left(\gamma^s_2 \pm \varkappa\sin\xi\sin^2 r\,\gamma^s_3\right)\\
&+\sin\xi\left(1+\varkappa^2\sin^4 r\right)n^s_{23} \mp \varkappa\sin r\cos r\cos\xi\left(n^s_{25} \pm \varkappa\sin\xi\sin^2 r\, n^s_{35}\right)\\
&+\cos r\cos\xi\left(n^s_{12} \pm \varkappa\sin^2 r\sin\xi\, n^s_{13}\right)\Big]\\
&-\partial_\pm\phi_2\Big[\sin r\sin\xi\left(i\frac{1}{2}\,\gamma^s_4 \mp \varkappa\cos r\, n^s_{45}\right)\\
&+\sin\xi\sin r\cos r\, n^s_{14} + \cos\xi(n^s_{34} \pm \varkappa\sin\xi\sin^2 r\, n^s_{24})\Big]\\
&+ f_s(r)\partial_\pm\phi_3\left[i\frac{1}{2}\cos r\,(\gamma^s_5 \pm \varkappa\sin r\,\gamma^s_1) - (1+\varkappa^2)\sin r\, n^s_{15}\right],
\end{aligned}
\tag{B.18}
$$

where we have introduced new functions defined as

$$
\begin{aligned}
f_a(\rho) &\equiv \frac{1}{1-\varkappa^2\sinh^2\rho}\,, \qquad & g_a(\rho,\zeta) &\equiv \frac{1}{1+\varkappa^2\sin^2\zeta\sinh^4\rho}\,,\\
f_s(r) &\equiv \frac{1}{1+\varkappa^2\sin^2 r}\,, \qquad & g_s(r,\xi) &\equiv \frac{1}{1+\varkappa^2\sin^2\xi\sin^4 r}\,.
\end{aligned}
\tag{B.19}
$$

The deformed currents in (B.17) and (B.18) enable us to compute (i) the metric and NS-NS two-form and (ii) the explicit form of the Lax pair.

Firstly, the resulting metric and NS-NS two-form are given by [52]

$$
\begin{aligned}
ds^2_{\mathrm{AdS}_5} &= \sqrt{1+\varkappa^2}\Big[\frac{1}{1-\varkappa^2\sinh^2\rho}\left(-\cosh^2\rho\, dt^2 + d\rho^2\right)\\
&\quad+\frac{1}{1+\varkappa^2\sin^2\zeta\sinh^4\rho}\sinh^2\rho\left(d\zeta^2+\cos^2\zeta\,(d\psi_1)^2\right)+\sinh^2\rho\sin^2\zeta\,(d\psi_2)^2\Big],\\
B_{\mathrm{AdS}_5} &= \varkappa\sqrt{1+\varkappa^2}\,\frac{\sinh^4\rho\sin 2\zeta}{1+\varkappa^2\sin^2\zeta\sinh^4\rho}\,d\psi_1\wedge d\zeta\,,
\end{aligned}
\tag{B.20}
$$

$$
\begin{aligned}
ds^2_{\mathrm{S}^5} &= \sqrt{1+\varkappa^2}\Big[\frac{1}{1+\varkappa^2\sin^2 r}\left(\cos^2 r(d\phi_3)^2 + dr^2\right)\\
&\quad\frac{\sin^2 r}{1+\varkappa^2\sin^2\xi\sin^4 r}\left(d\xi^2+\cos^2\xi\,(d\phi_1)^2\right)+\sin^2 r\sin^2\xi\,(d\phi_2)^2\Big],\\
B_{\mathrm{S}^5} &= \varkappa\sqrt{1+\varkappa^2}\,\frac{\sin^4 r\sin 2\xi}{1+\varkappa^2\sin^2\xi\sin^4 r}\,d\phi_1\wedge d\xi\,.
\end{aligned}
\tag{B.21}
$$

Here total derivative terms in the NS-NS two-form have been ignored.

Secondly, the Lax pair (B.12) is also decomposed into two parts: $\mathcal{L}_\pm = \mathcal{L}^a_\pm + \mathcal{L}^s_\pm$. Then the explicit forms of $\mathcal{L}^a_\pm$ and $\mathcal{L}^s_\pm$ turn out to be

$$\begin{aligned}
\mathcal{L}^a_\pm = &-f_a(\rho)\partial_\pm\rho\left[\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}(\gamma^a_1 \pm i\varkappa\sinh\rho\,\gamma^a_5)\pm i\varkappa\cosh\rho\, n^a_{15}\right]\\
&+f_a(\rho)\partial_\pm t\left[\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}\cosh\rho\,(i\gamma^a_5\pm\varkappa\sinh\rho\,\gamma^a_1)+i(1+\varkappa^2)\sinh\rho\, n^a_{15}\right]\\
&-g_a(\rho\,,\zeta)\partial_\pm\zeta\left[\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}\sinh\rho\left(\gamma^a_3\pm\varkappa\sin\zeta\sinh^2\rho\gamma^a_2\right)\right.\\
&\pm i\varkappa\sinh\rho\cosh\rho\left(n^a_{35}\pm\varkappa\sin\zeta\sinh^2\rho\, n^a_{25}\right)\mp\varkappa\cos\zeta\sinh^2\rho\, n^a_{23}\\
&\left.+\cosh\rho\left(n^a_{13}\pm\varkappa\sin\zeta\sinh^2\rho\, n^a_{12}\right)\right]\\
&-g_a(\rho\,,\zeta)\partial_\pm\psi_1\left[\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}\cos\zeta\sinh\rho(\gamma^a_2\mp\varkappa\sin\zeta\sinh^2\rho\,\gamma^a_3)\right.\\
&+\sin\zeta(1+\varkappa^2\sinh^4\rho)\, n^a_{23}+\cos\zeta\cosh\rho\,(n^a_{12}\mp\varkappa\sin\zeta\sinh^2\rho\, n^a_{13})\\
&\left.\pm i\varkappa\cos\zeta\sinh\rho\cosh\rho(n^a_{25}\mp\varkappa\sin\zeta\sinh^2\rho n^a_{35})\right]\\
&+\partial_\pm\psi_2\left[i\sin\zeta\sinh\rho\left(\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}\gamma^a_0\pm i\varkappa\cosh\rho\, n^a_{05}\right)\right.\\
&\left.-i\sin\zeta\cosh\rho\, n^a_{01}-i\cos\zeta(n^a_{03}\mp\varkappa\sin\zeta\sinh^2\rho\, n^a_{02})\right], \qquad\qquad \text{(B.22)}\\
\mathcal{L}^s_\pm = &-f_s(r)\partial_\pm r\left[i\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}(\gamma^s_1\mp\varkappa\sin r\,\gamma^s_5)\mp\varkappa\cos r\, n^s_{15}\right]\\
&-g_s(r,\xi)\partial_\pm\xi\left[i\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}\sin r\left(\gamma^s_3\mp\varkappa\sin\xi\sin^2 r\,\gamma^s_2\right)\right.\\
&\pm\varkappa\cos\xi\sin^2 r\, n^s_{23}\mp\varkappa\sin r\cos r\left(n^s_{35}\mp\varkappa\sin\xi\sin^2 r\, n^s_{25}\right)\\
&\left.+\cos r\left(n^s_{13}\mp\varkappa\sin\xi\sin^2 r\, n^s_{12}\right)\right]\\
&-g_s(r,\xi)\partial_\pm\phi_1\left[i\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}\sin r\cos\xi\left(\gamma^s_2\pm\varkappa\sin\xi\sin^2 r\,\gamma^s_3\right)\right.\\
&+\sin\xi\left(1+\varkappa^2\sin^4 r\right) n^s_{23}\mp\varkappa\sin r\cos r\cos\xi\left(n^s_{25}\pm\varkappa\sin\xi\sin^2 r\, n^s_{35}\right)\\
&\left.+\cos r\cos\xi\left(n^s_{12}\pm\varkappa\sin^2 r\sin\xi\, n^s_{13}\right)\right]\\
&-\partial_\pm\phi_2\left[\sin r\sin\xi\left(i\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}\,\gamma^s_4\mp\varkappa\cos r\, n^s_{45}\right)\right.\\
&\left.+\sin\xi\sin r\cos r\, n^s_{14}+\cos\xi(n^s_{34}\pm\varkappa\sin\xi\sin^2 r\, n^s_{24})\right]\\
&+f_s(r)\partial_\pm\phi_3\left[i\frac{\lambda^{\pm1}}{2}\sqrt{1+\varkappa^2}\cos r\,(\gamma^s_5\pm\varkappa\sin r\,\gamma^s_1)-(1+\varkappa^2)\sin r\, n^s_{15}\right]. \qquad \text{(B.23)}
\end{aligned}$$

The expression (B.22) is utilized in section 3 in order to reproduce the desired Lax pair as a scaling limit.

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