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Cosmological disformal invariance

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Abstract. The invariance of physical observables under disformal transformations is considered. It is known that conformal transformations leave physical observables invariant. However, whether it is true for disformal transformations is still an open question. In this paper, it is shown that a pure disformal transformation without any conformal factor is equivalent to rescaling the time coordinate. Since this rescaling applies equally to all the physical quantities, physics must be invariant under a disformal transformation, that is, neither causal structure, propagation speed nor any other property of the fields are affected by a disformal transformation itself. This fact is presented at the action level for gravitational and matter fields and it is illustrated with some examples of observable quantities. We also find the physical invariance for cosmological perturbations at linear and high orders in perturbation, extending previous studies. Finally, a comparison with Horndeski and beyond Horndeski theories under a disformal transformation is made.

Keywords: Disformal transformations, cosmological perturbations

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Contents

1 Introduction

Observational cosmology reveals how small our knowledge of the Universe is. For example, two of the main unsolved mysteries are periods of accelerated expansion of the Universe, inflation and dark energy, whose underlying sources remain unknown. Plenty of theoretical models (for an extensive review see [1]) point towards a scalar field as the main responsible for such an accelerated expansion. Interestingly, despite the good agreement of the standard ΛCDM model with current data [2, 3], some tension appears from direct measurements of the time evolution of the Hubble parameter [4]. Furthermore, a recent release of the CMB observation by Planck [5], which supports inflation, seems to favour those inflationary models with a non-minimal coupling, a.k.a. the scalar-tensor theory of gravity [6–8] that includes the so-called $f(R)$ theory [9, 10].

Conformal transformations have been proved to play a very important role in physics [11]. In particular, in the scalar-tensor theory they are not only a useful mathematical tool but they leave observational physics invariant at a classical level (for a discussion supporting this point of view see [12–21]; also see [22–25] for a discussion in a different point of view in the context of $F(R)$ gravity). Consequently, the notion of conformally related frames naturally appears. The Jordan frame or matter frame, where the scalar field is non-minimally coupled to the metric $\tilde{g}_{\mu\nu}$ but matter is minimally coupled, and the Einstein frame or gravitational frame, where the scalar field is minimally coupled to the metric $g_{\mu\nu}$ but matter is dilatonically coupled to the scalar field. An alternative way to understand conformal transformations in the scalar-tensor theory is the following. One has a gravitational metric $g_{\mu\nu}$ which satisfies the Einstein equations and a matter metric related to the former by a field dependent rescaling, say $\tilde{g}_{\mu\nu} = \Omega(\phi)^2 g_{\mu\nu}$.

From this point of view, one may wonder whether there are more general transformations that lead to a new form of the matter metric. In other words, is there any other way that matter can non-trivially couple to gravity through a scalar field? This question was studied by Bekenstein in [26] where a new class of transformations, called disformal transformations, were proposed. The idea behind such transformations is that matter is coupled to a metric which is not just a rescaling of the gravitational metric but it is stretched in a particular direction, given by the gradient of a scalar field.

Disformal transformations can be motivated from brane world models and from massive gravity theories (see [27, 28] and references therein) and have been applied to inflation [29], dark energy [30–32], varying speed of light models [33–36], atomic physics [37] and mimetic gravity [38]. Consequently, considerable effort has been made to look for constraints and how to avoid them, e.g. screening mechanisms, for disformally coupled matter models [27, 28, 31, 32, 39–44]. In addition, with the recent rediscovery of the Horndeski theory [45–47], which is the general scalar-tensor theory with second order field equations of motion, and its generalization, known as beyond Horndeski or GLPV theory [48–51] (see also its further extension [52]), attention has been paid to its mathematical invariance under disformal transformations in those theories [53, 54]. If observational physics turns out to be invariant under disformal transformations that will provide us with a powerful mathematical and physical tool to classify or work within the Horndeski theory (for example [28, 55]), and to make progress in our understanding of the symmetries of gravity. In this direction, refs. [56–58] study the disformal invariance of cosmological perturbations at the linear level with a positive answer. For a recent development see references [59–61]. In particular, see [59] for a multi-field extension of disformal transformations.

That being said, a note is in order. It is generally believed that physics does not change under a non-singular field redefinition. However, a field redefinition of a metric is more subtle when it comes to interpretations. A simple example would be a conformal transformation which could lead us from an homogeneous and isotropic expanding universe to a static spacetime (see Deruelle & Sasaki [14]). The invariance of physically observable quantities before and after the transformation is not immediately clear. Therefore, and also to clearly understand how to interpret the results, one has to *explicitly* check the invariance.

Turning to the case of a disformal transformation, it is no longer just a simple field redefinition, because it involves derivatives of the scalar field. Thus there may be a higher level of subtlety than in the case of a conformal transformation. For this reason, we dedicate this work to study the effects of a pure disformal transformation, its interpretations and the invariance of physical observables in cosmology.

For the sake of simplicity and clarity, we restrict ourselves to work in the comoving, or the uniform ϕ slicing, on which the scalar field is homogeneous. In this sense, we discuss *cosmological* disformal transformations by implicitly assuming the existence of ϕ-constant spacelike hyper-surfaces, which is a common and reasonable assumption in cosmology. We show that a pure cosmological disformal transformation without any conformal factor is equivalent to rescaling the time coordinate, which could even be regarded as a time coordinate transformation. This is checked at the action level for gravitational and matter fields, which ensures that physics is invariant under disformal transformations. Consequently, one can work in the frame one considers most suitable for either computations or interpretations.

The paper is organized as follows. In section 2 we briefly review disformal transformations and we focus on pure disformal transformations, which are intuitively shown to be equivalent to rescaling the time coordinate when applied to cosmology. Afterwards, in section

3 we proceed to check that this is the case for gravitational and matter fields at the action level, with a comparison to Horndeski theory. Let us stress that throughout sections 2 and 3 we focus on the effects of a disformal transformation *itself*, i.e. no comparison between fields, c.f. scalar and matter fields is made. Afterwards, in section 4 we explicitly compute the disformal invariance of some observable quantities considering the system as a whole, i.e. gravitational and matter sector altogether, and we emphasise on the choice of frames and its invariance. Finally, in section 5 we summarize our results.

2 Pure disformal transformation

The general form of a disformal transformation is given by [26]

$$\bar{g}_{\mu\nu} = G(\phi, X)g_{\mu\nu} + F(\phi, X)\phi_{,\mu}\phi_{,\nu}, \tag{2.1}$$

where $X = -\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$ and $\phi_{,\mu} \equiv \partial_\mu\phi$. The first term in the right hand side corresponds to a conformal transformation, i.e. a rescaling of the metric, whereas the latter is a pure disformal transformation, i.e. the metric is stretched in the direction of $\phi_{,\mu}$. In this work, we restrict ourselves to the case of a pure disformal transformation, that is $G(\phi, X) = 1$, as any conformal transformation can be *generally* done afterwards.

In cosmology it is a reasonable assumption that $\phi_{,\mu}$ is regular and timelike everywhere, and hence one can choose the comoving, or the uniform ϕ slicing on which the scalar field is homogeneous. Throughout this paper, we take the comoving slicing to simplify the argument and make clear physical interpretations and consequences. In fact one immediately notices from (2.1) that under this pure cosmological disformal transformation only the time-component of the metric, namely the lapse function, is modified. To see this explicitly, let us express a pure "disformation" in the form of the $(3+1)$-decomposition,

$$\begin{aligned} & g \equiv ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \\ \rightarrow \quad & \bar{g} \equiv d\bar{s}^2 = -\bar{N}^2dt^2 + \bar{g}_{ij}(dx^i + \bar{N}^i dt)(dx^j + \bar{N}^j dt)\,, \end{aligned} \tag{2.2}$$

where

$$\bar{N}^2 = N^2\alpha^2\,, \quad \bar{N}^i = N^i\,, \quad \bar{g}_{ij} = g_{ij}\,, \tag{2.3}$$

where for the sake of simplicity we defined

$$\alpha^2 = 1 - 2F(\phi, X)X. \tag{2.4}$$

It should be noted that now X reduces to $-(1/2)g^{tt}\dot{\phi}^2$ where the scalar field only depends on the time coordinate but $g^{tt} = -N^{-2}$ has spatial dependence in general, which implies the spatial dependence of the disformal factor, F, as well as of α. From (2.3) we infer that in order to preserve the Lorentzian signature of the metric one must require $\alpha^2 > 0$ [26], which is assumed throughout this paper. Due to the fact that just a particular component of the metric is modified, one may naively think that the causal structure and the propagation speed are altered as well. In fact, it is rather the opposite as shown later.

A note is in order. Deruelle and Rua [38] showed that a disformal transformation does not alter Einstein's equations in general, although the form of such is quite different when expressed in terms of the transformed metric. One may naively expect this result due to the fact that a disformal transformation is a re-parametrization of the metric.

However, there is a particular class of disformal transformations which leads to a departure from usual General Relativity (GR). In fact, its effect is to give rise to a source term in Einstein's equations even in the absence of matter fields. This particular class includes the Mimetic Dark Matter model [62, 63] where this new source term imitates a dark matter component. Concretely, the mimetic gravity condition gives a constraint between the conformal and disformal factors. In our notation it reads

$$F(\phi, X) = \frac{1}{2X} G(\phi, X) - f(\phi), \tag{2.5}$$

where $f(\phi)$ is an arbitrary positive function of the scalar field alone. For a pure disformal transformation (G=1) it reduces to

$$F(\phi, X) = \frac{1}{2X} - f(\phi), \tag{2.6}$$

which plugged back in (2.3) leads us to a degenerate metric without lapse, that is

$$d\bar{s}^2 = -f(\phi)(\partial_t \phi)^2 dt^2 + \bar{g}_{ij}(dx^i + \bar{N}^i dt)(dx^j + \bar{N}^j dt)\,. \tag{2.7}$$

Note that there will no longer be any constraint equation from the variation with respect to the lapse. The absence of this constraint seems to be the origin of a new "mimetic" degree of freedom. In this work, we will not pursue this particular case further, leaving its interesting implications for future work.

In the next subsection 2.1 we shall see how the physics can be changed/unchanged under a disformal transformation focusing first on the homogeneous and isotropic background. And then the discussion is extended to deal with the general situation in the next subsection 2.2 obtaining the same conclusion.

2.1 Cosmological background

Before going in depth let us clarify our starting point in order to avoid any confusion. We begin with a given action for a given metric and we are interested in the change of physics under a transformation of the metric. Obviously, we do not want to alter our model, i.e. our action. For this reason, we must take a passive approach. What we mean by a passive transformation is the following. For simplicity, let us consider two metrics $\bar{g} = d\bar{s}^2$ and $g = ds^2$ which are related by a conformal transformation $\bar{g} = \Omega^2 g$. Let us assume that we are given a model with the metric $\bar{g}$ and the action $S[\bar{g}]$. Now, we perform the transformation to the metric. Basically, we have two options. We can rewrite the action in terms of the transformed metric $S[\bar{g}] = S[\Omega^2 g]$, or replace the metric $\bar{g} \to g = \Omega^{-2}\bar{g}$ while keeping the same functional form of the action, which yields a different value of the action $S[\bar{g}] \neq S[g]$. We call the former a passive transformation and the latter an active one. As we stated above, we consider the former. From this point of view, we should expect at most a change in the interpretation of the physics but not a change in observational results. We will come back to this point later with an explicit form of the action in section 3.

That being said, we can readily have an idea of the effect of a disformal transformation by expressing our original line element in terms of the disformally transformed one. To be specific, we consider a model with a metric $\bar{g}$ and by means of a disformal transformation we choose to work in terms of g, i.e. we passively transform the barred frame to the unbarred frame by $\bar{g} = \bar{g}(g)$.

Let us focus on a spatially homogeneous and isotropic background with the metric,

$$ds^2 = g_{\mu\nu}(t)dx^\mu dx^\nu = -dt^2 + g_{ij}dx^i dx^j\,, \tag{2.8}$$

where we have chosen the cosmic proper time, $g_{tt} = -1$ ($N = 1$), and $g_{ij} = a^2(t)\Omega_{ij}$ with Ω_{ij} being the metric of a homogeneous and isotropic 3-space. The disformal transformation (2.1) ,

$$d\bar{s}^2 = \bar{g}_{\mu\nu}(t)dx^\mu dx^\nu = \Big(g_{\mu\nu} + F(\phi, X)\phi_{,\mu}\phi_{,\nu}\Big)dx^\mu dx^\nu\,, \tag{2.9}$$

with $N = 1$ (2.3) can be read as

$$d\bar{s}^2 = ds^2 + \Big(1 - \alpha^2(t)\Big)dt^2 = -\alpha^2(t)dt^2 + g_{ij}dx^i dx^j\,. \tag{2.10}$$

It should be noted that α here depends only on the time coordinate. This enables us to perform a time coordinate transformation given by

$$d\bar{t} = \alpha(t)dt\,. \tag{2.11}$$

With this new time coordinate, the barred metric is expressed simply as

$$d\bar{s}^2 = -d\bar{t}^2 + g_{ij}dx^i dx^j\,. \tag{2.12}$$

As clear from this, the time coordinate $\bar{t}$ is in fact the cosmic proper time in the barred frame.

One readily see that the above form of the metric is exactly in the same form as the unbarred metric with the replacement $t \to \bar{t}$, with the understanding that the scale factor in the transformed frame, say $A(\bar{t})$, is regarded as a function of $\bar{t}$ through its t-dependence, $A(\bar{t}) = a\big(t(\bar{t})\big)$. In the sense of the foregoing discussion, if one were to start from an action with the barred metric $\bar{g}_{\mu\nu}$, one would find that the form of the action in terms of the unbarred metric $g_{\mu\nu}$ can be interpreted as a rescaling of time from t to $\bar{t}$, or in a symbolic form $S[\bar{g}(g); t] = S[g; \bar{t}]$. See section 3 for an explicit proof.

This indicates an important fact that a (cosmological) disformal transformation (2.3) is essentially equivalent to a rescaling of the time coordinate (2.11). Since this rescaling applies to all the physical quantities equally in the transformed frame, it implies the invariance of the physics under the disformal transformation. In other words, since we work with the same action but expressed in terms of new variables, observed physics should not change. If this reasoning is valid, it is straightforward that in particular neither the causal structure nor the propagation speed are modified under the disformal transformation.

Wave propagation. We now consider the effect of a disformal transformation on the propagation of waves. For simplicity, let us focus on a scalar wave of comoving wavenumber k living in the barred frame,

$$\left[\frac{1}{a^3}\frac{d}{\bar{N}dt}\left(a^3\frac{d}{\bar{N}dt}\right) + \frac{\bar{c}_s^2 k^2}{a^2}\right]\bar{\phi}_k(t) = 0\,, \tag{2.13}$$

where $\bar{c}_s$ is the sound velocity. Applying the passive disformal transformation (2.3), it becomes

$$\left[\frac{1}{a^3\alpha}\frac{d}{Ndt}\left(\frac{a^3}{\alpha}\frac{d}{Ndt}\right) + \frac{\bar{c}_s^2 k^2}{a^2}\right]\bar{\phi}_k(t) = \frac{1}{\alpha^2}\left[\frac{\alpha}{a^3}\frac{d}{dt}\left(\frac{a^3}{\alpha}\frac{d}{dt}\right) + \frac{\alpha^2\bar{c}_s^2 k^2}{a^2}\right]\bar{\phi}_k(t) = 0\,, \tag{2.14}$$

where in the last step we set $N = 1$ so that we chose t to be the cosmic proper time of the unbarred frame. Thus it appears that the sound velocity is changed to $c_{s,ap} = \alpha \bar{c}_s$, where $c_{s,ap}$ stands for apparent sound speed. However, recalling that the cosmic proper time in the barred frame is given by $d\bar{t} = \alpha dt$, the actual sound velocity in the transformed frame should be read off from the equation rewritten in terms of $\bar{t}$,

$$\left[\frac{1}{a^3}\frac{d}{d\bar{t}}\left(a^3\frac{d}{d\bar{t}}\right) + \frac{\bar{c}_s^2 k^2}{a^2}\right]\phi_k(\bar{t}) = 0\,, \tag{2.15}$$

where $\phi_k(\bar{t}) = \bar{\phi}_k\big(t(\bar{t})\big)$ or equivalently $\bar{\phi}_k(t) = \phi_k(\bar{t}(t))$. Here it is important to note that no scalar function is modified by such a passive disformal transformation. Essentially, the functional form has apparently changed but not its value as indicated above. It is apparent that the physical sound velocity $\bar{c}_s$ is the same in both frames. In fact, this is similar to that pointed out by Ellis and Uzan in [64, 65]. We further use this result in section 4.

Causal structure. Bearing in mind the above result, let us discuss the causal structure. Consider the norm of a vector k^μ in the barred frame, i.e.

$$k^2 \equiv \bar{g}_{\mu\nu}\bar{k}^\mu\bar{k}^\nu = -\bar{N}^2(\bar{k}^t)^2 + g_{ij}\bar{k}^i\bar{k}^j, \tag{2.16}$$

where $k^2 > 0$, $= 0$ or < 0 for a spacelike, null or time-like vector, respectively. Actually, k^2 is a scalar and consequently is invariant under passive disformal transformations. By imposing such a condition, the invariance of the causal structure is automatic. Conversely, the vector components must change in order to balance the transformation of the metric and to keep k^2 invariant.

Let us briefly discuss its consequences. The passive disformal transformation (2.3) in k^2 leads us to

$$\begin{aligned} k^2 &= \bar{g}_{\mu\nu}\bar{k}^\mu\bar{k}^\nu \\ &= \left(g_{\mu\nu} + F(\phi, X)\phi_{,\mu}\phi_{,\nu}\right)\bar{k}^\mu\bar{k}^\nu = -\alpha^2(\bar{k}^t)^2 + g_{ij}\bar{k}^i\bar{k}^j\,, \end{aligned} \tag{2.17}$$

where again we have set $N = 1$ so that t is the proper time in the unbarred frame. From the above we can identify the transformation rule for the vector components under the disformal transformation, that is

$$\bar{k}^t = \alpha^{-1}k^t \quad \text{and} \quad \bar{k}^i = k^i. \tag{2.18}$$

The covariant components transform according to

$$\bar{k}_t = \alpha k_t \quad \text{and} \quad \bar{k}_i = k_i. \tag{2.19}$$

A straightforward consequence is that the sound speed is seemingly modified, i.e.

$$c_s^{-1} \equiv \frac{dk^t}{d|k|} = \alpha\frac{d\bar{k}^t}{d|k|} = \alpha c_{s,ap}^{-1}\,, \tag{2.20}$$

where $c_{s,ap}$ is the apparent sound speed, similar to the case of a scalar wave equation. Actually, this change in the sound speed is an artifact of working with time t not proper to the barred frame. With the proper time $\bar{t}$ in the barred frame given by (2.11), we can rewrite k^2 as

$$k^2 = -(\bar{k}^{\bar{t}})^2 + g_{ij}\bar{k}^i\bar{k}^j\,, \tag{2.21}$$

where $\bar{k}^{\bar{t}} = \alpha \bar{k}^t$. Notice that the physical sound speed is frame independent as well, namely

$$\bar{c}_s^{-1} \equiv \frac{d\bar{k}^{\bar{t}}}{d|k|} = c_s^{-1} \ . \tag{2.22}$$

Before ending this subsection, let us stress again that we consider passive disformal transformations throughout this paper unless otherwise stated.

2.2 Non-linear considerations

In this subsection we shall extend the previous discussion by taking into account the spatial dependence of the scalar field, which leads to the spatial dependence of the metric and the disformal factor. Here we assume that the perturbation expansion is valid, i.e., we assume the existence of a spatially homogeneous background solution on which the physical spacetime can be constructed. Note that, however, the perturbation can be fully nonlinear. In this case, it is crucial to require the existence of a comoving slicing, i.e. the normal vector of constant ϕ hyper-surfaces is assumed to be time-like everywhere, which ensures that time is the only component stretched by a disformal transformation. Even so, there is still a spatial dependence of the disformal factor through X due to the fluctuations of the metric.

Under this assumption, the previous discussion can be generalized, in practice, by introducing a local Lorentz frame where the spatial dependence of the disformal factor can be neglected since the scalar field can be considered spatially homogeneous in this frame. Consequently, the essential effect due to a disformal transformation is a mere change of the local Lorentz lapse function, which in turn can be absorbed by a redefinition of the local time coordinate. Hence, it implies the equivalence between a disformal transformation and a rescaling of time. In this way, the previous discussion for the background can be generalised a priori to include non-linear perturbations without any difficulty.

3 Action invariance

Thus far, we have shown that at the metric level a disformal transformation is equivalent to rescaling time. The next point to deal with is to show whether this expectation holds at the action level for gravitational and matter fields. Before going into details, let us clarify the notation. We start from a metric $\bar{g}_{\mu\nu}$, which we refer to it as the barred frame, and we consider the change of the action under a passive disformal transformation, that is we re-express the action in terms of the disformally related metric $g_{\mu\nu}$, which we call the unbarred frame. We shall do the same procedure for all fields. In the next subsection we consider the Einstein-scalar action in the (3+1)-decomposition including nonlinear cosmological perturbations. Afterwards, we consider the action for several kinds of matter fields in the same decomposition of the metric.

3.1 Einstein-scalar action

Let us address the disformal transformation of the gravitational action, for the sake of simplicity the Einstein-Hilbert action with a canonical scalar field. At the end of this section we clearly show that the same procedure applies to its scalar-tensor theory extensions. For this purpose, a crucial point is to work in the *comoving*, or the *uniform ϕ slicing* ($\delta\phi = 0$) which ensures that only the time derivative of the scalar field plays a role in the disformal transformation as no perturbations of the scalar field are present on this slicing.[1] It is appropriate to

[1]This is always possible in the single field case.

work within the (3+1)-decomposition of the unbarred/barred metrics which are given by

$$ds^2 = -N^2(t,\boldsymbol{x})dt^2 + h_{ij}(t,\boldsymbol{x})dx^i dx^j \quad \text{and} \quad d\bar{s}^2 = -\bar{N}^2(t,\boldsymbol{x})dt^2 + h_{ij}(t,\boldsymbol{x})dx^i dx^j, \tag{3.1}$$

where N, $\bar{N}$ and h_{ij} are the unbarred/barred lapse and the spatial metric, respectively.[2] Note that we set to zero the shift vector, $\bar{N}^i = N^i = 0$, by choosing a particular set of spatial coordinates. This fact will make the disformal transformation much clearer afterwards, though one can include a non-vanishing shift vector without essential difficulties. In this decomposition the action reads

$$S_g = \frac{1}{2}\int d^3x dt \bar{N}\sqrt{h}\left\{R^{(3)} + \bar{K}_{ij}\bar{K}^{ij} - \bar{K}^2 + \bar{N}^{-2}\left(\partial_t\phi\right)^2 - 2V(\phi) + 2\bar{\nabla}_\mu\left(\bar{n}^\mu\bar{\nabla}_\nu\bar{n}^\nu - \bar{n}^\nu\bar{\nabla}_\nu\bar{n}^\mu\right)\right\}, \tag{3.2}$$

where ${}^{(3)}R$, $\bar{K}_{ij}$ and $\bar{n}_\mu dx^\mu = -\bar{N}dt$ respectively are the spatial Ricci scalar, the extrinsic curvature and the normal vector of the spatial hyper-surface. The extrinsic curvature and the gradient of the normal vector given in terms of (3.1) read

$$K_{ij} = \tfrac{1}{2N}\partial_t h_{ij}, \qquad \bar{K} = h^{ij}\bar{K}_{ij}, \tag{3.3}$$

and

$$\bar{\nabla}_\mu\bar{n}_\nu = \delta^0_\mu\delta^i_\nu\bar{N}_{,i} + \delta^i_\mu\delta^j_\nu\bar{K}_{ij}. \tag{3.4}$$

In this way, we can explicitly express the action in terms of the $\bar{N}$ and h_{ij}, i.e.

$$\begin{aligned} S_g = \frac{1}{2}\int d^3x dt \bar{N}\sqrt{h}\Big\{&R^{(3)} + \bar{K}_{ij}\bar{K}^{ij} - \bar{K}^2 + \bar{N}^{-2}\left(\partial_t\phi\right)^2 - 2V(\phi) \\ &+\frac{2}{\sqrt{h}\bar{N}}\partial_t(\sqrt{h}\bar{K}) - \frac{2}{\sqrt{h}\bar{N}}\partial_i(\sqrt{h}h^{ij}\partial_j\bar{N})\Big\}, \end{aligned} \tag{3.5}$$

where we have kept the last two total derivative terms as they contribute if a non-minimal coupling is present.

Let us now perform a disformal transformation to the unbarred frame where the action takes the form,

$$\begin{aligned} S_g = \frac{1}{2}\int d^3x dt \alpha N\sqrt{h}\Big\{&R^{(3)} + \alpha^{-2}\left(K_{ij}K^{ij} - K^2 + N^{-2}\left(\partial_t\phi\right)^2\right) - 2V(\phi) \\ &+ \frac{2}{\sqrt{h}\alpha N}\partial_t(\sqrt{h}\alpha^{-1}K) - \frac{2}{\sqrt{h}\alpha N}\partial_i\left(\sqrt{h}h^{ij}\partial_j(\alpha N)\right)\Big\}, \end{aligned} \tag{3.6}$$

where we used the fact that $\bar{N}(t,x^i) = \alpha(\phi,X)N(t,x^i)$ from Eq. (2.3) and that $K_{ij} = \frac{1}{2N}\partial_t h_{ij}$. At this point, we have to be careful due to the hidden dependence of the old lapse in the disformal factor, i.e.

$$\alpha(\phi,X) = \alpha(t,N) = \alpha(t,x^i), \tag{3.7}$$

[2] We are using the fact that a disformal transformation does not affect the spatial components. We use h_{ij} not to overload notation.

which does not allow us to do a straightforward time redefinition. However, we can split the lapse and disformal factor into the background and the perturbed values. In our notation they are defined by

$$N(t,x^i) = \mathrm{e}^{n(t,x^i)} \qquad \text{and} \qquad \alpha(t,x^i) = \alpha_0(t)\mathrm{e}^{\sigma(t,x^i)}, \tag{3.8}$$

where $\alpha_0(t)$ is the background value of the disformal factor which is only a function of time. Likewise, the barred lapse is decomposed by $\bar{N}(t,x^i) = \alpha_0(t)\mathrm{e}^{\bar{n}(t,x^i)}$. Note that thanks to this decomposition we can absorb the background disformal transformation in a time redefinition (2.11) given by

$$d\bar{t} = \alpha_0(t)dt. \tag{3.9}$$

As a result, physics is invariant *at the background level*. It should be noted that this is valid as long as perturbation theory applies.

Implications for cosmological perturbations. For simplicity we do not consider a non-minimal coupling here and, therefore, we drop the total derivative terms in (3.6). In this way, the action in the unbarred frame is now given by

$$S_g = \frac{1}{2}\int d^3x\, d\bar{t}\sqrt{h}\,\mathrm{e}^{n(t,x^i)+\sigma(t,x^i)}\left\{R^{(3)} + \mathrm{e}^{-2[n(t,x^i)+\sigma(t,x^i)]}\left(E_{ij}E^{ij} - E^2 + (\partial_{\bar{t}}\phi)^2\right) - 2V(\phi)\right\}, \tag{3.10}$$

where we defined $E_{ij} \equiv \frac{1}{2}\partial_{\bar{t}}h_{ij}$ and $E = h^{ij}E_{ij}$, following the notation of Maldacena [66], so that every quantity is expressed in terms of the new time $\bar{t}$ but for the perturbed lapse $n(t,x^i)$ and the perturbed disformal factor $\sigma(t,x^i)$. We deliberately kept the old time dependence in the latter for the reason shown below. In the Hamiltonian formalism, the lapse is a Lagrange multiplier and, hence, a redefinition of it has no effect in the dynamics. For this reason, let us redefine the lapse so that it absorbs the perturbed disformal factor, i.e.

$$n(t,x^i) + \sigma(t,x^i) = \bar{n}(t,x^i). \tag{3.11}$$

In terms of this lapse, the Hamiltonian constraint is given by

$$R^{(3)} - \mathrm{e}^{-2\bar{n}(t,x^i)}\left[E_{ij}E^{ij} - E^2 + (\partial_{\bar{t}}\phi)^2\right] - 2V(\phi) = 0, \tag{3.12}$$

which at the background level has the same solution as (3.2) but given in terms of the redefined time $\bar{t}$. Due to this fact, the perturbed barred lapse must be exactly equal to the unbarred one but expressed in terms of the redefined time $\bar{t}$, in other words

$$\bar{n}(t,x^i) = n(\bar{t},x^i). \tag{3.13}$$

Consequently, the action (3.10) takes exactly the same form as (3.2) but in terms of the redefined time $\bar{t}$.

We may go further and consider a non-minimal coupling or a Horndeski-type Lagrangian. The lapse N is always accompanied by a factor α which at the background level is successfully absorbed by the time redefinition and at the perturbation level gives rise to a factor $\mathrm{e}^{\bar{n}(t,x^i)}$. For example, consider one of the total derivative terms in (3.6) which in presence of a non-minimal coupling gives a non-trivial contribution to the action as

$$S_g \supset \int d^3x dt\, \alpha N\sqrt{h}\,\Omega(\phi,X)\frac{1}{\sqrt{h}\alpha N}\partial_t(\sqrt{h}\alpha^{-1}K), \tag{3.14}$$

where $\Omega(\phi, X)$ is the non-minimal coupling. Absorbing the background value of the disformal function and integrating by parts we are led to

$$S_g \supset -\int d^3x d\bar{t}\sqrt{h}\, \mathrm{e}^{-\bar{n}(t,x^i)} E\ \partial_{\bar{t}}\Omega, \tag{3.15}$$

where the factor X changes as well according to

$$X = \frac{1}{2\alpha^2 N^2}(\partial_t\phi)^2 = \frac{\mathrm{e}^{-2\bar{n}(t,x^i)}}{2}(\partial_{\bar{t}}\phi)^2. \tag{3.16}$$

In this way, the function $\bar{n}(t, x^i)$ appears exactly in the same place where the original lapse $n(t, x^i)$ is in the original action (3.2). The only difference is the time coordinate $\bar{t}$ which is used in any other variables except for the perturbed lapse, which gives us no other choice but to conclude that $\bar{n}(t, x^i) = n(\bar{t}, x^i)$.

In summary, a cosmological disformal transformation in the *comoving*, or the *uniform ϕ slicing* is equivalent to a rescaling of the time coordinate even at *higher orders* in perturbative expansion. Therefore, as long as perturbation theory is valid, cosmological perturbations are invariant under a disformal transformation. This result is in agreement with [56–58] where it was found that scalar and tensor power spectra are frame independent at leading order in the perturbation.

3.2 Towards a Horndeski Lagrangian

Once we know how the gravitational action transforms, it is appropriate to take a look at the resulting Horndeski-type action if we stick to the original time, i.e., without rescaling the time coordinate. The beyond-Horndeski or GLPV Lagrangian can be written in terms of geometrical quantities [48–50] and it consists of the following four terms:

$$\mathcal{L}_2 = A_2(\phi, X), \tag{3.17}$$

$$\mathcal{L}_3 = A_3(\phi, X)K, \tag{3.18}$$

$$\mathcal{L}_4 = A_4(\phi, X)R^{(3)} + B_4(\phi, X)(K_{ij}K^{ij} - K^2), \tag{3.19}$$

$$\mathcal{L}_5 = A_5(\phi, X)\left(R^{(3)}_{ij}K^{ij} - \frac{R^{(3)}K}{2}\right) + B_5(\phi, X)\left(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K^j_k\right). \tag{3.20}$$

The GLPV Lagrangian reduces to the Horndeski Lagrangian when the functions B_4 and B_5 are respectively related to A_4 and A_5 by

$$B_4(\phi, X) = A_4 - 2XA_{4,X} \quad \text{and} \quad B_5(\phi, X) = -\frac{1}{2}XA_{5,X}. \tag{3.21}$$

A glance at the disformally transformed action in terms of the original time (3.6) tells us that the existence of an Einstein frame from the Horndeski theory point of view is closely related to the absence of the term $\mathcal{L}_5$. In addition, we can identify the following terms for a cosmological background with a canonical scalar field with a potential:

$$A_2(\phi, X) = \alpha^{-1}X - \alpha V\,, \quad A_3(\phi, X) = 0\,, \quad A_4(\phi, X) = \alpha \quad \text{and} \quad B_4(\phi, X) = \alpha^{-1}, \tag{3.22}$$

where we remind the reader that $\alpha(\phi, X) = \sqrt{1 - 2XF(\phi, X)}$. The first condition in (3.21), which a Horndeski Lagrangian must satisfy, implies that the disformal factor must fulfil

$$\alpha^{-1} = \alpha - 2X\alpha_{,X}, \tag{3.23}$$

which means $F(\phi, X) = F(\phi)$. As a result, only a *field dependent* disformal transformation leads us to a Horndeski Lagrangian. This is also pointed out by Bettoni and Liberati in [54] with a general treatment. In other words, a general disformal transformation of the Einstein-scalar theory does not satisfy the condition that the field equations are at most second order in time derivatives. This implies the existence of higher time derivatives and hence of Ostrogradsky ghosts. Nevertheless, despite the apparent appearance of Ostrogradsky ghosts for a X dependent disformal transformation, no real ghost should actually be present as the theory in the transformed frame is equivalent to the healthy Einstein-scalar theory. This is discussed by Zumalacárregui and García-Bellido in [53] where they show the existence of hidden constraints. We leave its application to cosmology for future work.

3.3 Matter action

Now let us consider matter fields. We work under the same decomposition of the metric given by (3.1) but only considering the background dynamics. As explained in section 2.2, the generalisation to include perturbations is straightforward.

Scalar field. Let us start with a scalar field χ with mass m whose action, in the barred frame, is given by

$$S = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\left(\bar{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + m^2\chi^2\right), \tag{3.24}$$

which in terms of the barred metric (3.1) yields

$$S = \frac{1}{2}\int d^3x\ dt\bar{N}\sqrt{\bar{h}}\left\{\bar{N}^{-2}(\partial_t\chi)^2 - \bar{h}^{ij}\partial_i\chi\partial_j\chi - m^2\chi^2\right\}. \tag{3.25}$$

By means of a disformal transformation (2.3) we find that the action in the unbarred frame reads

$$S = \frac{1}{2}\int d^3x\ dt\ \alpha N\sqrt{h}\left\{N^{-2}\alpha^{-2}(\partial_t\chi)^2 - h^{ij}\partial_i\chi\partial_j\chi - m^2\chi^2\right\}, \tag{3.26}$$

where we used that $\bar{N}(t) = \alpha(t)N(t)$. Note that one may argue that the sound speed in the unbarred frame is modified compared to that in the barred frame. However, there is a factor α in front of every dt which can be successfully absorbed by the time redefinition (2.11) that leads us to

$$S = \frac{1}{2}\int d^3x\, d\bar{t}\, N\sqrt{h}\left\{N^{-2}(\partial_{\bar{t}}\chi)^2 - h^{ij}\partial_i\chi\partial_j\chi - m^2\chi^2\right\}, \tag{3.27}$$

which is the action one would expect in the unbarred frame (3.1) with the relabelling $t \to \bar{t}$. In other words, if we denote a solution of the field equation in the barred frame by $\bar{\chi}(t)$, the corresponding solution in the unbarred frame $\chi(t)$ is related to it as $\chi(\bar{t}) = \bar{\chi}(t)$.

Electromagnetic field. Let us move towards the disformal transformation of the electromagnetic field. It is known that the Maxwell field is conformally invariant [67] but in general it may not be disformally invariant. This is an important point since observations mainly use photons. However, we want to stress that as long as matter is universally coupled to a unique metric, no issue arises (see appendix B for an example). The action for the electromagnetic field A_μ in the barred frame reads

$$S = -\frac{1}{4}\int d^4x\sqrt{-\bar{g}}\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}, \tag{3.28}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. We expand the action (3.28) in terms of the barred metric (3.1) to explicitly split the time components from the spatial ones, i.e.

$$S = \frac{1}{4}\int d^3x\, dt\, \bar{N}\sqrt{\bar{h}}\left(2\bar{N}^{-2}\bar{h}^{ij}F_{ti}F_{tj} - \bar{h}^{ij}\bar{h}^{kl}F_{ik}F_{jl}\right), \tag{3.29}$$

and perform the disformal transformation (2.3) to obtain

$$S = \frac{1}{4}\int d^3x\, dt\alpha N\sqrt{h}\left(2\alpha^{-2}N^{-2}h^{ij}F_{ti}F_{tj} - h^{ij}h^{kl}F_{ik}F_{jl}\right). \tag{3.30}$$

Note that $F_{ti} = \partial_t A_i - \partial_i A_t$ and, therefore, in order to successfully absorb the α factor in the time redefinition (2.11) the time component of the electromagnetic field must be transformed as

$$A_{\bar{t}} = \alpha^{-1}A_t\,. \tag{3.31}$$

The resulting action is given by

$$S = \frac{1}{4}\int d^3x\, d\bar{t}\, N\sqrt{h}\left(2N^{-2}h^{ij}F_{\bar{t}i}F_{\bar{t}j} - h^{ij}h^{kl}F_{ik}F_{jl}\right), \tag{3.32}$$

where we have used the fact that $\partial_i A_{\bar{t}} = \alpha^{-1}\partial_i A_t$ for a spatially homogeneous $\alpha = \alpha(t)$. Again this action is the one for the electromagnetic field in the metric (3.1) but labelled in terms of $\bar{t}$.

The redefinition of the electromagnetic field given by equation (3.31) is not surprising within the differential form approach, where the 1-form is given by

$$\begin{aligned}\mathbb{A} &= A_\mu dx^\mu = A_t dt + A_i dx^i \\ &= A_t\alpha^{-1}d\bar{t} + A_i dx^i = A_{\bar{t}}d\bar{t} + A_i dx^i.\end{aligned} \tag{3.33}$$

Thus, the 1-form $\mathbb{A}$ is invariant under cosmological disformal transformations.

Dirac field. For the sake of completeness let us quickly consider the disformal transformation of a Dirac field, e.g. an electron. The action for a fermion field Ψ with mass m and charge e in curved space-time is given by [68]

$$S = -\frac{i}{2}\int d^4x\sqrt{-\bar{g}}\left(\breve{\Psi}\bar{\gamma}^\mu\bar{D}_\mu\Psi + m\breve{\Psi}\Psi\right), \tag{3.34}$$

where $\breve{\Psi} = \Psi^\dagger\gamma^0$ is the adjoint spinor, $\bar{D}_\mu = \partial_\mu + ieA_\mu + \bar{\Gamma}_\mu$ and the tetrad components $\bar{e}^\mu_{(a)}$ are defined by

$$\eta_{ab} = \bar{e}^\mu_{(a)}\bar{e}^\nu_{(b)}\bar{g}_{\mu\nu}, \tag{3.35}$$

which relate the gamma matrices and the spin connection in curved space-time to the gamma matrices in flat space-time, i.e. $\bar{\gamma}^\mu = \gamma^a \bar{e}^\mu_{(a)}$ and $\bar{\Gamma}_\mu = \frac{1}{8}[\gamma^a, \gamma^b]\bar{e}^\lambda_{(a)}\bar{\nabla}_\mu \bar{e}_{(b)\lambda}$. In the (3+1)-decomposition the action reads

$$S = -\frac{i}{2}\int d^3x\ dt\ \bar{N}\sqrt{\bar{h}}\left(\breve{\Psi}\gamma^0\bar{N}^{-1}\bar{D}_t\Psi + \breve{\Psi}\bar{\gamma}^i\bar{D}_i\Psi + m\breve{\Psi}\Psi\right), \tag{3.36}$$

where $\bar{e}^t_{(a)} = \delta^0_{(a)}\bar{N}^{-1}$ and $\bar{e}^i_{(a)}\bar{e}^j_{(b)} = \delta_{ab}\bar{h}^{ij}$ so that $\bar{\gamma}^t = \gamma^0\bar{N}^{-1}$. The effect of the disformal transformation (2.3) can be summarised as follows. The tetrad is modified according to (3.35),

$$\bar{e}^t_{(a)} = e^t_{(a)}\alpha^{-1} \quad \text{and} \quad \bar{e}^i_{(a)} = e^i_{(a)}, \tag{3.37}$$

which yields a similar transformation for the gamma matrices,

$$\bar{\gamma}^t = \gamma^t\alpha^{-1}, \qquad \bar{\gamma}^i = \gamma^i. \tag{3.38}$$

The modification of the spin connection $\bar{\Gamma}_\mu$ is slightly more involved. Essentially, there is a contribution not only from the transformation of the tetrad $\bar{e}^\lambda_{(a)}$ but also from its covariant derivative $\bar{\nabla}_\mu\bar{e}_{(b)\lambda}$. The latter is due to the change of the Christoffel symbols, which are defined by $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\bar{g}^{\lambda\sigma}\left(\partial_\mu\bar{g}_{\sigma\nu} + \partial_\nu\bar{g}_{\mu\sigma} - \partial_\sigma\bar{g}_{\mu\nu}\right)$. Specifically, the non-vanishing Christoffel symbols for the metric (2.10) which are affected by the disformal transformation are explicitly given by

$$\begin{aligned}
\bar{\Gamma}^t_{tt} &= \tfrac{1}{2}\bar{N}^{-2}\partial_t\bar{N}^2, & \bar{\Gamma}^i_{tt} &= \tfrac{1}{2}\bar{h}^{ij}\partial_j\bar{N}^2, \\
\bar{\Gamma}^t_{it} &= \tfrac{1}{2}\bar{N}^{-2}\partial_i\bar{N}^2, & \bar{\Gamma}^t_{ij} &= \tfrac{-1}{2}\bar{N}^{-2}\partial_t\bar{h}_{ij}, \\
\bar{\Gamma}^i_{jt} &= \tfrac{1}{2}\bar{h}^{ik}\partial_t\bar{h}_{kj},
\end{aligned} \tag{3.39}$$

where the spatial component $\bar{\Gamma}^i_{jk}$ is not modified under such a disformal transformation. This result greatly simplifies the expression for the spin connection. The time component $\bar{\Gamma}_t$ is proportional to

$$\bar{e}^\lambda_{(a)}\bar{\nabla}_t\bar{e}_{(b)\lambda} = \bar{e}^t_{(a)}\partial_t\bar{e}_{(b)t} + \bar{e}^i_{(a)}\partial_t\bar{e}_{(b)i} - \bar{e}^t_{(a)}\bar{\Gamma}^t_{tt}\bar{e}_{(b)t} - \bar{e}^i_{(a)}\bar{\Gamma}^j_{ti}\bar{e}_{(b)j} - \bar{e}^t_{(a)}\bar{\Gamma}^i_{tt}\bar{e}_{(b)i} - \bar{e}^i_{(a)}\bar{\Gamma}^t_{ti}\bar{e}_{(b)t}, \tag{3.40}$$

and the spatial component $\bar{\Gamma}_j$ is proportional to

$$\bar{e}^\lambda_{(a)}\bar{\nabla}_j\bar{e}_{(b)\lambda} = \bar{e}^t_{(a)}\partial_j\bar{e}_{(b)t} + \bar{e}^i_{(a)}\partial_j\bar{e}_{(b)i} - \bar{e}^t_{(a)}\bar{\Gamma}^t_{jt}\bar{e}_{(b)t} - \bar{e}^i_{(a)}\bar{\Gamma}^k_{ji}\bar{e}_{(b)k} - \bar{e}^i_{(a)}\bar{\Gamma}^t_{ij}\bar{e}_{(b)t} - \bar{e}^t_{(a)}\bar{\Gamma}^i_{jt}\bar{e}_{(b)i}. \tag{3.41}$$

The dependence of Γ_μ on α after the disformal transformation (2.3) can be seen from the above two equations. Let us first look at the time component, i.e.

$$\bar{\Gamma}_t = \frac{1}{8}[\gamma^a, \gamma^b]\,\bar{e}^\lambda_{(a)}\bar{\nabla}_t\bar{e}_{(b)\lambda}. \tag{3.42}$$

From equation (3.40) we see that any term containing a time derivative, namely the first four terms on the right hand side of it, has no α dependence. In particular, the $\partial_t\alpha$ term arising from $\partial_t\bar{e}_{(b)t}$ is canceled with that coming from the $\bar{\Gamma}^t_{tt}$ term. The last two terms on the right hand side contain spatial derivatives and each spatial derivative is accompanied by a factor α. Next let us move to the spatial component, i.e.

$$\bar{\Gamma}_j = \frac{1}{8}[\gamma^a, \gamma^b]\, \bar{e}^{\lambda}_{(a)} \bar{\nabla}_j \bar{e}_{(b)\lambda}. \tag{3.43}$$

From (3.41) we realise that the first four terms on the right hand side of it which contain spatial derivatives has no α dependence. On the other hand, those which contain time derivatives, i.e. the last two terms, have a factor α^{-1} for each time derivative. At the end of the day, we can schematically express the effect of a disformal transformation on the spin connection as

$$\bar{\Gamma}_t[\partial_t, \partial_i] = \Gamma_t[\partial_t, \alpha\partial_i] = \alpha\Gamma_t[\alpha^{-1}\partial_t, \partial_i] \quad \text{and} \quad \bar{\Gamma}_i[\partial_t, \partial_i] = \Gamma_i[\alpha^{-1}\partial_t, \partial_i], \tag{3.44}$$

where $\Gamma_\mu = \frac{1}{8}[\gamma^a, \gamma^b]\, e^{\lambda}_{(a)} \nabla_\mu e_{(b)\lambda}$ is the spin connection in the unbarred frame. It is clear now that the factor α^{-1} in $\bar{N}^{-1}$ in front of $\bar{D}_t$ in (3.36) cancels the factor α from $\bar{\Gamma}_t$, and the factor α^{-1} associated with each ∂_t is successfully absorbed in the time redefinition (2.11). As a result, the disformal transformation of the full action can be rewritten as

$$S = -\frac{i}{2}\int d^3x\, d\bar{t}\, N\sqrt{h}\left(\breve{\Psi}\gamma^0 N^{-1} D_{\bar{t}}\Psi + \breve{\Psi}\gamma^i D_i\Psi + m\breve{\Psi}\Psi\right), \tag{3.45}$$

where $D_{\bar{t}} = \partial_{\bar{t}} + ieA_{\bar{t}} + \Gamma_{\bar{t}}$, $\Gamma_{\bar{t}} \equiv \Gamma_t[\partial_{\bar{t}}, \partial_i]$ and we have used the previous result for the electromagnetic field (3.31). In this form, the disformal invariance is certainly manifest. Lastly, we note that neither the charge e nor the mass m is affected by a disformal transformation.

4 Frame independence

In this section, we shall consider the system, gravity plus matter, as a whole. We assume that matter is *universally coupled* to $\bar{g}_{\mu\nu}$, which we call the matter frame, and is related by a disformal transformation to $g_{\mu\nu}$ which we call the gravitational frame. For example, we may have the Einstein-Hilbert action for the metric $g_{\mu\nu}$. Thus, the weak equivalence principle is preserved. This is in contrast with [42, 43] where different disformal couplings for matter and radiation are considered.

4.1 Gravity plus matter

In the preceding section we explicitly showed the invariance under disformal transformations field by field. In this subsection, we consider the whole system. For example, the simplest and commonly used action [34–36, 39–43] is given by

$$S = \int d^4x \left\{\sqrt{-g}\Big(R[g] + \mathcal{L}_\phi(g,\phi)\Big) + \sqrt{-\gamma}\mathcal{L}_m(\gamma, \psi_I)\right\}, \tag{4.1}$$

where the gravitational sector is the Einstein-Hilbert action in terms of the metric g plus and a scalar field ϕ,[3] the latter being responsible for a disformal coupling with matter fields ψ_I through

$$\gamma_{\mu\nu} = g_{\mu\nu} + H(\phi, X)\phi_{,\mu}\phi_{,\nu}\,. \tag{4.2}$$

[3]In principle, one could choose any form for the gravitational sector [28].

Under the cosmological assumption we adopt throughout this paper, this amounts to the replacement of the lapse function in the matter sector by

$$N_\gamma^2 = N_g^2\Big(1 - 2XH(\phi, X)\Big) \equiv N_g^2\beta^2(\phi, X)\,. \tag{4.3}$$

One may wonder whether the system is still disformally invariant as a whole. Let us show that this is the case.

Let us perform a disformal transformation given by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + F(\phi, \bar{X})\phi_{,\mu}\phi_{,\nu}\,, \tag{4.4}$$

which introduces a new lapse function related to the original one by

$$N_g^2 = \bar{N}_g^2(1 - 2XF(\phi, \bar{X})) \equiv \bar{N}_g^2\alpha^2(\phi, \bar{X})\,, \tag{4.5}$$

where $\bar{X} = (\partial_t\phi)^2/2\bar{N}_g^2$. As before, we introduce a new time coordinate by $d\bar{t} = \alpha_0(t)dt$ and absorb the perturbation of α into $\bar{N}_g$ by redefining it. This means

$$N_g dt = \bar{N}_g\alpha dt = e^{\bar{n}_g}\alpha_0(t)dt = e^{\bar{n}_g}d\bar{t}\,, \tag{4.6}$$

where we used equation (3.8) to define $e^{\bar{n}_g} \equiv \bar{N}_g e^{\sigma(t,x^i)}$. To apply the passive disformal transformation to the action we first express the matter sector in terms of N_g,

$$S = \int d^3x\; dt N_g\sqrt{h}\left\{R[g] + \mathcal{L}_\phi(g,\phi) + \beta(\phi, X)\mathcal{L}_m\big(h, (\beta N_g)^{-1}\partial_t\psi_I, \cdots\big)\right\}. \tag{4.7}$$

Using (4.6), we can then rewrite the action in terms of the barred time as

$$S = \int d^3x\; d\bar{t}\bar{N}_g\sqrt{h}\left\{R[\bar{g}] + \mathcal{L}_\phi(\bar{g},\phi) + \beta(\phi, X_{\bar{g}})\mathcal{L}_m\big(h, (\beta\bar{N}_g)^{-1}\partial_{\bar{t}}\psi_I, \cdots\big)\right\}, \tag{4.8}$$

where it should be noted that $X_{\bar{g}}$ is in fact equal to X but rewritten in terms of barred quantities,

$$X_{\bar{g}} \equiv \frac{2}{\bar{N}_g^2}\left(\frac{\partial\phi}{\partial\bar{t}}\right)^2 = X\,. \tag{4.9}$$

Once the disformal transformation is done we can go back to the original form of the action but in terms of barred quantities, i.e.

$$S = \int d^3x\; d\bar{t}\left\{\sqrt{-\bar{g}}\Big(R[\bar{g}] + \mathcal{L}_\phi(\bar{g},\phi)\Big) + \sqrt{-\bar{\gamma}}\mathcal{L}_m(\bar{\gamma}, \psi_I)\right\}, \tag{4.10}$$

by defining $\bar{\gamma}$ through $\bar{N}_\gamma \equiv \bar{N}_g\beta(\phi, X_{\bar{g}})$ or alternatively

$$\bar{\gamma}_{\mu\nu} = \bar{g}_{\mu\nu} + H(\phi, X_{\bar{g}})\phi_{,\mu}\phi_{,\nu}\;, \tag{4.11}$$

which is identical to the original relation (4.2) but in terms of barred quantities.

4.2 Propagation speed of gravitons and photons

One effect that we may be able to detect which is not present in the case of a conformal coupling was already pointed out by Bekenstein [26]. It was argued that gravitons may travel faster or slower than light, depending on the sign of the disformal factor (2.1). This is true when matter and gravity are expressed in terms of two different metrics which are related by a disformal transformation like (4.2). However, as we have seen, we should keep in mind that the disformal transformation itself does not change the propagation speed. The difference in the propagation speed exists independent of frames.

To illustrate this fact let us consider the previous action given by

$$S = \int d^4x \left\{ \sqrt{-g}\Big(R[g] + \mathcal{L}_\phi(g,\phi)\Big) + \sqrt{-\bar{g}}\mathcal{L}_m(\bar{g},\psi_I) \right\}, \tag{4.12}$$

where the propagation speed of gravitons may be different from that of photons, for example, due to the disformal coupling. Note that we renamed γ as $\bar{g}$ in order to be consistent with notation in section 3. Let us express the gravitational sector in terms of the matter metric as

$$S = \int d^4x \left\{ \sqrt{-g[\bar{g}]}\Big(R[g[\bar{g}]] + \mathcal{L}_\phi(g[\bar{g}],\phi)\Big) + \sqrt{-\bar{g}}\mathcal{L}_m(\bar{g},\psi_I) \right\}, \tag{4.13}$$

as given by the disformal transformation from (3.5) to (3.6), interchanging the roles of the barred and unbarred metrics (see appendix A). This gives rise to a Horndeski action schematically given by

$$S = \int d^4x \sqrt{-\bar{g}} \left\{ \mathcal{L}_{horn}(\bar{g},\bar{K},\bar{R},\phi,\bar{X}) + \mathcal{L}_m(\bar{g},\psi_I) \right\}. \tag{4.14}$$

The interpretation in this form is different. Now, the relative difference of the propagation speed between gravitons and photons is not due to the disformal coupling of matter but due to a modification of gravity. The resulting Horndeski action in terms of the matter metric has a tensor propagation speed given by $c_T = \alpha^{-1}$.[4] However, we emphasize that this is not an observable feature in the primordial power spectrum [56], which may be understood by the fact that the coupling in the matter sector is irrelevant during inflation.

The above discussion has an important implication. It means that as long as the tensor power spectrum is concerned, the tensor propagation speed may be always set to unity by a disformal transformation, a definition for the Einstein frame (at least perturbatively). This consequence has been also discussed by Creminelli et al. [56]. In any case, a disformal coupling with matter should be in principle observable if we could measure the relative difference in the propagation speed between gravitons and matter, which is frame independent.

In this sense, if one is interested in the causal structure of the whole system, by means of a disformal transformation one may go to the frame where the propagation speed of the fastest species (e.g. either gravitons or photons) is equal to unity, i.e. equal to the geometrical factor c which defines the causal structure. In this frame, any other field will have a propagation speed less than unity and hence the standard causal structure is preserved.

5 Conclusions

Disformal transformations play an important role as a generalization of conformal transformations. The latter is widely used in modifications of gravity as well as in cosmology and

[4]Alternatively, it can be inferred from the relative factor α^2 relating both metrics (2.3).

it has been proven to leave observable physics invariant. In this paper, we focused on pure disformal transformations, i.e. those without any transformation in the conformal factor. Further we assumed a cosmological setting, i.e. under the assumption that a scalar field responsible for the disformal transformation is regular everywhere and its derivative is time-like, which allows us to choose the time slicing on which the scalar field is homogeneous. Any posterior conformal transformation can be independently done in general. Our main result is that such a disformal transformation, which we call a cosmological disformal transformation, is equivalent to a rescaling of the time coordinate, which is valid full non-linearly, provided that perturbation theory is applicable. From this result, it follows that observable physics must be invariant under cosmological disformal transformations. In this sense, we extended the work for cosmological perturbations in the literature [57, 58] which showed the disformal invariance at leading order in the perturbation. We placed emphasis on the fact that neither the causal structure, propagation speed nor any other property of the fields is affected by a disformal transformation itself, although it may seem so if an appropriate time is not used.

The physical invariance under disformal transformations is an interesting result. It may help us to better understand Horndeski or Galileon theory. It may simplify calculations by going to the frame where the speed of sound of the tensor modes is equal to unity, i.e. to the Einstein frame. For example, if applied to inflation, the subset of Horndeski theory with coefficients (3.22) satisfying (3.23) is observationally indistinguishable from the corresponding Einstein-Hilbert action with a canonical scalar field with a potential.

On the other hand, if applied to dark energy where gravity and matter may have different disformal couplings, there are strong constraints from the solar system experiments due to the appearance of a fifth force [39]. Nevertheless, a slowly rolling field may be able to pass those constraints as the disformal transformation depends on its first time derivative [40]. Another observationally distinguishable signature is the relative difference of the propagation speed between gravitons and photons. This may become very important in the near future when gravitational waves from distant sources will begin to be detected.

In summary, observable physics is invariant under disformal transformations, when applied to cosmology. We leave as future work the study of the appearance of false Ostrogradsky ghosts when the conformal or disformal factors include a kinetic term dependence, where hidden constraints should arise [53].

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APPENDICES

A Inverse disformal transformation

In the main text we considered a disformal transformation $g \to \bar{g}$ from the passive point of view. Namely we start from the barred frame $\bar{g} = d\bar{s}^2$ and express the barred quantities in terms of the quantities in the unbarred frame $\bar{g} = \bar{g}(g)$. Here we briefly discuss the inverse case where we start from a model in the unbarred frame with the metric g and perform an inverse passive disformal transformation to work with $\bar{g}$. In other words, we begin with $g = ds^2$ and express the unbarred quantities in terms of the barred one as $g = g(\bar{g})$.

Let us fix that the barred metric is given by

$$d\bar{s}^2 = -dt^2 + g_{ij}dx^i dx^j \,, \tag{A.1}$$

namely we want to work with the proper cosmic time ($\bar{N} = 1$) of $\bar{g}_{\mu\nu}$. By means of a disformal transformation,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\bar{g}_{\mu\nu} - F(\phi, X)\phi_{,\mu}\phi_{,\nu})dx^\mu dx^\nu \,, \tag{A.2}$$

the unbarred line element reads

$$ds^2 = d\bar{s}^2 + (1 - \alpha^{-2}(t))dt^2 = -\alpha^{-2}(t)dt^2 + g_{ij}dx^i dx^j . \tag{A.3}$$

This time, the suitable proper time redefinition in the unbarred frame is given by

$$d\tilde{t} = \alpha^{-1}(t)dt \tag{A.4}$$

which leads us to

$$ds^2 = -d\tilde{t}^2 + g_{ij}dx^i dx^j . \tag{A.5}$$

In this form, the metric components are the same as those in the barred frame but with a rescaling of time $t \to \tilde{t}$. To summarize, starting from an action with the unbarred metric $g_{\mu\nu}$ and rewriting it in terms of the barred metric $\bar{g}_{\mu\nu}$ is equivalent to the rescaling of time from t to $\tilde{t}$, i.e. $S[g(\bar{g}); t] = S[\bar{g}, \tilde{t}]$.

B Example of an observable: Redshift

Here we examine the frame independence of the measured redshift as a simple example of an observable quantity. The main point is that once we are given the action of a system, we can compute observable quantities in any frame and obtain the same result.

Let us assume that the background in the gravitational frame, where the gravitational action is given by the Einstein-Hilbert one, is well described by a flat FLRW background,

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\mathbf{x}^2\right) , \tag{B.1}$$

where $a(\eta)$ is the scale factor as a function of the conformal time defined by $d\eta = dt/a$. We assume the matter is coupled to a disformally transformed metric, given by the transformation (2.3), which we call the matter frame metric. It is expressed as

$$d\bar{s}^2 = -a^2(\eta)\alpha^2(\eta)d\eta^2 + a^2(\eta)d\mathbf{x}^2 = \bar{a}^2(\bar{\eta})\left(-d\bar{\eta}^2 + d\mathbf{x}^2\right) , \tag{B.2}$$

where $\bar{a}(\bar{\eta}) = a(\eta(\bar{\eta}))$ and in the last step we used the time redefinition $d\bar{\eta} = \alpha d\eta$. Naturally, photons follow null geodesics of $\bar{g}_{\mu\nu}$, i.e. $d\bar{s}^2 = 0$. The energy of a photon with four-momentum $\bar{k}^\mu = (\bar{k}^\eta, \bar{\boldsymbol{k}})$ measured by a comoving observer with four-momentum $\bar{u}_\mu = (-\bar{a}(\bar{\eta}), 0)$ is given by

$$\bar{\mathcal{E}} = -\bar{k}^\mu \bar{u}_\mu = \bar{a}(\bar{\eta})\bar{k}^{\bar{\eta}}. \tag{B.3}$$

One of the important points to be remembered when performing a passive disformal transformation is that scalar quantities are invariant under such a transformation. This applies, in particular, to the above energy measured by a comoving observer. Alternatively, the same conclusion may be obtained by considering the transformation rules for a vector (2.18), which implies $\bar{k}^\eta = \alpha^{-1} k^\eta$. We can compute the energy in the unbarred frame as

$$\mathcal{E} = -k^\mu u_\mu = a(\eta) k^\eta. \tag{B.4}$$

Now since we have $\bar{k}^{\bar{\eta}} = \alpha \bar{k}^\eta = k^\eta$, the measured energy is frame independent, $\bar{\mathcal{E}} = \mathcal{E}$. Consequently the redshift, which is defined as the ratio between the measured photon energy at emission and observation,

$$1 + z \equiv \mathcal{E}_{emit}/\mathcal{E}_{obs} = \frac{a(\eta_{obs})}{a(\eta_{emit})} = \frac{\bar{a}(\bar{\eta}_{obs})}{\bar{a}(\bar{\eta}_{emit})}, \tag{B.5}$$

is frame independent as well. Once more the result is consistent with the fact that a cosmological pure disformal transformations is equivalent to a rescaling of the time.

Finally, note that the physical speed of light is the same in both frames for a *matter observer*, as long as matter and radiation are universally coupled to a single metric. This can been easily seen from the condition $d\bar{s}^2 = 0$ which implies that the dispersion relation is $\bar{k}^{\bar{\eta}} = |\bar{\mathbf{k}}| = k^\eta$. We can further extend this results by noting that since the observed redshift is frame independent and the scale factor is not affected by the disformal transformation, the luminosity distance is frame independent as well. This is in agreement with the work by Brax et al. [43] when matter is universally coupled. Other possible imprints in the Cosmological Microwave Background (CMB), such as a modification of the distribution function, are considered in [42], where it is found that if matter and radiation are universally coupled to the disformal metric there is no observable difference in the CMB.

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