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Gevrey class or ultradistribution solutions to Cauchy and boundary value problems for systems with irregularities

By

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Abstract

Consider the Gevrey ultradifferentiable function or ultradistribution solution sheaf complexes to a system of analytic linear differential equations. Then a bound of their microsupports is obtained for under an irregularity condition along a regular involutive submanifold. As applications, the unique solvability theorems are obtained for non-characteristic Cauchy and boundary value problems under conditions of the irregularity above and a (weak) hyperbolicity. Details will be appeared in a forthcoming paper.

Introduction

In *Algebraic Analysis*, a system of holomorphic linear differential equations on a complex manifold X is exactly a coherent Module $\mathscr{M}$ over the Ring $\mathscr{D}_X$ of *holomorphic linear differential operators* (in this paper, we shall write a *Ring* or a *Module* with capital letters respectively, instead of a *sheaf of rings* or a *sheaf of left modules*). Especially in this theory, we can study the non-characteristic Cauchy and boundary value problems satisfactorily in the case of hyperfunction solutions. For example, let us recall known-results concerning the boundary value problems. Suppose that the boundary is real analytic and non-characteristic for the system. Then all the hyperfunction solutions have their boundary values as hyperfunction solutions to the *inverse image* on the boundary, and the local uniqueness (the Holmgren type) theorem holds (see for example Komatsu-Kawai [22] and Schapira [27] for a single equation case with one-codimensional boundary, and Oaku-Yamazaki [23] for a system case with higher-codimensional boundary). Moreover, the unique solvability theorems are obtained for

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Cauchy and boundary value problems under a (weak) hyperbolicity condition (see for example Kashiwara-Schapira [15]).

On the other hand, the *microsupport* theory due to Kashiwara-Schapira gave a new insight and powerful method to algebraic analysis: Let M be a paracompact real analytic manifold, and X a complexification of M. Let us denote by $\mathrm{SS}(*)$ the *microsupport*. Then it follows that

$$\mathrm{SS}\big(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M)\big) \subset T^*M \cap C_{T^*_M X}(\mathrm{Ch}\,\mathscr{M}). \tag{0.1}$$

Here $\mathscr{B}_M$ denotes the sheaf on M of *Sato hyperfunctions*, $\mathrm{Ch}\,\mathscr{M}$ the *characteristic variety* of $\mathscr{M}$, and $C_{T^*_M X}(\mathrm{Ch}\,\mathscr{M})$ the *normal cone* of $\mathrm{Ch}\,\mathscr{M}$ along $T^*_M X$ (see § 2 for the notation). Many results (such as above-mentioned facts) can be derived from the estimate (0.1) by making use of functorial and geometrical arguments developed by Kashiwara-Schapira (see [16]).

Next, we replace $\mathscr{B}_M$ by the sheaf $\mathscr{D}\!\ell_M$ of *Schwartz distributions* on M. Then although an estimate corresponding to (0.1) does not hold in general, Kashiwara-Monteiro Fernandes-Schapira [13] showed that if $\mathscr{M}$ has regular singularities along a $\mathbb{C}^\times$-conic closed regular involutive submanifold $\varLambda$ of T^*X in the sense of Kashiwara-Oshima [14] and $\mathrm{Ch}\,\mathscr{M} \subset \varLambda$ (in particular, $\mathrm{supp}\,\mathscr{M} \subset \varLambda \cap T^*_X X$), then:

$$\mathrm{SS}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}\!\ell_M)) \subset T^*M \cap C_{T^*_M X}(\varLambda). \tag{0.2}$$

In this paper, we state the counterpart of (0.2) for *Gevrey* (*ultradifferentiable*) *functions* or *ultradistributions*, replacing the condition of regular singularities by that of *irregularities*. Here we remark that several definitions of irregularities are proposed for coherent $\mathscr{D}$-Modules. In this paper, we shall follow the definition of Honda [8], which is a natural generalization of the notion of regular singularities in the sense of [14] and Kashiwara-Kawai [12]. Further as applications, we consider non-characteristic Cauchy and boundary value problems for Gevrey function or ultradistribution solutions, and prove the unique solvability theorem under conditions of the irregularity above and a (weak) hyperbolicity.

In order to treat distributions and C^∞ functions from the microlocal point of view, Andronikof [1] and Colin [6] defined the microlocalization functors respectively in the framework of subanalytic categories (cf. Prelli [24]). However at this stage, to Gevrey functions or ultradistributions, the corresponding theories of microlocalization do not exist in this framework for technical difficulties (cf. Honda-Morando [9], Hörmander [10]). Therefore we estimate directly the microsupports of Gevrey function or ultradistribution solution sheaf complexes. In the last section, we treat distribution solutions specially. Details of this article will be appeared in a forthcoming paper ([30]).

Finally it should be mentioned that in a single equation case, Uchikoshi [29] introduced a precise irregularity and proved the unique solvability theorem for hyperbolic

Cauchy problem in the category of ultradistributions from the microlocal point of view (see also references cited in [29]).

§ 1. Preliminaries

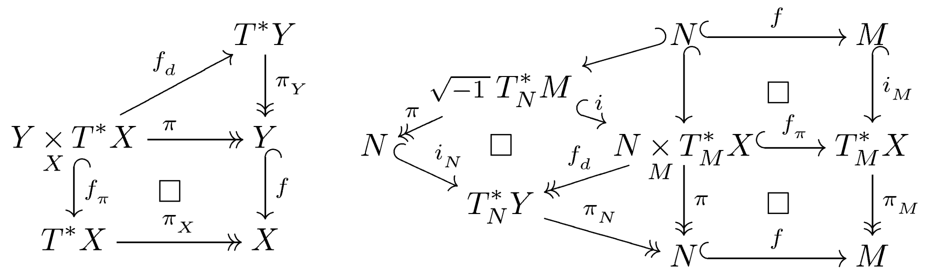
Our general references in this paper are Kashiwara [11] and Kashiwara-Schapira [16].

We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all the integers, rational numbers, real numbers and complex numbers respectively. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z};\, n \geqslant 1\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_{>0} := \{r \in \mathbb{R};\, r > 0\}$ and $\mathbb{C}^\times := \mathbb{C} \smallsetminus \{0\}$.

Let $\mathcal{A}$ be a Ring on a topological space. We denote by $\mathfrak{Mod}(\mathcal{A})$ the category of $\mathcal{A}$-Modules, and by $\mathfrak{Mod}_{\mathrm{coh}}(\mathcal{A})$ the full subcategory of $\mathfrak{Mod}(\mathcal{A})$ consisting of coherent $\mathcal{A}$-Modules. Further we denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of complexes of $\mathcal{A}$-Modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ consisting of objects with coherent cohomologies. Let $\mathcal{C}$ be any of $\mathfrak{Mod}(\mathcal{A})$, $\mathfrak{Mod}_{\mathrm{coh}}(\mathcal{A})$, $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ or $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$. By the abuse of the notation, we write simply $\mathcal{F} \in \mathcal{C}$ if $\mathcal{F}$ is an object of $\mathcal{C}$ on an open subset.

In this paper, all the manifolds are assumed to be *paracompact*. If $p\colon E \to Z$ is a vector bundle over a manifold Z, we set $\dot{p}\colon \dot{E} := E \smallsetminus Z \to Z$ (the zero-section removed). Moreover for any conic subset $V \subset E$, we set $\dot{V} := V \cap \dot{E}$. Throughout this paper, M denotes a real analytic manifold, and X a complexification of M. Let $\pi_X\colon T^*X \to X$ be the cotangent bundle. We denote by $\mathscr{O}_X$ the *Ring of holomorphic functions*, and by $\mathscr{D}_X$ the *Ring of holomorphic linear differential operators* on X respectively. Let $\mathscr{E}_X$ be the *Ring of microdifferential operators* on T^*X and $\{\mathscr{E}_X(m)\}_{m\in\mathbb{Z}}$ the usual *order filtration* on $\mathscr{E}_X$ (see [26] or [28]). In particular, $\mathscr{D}_X(m) := \mathscr{E}_X(m)\big|_X$.

Let N be a d-codimensional real analytic closed submanifold of M, and $f\colon N \hookrightarrow M$ the embedding. Let Y denote a complexification of N in X. We keep the same notation f to stand for the embedding $Y \hookrightarrow X$. Then f induces mappings:



Here π etc. are canonical projections, i_N, i_M and i are zero-section embeddings, and $\square$ means that the square is *Cartesian*.

Let $\mathscr{F} \in \mathbf{D}^{\mathrm{b}}(\mathscr{D}_X)$. We define the *inverse image* in $\mathscr{D}$-Module theory by

$$\boldsymbol{D}f^*\mathscr{F} := \mathscr{D}_{Y\to X} \overset{\boldsymbol{L}}{\underset{f^{-1}\mathscr{D}_X}{\otimes}} f^{-1}\mathscr{F}.$$

Further, we set $D^\nu f^*\mathscr{F} := H^\nu \boldsymbol{D} f^*\mathscr{F}$, in particular, set $Df^*\mathscr{F} := H^0\boldsymbol{D}f^*\mathscr{F}$. Here $\mathscr{D}_{Y\to X} := \mathscr{O}_Y \underset{f^{-1}\mathscr{O}_X}{\otimes} f^{-1}\mathscr{D}_X$ is the *transfer* $(\mathscr{D}_Y, f^{-1}\mathscr{D}_X)$ *bi-Module* attached to $f\colon Y\to X$. Further we set

$$^{\mathscr{E}}\mathscr{F} := \mathscr{E}_X \underset{\pi_X^{-1}\mathscr{D}_X}{\overset{\boldsymbol{L}}{\otimes}} \pi_X^{-1}\mathscr{F} = \mathscr{E}_X \underset{\mathscr{D}_X}{\overset{\boldsymbol{L}}{\otimes}} \mathscr{F}.$$

Here and in what follows, we often omit the symbol π_X^{-1} etc. by the abuse of the notation. Note that since $\mathscr{E}_X$ is flat over $\pi_X^{-1}\mathscr{D}_X$, we have $H^\nu({}^{\mathscr{E}}\mathscr{F}) = {}^{\mathscr{E}}H^\nu(\mathscr{F})$. In particular, if $\mathscr{F}\in \mathfrak{Mod}(\mathscr{D}_X)$, then ${}^{\mathscr{E}}\mathscr{F} = \mathscr{E}_X \underset{\mathscr{D}_X}{\otimes}\mathscr{F}$.

For any $\mathscr{M}\in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, its *characteristic variety* is defined by

$$\operatorname{Ch}\mathscr{M} := \bigcup_\nu \operatorname{supp}{}^{\mathscr{E}}H^\nu(\mathscr{M}).$$

For a subset $A\subset T^*X$, we denote by $C_{T^*_MX}(A)$ the *normal cone of* A *along* T^*_MX. Then $C_{T^*_MX}(A)$ is a closed cone of $T_{T^*_MX}T^*X$. Take a local coordinate system

$$(z;\zeta) = (x+\sqrt{-1}\,y;\xi+\sqrt{-1}\,\eta)$$

in T^*X, where T^*_MX is defined by $\{y=0,\ \xi=0\}$. By this coordinate system, we identify $T_{T^*_MX}T^*X$ with T^*X, hence we use $(x,\sqrt{-1}\,y;\xi,\sqrt{-1}\,\eta)$ as the coordinate system of $T_{T^*_MX}T^*X$. By [16, Proposition 4.1.2], we see that $(x_0,\sqrt{-1}\,y_0;\xi_0,\sqrt{-1}\,\eta_0)\in C_{T^*_MX}(A)$ if and only if there exist sequences $\{(z_n;\zeta_n)\}_{n=1}^\infty\subset A$ and $\{c_n\}_{n=1}^\infty\subset\mathbb{R}_{>0}$ such that

$$(z_n;\zeta_n)\underset{n}{\to}(x_0,\sqrt{-1}\,\eta_0),\quad c_n(y_n,\xi_n)\underset{n}{\to}(y_0,\xi_0).$$

There exists a canonical inclusion $T^*M\hookrightarrow T_{T^*_MX}T^*X$ which is described by $(x;\xi)\mapsto(x,0;\xi,0)$ in local coordinates above (see [16]). Then $(x_0;\xi_0)\in T^*M\cap C_{T^*_MX}(A)$ if and only if there exists a sequence $\{(z_n;\zeta_n)\}_{n=1}^\infty\subset A$ such that

$$(x_n+\sqrt{-1}\,y_n;\xi_n)\underset{n}{\to}(x_0,0;\xi_0),\quad |y_n|\,|\eta_n|\underset{n}{\to}0.$$

Let V be a $\mathbb{C}^\times$-conic closed subset of T^*X. Recall that $f\colon N\hookrightarrow M$ (or simply N) is said to be *hyperbolic for* V if

$$\dot{T}^*_NM\cap C_{T^*_MX}(V)=\emptyset. \tag{1.1}$$

1.1. Remark. (1) $(N\underset{Y}{\times}T^*_YX)\cap\dot{V}=\emptyset$ holds under the assumption (1.1). Since both $N\underset{Y}{\times}T^*_YX$ and V are closed, we may assume that $T^*_YX\cap\dot{V}=\emptyset$ on a neighborhood of $N\underset{X}{\times}T^*X$.

(2) It is well known ([26], [28]) that if Y is non-characteristic for $\mathscr{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$; that is, if $\dot{T}^*_YX\cap f_\pi^{-1}\operatorname{Ch}\mathscr{M}=\emptyset$, it follows that $\boldsymbol{D}f^*\mathscr{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ and

$$D^\nu f^*\mathscr{M} = Df^*H^\nu(\mathscr{M}).$$

Hence, if moreover $\mathscr{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$, then $\boldsymbol{D}f^*\mathscr{M} = Df^*\mathscr{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_Y)$. If Y is non-characteristic for $\mathfrak{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X)$; that is, if $\dot{T}^*_Y X \cap f_\pi^{-1} \operatorname{supp} \mathfrak{M} = \emptyset$, it follows that

$$\mathfrak{M}_Y := f_{d*}(\mathscr{E}_{Y \to X} \underset{f_\pi^{-1}\mathscr{E}_X}{\otimes} f_\pi^{-1}\mathfrak{M}) \simeq \boldsymbol{R}f_{d*}(\mathscr{E}_{Y \to X} \underset{f_\pi^{-1}\mathscr{E}_X}{\overset{\boldsymbol{L}}{\otimes}} f_\pi^{-1}\mathfrak{M})$$

is a coherent $\mathscr{E}_Y$-Module. Moreover if Y is non-characteristic for $\mathscr{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$, then ${}^{\mathscr{E}}(Df^*\mathscr{M}) \simeq ({}^{\mathscr{E}}\mathscr{M})_Y$.

§ 2. Definition of Irregularities

Let $\mathscr{O}_{T^*X}(i)$ denote the subsheaf of $\mathscr{O}_{T^*X}$ of sections with homogenous of degree i for fiber variables. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}_0^l$, we set $|\alpha| := \sum_{j=1}^{l} \alpha_j$. Take $\kappa \in \mathbb{Q}$ with $1 \leqslant \kappa < \infty$. Let V be a $\mathbb{C}^\times$-conic closed regular or maximally degenerate involutive submanifold of $\dot{T}^*X$. We set

$$I_V := \{f \in \mathscr{O}_{T^*X};\ f\big|_V \equiv 0\},$$
$$\mathcal{I}_V := \{P \in \mathscr{E}_X(1);\ \sigma_1(P) \in I_V \cap \mathscr{O}_{T^*X}(1)\}.$$

Here $\sigma_m(P) \in \mathscr{O}_{T^*X}(m)$ denotes the *principal symbol* of $P \in \mathscr{E}_X(m)$.

2.1. Definition ([8]). We set:

$$\mathscr{E}_V^{(\kappa)} := \sum_{\substack{(m,i)\in\mathbb{Z}^2 \\ m\kappa \leqslant i}} \mathscr{E}_X(m-i)\,\mathcal{I}_V^i \subset \mathscr{E}_X .$$

Here if $i \leqslant 0$, we set $\mathcal{I}_V^i := \mathscr{E}_X(0)$.

We recall an explicit expression of $\mathscr{E}_V^{(\kappa)}$:

2.2. Definition. Let V be a $\mathbb{C}^\times$-conic closed regular or maximally degenerate involutive submanifold of $\dot{T}^*X$ and $p_0 \in V$. We say that an l-tuple $(P_1, \ldots, P_l)$ is a system of *involutive coordinate operators* (at p_0) if

(1) $P_j \in \mathscr{E}_X(1) \smallsetminus \mathscr{E}_X(0)$ for each j, and $[P_j, P_k] = 0$ for any $1 \leqslant j, k \leqslant l$;
(2) $(\sigma_1(P_1), \ldots, \sigma_1(P_l)) \in (\mathscr{O}_{T^*X}(1))^l$ forms a basis of I_V in a neighborhood of p_0.

Using a system of involutive coordinate operators $(P_1, \ldots, P_l)$, we can represent $\mathscr{E}_V^{(\kappa)}$ in a neighborhood of $p_0 \in V$ as

$$\mathscr{E}_V^{(\kappa)} = \sum_{\alpha\in\mathbb{N}_0^l} \mathscr{E}_X(\kappa^*[|\alpha|]-|\alpha|)P^\alpha. \tag{2.1}$$

Here $P^\alpha := P_1^{\alpha_1}\cdots P_l^{\alpha_l}$, and $\kappa^*[\nu] := \max\{m \in \mathbb{N}_0;\, m\kappa \leqslant \nu\}$. The right-hand side of (2.1) does not depend on the choice of a system of involutive coordinate operators. Indeed, if $(\widetilde{P}_1,\dots,\widetilde{P}_l)$ is another system of involutive coordinate operators at p_0, then by the division theorem there are Q_{jk}, $R_j \in \mathscr{E}_X(0)$ $(1 \leqslant j,k \leqslant l)$ such that

$$P_j = \sum_{k=1}^{l} Q_{jk}\widetilde{P}_k + R_j$$

and an $(l\times l)$-matrix $(\sigma_0(Q_{jk}))_{j,k=1}^l$ is invertible at p_0. Hence it is easy to see

$$\sum_{\alpha\in\mathbb{N}_0^l} \mathscr{E}_X(\kappa^*[|\alpha|]-|\alpha|)P^\alpha = \sum_{\alpha\in\mathbb{N}_0^l} \mathscr{E}_X(\kappa^*[|\alpha|]-|\alpha|)\widetilde{P}^\alpha.$$

We can see that any (germ of) quantized contact transformation acts on $\mathscr{E}_V^{(\kappa)}$ as a Ring isomorphism.

2.3. Remark. The following properties are easy to prove:

(1) $\mathscr{E}_V^{(1)}$ coincides with $\mathscr{E}_V$ due to Kashiwara-Oshima [14].

(2) If $\kappa \leqslant \kappa'$, $\mathscr{E}_X(0)|_V \subset \mathscr{E}_V^{(\kappa')} \subset \mathscr{E}_V^{(\kappa)} \subset \mathscr{E}_X|_V$ (sub-Rings), and $\mathscr{E}_V^{(\kappa)}$ is a left and right $\mathscr{E}_X(0)$-Module.

Further, we can prove:

2.4. Proposition. (1) $\mathscr{E}_V^{(\kappa)}$ *is a Noetherian Ring in the sense of* [16, Definition 11.1.1] (*hence coherent*).

(2) *Any coherent* $\mathscr{E}_X$*-Module is pseudocoherent as an* $\mathscr{E}_V^{(\kappa)}$*-Module.*

2.5. Proposition. *Let* $\mathfrak{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X)$ *on an open subset* Ω *of* $\dot{T}^*X$, *and* V *a* $\mathbb{C}^\times$*-conic closed regular or maximally degenerate involutive submanifold of* $\dot{T}^*X$*. Then the following conditions are equivalent*:

(1) *For any* $p^* \in \Omega$, *there exist an open neighborhood* U *of* p^* *and an* $\mathscr{E}_V^{(\kappa)}|_U$ *sub-Module* $\mathfrak{L} \subset \mathfrak{M}|_U$ *such that* $\mathfrak{L} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X(0))$ *and* $\mathscr{E}_X\mathfrak{L} = \mathfrak{M}$.

(2) *For any open subset* $U \subset \Omega$ *and any coherent* $\mathscr{E}_X(0)|_U$ *sub-Module* $\mathfrak{N} \subset \mathfrak{M}|_U$, *it follows that* $\mathscr{E}_V^{(\kappa)}\mathfrak{N} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X(0))$.

(3) *For any open subset* $U \subset \Omega$, *every coherent* $\mathscr{E}_V^{(\kappa)}|_U$ *sub-Module* $\mathfrak{N} \subset \mathfrak{M}|_U$ *is* $\mathscr{E}_X(0)$*-coherent.*

The proof is same as in that of [14, Theorem 1.7].

2.6. Remark. The equivalent conditions of Proposition 2.5 are invariant under any quantized contact transformation, since $\mathscr{E}_V^{(\kappa)}$ is stable under any quantized contact transformation.

2.7. Definition ([8], cf. [3]). Let V be a $\mathbb{C}^\times$-conic closed regular or maximally degenerate involutive submanifold of $\dot{T}^*X$.

(1) Let $\mathfrak{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X)$ on an open subset of $\dot{T}^*X$. Then we say that $\mathfrak{M}$ *has irregularities at most* κ *along* V if the equivalent conditions of Proposition 2.5 are satisfied.

(2) Let $\mathfrak{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{E}_X)$ on an open subset of $\dot{T}^*X$. Then we say that $\mathfrak{M}$ has irregularities at most κ along V if so does each cohomology $H^j(\mathfrak{M})$.

2.8. Remark (see [8]). We collect some results about irregularities:

(1) If $\mathfrak{M}$ has irregularities at most κ along V, $\operatorname{supp}\mathfrak{M} \subset V$ in $\dot{T}^*X$ (see [12, Lemma 1.1.13]).

(2) Let $V \subset V_1$ be two $\mathbb{C}^\times$-conic closed regular or maximally degenerate involutive submanifolds of $\dot{T}^*X$. Assume that $\mathfrak{M}$ has irregularities at most κ along V. Then $\mathfrak{M}$ has irregularities at most κ along V_1 since $\mathscr{E}_{V_1}^{(\kappa)} \subset \mathscr{E}_V^{(\kappa)}$.

(3) Let $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ be an exact sequence in $\mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X)$. Then $\mathfrak{M}$ has irregularities at most κ along V if and only if both $\mathfrak{M}'$ and $\mathfrak{M}''$ have irregularities at most κ along V (see [12, Proposition 1.1.14]).

In general, let $\mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \xrightarrow{+1}$ be a distinguished triangle in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{E}_X)$. Then if any two objects have irregularities at most κ along V, so does the rest one.

2.9. Definition (cf. [13]). Let V be a $\mathbb{C}^\times$-conic closed regular or maximally degenerate involutive submanifold of $\dot{T}^*X$, and $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Then we say that $\mathscr{M}$ has irregularities at most κ along V if so does ${}^{\mathscr{E}}\mathscr{M}$.

For any object F of $\mathbf{D}^{\mathrm{b}}(M)$, we denote by $\mathrm{SS}(F)$ the *microsupport* of F which is a closed conic involutive subset of T^*M and defined as follows:

Let p be a point of T^*M. Then $p \notin \mathrm{SS}(F)$ if the following condition holds: there exists a neighborhood U of p in T^*M such that for any $x_0 \in M$ and any real valued real analytic function ψ defined on a sufficiently small neighborhood of x_0 satisfying $\psi(x_0) = 0$ and $(x_0; d\psi(x_0)) \in U$, we have

$$\boldsymbol{R}\Gamma_{\{x;\psi(x)\geqslant 0\}}(F)_{x_0} = 0.$$

Note that $\mathrm{SS}(F) \cap T^*_M M = \operatorname{supp} F$.

2.10. Remark. (1) Assume that $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ has irregularities at most κ along V. Then by Remark 2.8 (1) and [16, Theorem 11.3.3], we have

$$\mathrm{SS}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)) = \mathrm{Ch}\,\mathscr{M} \subset V \cup \operatorname{supp}\mathscr{M}.$$

(2) Let $\Lambda \subset T^*X$. Note that $\mathscr{M}$ is said to be regular along Λ in the sense [13] if $\mathscr{M}$ has irregularities at most 1 along $\dot{\Lambda}$ and $\mathrm{Ch}\,\mathscr{M} \subset \Lambda$ (i.e. $\operatorname{supp}\mathscr{M} \subset \Lambda \cap T^*_X X$).

2.11. Example. Let us set $X := \mathbb{C}^n$ and $p_0 := (0; 1, 0, \dots, 0)$. Moreover, we set

$$V := \{(z; \zeta) \in T^*X;\ \zeta_1 \neq 0,\ \zeta_n = 0\}.$$

For any $P = \partial_{z_n}^m + \sum_{j=0}^{m-1} P_j(z, \partial'_z)\, \partial_{z_n}^j \in \mathscr{E}_X(m)_{p_0}$ with $\partial' := (\partial_{z_1}, \dots, \partial_{z_{n-1}})$, set

$$\operatorname{Irr}_V P := \max_{0 \leqslant j \leqslant m-1} \Bigl\{1, \frac{m-j}{m-j-\operatorname{ord} P_j}\Bigr\}.$$

Then $\mathscr{E}_X / \mathscr{E}_X P$ has irregularities at most $\operatorname{Irr}_V P$ along V.

§ 3. Irregularities and Inverse Image

In this section, we shall investigate the relation of irregularities and inverse image for coherent $\mathscr{E}$-Modules. First, we state the following:

3.1. Condition. Λ is a subset of T^*X such that $\dot{\Lambda}$ is a $\mathbb{C}^\times$-conic closed regular involutory complex submanifold of $\dot{T}^*X$.

3.2. Condition. $\Lambda \subset T^*X$ satisfies Condition 3.1, and if we set $\Lambda_M := \Lambda \cap T^*_M X$, $\dot{\Lambda}_M$ is a closed regular involutory submanifold of $\dot{T}^*_M X$, and $\dot{\Lambda}$ is a complexification of $\dot{\Lambda}_M$.

Then we quote

3.3. Proposition ([7, Lemma 2.10], [28, Corollary A.4.5]). *Let $\Lambda \subset T^*X$ be as in Conditions* 3.1 *and* 3.2, *and N a d-codimensional closed real analytic submanifold of M. Suppose that $f\colon N \hookrightarrow M$ is hyperbolic for Λ. Set $n := \dim M = \dim_{\mathbb{C}} X$ and $l := \operatorname{codim}^{\mathbb{C}}_{T^*X} \Lambda$. Then for any $p_0 \in N \underset{M}{\times} \dot{\Lambda}_M$, there exist a conic neighborhood U_{p_0} of p_0 in $\dot{T}^*X$ and a homogeneous symplectic transformation χ_{p_0} on U_{p_0} such that*:

(1) *χ is a complexification of real homogeneous symplectic transformation on the set $U_{p_0} \cap T^*_M X$;*

(2) *local coordinates in $\chi_{p_0}(U_{p_0})$ can be chosen to be induced by the following coordinates*:

(3.1)
$$\begin{array}{ccc} N = \mathbb{R}^{n-d} \times \{0\} & \overset{f}{\hookrightarrow} & M = \mathbb{R}^{n-d} \times \mathbb{R}^d \\ \downarrow & & \downarrow \\ Y = \mathbb{C}^{n-d} \times \{0\} & \overset{f}{\hookrightarrow} & X = \mathbb{C}^{n-d} \times \mathbb{C}^d \end{array}$$

and under these coordinates p_0 and Λ are written as

(3.2)
$$p_0 = (0; \sqrt{-1}\, dx_1) \in \Lambda = \{(z; \zeta);\ \zeta_{n-l+1} = \dots = \zeta_n = 0\}.$$

Moreover it follows that $l \geqslant d$.

By Remark 1.1 (1), we see that Y is non-characteristic for $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ on a neighborhood of N under the assumptions of Proposition 3.3 and $\mathrm{Ch}\,\mathscr{M} \cap \dot{T}^*X \subset \dot{\Lambda}$.

Throughout this section, we suppose that all the assumptions in Proposition 3.3 are satisfied. Therefore, we can use coordinates of (3.1) and (3.2). Set $\Lambda_1 := f_d(f_\pi^{-1}\Lambda) \subset T^*Y$. Note that $f_d\colon f_\pi^{-1}\Lambda \xrightarrow{\sim} \Lambda_1$.

3.4. Proposition. *Assume that $\mathfrak{M}$ has irregularities at most κ along $\dot{\Lambda}$, and $l = \mathrm{codim}^{\mathbb{C}}_{T^*X}\Lambda > d = \dim_{\mathbb{C}} Y$. Then $\mathfrak{M}_Y$ has irregularities at most κ along $\dot{\Lambda}_1$.*

§ 4. Estimate of Microsupport

Let $\mathscr{D}\!\mathscr{f}^*_M$ and $\mathscr{D}\!\mathscr{b}^*_M$ be the sheaves on M of *Gevrey* (*ultradifferentiable*) *functions* and of *ultradistributions* of class $*$, respectively. Here and in what follows, $*$ stands for $\{s\}$ with $1 < s < \infty$ or (s) with $1 < s \leqslant \infty$ to indicate the Gevrey growth order, and we understand that $\mathscr{D}\!\mathscr{f}^{(\infty)}_M := \mathcal{C}^\infty_M$ is the sheaf on M of functions of class C^∞. In particular, $\mathscr{D}\!\mathscr{b}^{(\infty)}_M = \mathscr{D}\!\mathscr{b}_M$. The sheaf $\mathscr{D}\!\mathscr{b}^{\{s\}}_M$ (resp. $\mathscr{D}\!\mathscr{b}^{(s)}_M$) is called the sheaf on M of *ultradistributions of Roumieu type* (resp. *of Beurling-Björck type*).

4.1. Remark. If $M = \mathbb{R}^n_x$, $\mathscr{D}\!\mathscr{f}^*_M$ and $\mathscr{D}\!\mathscr{b}^*_M$ are defined as follows: Let $U \subset M$ be an open set. For $u(x) \in \Gamma(U; \mathcal{C}^\infty_M)$, compact set $K \Subset U$ and $h > 0$, we set

$$\mathfrak{p}^{\{s\}}_{h,K}(u) := \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial_x{}^\alpha u(x)|}{h^{|\alpha|}|\alpha|!^s}\,.$$

Then $u(x) \in \Gamma(U; \mathscr{D}\!\mathscr{f}^{\{s\}}_M)$ (resp. $\Gamma(U; \mathscr{D}\!\mathscr{f}^{(s)}_M)$) if for any $K \Subset U$, there exist $h > 0$ such that (resp. for any $K \Subset U$ and $h > 0$) $\mathfrak{p}^{\{s\}}_{h,K}(u) < \infty$. By the system of semi-norms $\{\mathfrak{p}^{\{s\}}_{h,K}(\,\cdot\,)\}_{h\in\mathbb{R}_{>0},\,K\Subset U}$, we can endow each $\Gamma(U; \mathscr{D}\!\mathscr{f}^*_M)$ with a natural locally convex topology. Let $\mathscr{V}^*_M$ be the sheaf on M of volume elements with coefficients in $\mathscr{D}\!\mathscr{f}^*_M$. Then

$$\Gamma(U; \mathscr{D}\!\mathscr{b}^*_M) := \Gamma_{\mathrm{c}}(U; \mathscr{V}^*_M)'.$$

Here the prime means the strong dual of a topological vector space, and the subscript c means the sections with compact support. We refer to [18], [19] (and [20] in Japanese) for an exposition of $\mathscr{D}\!\mathscr{f}^*_M$ and $\mathscr{D}\!\mathscr{b}^*_M$ from the viewpoint of Algebraic Analysis.

As usual, we denote by $\mathscr{C}_M$ the sheaf of *Sato's microfunctions* on T^*_MX. Let $\mathscr{C}^{d,*}_M$ and $\mathscr{C}^*_M$ be subsheaves of $\mathscr{C}_M$ of microfunctions of regular class $*$ and of class $*$ respectively; that is,

$$\mathscr{C}^{d,*}_M := \mathrm{Image}(\mathrm{sp}_M\colon \pi_M^{-1}\mathscr{D}\!\mathscr{f}^*_M \to \mathscr{C}_M),$$

$$\mathscr{C}^*_M := \mathrm{Image}(\mathrm{sp}_M : \pi_M^{-1}\mathscr{D}\!\ell^*_M \to \mathscr{C}_M).$$

Here $\mathrm{sp}_M : \pi_M^{-1}\mathscr{B}_M \to \mathscr{C}_M$ is the *spectral morphism.* In particular, $\mathscr{C}^f_M := \mathscr{C}^{(\infty)}_M$ is the sheaf of temperate microfunctions, and $\mathscr{C}^d_M := \mathscr{C}^{d,(\infty)}_M$ is the sheaf of microfunctions of differentiable class (see [1], Bony [5]). Note that $\mathscr{C}^{d,*}_M$ and $\mathscr{C}^*_M$ are conically soft (even conically supple) on $\dot{T}^*_M X$; that is, the direct images of $\mathscr{C}^{d,*}_M|_{\dot{T}^*_M X}$ and $\mathscr{C}^*_M|_{\dot{T}^*_M X}$ on $\dot{T}^*_M X/\mathbb{R}_{>0}$ are soft (supple) sheaves. If $1 < s < t$, as subsheaves we have

$$\mathscr{C}^{d,(s)}_M \subset \mathscr{C}^{d,\{s\}}_M \subset \mathscr{C}^{d,(t)}_M \subset \mathscr{C}^{d,\{t\}}_M \subset \mathscr{C}^d_M \subset \mathscr{C}^f_M \subset \mathscr{C}^{\{t\}}_M \subset \mathscr{C}^{(t)}_M \subset \mathscr{C}^{\{s\}}_M \subset \mathscr{C}^{(s)}_M \subset \mathscr{C}_M .$$

Let $\mathscr{A}_M := \mathscr{O}_X|_M$ be the sheaf of *real analytic functions* on M. Then, we have

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{A}_M & \to & \mathscr{D}\!\!\!/^*_M & \to & \dot{\pi}_{M*}\mathscr{C}^{d,*}_M & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & \mathscr{A}_M & \to & \mathscr{D}\!\ell^*_M & \to & \dot{\pi}_{M*}\mathscr{C}^*_M & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & \mathscr{A}_M & \to & \mathscr{B}_M & \to & \dot{\pi}_{M*}\mathscr{C}_M & \to & 0 \end{array}$$

and $R^\nu\dot{\pi}_{M*}\mathscr{C}^{d,*}_M = R^\nu\dot{\pi}_{M*}\mathscr{C}^*_M = 0$ $(\nu \neq 0)$. For the fundamental properties of $\mathscr{C}^{d,*}_M$ and $\mathscr{C}^*_M$, see for example [25] and [21].

From now on we set

$$r_\kappa := \begin{cases} \dfrac{\kappa}{\kappa - 1} \in \mathbb{Q} & (\kappa > 1), \\ \infty & (\kappa = 1). \end{cases}$$

Let $\varpi : T_{T^*_M X}T^*X \simeq T^*T_M X \to T^*_M X$ be the canonical projection. Then the following theorem is fundamental in this paper:

4.2. Theorem. *Let $\Lambda \subset T^*X$ be as in Conditions* 3.1 *and* 3.2, *and $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Suppose that $\mathscr{M}$ has irregularities at most κ along $\dot{\Lambda}$. Then*

$$\mathrm{SS}\bigl(\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^{d,*}_M)\bigr) \cap \varpi^{-1}(\dot{T}^*_M X) \subset C_{\dot{T}^*_M X}(\Lambda),$$

$$\mathrm{SS}\bigl(\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^*_M)\bigr) \cap \varpi^{-1}(\dot{T}^*_M X) \subset C_{\dot{T}^*_M X}(\Lambda),$$

provided that $* = \{s\}$ with $1 \leqslant s < r_\kappa$ or $* = (s)$ with $1 < s \leqslant r_\kappa$.

For the proof, we use

(a) a real quantized contact transformation (see Proposition 3.3),
(b) a result due to Aoki [2].

4.3. Corollary. *Let $\Lambda \subset T^*X$ be as in Conditions* 3.1 *and* 3.2, *and $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Suppose that $\mathscr{M}$ has irregularities at most κ along $\dot{\Lambda}$. Then*

$$\mathrm{SS}(\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\!\!\!/^*_M)) \subset T^*M \cap C_{T^*_M X}(\dot{\Lambda} \cup \mathrm{supp}\,\mathscr{M}),$$

$$\mathrm{SS}(\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\!\ell^*_M)) \subset T^*M \cap C_{T^*_M X}(\dot{\Lambda} \cup \mathrm{supp}\,\mathscr{M}),$$

provided that $* = \{s\}$ with $1 \leqslant s < r_\kappa$ or $* = (s)$ with $1 < s \leqslant r_\kappa$.

§ 5. Cauchy and Boundary Value Problems

As in § 1, let N be a d-codimensional real analytic closed submanifold of M, Y a complexification of N in X, and $f\colon N \hookrightarrow M$ (or $f\colon Y \hookrightarrow X$) the embedding. Under the non-characteristic condition, we can construct restriction morphisms

$$\boldsymbol{R}f_{d*}\, f_\pi^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M^{d,*}) \to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{C}_N^{d,*}),$$
$$\boldsymbol{R}f_{d*}\, f_\pi^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M^{*}) \to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{C}_N^{*}).$$

under the non-characteristic condition. By virtue of these morphisms and the Cauchy-Kovalevskaja-Kashiwara theorem, we obtain:

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M)|_N & \xrightarrow{\ \sim\ } & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{A}_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\!\mathscr{f}_M^{*})|_N & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}\!\mathscr{f}_N^{*}) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \boldsymbol{R}\dot{\pi}_{M*}\mathscr{C}_M^{d,*})|_N & \to & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \boldsymbol{R}\dot{\pi}_{N*}\mathscr{C}_N^{d,*}) \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1}
\end{array}$$

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M)|_N & \xrightarrow{\ \sim\ } & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{A}_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\!\mathscr{b}_M^{*})|_N & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}\!\mathscr{b}_N^{*}) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \boldsymbol{R}\dot{\pi}_{M*}\mathscr{C}_M^{*})|_N & \to & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \boldsymbol{R}\dot{\pi}_{N*}\mathscr{C}_N^{*}) \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1}
\end{array}$$

5.1. Theorem (Cauchy Problem)**.** *Let* $\Lambda \subset T^*X$ *be as in Conditions* 3.1 *and* 3.2*, and* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$*. Suppose that* $\mathscr{M}$ *has irregularities at most* κ *along* $\dot{\Lambda}$*, and* $f\colon N \hookrightarrow M$ *is hyperbolic for* Λ*. Let* $* = \{s\}$ *with* $1 \leqslant s < r_\kappa$ *or* $* = (s)$ *with* $1 < s \leqslant r_\kappa$ *. Then*:

$$\boldsymbol{R}f_{d*}\, f_\pi^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M^{d,*}) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{C}_N^{d,*}),$$
$$\boldsymbol{R}f_{d*}\, f_\pi^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M^{*}) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{C}_N^{*}).$$

In particular,

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\!\mathscr{f}_M^{*})|_N \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}\!\mathscr{f}_N^{*}),$$
$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\!\mathscr{b}_M^{*})|_N \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}\!\mathscr{b}_N^{*}).$$

5.2. Remark. In distribution case ($\kappa = 1$ and $* = (\infty)$), Theorem 5.1 is proved in [7] (see § 6), and in hyperfunction case, we have only to assume the hyperbolicity condition ([15], [16]). Moreover, we refer to [4] for the propagation of Gevrey singularities.

We denote by $\tau_N : T_N M \to N$ *the normal bundle to* N *in* M. Set for short

$$\tau := f \circ \tau_N : T_N M \to M.$$

Let $\nu_N(*)$ and $\mu_N(*)$ be the *specialization and microlocalization functors along* N respectively. Then, we can obtain

5.3. Theorem. *Let $\Lambda \subset T^*X$ be as in Conditions* 3.1 *and* 3.2, *and $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Suppose that $\mathscr{M}$ has irregularities at most κ along $\dot{\Lambda}$ and $f : N \hookrightarrow M$ is hyperbolic for Λ. Let $* = \{s\}$ with $1 \leqslant s < r_\kappa$ or $* = (s)$ with $1 < s \leqslant r_\kappa$. Then*:

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_N(\mathscr{D}f^*_M))\big|_{\dot{T}^*_N M} = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_N(\mathscr{D}b^*_M))\big|_{\dot{T}^*_N M} = 0,$$

and

$$\begin{aligned} \tau^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}f^*_M) &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{D}f^*_M)), \\ \tau^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}b^*_M) &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{D}b^*_M)). \end{aligned} \tag{5.1}$$

By Corollary 4.3 and (5.1), we obtain:

5.4. Theorem (Boundary Value Problem). *Let $\Lambda \subset T^*X$ be as in Conditions* 3.1 *and* 3.2, *and $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Suppose that $\mathscr{M}$ has irregularities at most κ along $\dot{\Lambda}$, and $f : N \hookrightarrow M$ is hyperbolic for Λ. Then it follows that*:

$$\begin{aligned} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{D}f^*_M)) &\simeq \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}f^*_N), \\ \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{D}b^*_M)) &\simeq \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}b^*_N), \end{aligned}$$

provided that $* = \{s\}$ with $1 \leqslant s < r_\kappa$ or $* = (s)$ with $1 < s \leqslant r_\kappa$.

§ 6. Remark on Regular-Singular Case

We inherit the notation form the preceding section. In this section, we treat the case where $\kappa = 1$ and $* = (\infty)$ under Condition 3.1 only. By Kashiwara-Monteiro Fernandes-Schapira [13], we obtain

6.1. Theorem. (1) *Let U be a conic open subset of $\dot{T}^*X$ and Λ a closed $\mathbb{C}^\times$-conic regular involutory submanifold of U. Let $\mathfrak{M} \in \mathfrak{Mod}_{\mathrm{coh}}(\mathscr{E}_X)$ on U. Suppose that $\mathfrak{M}$ has irregularities at most 1 along Λ. Then*

$$\mathrm{SS}(\boldsymbol{R}\mathscr{H}om_{\mathscr{E}_X}(\mathfrak{M}, \mathscr{C}^f_M)) \cap \varpi^{-1}(U) \subset C_{T^*_M X}(\Lambda)$$

(*recall that $\varpi\colon T_{T^*_M X}T^*X \simeq T^*T_M X \to T^*_M X$ is the canonical projection*).

(2) *Let $\Lambda \subset T^*X$ be as in Condition 3.1, and $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Suppose that $\mathscr{M}$ has irregularities at most 1 along $\dot{\Lambda}$. Then*

$$\mathrm{SS}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\ell_M)) \subset T^*M \cap C_{T^*_M X}(\dot{\Lambda} \cup \operatorname{supp} \mathscr{M}).$$

Let $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)$ be the bounded derived category *of* $\mathbb{R}$-*constructible sheaves* on M. We denote by

$$T\text{-}\nu_N \mathscr{T}hom(*, \mathscr{D}\ell_M) = T\text{-}\nu_N TH_M(*)\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)^\circ \to \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_N M)$$

the *temperate specialization functor of distribution along N* due to Andronikof [1]. Here, $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)^\circ$ is the dual of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(M)$, and $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_N M)$ is the subcategory of the bounded derived category $\mathbf{D}^{\mathrm{b}}(T_N M)$ of sheaves such that each cohomology is conic. In particular, we set:

$$\nu^t_N(\mathscr{D}\ell_M) := T\text{-}\nu_N \mathscr{T}hom(\mathbb{C}_M, \mathscr{D}\ell_M),$$

and denote by $\mu^t_N(\mathscr{D}\ell_M)$ the Fourier-Sato transform of $\nu^t_N(\mathscr{D}\ell_M)$. Then, by Theorem 6.1, we can obtain:

6.2. Theorem. *Let $\Lambda \subset T^*X$ be as in Condition 3.1, and $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Suppose that $\mathscr{M}$ has irregularities at most 1 along $\dot{\Lambda}$, and $f\colon N \hookrightarrow M$ is hyperbolic for Λ. Then*:

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_N(\mathscr{D}\ell_M))|_{\dot{T}^*_N M} = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu^t_N(\mathscr{D}\ell_M))|_{\dot{T}^*_N M} = 0.$$

and there exist the following isomorphisms:

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\ell_M)|_N & \xrightarrow{\ \sim\ } & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{D}\ell_M)) \otimes \omega^{\otimes -1}_{N/M} \\
\downarrow\wr & & \downarrow\wr \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}\ell_N) & \xrightarrow{\ \sim\ } & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \boldsymbol{R}\Gamma_N(\mathscr{D}\ell_M)) \otimes \omega^{\otimes -1}_{N/M},
\end{array}$$

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu^t_N(\mathscr{D}\ell_M)) & \xrightarrow{\ \sim\ } & \tau_N^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{D}\ell_N) \\
\downarrow\wr & & \downarrow\wr \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{D}\ell_M)) & \xrightarrow{\ \sim\ } & \tau_N^{-1}\boldsymbol{R}\Gamma_N \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}\ell_M) \otimes \omega^{\otimes -1}_{N/M}.
\end{array}$$

Here $\omega^{\otimes -1}_{N/M}$ is the dual of the relative dualizing complex $\omega_{N/M}$.

6.3. Remark. (1) We can prove same results as in Theorem 6.2 replacing $\mathscr{D}\mathscr{b}_M$ by $\mathcal{C}^\infty_M$ ([31]).

(2) If N is a hypersurface, each section of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu^t_N(\mathscr{D}\mathscr{b}_M))|_{\dot{T}_N M}$ is an extensible mild distribution in the sense of Kataoka [17].

References

[1] Andronikof, A., Microlocalisation Tempérée, *Mém. Soc. Math. Fr. (N.S.)* **57**, 1994.

[2] Aoki, T, Growth order of microdifferential operators of infinite order, *J. Fac. Univ. Tokyo IA Math.* **29** (1982), 143–159.

[3] ———, An invariant measuring the irregularity of a differential operator and a microdifferential operator, *J. Math. Pures Appl.* **61** (1982), 131–148.

[4] Arisumi, T., Propagation of Gevrey singularities for a class of microdifferential operators, *Microlocal Analysis and its Applications*, *RIMS Kôkyûroku* **750**, 1991, pp. 24–37.

[5] Bony, J.-M., Propagation des singularités différentiables pour une classe d'opérateurs différentiels à coefficients analytiques, *Équ. Dériv. Partielles de Rennes* (1975), *Astérisque* **34**–**35**, 1976, pp. 43–91.

[6] Colin, V., Formal microlocalization, *C. R. Acad. Sci. Paris Sér. I Math.* **327** (1998), 289–293.

[7] D'Agnolo, A. and Tonin, F., Cauchy problem for hyperbolic $\mathcal{D}$-modules, *Pacific J. Math.* **184** (1998), 1–22.

[8] Honda, N., On the reconstruction theorem of holonomic modules in the Gevrey classes, *Publ. Res. Inst. Math. Sci.* **27** (1991), 923–943.

[9] Honda, N. and Morando, G., Stratified Whitney jets and tempered ultradistributions on the subanalytic site, *Bull. Soc. Math. France* **139** (2011), 389–435.

[10] Hörmander, L, Between distributions and hyperfunctions, *Colloque en L'honnur de Laurent Schwartz,* Vol. 1, *Astérisque* **131**, 1985, pp. 89–106.

[11] Kashiwara, M., *General Theory of Algebraic Analysis*, Iwanami Shoten, Tokyo, 2000 (Japanese); English translation; *D-modules and Microlocal Calculus*, AMS, 2003.

[12] Kashiwara, M. and Kawai, T., On holonomic systems of microdifferential equations. III. Systems with regular singularities, *Publ. Res. Inst. Math. Sci.* **17** (1981), 813–979.

[13] Kashiwara, M., Monteiro Fernandes, T. and Schapira, P., Micro-support and Cauchy problem for temperate solutions of regular $\mathcal{D}$-modules, *Portugal. Math.* **58** (2001), 485–504.

[14] Kashiwara, M. and Oshima, T., Systems of differential equations with regular singularities and their boundary value problems, *Ann. Math.* (2) **106** (1977), 145–200.

[15] Kashiwara, M. and Schapira, P., Micro-hyperbolic systems, *Acta Math.* **142** (1979), 1–55.

[16] ———, *Sheaves on Manifolds*, Grundlehren Math. Wiss. **292**, Springer, Berlin–Heidelberg–New York, 1990.

[17] Kataoka, K., Some properties of mild hyperfunctions and an application of microanalyticity in boundary value problems, *J. Fac. Sci. Univ. Tokyo Sect. IA* **30** (1983), 279–296.

[18] Komatsu, H., Ultradistributions, I, Structure theorems and a characterization, *J. Fac. Sci. Univ. Tokyo Sect. IA* **20** (1973), 25–105.

[19] ———, Ultradistributions, II, Kernel theorem and ultradistributions with support in a submanifold, *J. Fac. Sci. Univ. Tokyo Sect. IA* **24** (1977), 607–628.

[20] ———, *An Introduction to Theory of Generalized Functions*, Iwanami, Tokyo, 1978 (Japanese).

[21] ———, Microlocal analysis in Gevrey classes and in Complex domains, *Microlocal Analysis and Applications, Proceedings Montecatini Terme*, 1989 (L. Cattabriga and L. Rodino, eds.), Lecture Notes in Math. **1495**, Springer, Berlin–Heidelberg–New York, 1991, pp. 161–236.

[22] Komatsu, H. and Kawai, T., Boundary values of hyperfunction solutions of linear partial differential equations, *Publ. Res. Inst. Math. Sci.* **7** (1971), 95–104.

[23] Oaku, T. and Yamazaki, S., Higher-codimensional boundary value problems and F-mild microfunctions, *Publ. Res. Inst. Math. Sci.* **34** (1998), 383–437.

[24] Prelli, L., Specialization and microlocalization of subanalytic sheaves, to appear in *Mém. Soc. Math. Fr. (N.S.)*.

[25] de Roever, J. W., Hyperfunctional singular support of ultradistributions, *J. Fac. Sci. Univ. Tokyo Sect. IA* **31** (1984), 585–631.

[26] Sato, M., Kawai, T. and Kashiwara, M., Microfunctions and pseudo-differential equations, *Hyperfunctions and Pseudo-Differential Equations, Proc. Conf. Katata* 1971*, dedicated to the memory of A. Martineau* (H. Komatsu, ed.), Lecture Notes in Math. **287**, Springer, Berlin–Heidelberg–New York, 1973, pp. 265–529.

[27] Schapira, P., Problème de Dirichlet et solutions hyperfonctions des équations elliptiques, *Boll. Un. Mat. Ital.*

[28] ———, *Microdifferential Systems in the Complex Domain*, Grundlehren Math. Wiss. **269**, Springer, Berlin–Heidelberg–New York, 1985.

[29] Uchikoshi, K., Irregularities of microhyperbolic operators, *J. Math. Soc. Japan* **58** (2006), 453–484.

[30] Yamazaki, S., Microsupports of Gevrey class or ultradistribution solutions to systems with irregularities and their applications, in preparation.

[31] ———, Microsupport of Whitney function solutions to systems with regular singularities and its applications, in preparation.