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Phase space Feynman path integrals via piecewise bicharacteristic paths and their semiclassical approximations

By

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Abstract

This paper is a rough survey of our paper [23]. Since the RIMS Kôkyûroku Bessatsu gives us a chance to introduce the ideas which are meaningful but are not suited for publication in ordinary journal, we introduce some ideas and some calculations of [23] using some figures.

§ 1. Introduction to phase space Feynman path integral

Let $U(T,0)$ be the fundamental solution for the Schrödinger equation

$$(1.1)\qquad (i\partial_T - \frac{1}{\hbar}H(T,x,\frac{\hbar}{i}\partial_x))U(T,0) = 0\,, \quad U(0,0) = I\,,$$

where $T > 0$, $x \in \mathbf{R}^d$ and $\hbar$ is the Planck parameter with $0 < \hbar < 1$. The Hamiltonian $H(T,x,\frac{\hbar}{i}\partial_x)$ can be written as a pseudo-differential operator:

$$(1.2)\qquad \begin{aligned} H(T,x,\frac{\hbar}{i}\partial_x)v(x) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} e^{ix\cdot\xi_0} H(T,x,\hbar\xi_0)\widehat{v}(\xi_0)d\xi_0 \\ &= \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} H(T,x,\xi_0)v(x_0)dx_0 d\xi_0\,. \end{aligned}$$

One may ask whether we can use the Fourier integral operator

$$(1.3)\qquad I(T,0)v(x) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0 - \frac{i}{\hbar}\int_0^T H(t,x,\xi_0)dt} v(x_0)dx_0 d\xi_0$$

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as an approximation of $U(T,0)v(x)$. In fact, we have the following:

Let $\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0$ be a division of the interval $[0,T]$. Set $t_j = T_j - T_{j-1}$ for $j = 1,2,\ldots,J,J+1$ and $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j$. Then, under a suitable condition (cf. [21]), we have

$$\begin{aligned}(1.4)\quad U(T,0)v(x) &= \lim_{|\Delta_{T,0}|\to 0} I(T,T_J)I(T_J,T_{J-1})\cdots I(T_2,T_1)I(T_1,0)v(x)\\ &= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{d(J+1)}\int_{\mathbf{R}^{2d(J+1)}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt}\\ &\quad \times v(x_0)\prod_{j=0}^{J} dx_j d\xi_j\end{aligned}$$

with $x = x_{J+1}$. In other words, if we consider the function $U(T,0,x,\xi_0)$ satisfying

$$U(T,0)v(x) = \left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)v(x_0)dx_0d\xi_0\,,$$

then we can write

$$\begin{aligned}(1.5)\, & e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)\\ &= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt}\prod_{j=1}^{J} dx_j d\xi_j\,.\end{aligned}$$

Now we introduce the position path $q(t)$ and the momentum path $p(t)$ with $q(T_j) = x_j$ for $j = 0,1,\ldots,J+1$ and $p(T_j) = \xi_j$ for $j = 0,1,\ldots,J$. Using the phase space path integral introduced by R. P. Feynman [9, Appendix B], we formally write

$$(1.6)\qquad e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) = \int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]\,.$$

Here $(q,p) : [0,T] \to \mathbf{R}^d \times \mathbf{R}^d$ are the paths with $q(0) = x_0$, $q(T) = x$ and $p(0) = \xi_0$ in the phase space, $\phi[q,p]$ is the action of Hamiltonian type defined by

$$(1.7)\qquad \phi[q,p] = \int_{[0,T)} p(t)\cdot dq(t) - \int_{[0,T)} H(t,q(t),p(t))dt\,,$$

and the phase space path integral $\int \sim \mathcal{D}[q,p]$ is a sum over all the paths (q,p) (see Fig. 1). The expression (1.5) of the phase space path integral (1.6) is called the time slicing approximation. However, in the sense of mathematics, the measure $\mathcal{D}[q,p]$ which weighs all the paths (q,p) equally, does not exist (cf. I. M. Gel'fand-N. Y. Vilenkin [14, Theorem 4, p. 359]). Furthermore, in the sense of quantum physics, we can not have the position $q(t)$ and the momentum $p(t)$ at the same time t.

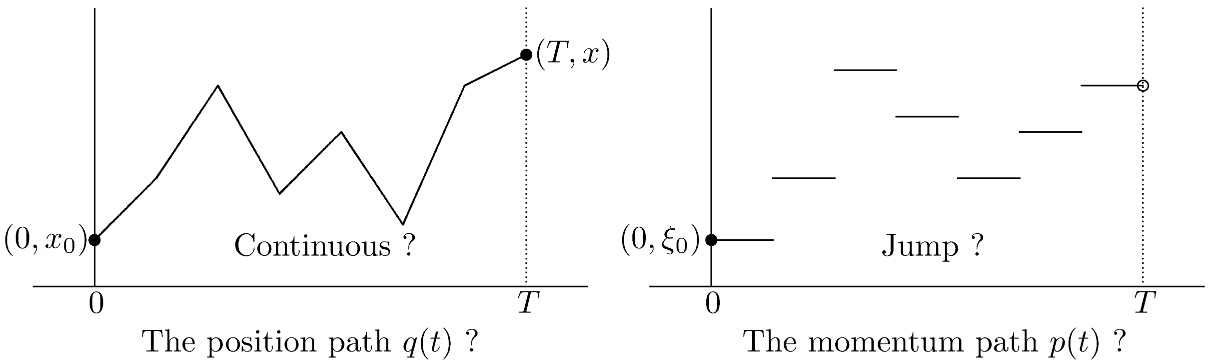


Figure 1.

In [23], when the time interval $[0,T]$ is small, using piecewise bicharacteristic paths, we proved the existence of the phase space Feynman path integrals

$$\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p] \tag{1.8}$$

with general functional $F[q,p]$ as integrand. More precisely, we gave a fairly general class $\mathcal{F}$ of functionals $F[q,p]$ such that for any $F[q,p]\in\mathcal{F}$, the time slicing approximation converges uniformly on compact subsets with respect to $(x,\xi_0,x_0)\in\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R}^d$.

Remark. The size of the time interval $[0,T]$ depends only on the dimension d and the constant κ_2 of Assumption 1. As an appendix, in §9.1, we will give an example which converges uniformly on compact subsets with respect to (x,ξ_0,x_0) when T is small. This example implies that Theorems 5 and 4 are not valid when T is large. In §9.2, we will show the convergence in the sense of operator when T is large and $F[q,p]\equiv 1$.

Remark. For the phase space path integral (1.6) via Fourier integral operators, see H. Kumano-go-H. Kitada [19], N. Kumano-go [21] and W. Ichinose [16]. We regard (1.6) as the particular case of (1.8) with $F[q,p]\equiv 1$. Using the piecewise linear paths $q(t)$ and the piecewise constant paths $p(t)$, W. Ichinose [16] gave some functionals $F[q,p]=\prod_{k=1}^{K}B_k(q(\tau_k),p(\tau_k))$, $0<\tau_1<\cdots<\tau_k<T$ of cylinder type which do not converge as an operator. Note that we will exclude $F[q,p]=B(t,q(t),p(t))$ at the time t from our class $\mathcal{F}$.

Remark. As we will see in §4, piecewise bicharacteristic paths naturally lead us to the semiclassical approximation of Hamiltonian type. Our use of jumps at $t=T_j$ was inspired by C. Garrod [12], L. S. Schulman [26, Chapter 31] and J. C. Zambrini [5, Part 2].

Remark. The phase space path integrals via Fourier integral operators are also used in other equations (cf. J. Le Rousseau [17], N. Kumano-go [22]).

Since Feynman [9], the phase space path integral has been rediscovered many times (cf. W. Tobocman [27], H. Davies [7], C. Garrod [12]) and various formulations have also been developed. C. DeWitt-Morette-A. Maheshwari-B. Nelson [8] (cf. [4, Chapter 3]) and M. M. Mizrahi [24] introduced the formulation without limiting procedure. K. Gawedzki [13] used the techniques analogous to those used by Ito in the configuration path integral. I. Daubechies-J. R. Klauder [6] presented the phase space path integral via analytic continuation from Wiener measure. S. Albeverio-G. Guatteri-S. Mazzucchi [2] (cf. [1, Chapter 10]) realized the phase space path integral as an infinite dimensional oscillatory integral. O. G. Smolyanov-A. G. Tokarev-A. Truman [28] formulated the phase space path integral via Chernoff formula. G. W. Johnson-M. Lapidus [18] and T. L. Gill–W. W. Zachary[15] developed Feynman's operational calculus of the main part of [9].

§ 2. Existence of phase space path integrals

In this section, we explain our result about the existence of the phase space path integrals (1.8) step by step.

§ 2.1. Assumption of the Hamilton function

Our assumption of the Hamilton function $H(t,x,\xi)$ of (1.1) are the following.

Assumption 1. *$H(t,x,\xi)$ is a real-valued function of $(t,x,\xi)\in\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d$, and for any multi-indices α, β, $\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)$ is continuous. For any non-negative integer k, there exists a positive constant κ_k such that*

$$|\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)|\le\kappa_k(1+|x|+|\xi|)^{\max(2-|\alpha+\beta|,0)} \tag{2.1}$$

for any multi-indices α, β with $|\alpha+\beta|=k$.

The typical examples of the Hamiltonian $H(t,x,\frac{\hbar}{i}\partial_x)$ of (1.1) are the following.

Example 1.

$$\begin{aligned}H(t,x,\frac{\hbar}{i}\partial_x)=&\sum_{j,k=1}^{d}(a_{j,k}(t)\frac{\hbar}{i}\partial_{x_j}\frac{\hbar}{i}\partial_{x_k}+b_{j,k}(t)x_j\frac{\hbar}{i}\partial_{x_k}+c_{j,k}(t)x_jx_k)\\&+\sum_{j=1}^{d}(a_j(t)\frac{\hbar}{i}\partial_{x_j}+b_j(t)x_j)+c(t,x)\,.\end{aligned}$$

Here $a_{j,k}(t)$, $b_{j,k}(t)$, $c_{j,k}(t)$, $a_j(t)$, $b_j(t)$ and $\partial_x^\alpha c(t,x)$ are real-valued continuous bounded functions.

§ 2.2. We can produce many $F[q,p] \in \mathcal{F}$

Typical examples of the functionals $F[q,p]$ in our class $\mathcal{F}$ are the following.

Example 2.

(1) *Let $m \geq 0$. Let $B(t,x)$ be a function of $(t,x) \in \mathbf{R}\times\mathbf{R}^d$ such that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous and satisfies $|\partial_x^\alpha B(t,x)| \leq C_\alpha(1+|x|)^m$ with a positive constant C_α. Then, the value at time t, $0 \leq t \leq T$,*

$$F[q,p] = B(t, q(t)) \in \mathcal{F}\,.$$

In particular, if $F[q,p] \equiv 1$, then $F[q,p] \in \mathcal{F}$.

(2) *Let $m \geq 0$ and $0 \leq T' \leq T'' \leq T$. Let $B(t,x,\xi)$ be a function of $(t,x,\xi) \in \mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d$ such that for any multi-indices α, β, $\partial_x^\alpha\partial_\xi^\beta B(t,x,\xi)$ is continuous and satisfies $|\partial_x^\alpha\partial_\xi^\beta B(t,x,\xi)| \leq C_{\alpha,\beta}(1+|x|+|\xi|)^m$ with a positive constant $C_{\alpha,\beta}$. Then*

$$F[q,p] = \int_{[T',T'')} B(t,q(t),p(t))dt \in \mathcal{F}\,.$$

Furthermore, if $m = 0$, then

$$F[q,p] = e^{\int_{[T',T'')} B(t,q(t),p(t))dt} \in \mathcal{F}\,.$$

We will define the class $\mathcal{F}$ in Definition 1 of §8. Because, even if we do not state the definition of $\mathcal{F}$ here, we can produce many functionals $F[q,p] \in \mathcal{F}$, applying Theorem 1 to Example 2.

Theorem 1 (Algebra). *If $F[q,p] \in \mathcal{F}$ and $G[q,p] \in \mathcal{F}$, then $F[q,p] + G[q,p] \in \mathcal{F}$ and $F[q,p]G[q,p] \in \mathcal{F}$.*

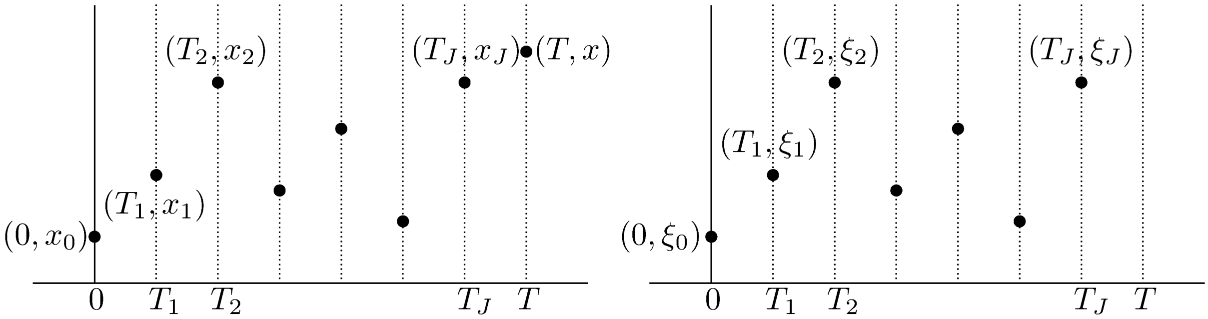
§ 2.3. Time slicing approximation

Our approach via piecewise bicharacteristic paths is a little different from known approaches. Therefore, in order to explain piecewise bicharacteristic paths, we begin with the time slicing approximation again.

Let $\Delta_{T,0} = (T_{J+1}, T_J, \ldots, T_1, T_0)$ be any division of the interval $[0,T]$, i.e.,

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0\,. \tag{2.2}$$

Let $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1\leq j\leq J+1} t_j$. Set $x_{J+1} = x$. Let $x_j \in \mathbf{R}^d$ and $\xi_j \in \mathbf{R}^d$ for $j = 1, 2, \ldots, J$ (see Fig. 2).



The time slicing on (t, q)-space The time slicing on (t, p)-space

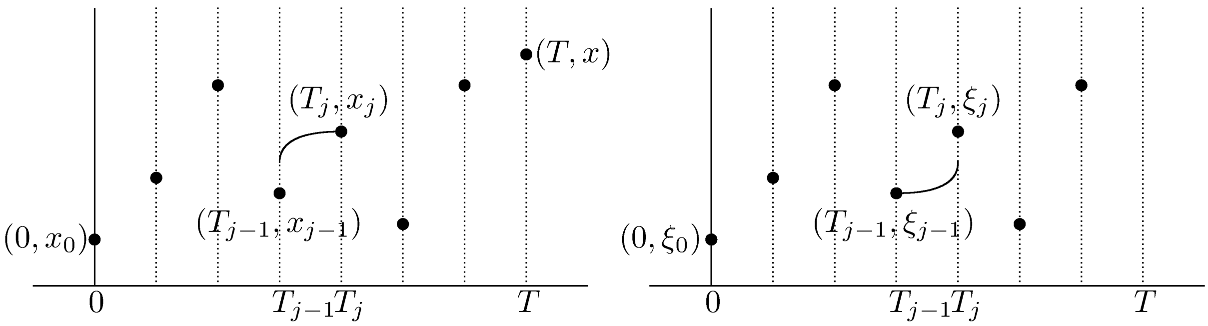
Figure 2.

§ 2.4. Bicharacteristic paths

Let $\kappa_2 d(T_j - T_{j-1}) < 1/2$. Then we can define the bicharacteristic paths $\bar{q}_{T_j,T_{j-1}} = \bar{q}_{T_j,T_{j-1}}(t, x_j, \xi_{j-1})$ and $\bar{p}_{T_j,T_{j-1}} = \bar{p}_{T_j,T_{j-1}}(t, x_j, \xi_{j-1})$, $T_{j-1} \le t \le T_j$ by the Hamilton canonical equation

$$
\begin{aligned}
&\partial_t \bar{q}_{T_j,T_{j-1}}(t) = (\partial_\xi H)(t, \bar{q}_{T_j,T_{j-1}}, \bar{p}_{T_j,T_{j-1}}),\\
&\partial_t \bar{p}_{T_j,T_{j-1}}(t) = -(\partial_x H)(t, \bar{q}_{T_j,T_{j-1}}, \bar{p}_{T_j,T_{j-1}}), \quad T_{j-1} \le t \le T_j,\\
&\bar{q}_{T_j,T_{j-1}}(T_j) = x_j, \quad \bar{p}_{T_j,T_{j-1}}(T_{j-1}) = \xi_{j-1}\,.
\end{aligned} \tag{2.3}
$$

Note that $\bar{q}_{T_j,T_{j-1}}(T_{j-1})$ and $\bar{p}_{T_j,T_{j-1}}(T_j)$ are independent of x_{j-1} and ξ_j (see Fig. 3).



The bicharacteristic path $\bar{q}_{T_j,T_{j-1}}$ The bicharacteristic path $\bar{p}_{T_j,T_{j-1}}$

Figure 3.

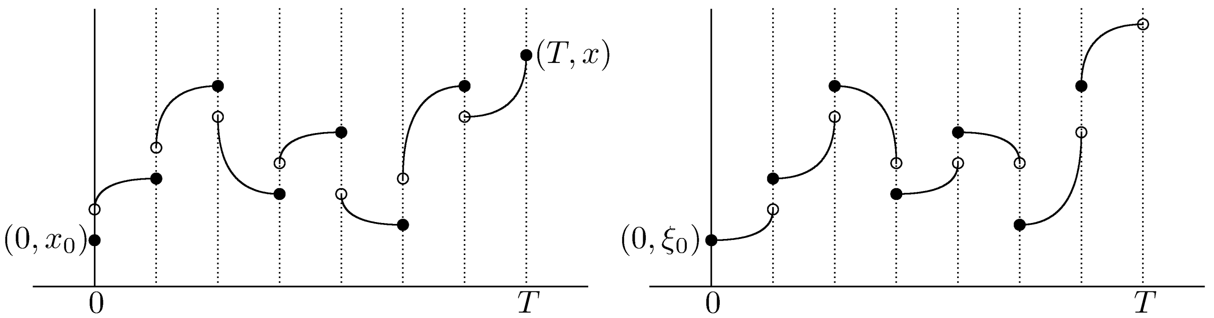
§ 2.5. Piecewise bicharacteristic paths

Using the bicharacteristic paths $\bar{q}_{T_j,T_{j-1}}$ and $\bar{p}_{T_j,T_{j-1}}$ of (2.3), we define the piecewise bicharacteristic paths $q_{\Delta_{T,0}} = q_{\Delta_{T,0}}(t, x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)$ and

$p_{\Delta_{T,0}} = p_{\Delta_{T,0}}(t, x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0)$ by

$$q_{\Delta_{T,0}}(t) = \bar{q}_{T_j,T_{j-1}}(t, x_j, \xi_{j-1}), \quad T_{j-1} < t \le T_j, \quad q_{\Delta_{T,0}}(0) = x_0, \tag{2.4}$$
$$p_{\Delta_{T,0}}(t) = \bar{p}_{T_j,T_{j-1}}(t, x_j, \xi_{j-1}), \quad T_{j-1} \le t < T_j$$

for $j = 1, 2, \dots, J, J+1$ (see Fig. 4).



The piecewise bicharacteristic path $q_{\Delta_{T,0}}$ The piecewise bicharacteristic path $p_{\Delta_{T,0}}$

Figure 4.

Then the functionals $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$, $F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ become functions, i.e.,

$$\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] = \phi_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0)\,, \tag{2.5}$$

$$F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] = F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0)\,. \tag{2.6}$$

§ 2.6. Phase space Feynman path integrals exist

Our result about the existence of phase space Feynman path integrals is the following.

Theorem 2. *Let T be sufficiently small. Then, for any $F[q,p] \in \mathcal{F}$,*

$$\int e^{\frac{i}{\hbar}\phi[q,p]} F[q,p]\mathcal{D}[q,p] \tag{2.7}$$
$$\equiv \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]} F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j d\xi_j$$

converges uniformly on compact sets of $\mathbf{R}^{3d}$ with respect to (x, ξ_0, x_0), i.e., (2.7) is well-defined.

Remark. There are two hurdles if we try to treat (2.7) mathematically. The first hurdle is that even when $F[q,p] \equiv 1$, each integral of the right-hand side of (2.7) does not converge absolutely, i.e.,

$$\int_{\mathbf{R}^{2d}} 1 dx_j d\xi_j = \infty\,.$$

In order to get over the first hurdle, we treat integrals of this type as oscillatory integrals. The second hurdle is that if $|\Delta_{T,0}| \to \infty$, the number J of integrals of the right-hand side of (2.7) tends to ∞, i.e.,

$$\infty \times \infty \times \infty \times \infty \times \cdots\cdots\cdots .$$

In order to get over the second hurdle, we treat the multiple integral of (2.7) directly to keep the functionals $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$, $F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ in the multiple integral.

Remark. In the case $F[q,p] \equiv 1$, one can regard the right-hand side of (1.4) as composition of many operators in $L^2(\mathbf{R^d})$. It is possible to discuss whether the composed operator converges or not as $|\Delta| \to 0$. This approach is very powerful. However it treats the integrals one by one as an operator and its convergence does not seem to distinguish between the position x_0 and the momentum ξ_0. On the other hand, (2.7) converges with respect to $q(T) = x$ and $p(0) = \xi_0$. When $F[q,p] \equiv 1$, note that $U(T, 0, x, \xi_0)$ of (1.6) is independent of x_0.

We will explain the outline of the proof of Theorems 1 and 2 in the later sections §5-§8. In the next two sections §3 and §4, we explain some applications.

§ 3. A Fubini-type theorem

As a merit to treat the phase space path integral (1.8) with general functional $F[q,p]$ as integrand, we state the perturbation expansion formula.

Theorem 3. *Let T be sufficiently small. Let $m \geq 0$ and $0 \leq T' \leq T'' \leq T$. Assume that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous on $[T', T''] \times \mathbf{R}^d$ and there exists a positive constant C_α such that $|\partial_x^\alpha B(t,x)| \leq C_\alpha(1+|x|)^m$. Then, for any functional $F[q,p] \in \mathcal{F}$ including $F[q,p] \equiv 1$, we have*

$$\int e^{\frac{i}{\hbar}\phi[q,p]} \left(\int_{[T',T'')} B(t,q(t))dt \right) F[q,p]\mathcal{D}[q,p] \tag{3.1}$$
$$= \int_{[T',T'')} \left(\int e^{\frac{i}{\hbar}\phi[q,p]} B(t,q(t))F[q,p]\mathcal{D}[q,p] \right) dt \,.$$

Remark. In (3.1), we do not treat $B(t,q(t),p(t))$ at the time t because of the uncertain principle.

Remark (Perturbation expansion formula). If $|\partial_x^\alpha B(t,x)| \leq C_\alpha$, we have

$$\begin{aligned}&\int e^{\frac{i}{\hbar}\phi[q,p]+\frac{i}{\hbar}\int_{[0,T)} B(\tau,q(\tau))d\tau}\mathcal{D}[q,p]\\&=\sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n\int_{[0,T)}d\tau_n\int_{[0,\tau_n)}d\tau_{n-1}\cdots\int_{[0,\tau_2)}d\tau_1\\&\times\int e^{\frac{i}{\hbar}\phi[q,p]}B(\tau_n,q(\tau_n))B(\tau_{n-1},q(\tau_{n-1}))\cdots B(\tau_1,q(\tau_1))\mathcal{D}[q,p]\,.\end{aligned}$$

§ 4. Semiclassical approximation of Hamiltonian type

As a merit of the use of piecewise bicharacteristic paths, we state the semiclassical approximation of Hamiltonian type.

Let $4\kappa_2 dT < 1/2$. Then, for any $(x_{J+1},\xi_0) \in \mathbf{R}^d \times \mathbf{R}^d$, there exists the stationary point $(x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*)$ of the phase function $\phi_{\Delta_{T,0}} = \phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$, i.e.,

$$(\partial_{(\xi_J,x_J,\ldots,\xi_1,x_1)}\phi_{\Delta_{T,0}})(x_{J+1},\xi_J^*,x_J^*,\ldots,\xi_1^*,x_1^*,\xi_0) = 0\,. \tag{4.1}$$

Pushing the stationary point $(x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*)$ into the Hessian matrix of $\phi_{\Delta_{T,0}}$, we define $D_{\Delta_{T,0}}(x_{J+1},\xi_0)$ by

$$D_{\Delta_{T,0}}(x_{J+1},\xi_0) = (-1)^{dJ}\det(\partial^2_{(\xi_J,x_J,\ldots,\xi_1,x_1)}\phi_{\Delta_{T,0}})(x_{J+1},x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*,\xi_0)\,. \tag{4.2}$$

Lemma 4.1. *For any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that*

$$\begin{aligned}&|\partial_x^\alpha\partial_{\xi_0}^\beta(D_{\Delta_{T,0}}(x,\xi_0)-1)| \leq C_{\alpha,\beta}T^2\,,\\&|\partial_x^\alpha\partial_{\xi_0}^\beta(D_{\Delta_{T,0}}(x,\xi_0)-D(T,x,\xi_0))| \leq C_{\alpha,\beta}|\Delta_{T,0}|T\end{aligned}$$

with a limit function $D(T,x,\xi_0) = \lim_{|\Delta_{T,0}|\to 0} D_{\Delta_{T,0}}(x,\xi_0)$.

We use this limit function $D(T,x,\xi_0)$ as a Hamiltonian version of the Morette-Van Vleck determinant [25]

Theorem 4 (Semiclassical approximation of Hamiltonian type as $\hbar \to 0$). *Let T be sufficiently small. Then, for any $F[q,p] \in \mathcal{F}$, we have*

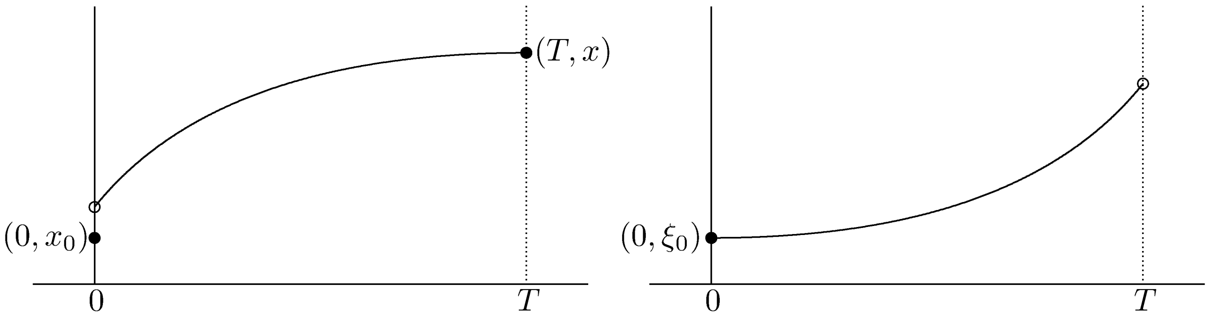
$$\begin{aligned}&\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p]\\&= e^{\frac{i}{\hbar}\phi[q_{T,0},p_{T,0}]}\Big(D(T,x,\xi_0)^{-1/2}F[q_{T,0},p_{T,0}]+\hbar\Upsilon(T,\hbar,x,\xi_0,x_0)\Big)\,.\end{aligned}$$

Here $q_{T,0} = q_{T,0}(t,x,\xi_0,x_0)$ and $p_{T,0} = p_{T,0}(t,x,\xi_0)$ are the piecewise bicharacteristic paths for the simplest division $0 < T$ (see Fig. 5). Furthermore, for any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ independent of $0 < \hbar < 1$ such that

$$|\partial_x^\alpha\partial_{\xi_0}^\beta\Upsilon(T,\hbar,x,\xi_0,x_0)| \leq C_{\alpha,\beta}(1+|x|+|\xi_0|+|x_0|)^m\,.$$

Remark. Using the notations (2.5) and (2.6), we can also write

$$\phi_{T,0}(x,\xi_0,x_0)=\phi[q_{T,0},p_{T,0}]\,,\quad F_{T,0}(x,\xi_0,x_0)=F[q_{T,0},p_{T,0}]\,. \tag{4.3}$$



The path $q_{T,0}$ for the division $0<T$ The path $p_{T,0}$ for the division $0<T$

Figure 5.

§ 5. Process of the proof of Theorems 1, 2 and 4

In this survey, we explain the process of the proof of [23]. For the proof, see [23].

In order to prove the convergence of the multiple integral

$$\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]\prod_{j=1}^{J}dx_j d\xi_j \tag{5.1}$$

as $|\Delta_{T,0}|\to 0$, we have only to add many assumptions for $F_{\Delta_{T,0}}=F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$. Because we gave no assumption for $F[q,p]\in\mathcal{F}$ until this section. The assumptions should be closed under addition and multiplication. Then $\mathcal{F}$ will be an algebra. Not to consider other things is better. Then $\mathcal{F}$ will become larger as a set. If lucky, $\mathcal{F}$ may contain at least one example $F[q,p]\equiv 1$ as the fundamental solution for the Schrödinger equation.

Our proof consists of 3 steps. In §6, we explain Lemma 6.1 as an estimate of H. Kumano-go-Taniguchi's type. Using Lemma 6.1, we can control the multiple integral (5.1) by C^J as $J\to\infty$ with a positive constant C. In §7, we explain Lemma 7.1 as an stationary phase method of Fujiwara's type. Using Lemma 7.1, we can control the multiple integral (5.1) by C independent of $J\to\infty$ with a positive constant C. In §8, we explain that the multiple integral (5.1) converges as $|\Delta_{T,0}|\to 0$.

§ 6. Estimate of H. Kumano-go-Taniguchi's type

We consider $q_{\Delta_{T,0}}(\hbar, x_{J+1}, \xi_0, x_0)$ defined by the multiple integral

$$\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)\prod_{j=1}^{J} dx_j d\xi_j \tag{6.1}$$
$$= e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)} q_{\Delta_{T,0}}(\hbar, x_{J+1}, \xi_0, x_0)$$

with $\phi_{T,0}(x, \xi_0, x_0)$ in (4.3).

Lemma 6.1 (Estimate of H. Kumano-go-Taniguchi's type)**.**
Let T be sufficiently small. Let $m \geq 0$. Assume that for any integer $M \geq 0$, there exist positive constants A_M, X_M such that for any multi-indices α_j, β_{j-1} with $|\alpha_j|$, $|\beta_{j-1}| \leq M$, $j = 1, 2, \ldots, J, J + 1$,

$$|(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}})F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)| \tag{6.2}$$
$$\leq A_M(X_M)^{J+1}(1 + \sum_{j=1}^{J+1}(|x_j| + |\xi_{j-1}|) + |x_0|)^m .$$

Then there exists a positive constant C such that

$$|q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0)| \leq C^J(1 + |x_{J+1}| + |\xi_0| + |x_0|)^m . \tag{6.3}$$

The case $m = 0$ of Lemma 6.1 is called H. Kumano-go-Taniguchi's theorem (cf. [20, pp. 359-360]).

§ 6.1. Integrate by parts over and over again

We explain the outline of the proof of Lemma 6.1 when $m = 0$ (cf. Fujiwara-N. Kumano-go-Taniguchi [11]).

Using some functions $\omega_{T_j,T_{j-1}}(x_j, \xi_{j-1})$, $j = 1, 2, \ldots, J, J + 1$, we can write

$$\phi_{\Delta_{T,0}} = \sum_{j=1}^{J+1}(x_j - x_{j-1})\xi_{j-1} + \sum_{j=1}^{J+1}\omega_{T_j,T_{j-1}}(x_j, \xi_{j-1}) . \tag{6.4}$$

We introduce the differential operators of the first order

$$M_j = \frac{1 - i(\partial_{\xi_j}\phi_{\Delta_{T,0}})\partial_{\xi_j}}{1 + \hbar^{-1}|\partial_{\xi_j}\phi_{\Delta_{T,0}}|^2}, \quad N_j = \frac{1 - i(\partial_{x_j}\phi_{\Delta_{T,0}})\partial_{x_j}}{1 + \hbar^{-1}|\partial_{x_j}\phi_{\Delta_{T,0}}|^2} \tag{6.5}$$

for $j = 1, 2, \ldots, J$. Note that

$$M_j e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} = e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} , \quad N_j e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} = e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} .$$

Integrating by parts over and over again, we write (6.1) as

$$
\begin{aligned}
(6.6)\qquad &\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}} \prod_{j=1}^{J} dx_j d\xi_j \\
&= \left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F^{\spadesuit}_{\Delta_{T,0}} \prod_{j=1}^{J} dx_j d\xi_j \,,
\end{aligned}
$$

where $F^{\spadesuit}_{\Delta_{T,0}}$ is the multi-product of differential operators given by

$$
(6.7)\qquad F^{\spadesuit}_{\Delta_{T,0}} = (N_J^*)^{d+1}\cdots(N_2^*)^{d+1}(N_1^*)^{d+1}(M_J^*)^{d+1}\cdots(M_2^*)^{d+1}(M_1^*)^{d+1}F_{\Delta_{T,0}}
$$

with the adjoint operators M_j^*, N_j^* of M_j, N_j.

§ 6.2. Expect different results

Generally speaking, we can not control multi-products of J differential operators by C^J as $J\to\infty$ with a positive constant C. However, by (6.4),

$$
\begin{aligned}
\partial_{\xi_j}\phi_{\Delta_{T,0}} &= -(x_j - x_{j+1}) + \partial_{\xi_j}\omega_{T_{j+1},T_j}(x_{j+1},\xi_j)\,,\\
\partial_{x_j}\phi_{\Delta_{T,0}} &= -(\xi_j - \xi_{j-1}) + \partial_{x_j}\omega_{T_j,T_{j-1}}(x_j,\xi_{j-1})
\end{aligned}
$$

are functions of $3d$-variables independent of J. Hence we can write

$$
\begin{aligned}
M_j^* &= a_j^1(x_{j+1},\xi_j,x_j)\partial_{\xi_j} + a_j^0(x_{j+1},\xi_j,x_j)\,,\\
N_j^* &= b_j^1(\xi_j,x_j,\xi_{j-1})\partial_{x_j} + b_j^0(\xi_j,x_j,\xi_{j-1})
\end{aligned}
$$

with some functions a_j^1, a_j^0, b_j^1 and b_j^0 of $3d$-variables independent of J. Therefore, in (6.7), only $\partial_{x_{j+1}}$, ∂_{ξ_j} and ∂_{x_j} differentiate M_j^*, and only ∂_{ξ_j} differentiates N_j^*. Therefore, in (6.7), only N_{j+1}^*, M_j^* and N_j^* differentiate M_j^* and only N_j^* differentiates N_j^* . Hence we can control the multi-product (6.7) by C^J as $J\to\infty$ with a positive constant C. Roughly speaking, from (6.5), the operation of M_j^* implies the multiplication of $C/(1+\hbar^{-1}|\partial_{\xi_j}\phi_{\Delta_{T,0}}|^2)^{1/2}$ with a positive constant C, and the operation of N_j^* implies the multiplication of $C/(1+\hbar^{-1}|\partial_{x_j}\phi_{\Delta_{T,0}}|^2)^{1/2}$ with a positive constant C.

§ 6.3. Change all variables at one time

Set $z_j = \partial_{\xi_j}\phi_{\Delta_{T,0}}$ and $\zeta_j = \partial_{x_j}\phi_{\Delta_{T,0}}$ for $j=1,2,\ldots,J$. From (6.7), we have

$$
|F^{\spadesuit}_{\Delta_{T,0}}| \le (C')^J \prod_{j=1}^{J} \frac{1}{(1+\hbar^{-1}|z_j|^2)^{(d+1)/2}}\frac{1}{(1+\hbar^{-1}|\zeta_j|^2)^{(d+1)/2}}
$$

with a positive constant C'. Furthermore, since T is sufficiently small, we can obtain

$$
\left|\det\frac{\partial(x_J,\ldots,x_1,\xi_J,\ldots,\xi_1)}{\partial(z_J,\ldots,z_1,\zeta_J,\ldots,\zeta_1)}\right| \le (C'')^J
$$

with a positive constant C''. Changing all variables at one time, we rewrite (6.6) as

$$(6.8)\quad \left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}}F^{\spadesuit}_{\Delta_{T,0}}\prod_{j=1}^{J}dx_j d\xi_j$$
$$=\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}}F^{\spadesuit}_{\Delta_{T,0}}\left|\det\frac{\partial(x_J,\ldots,x_1,\xi_J,\ldots,\xi_1)}{\partial(z_J,\ldots,z_1,\zeta_J,\ldots,\zeta_1)}\right|\prod_{j=1}^{J}dz_j d\zeta_j\,.$$

Integrating (6.8) with respect to $(z_J,\ldots,z_1,\zeta_J,\ldots,\zeta_1)$, we can control (6.1) by C^J as $J\to\infty$ with a positive constant C. □

§ 7. Stationary phase method of Fujiwara's type

We consider the remainder term $\Upsilon_{\Delta_{T,0}}(\hbar,x,\xi_0,x_0)$ of the multiple integral

$$(7.1)\quad \left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}}F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0)\prod_{j=1}^{J}dx_j d\xi_j$$
$$=e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}\left(D_{\Delta_{T,0}}(x,\xi_0)^{-1/2}F_{T,0}(x,\xi_0,x_0)+\hbar\Upsilon_{\Delta_{T,0}}(\hbar,x,\xi_0,x_0)\right)$$

with $\phi_{T,0}(x,\xi_0,x_0)$, $F_{T,0}(x,\xi_0,x_0)$ in (4.3) and $D_{\Delta_{T,0}}(x,\xi_0)$ in (4.2).

Lemma 7.1 (Stationary phase method of Fujiwara's type). *Let T be sufficiently small. Let $m\geq 0$. Assume that for any integer $M\geq 0$, there exist positive constants A_M, X_M such that for any $\Delta_{T,0}$ and any multi-indices α_j, β_{j-1} with $|\alpha_j|$, $|\beta_{j-1}|\leq M$, $j=1,2,\ldots,J,J+1$,*

$$(7.2)\quad |(\prod_{j=1}^{J+1}\partial_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}})F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0)|$$
$$\leq A_M(X_M)^{J+1}(\prod_{j=1}^{J+1}(t_j)^{\min(|\beta_{j-1}|,1)})(1+\sum_{j=1}^{J+1}(|x_j|+|\xi_{j-1}|)+|x_0|)^m\,.$$

Then there exists a positive constant C independent of $\Delta_{T,0}$ such that

$$(7.3)\quad |\Upsilon_{\Delta_{T,0}}(\hbar,x,\xi_0,x_0)|\leq CT(1+|x_{J+1}|+|\xi_0|+|x_0|)^m\,.$$

Remark. The remainder term of multiple integrals for configuration space path integrals was estimated by D. Fujiwara [10]. Though the present paper treats multiple integrals for phase space path integrals, the proof of Lemma 7.1 follows the rule of [10].

Remark. In order to control the remainder term $\Upsilon_{\Delta_{T,0}}(\hbar,x,\xi_0,x_0)$, we added the small term t_j for the differentiation $\partial_{\xi_{j-1}}$. Since $q_{\Delta_{T,0}}(t)\approx x_j-t_j\xi_{j-1}$ when $T_{j-1}<t\leq T_j$, the functional $F[q,p]=q(t)$ satisfies (7.2). However we give up treating the functional $F[q,p]=p(t)$.

§ 7.1. Distinguish the main term from the remainder term

We explain the outline of the proof of Lemma 7.1 when $m = 0$.

We must integrate (7.1) with respect to $(\xi_1, x_1), (\xi_2, x_2), \ldots, (\xi_J, x_J)$. First we integrate (7.1) with respect to (ξ_1, x_1). By the stationary phase method, we distinguish the main term $(\mathcal{M}_1 F_{\Delta_{T,0}})$ from the remainder term $(\mathcal{R}_1 F_{\Delta_{T,0}})$.

$$\begin{aligned}(7.4)\qquad &\left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}(\ldots, x_2, \xi_1, x_1, \xi_0, x_0) dx_1 d\xi_1 \\ &= e^{\frac{i}{\hbar}\phi_{(\Delta_{T,T_2},0)}} (\mathcal{M}_1 F_{\Delta_{T,0}})(\ldots, \xi_2, x_2, \xi_0, x_0) \\ &+ e^{\frac{i}{\hbar}\phi_{(\Delta_{T,T_2},0)}} (\mathcal{R}_1 F_{\Delta_{T,0}})(\ldots, \xi_2, x_2, \xi_0, x_0)\,.\end{aligned}$$

The main term $(\mathcal{M}_1 F_{\Delta_{T,0}})$ is 'simple' as a function of (ξ_2, x_2) and given by

$$\begin{aligned}(7.5)\qquad &(\mathcal{M}_1 F_{\Delta_{T,0}})(\ldots, \xi_2, x_2, \xi_0, x_0) \\ &= D_{\Delta_{T_2,0}}(x_2, \xi_0)^{-1/2} F_{\Delta_{T,0}}(\ldots, \xi_2, x_2, \xi_1^*, x_1^*, \xi_0, x_0) \\ &= D_{\Delta_{T_2,0}}(x_2, \xi_0)^{-1/2} F_{(\Delta_{T,T_2},0)}(\ldots, \xi_2, x_2, \xi_0, x_0)\,.\end{aligned}$$

Here $(\Delta_{T,T_2}, 0)$ be the division given by

$$(7.6)\qquad (\Delta_{T,T_2}, 0) : T = T_{J+1} > T_J > \cdots > T_2 > T_0 = 0\,,$$

and the stationary point (ξ_1^*, x_1^*) defined by $(\partial_{(\xi_1,x_1)}\phi_{\Delta_{T,0}})(x_2, \xi_1^*, x_1^*, \xi_0) = 0$ satisfies $x_1^* = \bar{q}_{T_2,0}(T_1)$, $\xi_1^* = \bar{p}_{T_2,0}(T_1)$ (see Fig. 6). The remainder term $(\mathcal{R}_1 F_{\Delta_{T,0}})$ is 'complicated' as a function of (ξ_2, x_2) but can be controlled by the small term $(t_2\hbar)$. For example, if $F[q,p] \equiv 1$, we have

$$(7.7)\qquad |(\mathcal{R}_1 F_{\Delta_{T,0}})| \le C(t_2\hbar)\,.$$

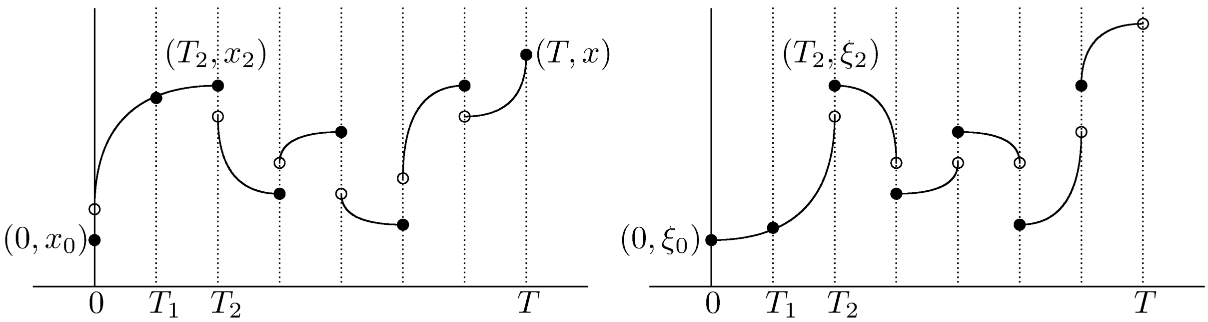
§ 7.2. Do only simple integrals

Since $(\mathcal{M}_1 F_{\Delta_{T,0}})$ is 'simple' as a function of (ξ_2, x_2), we integrate it further with respect to (ξ_2, x_2). By the stationary phase method, we have

$$\begin{aligned}&\left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}\phi_{(\Delta_{T,T_2},0)}} (\mathcal{M}_1 F_{\Delta_{T,0}})(\ldots, x_3, \xi_2, x_2, \xi_0, x_0) dx_2 d\xi_2 \\ &= e^{\frac{i}{\hbar}\phi_{(\Delta_{T,T_3},0)}} (\mathcal{M}_2\mathcal{M}_1 F_{\Delta_{T,0}})(\ldots, \xi_3, x_3, \xi_0, x_0) \\ &+ e^{\frac{i}{\hbar}\phi_{(\Delta_{T,T_3},0)}} (\mathcal{R}_2\mathcal{M}_1 F_{\Delta_{T,0}})(\ldots, \xi_3, x_3, \xi_0, x_0)\,.\end{aligned}$$

Here $(\Delta_{T,T_3}, 0)$ be the division given by

$$(7.8)\qquad (\Delta_{T,T_3}, 0) : T = T_{J+1} > T_J > \cdots > T_3 > T_0 = 0\,.$$



The position path of $(\mathcal{M}_1 F_{\Delta_{T,0}})$ The momentum path of $(\mathcal{M}_1 F_{\Delta_{T,0}})$

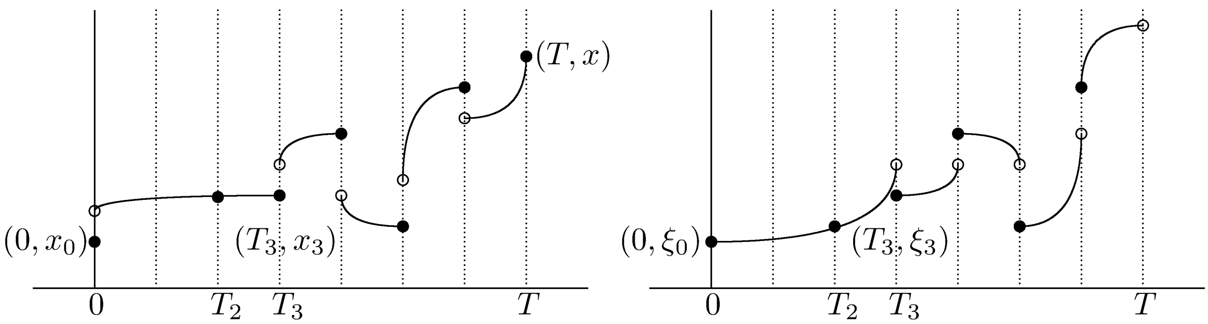
Figure 6.

The main term $(\mathcal{M}_2\mathcal{M}_1 F_{\Delta_{T,0}})$ is 'simple' as a function of (ξ_3, x_3) and given by

$$
\begin{aligned}
&(\mathcal{M}_2\mathcal{M}_1 F_{\Delta_{T,0}})(\dots, \xi_3, x_3, \xi_0, x_0) \\
&= D_{\Delta_{T_3,0}}(x_3, \xi_0)^{-1/2} F_{(\Delta_{T,T_3},0)}(\dots, \xi_3, x_3, \xi_0, x_0)
\end{aligned} \tag{7.9}
$$

(see Fig. 7). The remainder term $(\mathcal{R}_2\mathcal{M}_1 F_{\Delta_{T,0}})$ is 'complicated' as a function of (ξ_3, x_3) but can be controlled by the small term $(t_3\hbar)$. For example, if $F[q,p] \equiv 1$, we have

$$
|(\mathcal{R}_2\mathcal{M}_1 F_{\Delta_{T,0}})| \le C(t_3\hbar)\,. \tag{7.10}
$$



The position path of $(\mathcal{M}_2\mathcal{M}_1 F_{\Delta_{T,0}})$ The momentum path of $(\mathcal{M}_2\mathcal{M}_1 F_{\Delta_{T,0}})$

Figure 7.

Since the main term $(\mathcal{M}_2\mathcal{M}_1 F_{\Delta_{T,0}})$ is 'simple' as a function of (ξ_3, x_3), we integrate it further with respect to (ξ_3, x_3). Repeating this simple process, we get the main term of (7.1) (see Fig. 5).

$$
(\mathcal{M}_J\mathcal{M}_{J-1}\dots\mathcal{M}_1 F_{\Delta_{T,0}}) = D_{\Delta_{T,0}}(x, x_0)^{-1/2} F_{T,0}(x, \xi_0, x_0)\,.
$$

§ 7.3. Skip all complicated integrals

Now, we go back to the remainder term $(\mathcal{R}_1 F_{\Delta_{T,0}})$. Since $(\mathcal{R}_1 F_{\Delta_{T,0}})$ is 'complicated' as a function of (ξ_2, x_2), we skip the integration with respect to (ξ_2, x_2) and integrate it with respect to (ξ_3, x_3) beforehand. By the stationary phase method, we have

$$\begin{aligned}
&\left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbf{R}^d} e^{\frac{i}{\hbar}\phi(\Delta_{T,T_2},0)} (\mathcal{R}_1 F_{\Delta_{T,0}})(\ldots, x_4, \xi_3, x_3, \xi_2, x_2, \xi_0, x_0) dx_3 d\xi_3 \\
&= e^{\frac{i}{\hbar}\phi(\Delta_{T,T_4},T_2,0)} (\mathcal{M}_3 \mathcal{R}_1 F_{\Delta_{T,0}})(\ldots, \xi_4, x_4, \xi_2, x_2, \xi_0, x_0) \\
&+ e^{\frac{i}{\hbar}\phi(\Delta_{T,T_4},T_2,0)} (\mathcal{R}_3 \mathcal{R}_1 F_{\Delta_{T,0}})(\ldots, \xi_4, x_4, \xi_2, x_2, \xi_0, x_0)\,,
\end{aligned}$$

where the main term $(\mathcal{M}_3 \mathcal{R}_1 F_{\Delta_{T,0}})$ is 'simple' as a function of (ξ_4, x_4) and the remainder term $(\mathcal{R}_3 \mathcal{R}_1 F_{\Delta_{T,0}})$ is 'complicated' as a function of (ξ_4, x_4) but can be controlled by the two small terms $(t_4\hbar)$ and $(t_2\hbar)$. For example, if $F[q,p] \equiv 1$, we have

$$|(\mathcal{R}_3 \mathcal{R}_1 F_{\Delta_{T,0}})| \leq C(t_4\hbar) C(t_2\hbar)\,.$$

Since the main term $(\mathcal{M}_3 \mathcal{R}_1 F_{\Delta_{T,0}})$ is 'simple' as a function of (ξ_4, x_4), we integrate it further with respect to (ξ_4, x_4). But since the remainder term $(\mathcal{R}_3 \mathcal{R}_1 F_{\Delta_{T,0}})$ is 'complicated' as a function of (ξ_4, x_4), we skip the integration with respect to (ξ_4, x_4) and integrate it with respect to (ξ_5, x_5) beforehand.

§ 7.4. The rule is the following

The rule of D. Fujiwara [10] is the following: Integrate with respect to (ξ_j, x_j). By the stationary phase method, we distinguish the main term from the remainder term. The main term is 'simple' as a function of (ξ_{j+1}, x_{j+1}). Therefore, we integrate it further with respect to (ξ_{j+1}, x_{j+1}). However, the remainder term is 'complicated' as a function of (ξ_{j+1}, x_{j+1}). Therefore, we skip the integration with respect to (ξ_{j+1}, x_{j+1}) and integrate it with respect to (ξ_{j+2}, x_{j+2}) beforehand.

§ 7.5. Carry out the rule until the end

Carrying out the rule until the end, we have

$$q_{\Delta_{T,0}}(\hbar, x_{J+1}, \xi_0, x_0) = q_0(x_{J+1}, \xi_0, x_0) + \sum{}' q_{j_K, j_{K-1}, \ldots, j_1}(x_{J+1}, \xi_0, x_0)\,.$$

Here $q_0(x_{J+1}, \xi_0, x_0) = D_{\Delta_{T,0}}(x, \xi_0)^{-1/2} F_{T,0}(x_{J+1}, \xi_0, x_0)$ is the main term of (7.1), the sum $\sum{}'$ means the summation over all the sequence of integers $(j_K, j_{K-1}, \ldots, j_1)$ such that

$$0 = j_0 < j_1 - 1 < j_1 < j_2 - 1 < j_2 < \cdots < j_K - 1 < j_K \leq J+1\,,$$

and the summand $q_{j_K,j_{K-1},\dots,j_1}(x_{J+1},\xi_0,x_0)$ is the complicated integrals which we skipped

$$\begin{aligned}
&e^{\frac{i}{\hbar}\phi_{T,0}(x_{J+1},\xi_0,x_0)}q_{j_K,j_{K-1},\dots,j_1}(x_{J+1},\xi_0,x_0)\\
&=\int_{\mathbf{R}^{dK}} e^{\frac{i}{\hbar}\phi_{T,T_{j_K},\dots,T_1,0}} b_{j_K,j_{K-1},\dots,j_1}(x_{J+1},\xi_{j_K},x_{j_K},\dots,\xi_{j_1},x_{j_1},\xi_0,x_0)\prod_{k=1}^{K} dx_{j_k}d\xi_{j_k}\,,
\end{aligned}$$

where

$$\begin{aligned}
&b_{j_K,j_{K-1},\dots,j_1}(x_{J+1},\xi_{j_K},x_{j_K},\dots,\xi_{j_1},x_{j_1},\xi_0,x_0)\\
&=(\mathcal{Q}_J\cdots\mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1F_{\Delta_{T,0}})(x_{J+1},\xi_{j_K},x_{j_K},\dots,\xi_{j_1},x_{j_1},\xi_0,x_0)
\end{aligned}\tag{7.11}$$

with

$$\mathcal{Q}_j=\begin{cases}\text{Identity} & \text{if } j=j_K,j_{K-1},\dots,j_1\\ \mathcal{R}_j & \text{if } j=j_K-1,j_{K-1}-1,\dots,j_1-1\\ \mathcal{M}_j & \text{otherwise}\end{cases}\,.$$

Therefore the integrand can be controlled by the many small terms $(t_{j_k}\hbar)$, i.e.,

$$|b_{j_K,j_{K-1},\dots,j_1}(x_{J+1},\xi_{j_K},x_{j_K},\dots,\xi_{j_1},x_{j_1},\xi_0,x_0)|\le C^K\Big(\prod_{k=1}^{K}(t_{j_k}\hbar)\Big)\,.$$

§ 7.6. Force all complicated integrals on others

We force on the estimate of H. Kumano-go-Taniguchi's type all the complicated integrals which we skipped. By Lemma 6.1, we have

$$|q_{j_K,j_{K-1},\dots,j_1}(x_{J+1},\xi_0,x_0)|\le (C')^K\Big(\prod_{k=1}^{K}(t_{j_k}\hbar)\Big)$$

with a positive constant C'. The remainder term is the sum of $q_{j_K,j_{K-1},\dots,j_1}(x_{J+1},\xi_0,x_0)$.

$$\Upsilon_{\Delta_{T,0}}(\hbar,x_{J+1},\xi_0,x_0)=\frac{1}{\hbar}\sum{}' q_{j_K,j_{K-1},\dots,j_1}(x_{J+1},\xi_0,x_0)\,.$$

Using $\sum_{j=1}^{J+1}t_j=T$, we take a sum. Note $0<\hbar<1$. Then we have

$$\begin{aligned}
|\Upsilon_{\Delta_{T,0}}(\hbar,x_{J+1},x_0)|&\le\frac{1}{\hbar}\sum{}'\Big((C')^K\prod_{k=1}^{K}(t_{j_k}\hbar)\Big)\\
&\le\frac{1}{\hbar}\Big(\prod_{j=1}^{J+1}(1+C't_j\hbar)-1\Big)\le(C'')T
\end{aligned}$$

with a positive constant C''. □

§ 8. Definition of the class $\mathcal{F}$

The definition of the class $\mathcal{F}$ of functionals $F[q,p]$ is the following:

Definition 1 (The class $\mathcal{F}$). *Let $F[q,p]$ be a functional whose domain contains all the piecewise bicharacteristic paths $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ of (2.4). We say that $F[q,p] \in \mathcal{F}$ if $F_{\Delta_{T,0}} = F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ satisfies Assumption 2.*

Assumption 2. *Let $m \geq 0$. Let $u_j \geq 0$, $j = 1, 2, \ldots, J, J+1$ are non-negative parameters depending on the division $\Delta_{T,0}$ such that $\sum_{j=1}^{J+1} u_j \equiv U < \infty$. For any integer $M \geq 0$, there exist positive constants A_M, X_M such that for any $\Delta_{T,0}$, any multi-indices α_j, β_{j-1} with $|\alpha_j|$, $|\beta_{j-1}| \leq M$, $j = 1, 2, \ldots, J, J+1$ and any $1 \leq k \leq J$,*

$$
\begin{aligned}
(8.1) \qquad & |(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}}) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)| \\
& \leq A_M (X_M)^{J+1} (\prod_{j=1}^{J+1} (t_j)^{\min(|\beta_{j-1}|, 1)})(1 + \sum_{j=1}^{J+1}(|x_j| + |\xi_{j-1}|) + |x_0|)^m ,
\end{aligned}
$$

$$
\begin{aligned}
(8.2) \qquad & |(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}}) \partial_{x_k} F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)| \\
& \leq A_M (X_M)^{J+1} u_k (\prod_{j \neq k} (t_j)^{\min(|\beta_{j-1}|, 1)})(1 + \sum_{j=1}^{J+1}(|x_j| + |\xi_{j-1}|) + |x_0|)^m .
\end{aligned}
$$

Under Assumption 2, we consider the multiple integral again.

$$
\begin{aligned}
(8.3) \qquad & \left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0) \prod_{j=1}^{J} dx_j d\xi_j \\
& = e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)} q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0) \\
& = e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)} \Big(D_{\Delta_{T,0}}(x, \xi_0)^{-1/2} F_{T,0}(x, \xi_0, x_0) + \hbar \Upsilon_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0) \Big) .
\end{aligned}
$$

Then the estimate in Lemma 7.1 becomes the following.

Lemma 8.1. *Let T be sufficiently small. Under Assumption 2, there exist positive constants C, C' such that*

$$
(8.4) \qquad |\Upsilon_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0)| \leq CT(T+U)(1 + |x_{J+1}| + |\xi_0| + |x_0|)^m ,
$$

$$
(8.5) \qquad |q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0)| \leq C'(1 + |x_{J+1}| + |\xi_0| + |x_0|)^m .
$$

§ 8.1. Consider integrals with paths

Using paths, we interpret Lemma 8.1. The multiple integral (8.3) implies Fig. 4. Hence, (8.5) implies that Fig. 4 can be controlled by C'. The main term $F_{T,0}(x, \xi_0, x_0)$

of (8.3) implies Fig. 5. Therefore, (8.4) implies that the difference between Fig. 4 and Fig. 5 can be controlled by $CT(T+U)$.

§ 8.2. Compare two multiple integrals by two paths

We have only to show that the sequence of multiple integrals (2.7) is a Cauchy sequence with respect to the division $\Delta_{T,0}$. For the two divisions

$$\begin{aligned} &\Delta_{T,0} &&: T = T_{J+1} > T_J > \cdots \quad \cdots \quad \cdots > T_1 > T_0 = 0\,, \\ &(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0}) &&: T = T_{J+1} > \cdots > T_{N+1} \; > \; T_{n-1} > \cdots > T_0 = 0\,, \end{aligned}$$

we compare the multiple integral

$$\begin{aligned} (8.6) \qquad &\int \cdots\cdots \int \cdots\cdots \int \cdots\cdots \int \qquad \sim \qquad \prod_{j=1}^{J} dx_j d\xi_j \\ &= e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)} q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0) \end{aligned}$$

and the multiple integral

$$\begin{aligned} (8.7) \qquad &\int \cdots\cdots \int \qquad \int \cdots\cdots \int \qquad \sim \qquad \prod_{j=N+1}^{J} dx_j d\xi_j \prod_{j=n-1}^{J} dx_j d\xi_j \\ &= e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)} q_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}(\hbar, x, \xi_0, x_0)\,. \end{aligned}$$

Note that the multiple integral (8.6) implies Fig. 4 and that the multiple integral (8.7) implies Fig. 8. By (8.5), we can control the multiple integral on the interval $[0, T_{n-1}]$

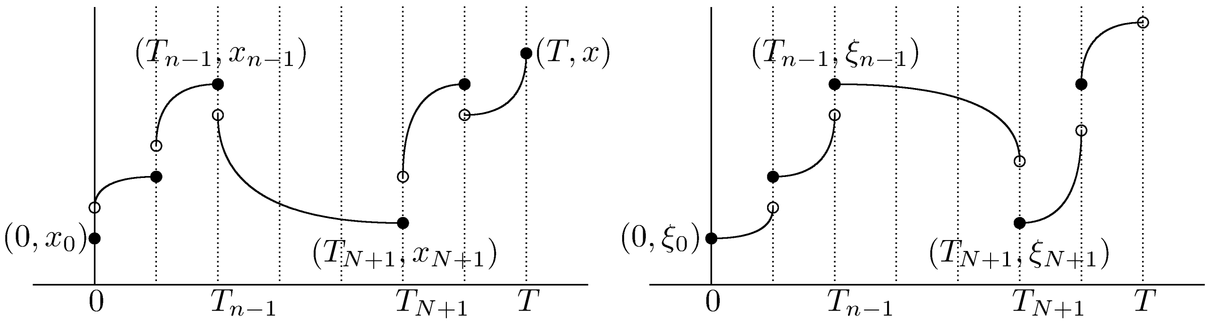


Figure 8.

and the multiple integral on the interval $[T_{N+1}, T]$ by C'. Furthermore, by (8.4), we can control the difference of the two multiple integrals on the interval $[T_{n-1}, T_{N+1}]$ by

$$C(T_{N+1} - T_{n-1})(T_{N+1} - T_{n-1} + U_{N+1} - U_{n-1})\,,$$

where $U_{N+1} = \sum_{j=1}^{N+1} u_j$ and $U_{n-1} = \sum_{j=1}^{n-1} u_j$. Therefore we can control the difference of the multiple integral (8.6) and the multiple integral (8.7) as follows.

$$\begin{aligned}(8.8)\quad &|q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0) - q_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}(\hbar, x, \xi_0, x_0)| \\ &\le C''(T_{N+1} - T_{n-1})(T_{N+1} - T_{n-1} + U_{N+1} - U_{n-1})(1 + |x| + |\xi_0| + |x_0|)^m\end{aligned}$$

with a positive constant C''.

§ 8.3. Phase space Feynman path integrals exist

Noting (8.8), we can obtain the following theorem which proves Theorem 2.

Theorem 5. *Let T be sufficiently small. For any multi-indices α, β, there exist positive constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ such that*

$$\begin{aligned}&|\partial_x^\alpha \partial_{\xi_0}^\beta q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0)| \le C_{\alpha,\beta}(1 + |x| + |\xi_0| + |x_0|)^m , \\ &|\partial_x^\alpha \partial_{\xi_0}^\beta (q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0) - q(T, \hbar, x, \xi_0, x_0))| \\ &\le C'_{\alpha,\beta}|\Delta_{T,0}|(T + U)(1 + |x| + |\xi_0| + |x_0|)^m\end{aligned}$$

with a limit function $q(T, \hbar, x, \xi_0, x_0) = \lim_{|\Delta_{T,0}|\to 0} q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0)$, i.e., the multiple integral (2.7) *converges on compact subsets of $\mathbf{R}^{3d}$ as $|\Delta_{T,0}| \to 0$.*

Remark. The class $\mathcal{F}$ is an algebra because we added assumptions closed under addition and multiplication. Furthermore, by accident, $\mathcal{F}$ contains the examples in Example 2.

§ 9. Appendix

§ 9.1. An example

We give an example which illustrates what happens if T is not small.

Let $d = 1$, $H(x, \xi) = x^2/2 + \xi^2/2$ and $F[q, p] \equiv 1$. Assume $|T_j - T_{j-1}| < \pi/2$. As in (2.3), by the canonical equation

$$\begin{aligned}&\partial_t \bar{q}_{T_j,T_{j-1}}(t) = \bar{p}_{T_j,T_{j-1}}(t) , \quad \partial_t \bar{p}_{T_j,T_{j-1}}(t) = -\bar{q}_{T_j,T_{j-1}}(t) , \quad T_{j-1} \le t \le T_j , \\ &\bar{q}_{T_j,T_{j-1}}(T_j) = x_j , \quad \bar{p}_{T_j,T_{j-1}}(T_{j-1}) = \xi_{j-1} ,\end{aligned}$$

we have the bicharacteristic paths

$$\begin{aligned}\bar{q}_{T_j,T_{j-1}}(t) &= \frac{x_j \cos(t - T_{j-1}) - \xi_{j-1} \sin(T_j - t)}{\cos(T_j - T_{j-1})}, \\ \bar{p}_{T_j,T_{j-1}}(t) &= \frac{-x_j \sin(t - T_{j-1}) + \xi_{j-1} \cos(T_j - t)}{\cos(T_j - T_{j-1})} .\end{aligned}$$

As in (2.4) and (2.5), using the piecewise bicharacteristic paths

$$q_{T_j,T_{j-1}}(t)=\bar q_{T_j,T_{j-1}}(t),\quad T_{j-1}<t\le T_j,\quad q_{T_j,T_{j-1}}(T_{j-1})=x_{j-1},$$
$$p_{T_j,T_{j-1}}(t)=\bar p_{T_j,T_{j-1}}(t),\quad T_{j-1}\le t<T_j\,,$$

we have the phase function

$$\begin{aligned}(9.1)\qquad &\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})\\ &=\int_{[T_{j-1},T_j)}p_{T_j,T_{j-1}}\cdot dq_{T_j,T_{j-1}}-\int_{[T_{j-1},T_j)}H(q_{T_j,T_{j-1}},p_{T_j,T_{j-1}})dt\\ &=(\bar q_{T_j,T_{j-1}}(T_{j-1})-x_{j-1})\cdot\xi_{j-1}\\ &\quad+\frac12\int_{T_{j-1}}^{T_j}\bar p_{T_j,T_{j-1}}\cdot d\bar q_{T_j,T_{j-1}}+\frac12\Big[\bar p_{T_j,T_{j-1}}\cdot\bar q_{T_j,T_{j-1}}\Big]_{T_{j-1}}^{T_j}\\ &\quad-\frac12\int_{T_{j-1}}^{T_j}\bar q_{T_j,T_{j-1}}\cdot d\bar p_{T_j,T_{j-1}}-\frac12\int_{T_{j-1}}^{T_j}(\bar q^2_{T_j,T_{j-1}}+\bar p^2_{T_j,T_{j-1}})dt\\ &=-x_{j-1}\cdot\xi_{j-1}+\frac{\bar p_{T_j,T_{j-1}}(T_j)x_j+\bar q_{T_j,T_{j-1}}(T_{j-1})\xi_{j-1}}{2}\\ &=-x_{j-1}\cdot\xi_{j-1}+\frac{2x_j\cdot\xi_{j-1}-(x_j^2+\xi_{j-1}^2)\sin(T_j-T_{j-1})}{2\cos(T_j-T_{j-1})}\,.\end{aligned}$$

First we consider the case for the division $T=T_2>T_1>T_0=0$. As in (6.1), set

$$(9.2)\qquad e^{\frac{i}{\hbar}\phi_{T,0}(x_2,\xi_0,x_0)}q_{T,T_1,0}=\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2}e^{\frac{i}{\hbar}\phi_{T,T_1,0}(x_2,\xi_1,x_1,\xi_0,x_0)}dx_1d\xi_1\,,$$

where

$$\phi_{T,T_1,0}=\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\phi_{T_1,T_0}(x_1,\xi_0,x_0)\,.$$

From (9.1), we have

$$\begin{aligned}&(-1)\det(\partial^2_{(\xi_1,x_1)}\phi_{T,T_1,0})\\ &=(-1)\det\begin{bmatrix}-\tan(T_2-T_1) & -1\\ -1 & -\tan(T_1-T_0)\end{bmatrix}\\ &=1-\frac{\sin(T_2-T_1)}{\cos(T_2-T_1)}\frac{\sin(T_1-T_0)}{\cos(T_1-T_0)}=\frac{\cos T}{\cos(T_2-T_1)\cos(T_1-T_0)}\,.\end{aligned}$$

Therefore, performing the integration (9.2), we get

$$q_{T,T_1,0}=\left(\frac{\cos(T_2-T_1)\cos(T_1-T_0)}{\cos T}\right)^{1/2}.$$

Next we consider the case for the general division $\Delta_{T,0}$. As in (6.1), set

$$(9.3)\quad e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}q_{\Delta_{T,0}}=\left(\frac{1}{2\pi\hbar}\right)^J\int_{\mathbf{R}^{2J}}e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\dots,\xi_1,x_1,\xi_0,x_0)}\prod_{j=1}^J dx_jd\xi_j\,.$$

Performing the integration (9.3) inductively, we get

$$q_{\Delta_{T,0}} = \left(\frac{\prod_{j=1}^{J+1} \cos(T_j - T_{j-1})}{\cos T} \right)^{1/2} .$$

Therefore, from (1.6) and (2.7), we can calculate the function $U(T, 0, x, \xi_0)$ of the fundamental solution $U(T, 0)$ for the Schrödinger equation as follows.

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} U(T, 0, x, \xi_0) = \int e^{\frac{i}{\hbar}\phi[q,p]} \mathcal{D}[q, p]$$
$$\equiv \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi\hbar}\right)^J \int_{\mathbf{R}^{2J}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]} \prod_{j=1}^{J} dx_j d\xi_j = \lim_{|\Delta_{T,0}|\to 0} e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)} q_{\Delta_{T,0}}$$
$$= \frac{1}{(\cos T)^{1/2}} \exp \frac{i}{\hbar}\Big(- x_0 \cdot \xi_0 + \frac{2x \cdot \xi_0 - (x^2 + \xi_0^2)\sin T}{2\cos T}\Big) .$$

Remark. Theorems 5 and 4 are not valid when $T = \pi/2$.

§ 9.2. Convergence in the uniform operator topology

As in (1.3), using the Fourier integral operator

$$I'(T_j, T_{j-1})v(x) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}\phi_{T_j,T_{j-1}}(x,\xi_0,x_0)} v(x_0) dx_0 d\xi_0 ,$$

we consider (2.7) with $F[q, p] \equiv 1$ in the sense of the operator $I'(\Delta_{T,0})$ given by

$$I'(\Delta_{T,0})v(x) = I'(T, T_J)I'(T_J, T_{J-1}) \cdots I'(T_2, T_1)I'(T_1, 0)v(x) .$$

Noting Theorem 5 under Assumption 2 with $m = 0$ and $u_j = 0$, $j = 1, 2, \ldots, J, J + 1$, we apply the L^2-boundedness theorem of Fourier integral operators to $I'(\Delta_{T,0})$ and $U(T, 0)$. Then there exist a small positive constant τ and positive constants C_1, C_2 such that if $0 < T < \tau$,

$$||I'(\Delta_{T,0})v||_{L^2} \le C_1||v||_{L^2} ,$$
$$\|I'(\Delta_{T,0})v - U(T, 0)v\|_{L^2} \le C_2|\Delta_{T,0}|T||v||_{L^2} .$$

Therefore, we have

$$||U(T, 0)v||_{L^2} \le C_1||v||_{L^2} .$$

Next we consider the case when $T \ge \tau$. Let $|\Delta_{T,0}| < \tau/2$. Using the number K with $K < 2T/\tau \le K+1$, we choose the numbers $0 = j_0 < j_1 < j_2 < \cdots < j_K < j_{K+1} = J+1$ such that $T_{j_k} \le kT/(K + 1) < T_{j_k+1}$ for $k = 1, 2, \ldots, K$.

Since $T_{j_{k+1}} - T_{j_k} < 2T/(K+1) \le \tau$, we have

$$
\begin{aligned}
||I'(\Delta_{T,0})v||_{L^2} &= ||I'(\Delta_{T,T_{j_K}})I'(\Delta_{T_{j_K},T_{j_{K-1}}})\cdots I'(\Delta_{T_{j_2},T_{j_1}})I'(\Delta_{T_{j_1},0})v||_{L^2} \\
&\le (\max(C_1,1))^{K+1}||v||_{L^2} \le C_1'||v||_{L^2}
\end{aligned}
$$

with $C_1' = (\max(C_1,1))^{2T/\tau+1}$. Furthermore, using

$$
U(T,0) = U(T,T_J)U(T_J,T_{J-1})\cdots U(T_2,T_1)U(T_1,0)\,,
$$

we obtain

$$
\begin{aligned}
&||I'(\Delta_{T,0})v - U(T,0)v||_{L^2} \\
&\le \sum_{k=1}^{K+1} \left\| I'(\Delta_{T,T_{j_k}})\left(I'(\Delta_{T_{j_k},T_{j_{k-1}}}) - U(T_{j_k},T_{j_{k-1}})\right)U(T_{j_{k-1}},0)v\right\|_{L^2} \\
&\le \sum_{k=1}^{K+1} C_1'C_2|\Delta_{T_{j_k},T_{j_{k-1}}}|(T_{j_k}-T_{j_{k-1}})C_1'||v||_{L^2} \le C_2'|\Delta_{T,0}|T||v||_{L^2}
\end{aligned}
$$

with $C_2' = C_1'C_2C_1'$. This implies that $I'(\Delta_{T,0})$ converges to $U(T,0)$ as $|\Delta_{T,0}| \to 0$ in the uniform operator topology even when T is large.

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