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On the number of the turning points of the second kind of the Noumi-Yamada systems with a large parameter

By

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§ 1. Introduction

The exact WKB analysis is a powerful method in studying both linear and non-linear differential equations which contain a large parameter in an appropriate way. The Noumi-Yamada system NY_l ($l = 2, 3, \ldots$) is one of important non-linear systems for which the exact WKB analysis works very well. Many important results (for example, see Y. Takei [4]) have been established, while some fundamental problems for the Stokes geometry still remain unsolved.

We were expected, as such a problem for the Noumi-Yamada systems, to show the existence of a 0-parameter formal solution of NY_l, or it is much better, to give a formula for the number of those solutions. Recently T. Aoki and N. Honda in [2] gave a complete answer for this problem. They have obtained, moreover, a formula for the number of turning points of the *first* kind of NY_l. Note that a turning point of NY_l consists of that of the first kind and that of the second kind, and the latter also plays an essential role in the Stokes geometry. In this paper, we present a formula for the number of turning points of the *second* kind of NY_l.

We emphasize that our formulas have not only their own theoretical interest but also practical meanings. In fact, when we calculate the concrete Stokes geometry numerically, these formulas confirm us that all the turning points are really obtained. This confirmation is inevitable because lack of an ordinary turning point generally makes a configuration of the Stokes geometry incomplete.

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Now we give a short description of our formulas whose detail forms are given in the last section: We first recall the explicit forms of the Noumi-Yamada systems NY_l ($l = 2, 3, \ldots$), which is a system of non-linear differential equations of $l+1$ unknown functions $u_0(t), \ldots, u_l(t)$ of the variable t with a large parameter η. The form of NY_l depends on the parity of l. The system NY_{2m} can be written in the form

$$\eta^{-1}\frac{du_j}{dt} = u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \alpha_j, \quad j = 0, 1, \ldots, 2m, \tag{1.1}$$

where α_j's are polynomials of η^{-1} with constant coefficients satisfying

$$\alpha_0 + \alpha_1 + \cdots + \alpha_{2m} = \eta^{-1}, \tag{1.2}$$

and the unknown functions u_j's satisfy the following normalization condition

$$u_0 + u_1 + \cdots + u_{2m} = t. \tag{1.3}$$

The system NY_{2m+1} can be written in the form

$$\eta^{-1}\frac{t}{2}\frac{du_j}{dt} = \begin{cases} u_j\left(\Pi_j + \dfrac{1}{2}\eta^{-1} - \alpha_{\text{even}}\right) + \dfrac{1}{2}\alpha_j t, & j = 0, 2, \ldots, 2m, \\ u_j\left(\Pi_j + \dfrac{1}{2}\eta^{-1} - \alpha_{\text{odd}}\right) + \dfrac{1}{2}\alpha_j t, & j = 1, 3, \ldots, 2m+1, \end{cases} \tag{1.4}$$

where we set

$$\begin{aligned} \Pi_j &= \sum_{1\le r\le s\le m} u_{j+2r-1}u_{j+2s} - \sum_{1\le s\le q\le m} u_{j+2s}u_{j+2q+1}, \\ \alpha_{\text{even}} &= \alpha_0 + \alpha_2 + \cdots + \alpha_{2m}, \quad \alpha_{\text{odd}} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2m+1}, \end{aligned} \tag{1.5}$$

and α_j's are polynomials of η^{-1} with constant coefficients satisfying

$$\alpha_0 + \alpha_1 + \cdots + \alpha_{2m+1} = \eta^{-1}, \tag{1.6}$$

and the unknown functions u_j's satisfy the following normalization conditions

$$u_0 + u_2 + \cdots + u_{2m} = \frac{t}{2}, \quad u_1 + u_3 + \cdots + u_{2m+1} = \frac{t}{2}. \tag{1.7}$$

In both cases the indices of α_j's and u_j's are considered to be elements of $\mathbb{Z}/(l+1)\mathbb{Z}$.

Then the main part of our theorem is summarized as follows (see also Theorems 4.1 and 4.2 in the last section):

Theorem 1.1. *For a generic parameter α, the number of turning points of the second kind of NY_l is exactly given by*

$$(1.8)\qquad \begin{cases} 2m(2m+1)\binom{2m}{m} - 3m2^{2m}, & \text{if } l = 2m, \\ (4m^2+6m+6)\binom{2m+1}{m} - 3(m+1)2^{2m+1}, & \text{if } l = 2m+1. \end{cases}$$

§ 2. Preparations

§ 2.1. The Stokes geometry of NY_l

We briefly recall several definitions related to the Stokes geometry for a non-linear equation. Let us first consider a formal solution $\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_l)$ of NY_l which is a formal power series of η^{-1} in the form

$$(2.1)\qquad \hat{u}(t) = \hat{u}^{(0)}(t) + \hat{u}^{(1)}(t)\eta^{-1} + \hat{u}^{(2)}(t)\eta^{-2} + \cdots,$$

where $\hat{u}^{(j)} = (\hat{u}_0^{(j)}, \hat{u}_1^{(j)}, \ldots, \hat{u}_l^{(j)})$, $j = 0, 1, \ldots$. The formal solution $\hat{u}$ is often called a 0-parameter solution of NY_l. We can easily see that the leading term $\hat{u}^{(0)}(t)$ satisfies the following algebraic equations of the variables $u = (u_0, u_1, \ldots, u_l)$:

- If l is even, then $\hat{u}^{(0)}(t)$ satisfies the normalization condition (1.3) and

$$(2.2)\qquad f_j = u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \alpha_j = 0, \qquad j = 0, 1, \ldots, 2m.$$

- If l is odd, then $\hat{u}^{(0)}(t)$ satisfies the normalization conditions (1.7) and

$$(2.3)\qquad f_j = \begin{cases} u_j(\Pi_j - \alpha_{\mathrm{even}}) + \frac{1}{2}\alpha_j t = 0, & j = 0, 2, \ldots, 2m, \\ u_j(\Pi_j - \alpha_{\mathrm{odd}}) + \frac{1}{2}\alpha_j t = 0, & j = 1, 3, \ldots, 2m+1. \end{cases}$$

Here we abbreviated $\hat{u}_j^{(0)}$ to u_j and $\alpha_{j,0}$ (the leading term of α_j) to α_j.

Let $\mathbb{C}_t \times \mathbb{C}_u^{l+1}$ be an affine complex space with a system of coordinates $(t; u_0, u_1, \ldots, u_l)$, and let $\mathcal{I}$ be the ideal defined by f_j's and the normalization conditions :

$$(2.4)\qquad \mathcal{I} = \begin{cases} \mathcal{O}_{\mathbb{C}_t \times \mathbb{C}_u^{l+1}}(f_0, f_1, \ldots, f_l, u_{\mathrm{total}} - t), & l : \text{even}, \\ \mathcal{O}_{\mathbb{C}_t \times \mathbb{C}_u^{l+1}}(f_0, f_1, \ldots, f_l, u_{\mathrm{even}} - \frac{t}{2}, u_{\mathrm{odd}} - \frac{t}{2}), & l : \text{odd}, \end{cases}$$

where $\mathcal{O}_{\mathbb{C}_t\times\mathbb{C}_u^{l+1}}$ denotes the sheaf of holomorphic functions on $\mathbb{C}_t\times\mathbb{C}_u^{l+1}$ and

$$u_{\text{total}}=\sum_{k=0}^{l}u_k,\quad u_{\text{even}}=\sum_{k=0}^{m}u_{2k},\quad u_{\text{odd}}=\sum_{k=0}^{m}u_{2k+1}. \tag{2.5}$$

We define the $\mathcal{O}_{\mathbb{C}_t\times\mathbb{C}_u^{l+1}}$-module $\mathcal{N}$ by

$$\mathcal{N}=\frac{\mathcal{O}_{\mathbb{C}_t\times\mathbb{C}_u^{l+1}}}{\mathcal{I}}, \tag{2.6}$$

and we set $V=\mathrm{Supp}(\mathcal{N})\subset\mathbb{C}_t\times\mathbb{C}_u^{l+1}$, i.e.,

$$V=\begin{cases}\{(t,\,u)\in\mathbb{C}_t\times\mathbb{C}_u^{l+1};\ f_0=0,\,\ldots,\,f_l=0,\ u_{\text{total}}=t\}, & l:\text{even},\\ \{(t,\,u)\in\mathbb{C}_t\times\mathbb{C}_u^{l+1};\ f_0=0,\,\ldots,\,f_l=0,\ u_{\text{even}}=\dfrac{t}{2},\ u_{\text{odd}}=\dfrac{t}{2}\}, & l:\text{odd}.\end{cases} \tag{2.7}$$

Let $(\triangle NY)_l$ denote the linearized equation of NY_l along a 0-parameter solution $\hat{u}(t)$, that is, by replacing u with $\hat{u}(t)+\triangle u$ in (1.1) (resp. (1.4)), we take its linear part with respect to the variable $\triangle u$ as the system $(\triangle NY)_l$ if l is even (resp. odd). Then the linear system $(\triangle NY)_l$ can be written in the form

$$\frac{\partial}{\partial t}\triangle u=\eta C(t,\,\eta)\triangle u,\quad C(t,\,\eta)=\sum_{k=0}^{\infty}\eta^{-k}C_k(t)=C_0(t)+\eta^{-1}C_1(t)+\cdots. \tag{2.8}$$

Here $C_k(t,\,\eta)$ is an $(l+1)\times(l+1)$ matrix whose entries are possibly multi-valued holomorphic functions of the variable t. Let J_u be the Jacobian matrix of functions $(f_0,\,f_1,\,\ldots,\,f_l)$ with respect to the variables u_i's:

$$J_u(f_0,\,f_1,\,\ldots,\,f_l)=\left(\frac{\partial f_i}{\partial u_j}\right)_{i=0,\cdots,l,j=0,\cdots,l.} \tag{2.9}$$

It is easy to see that the leading matrix C_0 coincides with the restriction $J_u|_V$ of J_u to V.

Let $\Lambda(\lambda,\,u)$ denote the characteristic polynomial of the leading matrix C_0:

$$\Lambda(\lambda,\,u)=\det(C_0-\lambda I_{l+1})=\det(J_u(f_0,\,f_1,\,\ldots,\,f_l)-\lambda I_{l+1}). \tag{2.10}$$

Definition 2.1. We say that $v=(t^*,\,u^*)\in V$ is a turning point if the discriminant of the characteristic polynomial $\Lambda(\lambda,\,u)$ vanishes at v.

It has been proved, in the paper [4], that $\Lambda(\lambda,\,u)$ has the following notable form

$$\Lambda(\lambda,\,u)=\begin{cases}\lambda\check{\Lambda}_{\text{even}}(\lambda,\,u), & l:\text{even},\\ (\lambda^2-\alpha_{\text{even}}^2)\check{\Lambda}_{\text{odd}}(\lambda,\,u), & l:\text{odd},\end{cases} \tag{2.11}$$

where $\check{\Lambda}_{\mathrm{even}}$ and $\check{\Lambda}_{\mathrm{odd}}$ are even functions of λ. Hence the equation $\check{\Lambda}_{\mathrm{even}}(\lambda, u) = 0$ (resp. $\check{\Lambda}_{\mathrm{odd}}(\lambda, u) = 0$) has m-pairs $(\lambda_i^+(u), \lambda_i^-(u))$ of roots with $\lambda_i^+(u) = -\lambda_i^-(u)$ $(i = 1, \ldots, m)$. At a turning point $v = (t^*, u^*) \in V$, as the discriminant of $\check{\Lambda}_{\mathrm{even}}(\lambda, u)$ (resp. $\check{\Lambda}_{\mathrm{odd}}(\lambda, u)$) vanishes, we can observe that one of the following situations occurs:

(i) For some index i, two roots λ_i^+ and λ_i^- merge at v. This is equivalent to saying that the condition $\check{\Lambda}_{\mathrm{even}}(0, u^*) = 0$ (resp. $\check{\Lambda}_{\mathrm{odd}}(0, u^*) = 0$) holds at v.

(ii) There exist mutually distinct indices i and j for which $\lambda_i^+ = \lambda_j^+$ and $\lambda_i^- = \lambda_j^-$ hold at v.

Therefore we have two kinds of turning points for non-linear equations. We call the former one a *turning point of the first kind*, and the latter one a *turning point of the second kind*. The following definition was introduced by Y.Takei [4].

Definition 2.2.

(i) A point $v = (t^*, u^*) \in V$ is said to be a turning point of the first kind if v satisfies $\check{\Lambda}_{\mathrm{even}}(0, u^*) = 0$ (resp. $\check{\Lambda}_{\mathrm{odd}}(0, u^*) = 0$).

(ii) A turning point other than the first kind is said to be of the second kind.

§ 2.2. The algebraic variety V associated with NY_l

In this subsection, we study the algebraic variety V associated with NY_l that was introduced in the previous subsection. This variety will be deeply investigated by T.Aoki and N.Honda in [2], and we review some results in the paper that are needed later.

Let $\mathbb{C}_t \times \mathbb{C}_\xi^{l+1}$ be an affine complex space with a system of coordinates $(t; \xi_0, \xi_1, \ldots, \xi_l)$. We define the $(l+1)$-polynomials g_k and ξ_{total} by

$$g_k = \frac{1}{4}(\xi_{k+1}^2 - \xi_k^2) + \alpha_k, \quad k = 0, 1, \ldots, l, \quad \xi_{\mathrm{total}} = \xi_0 + \xi_1 + \cdots + \xi_l. \tag{2.12}$$

Here the indices of ξ_i's are considered to be elements of $\mathbb{Z}/(l+1)\mathbb{Z}$. Let $\mathcal{I}_{t,\xi}$ (resp. $\mathcal{N}_{t,\xi}$) denote

$$\mathcal{O}_{\mathbb{C}_t \times \mathbb{C}_\xi^{l+1}}(g_0, g_1, \ldots, g_l, \xi_{\mathrm{total}} - t) \quad \left(\text{resp. } \frac{\mathcal{O}_{\mathbb{C}_t \times \mathbb{C}_\xi^{l+1}}}{\mathcal{I}_{t,\xi}}\right), \tag{2.13}$$

and we set $V_{t,\xi} = \mathrm{Supp}\,(\mathcal{N}_{t,\xi})$, i.e.,

$$V_{t,\xi} = \{g_0 = 0, \ldots, g_l = 0, \xi_{\mathrm{total}} = t\} \subset \mathbb{C}_t \times \mathbb{C}_\xi^{l+1}. \tag{2.14}$$

We also define the analytic map $\Psi : \mathbb{C}_t \times \mathbb{C}_\xi^{l+1} \to \mathbb{C}_t \times \mathbb{C}_u^{l+1}$ by

$$\Psi(t;\, \xi_0,\, \xi_1,\, \ldots,\, \xi_l) = \Bigl(t;\, \frac{\xi_0 + \xi_1}{2},\, \ldots,\, \frac{\xi_k + \xi_{k+1}}{2},\, \ldots,\, \frac{\xi_l + \xi_0}{2}\Bigr), \tag{2.15}$$

and the pull-back $\Psi^*(h)$ of a function h on $\mathbb{C}_t \times \mathbb{C}_u^{l+1}$ by

$$\Psi^*(h)(t,\, \xi) = h\bigl(\Psi(t,\, \xi)\bigr), \quad h \in \mathcal{O}_{\mathbb{C}_t \times \mathbb{C}_u^{l+1}}. \tag{2.16}$$

Note that Ψ gives an isomorphism if l is even, while it is not an isomorphism if l is odd. Therefore we need to restrict Ψ to a smaller space so that Ψ gives an isomorphism. Let $U_{t,u}$ (resp. $U_{t,\xi}$) be the open set

$$\{(t,\, u) \in \mathbb{C}_t \times \mathbb{C}_u^{l+1};\, t \neq 0\} \quad \Bigl(\text{resp. } \{(t,\, \xi) \in \mathbb{C}_t \times \mathbb{C}_\xi^{l+1};\, t \neq 0\}\Bigr). \tag{2.17}$$

Note that, since $t = 0$ is a singular point of NY_{2m+1}, we always exclude $t = 0$ if we consider the odd case. We set

$$W_{t,u} = \{(t,\, u) \in U_{t,u};\, u_{\text{even}} = \frac{t}{2},\, u_{\text{odd}} = \frac{t}{2}\}, \tag{2.18}$$

$$W_{t,\xi} = \{(t,\, \xi) \in U_{t,\xi};\, \xi_{\text{total}} - t = 0,\, \tau(\xi) - 4\alpha_{\text{even}} = 0\}, \tag{2.19}$$

where

$$\tau(\xi) = \sum_{0 \le k \le m} \xi_{2k}^2 - \sum_{0 \le k \le m} \xi_{2k+1}^2. \tag{2.20}$$

Then, if l is odd, the morphism Ψ gives an isomorphism between $W_{t,u}$ and $W_{t,\xi}$. We refer the readers to [2] for details. As a consequence, we have the following theorems that play an important role in studying our problem. Let $\pi : \mathbb{C}_t \times \mathbb{C}_\xi^{l+1} \to \mathbb{C}_t$ denote the canonical projection.

Theorem 2.3 ([2]). *We have the following equivalence*:

(i) *If $l = 2m$, the analytic set V is isomorphic to $V_{t,\xi}$. The morphism Ψ^* gives an isomorphism of $\pi^{-1}\mathcal{O}_{\mathbb{C}_t}$-module between $\Psi^{-1}\mathcal{N}$ and $\mathcal{N}_{t,\xi}$*

(ii) *If $l = 2m + 1$, the analytic set $V \cap U_{t,u}$ is isomorphic to $V_{t,\xi} \cap U_{t,\xi}$. The morphism Ψ^* gives an isomorphism of $\pi^{-1}\mathcal{O}_{\mathbb{C}_t}$-module between $\Psi^{-1}\mathcal{N}|_{W_{t,\xi}}$ and $\mathcal{N}_{t,\xi}|_{W_{t,\xi}}$.*

Theorem 2.4 ([4]). *The pull-back of the characteristic polynomial $\Lambda(\lambda,\, u)$ has the following form*:

(i) *If $l = 2m$, we have in $\mathbb{C}_t \times \mathbb{C}_\xi^{l+1}$:*

$$\Psi^*(\Lambda)(\lambda,\, \xi) = \frac{1}{2}\bigl(\Lambda^+(\lambda,\, \xi) + \Lambda^-(\lambda,\, \xi)\bigr), \tag{2.21}$$

$$\Lambda^+(\lambda,\, \xi) = \prod_{0 \le i \le 2m} (\lambda + \xi_i), \quad \Lambda^-(\lambda,\, \xi) = \prod_{0 \le i \le 2m} (\lambda - \xi_i). \tag{2.22}$$

(ii) *If $l = 2m+1$, we have in $W_{t,\xi}$:*

$$\Psi^*(\Lambda)(\lambda,\, \xi) = \frac{(\lambda^2 - \alpha_{\text{even}}^2)}{t^2\lambda}(\Lambda^+(\lambda,\, \xi) - \Lambda^-(\lambda,\, \xi)), \tag{2.23}$$

$$\Lambda^+(\lambda,\, \xi) = \prod_{0\le i\le 2m+1}\Big(\lambda + \frac{t\xi_i}{2}\Big), \quad \Lambda^-(\lambda,\, \xi) = \prod_{0\le i\le 2m+1}\Big(\lambda - \frac{t\xi_i}{2}\Big). \tag{2.24}$$

§ 2.3. The discriminant $\hat{D}(\xi)$ on the ξ-plane

It follows from Theorems 2.3 and 2.4 that we can reduce the algebraic equations on the space $\mathbb{C}_t \times \mathbb{C}_u^{l+1}$ to those on the space $\mathbb{C}_t \times \mathbb{C}_\xi^{l+1}$. In what follows, we consider the problems in the latter $(t,\, \xi)$ coordinates space, and we introduce a function $C(\lambda,\, \xi)$ on $\mathbb{C}_t \times \mathbb{C}_\xi^{l+1}$ that corresponds to the characteristic polynomial on the original space as follows:

$$C(\lambda,\, \xi) = \begin{cases} \dfrac{1}{2\lambda}\big(\Lambda^+(\lambda,\, \xi) + \Lambda^-(\lambda,\, \xi)\big), & l : \text{even}, \\[2ex] \dfrac{1}{t^2\lambda}(\Lambda^+(\lambda,\, \xi) - \Lambda^-(\lambda,\, \xi)), & l : \text{odd}. \end{cases} \tag{2.25}$$

Since the set of zero points of the discriminant of $C(\lambda,\, \xi)$ contains turning points of the first and the second kind, to obtain the number of those of the second kind correctly, we need to consider $C(\sqrt{\lambda},\, \xi)$ instead of $C(\lambda,\, \xi)$. We set

$$\hat{C}(\lambda,\, \xi) = \begin{cases} \dfrac{1}{2\sqrt{\lambda}}\Big(\displaystyle\prod_{0\le i\le 2m}(\sqrt{\lambda} + \xi_i) + \prod_{0\le i\le 2m}(\sqrt{\lambda} - \xi_i)\Big), & l = 2m, \\[2ex] \dfrac{1}{2t^2\sqrt{\lambda}}\Big(\displaystyle\prod_{0\le i\le 2m+1}(\sqrt{\lambda} + t\xi_i) - \prod_{0\le i\le 2m+1}(\sqrt{\lambda} - t\xi_i)\Big), & l = 2m+1. \end{cases} \tag{2.26}$$

Note that, as we always consider the problem on V (where $t = \xi_{\text{total}}$ holds), we may regard $\hat{C}$ as a function of the variables λ and ξ.

Let $\hat{D}(\xi)$ be the discriminant of $\hat{C}(\lambda,\, \xi)$. We first study several properties of the discriminant $\hat{D}(\xi)$.

Lemma 2.5. *The discriminant $\hat{D}(\xi)$ is a homogeneous polynomial of ξ with degree $4m(m-1)$ (resp. $2m(m-1)$) if $l = 2m+1$ (resp. $l = 2m$).*

Proof. We first consider the odd case, that is, $l = 2m + 1$ is odd. We have

(2.27)
$$\begin{aligned}\hat{C}(\lambda,\,\xi) &= \frac{1}{t^2}\{tS_1\lambda^m + t^3S_3\lambda^{m-1} + t^5S_5\lambda^{m-2} + \cdots + t^{2m-1}S_{2m-1}\lambda + t^{2m+1}S_{2m+1}\} \\ &= \lambda^m + \xi_{\mathrm{total}}S_3\lambda^{m-1} + \cdots + \xi_{\mathrm{total}}^{2k-1}S_{2k+1}\lambda^{m-k} + \cdots + \xi_{\mathrm{total}}^{2m-3}S_{2m-1}\lambda \\ &+ \xi_{\mathrm{total}}^{2m-1}S_{2m+1}.\end{aligned}$$

Here

$$S_{2k+1} = \sum_{0\le i_1<\cdots<i_{2k+1}\le 2m+1} \xi_{i_1}\cdots\xi_{i_{2k+1}}, \quad t = \xi_{\mathrm{total}}. \tag{2.28}$$

Hence $\hat{C}(\lambda,\,\xi)$ can be written in the form

$$\hat{C}(\lambda,\,\xi) = \lambda^m + a_2(\xi)\lambda^{m-1} + a_4(\xi)\lambda^{m-2} + \cdots + a_{2m-2}(\xi)\lambda + a_{2m}(\xi), \tag{2.29}$$

where $a_k(\xi)$ is a homogeneous polynomial of ξ with degree $2k$. Let $\gamma_i(\xi)$ be a root of the algebraic equation $\hat{C}(\lambda,\,\xi) = 0$. We easily obtain $\gamma_i(c\xi) = c^4\gamma_i(\xi)$ for any $c \in \mathbb{C}$. Therefore we find

$$\hat{D}(c\xi) = \prod_{i<j}(\gamma_i(c\xi) - \gamma_j(c\xi))^2 = \prod_{i<j} c^8(\gamma_i(\xi) - \gamma_j(\xi))^2 = c^{4m(m-1)}\hat{D}(\xi). \tag{2.30}$$

The proof has been completed for the odd case. Similarly we can prove the lemma for the even case. □

Noticing that, for an odd $l = 2m + 1$, we have

$$\hat{C}(t^2\lambda,\,\xi) = t^{2m-1}\frac{1}{2\sqrt{\lambda}}\Big(\prod_{0\le i\le 2m+1}(\sqrt{\lambda} + \xi_i) - \prod_{0\le i\le 2m+1}(\sqrt{\lambda} - \xi_i)\Big), \tag{2.31}$$

we set

$$G(\lambda,\,\xi) = \frac{\hat{C}(t^2\lambda,\,\xi)}{t^{2m-1}} = \frac{1}{2\sqrt{\lambda}}\Big(\prod_{0\le i\le 2m+1}(\sqrt{\lambda} + \xi_i) - \prod_{0\le i\le 2m+1}(\sqrt{\lambda} - \xi_i)\Big). \tag{2.32}$$

The function $G(\lambda,\,\xi)$ is often used in subsequent arguments.

§ 2.4. The compactification Z_ξ of V

Let $\mathbb{P}^{l+2}_{t,\xi}$ be a projective space with a system of homogeneous coordinates $(t,\,\xi_0,\,\ldots,\,\xi_l;\,\eta)$ where we identify $\mathbb{C}_t \times \mathbb{C}^{l+1}_\xi$ with $\{\eta \neq 0\}$. We set

(2.33)
$$\begin{aligned} g_k &= \frac{1}{4}(\xi_{k+1}^2 - \xi_k^2) + \alpha_k\eta^2, \qquad k = 0,\,1,\,\ldots,\,l, \\ \xi_{\mathrm{total}} &= \xi_0 + \xi_1 + \cdots + \xi_l.\end{aligned}$$

We always assume that the indices of ξ_j's are considered to be elements of $\mathbb{Z}/(l+1)\mathbb{Z}$. Let $\check{V}_{t,\xi} \subset \mathbb{P}^{l+2}_{t,\xi}$ be the analytic subset defined by

$$\check{V}_{t,\xi} = \{g_0 = 0,\, g_1 = 0,\, \ldots,\, g_l = 0,\, \xi_{\text{total}} = t\}. \tag{2.34}$$

We denote by $H_{t,\xi} \subset \mathbb{P}^{l+2}_{t,\xi}$ the hyperplane $\xi_{\text{total}} - t = 0$. Note that we have $\check{V}_{t,\xi} \subset H_{t,\xi}$.

We also consider the projective space $\mathbb{P}^{l+1}_{\xi}$ with a system of homogeneous coordinates $(\xi_0,\, \xi_1,\, \ldots,\, \xi_l;\, \eta)$ where we identify $\mathbb{C}^{l+1}_{\xi}$ with $\{\eta \neq 0\}$. Let $Z_\xi \subset \mathbb{P}^{l+1}_{\xi}$ denote the analytic subset defined by

$$Z_\xi = \{g_0 = 0,\, g_1 = 0,\, \ldots,\, g_l = 0\}. \tag{2.35}$$

For any fixed $\hat{t} \in \mathbb{C}$, we define the hyperplane $H_{\hat{t}}$ by

$$H_{\hat{t}} = \{(\xi\,;\,\eta) \in \mathbb{P}^{l+1}_{\xi} \mid \xi_{\text{total}} = \hat{t}\eta\}, \tag{2.36}$$

and, in particular, we set

$$H_0 = \{(\xi\,;\,\eta) \in \mathbb{P}^{l+1}_{\xi} \mid \xi_{\text{total}} = 0\} \quad \text{and} \quad H_\infty = \{(\xi\,;\,\eta) \in \mathbb{P}^{l+1}_{\xi} \mid \eta = 0\}. \tag{2.37}$$

Since the morphism

$$\begin{array}{rcl} \iota : \mathbb{P}^{l+1}_{\xi} & \longrightarrow & H_{t,\xi} \subset \mathbb{P}^{l+2}_{t,\xi} \\ (\xi;\, \eta) & \longmapsto & (\xi_{\text{total}},\, \xi;\, \eta) \end{array} \tag{2.38}$$

is an isomorphism, we get $\check{V}_{t,\xi} \simeq Z_\xi$ (resp. $\check{V}_{t,\xi}\backslash\{t = 0\} \simeq Z_\xi\backslash H_0$) if l is even (resp. odd). Our geometric situation in $\mathbb{P}^{l+2}_{t,\xi}$ can be easily reduced to that in $\mathbb{P}^{l+1}_{\xi}$ by this morphism. Hence, in what follows, we consider the problems in $\mathbb{P}^{l+1}_{\xi}$. In particular, as the discriminant $\hat{D}(\xi)$ of $\hat{C}(\lambda,\, \xi)$ is a homogeneous polynomial of ξ, we can regard $\hat{D}(\xi)$ as a divisor on $\mathbb{P}^{l+1}_{\xi}$.

Now we briefly recall some properties of the analytic subset $Z_\xi \subset \mathbb{P}^{l+1}_{\xi}$ that will be needed later. Let P be a parameter space defined by

$$P = \{(\alpha_0,\, \alpha_1,\, \ldots,\, \alpha_l) \in \mathbb{C}^{l+1};\, \alpha_0 + \alpha_1 + \cdots + \alpha_l = 0\}. \tag{2.39}$$

For any natural number l, we set

$$E^l_i = \{(\alpha_0,\, \ldots,\, \alpha_l) \in P;\, \alpha(i\,;\,0)\alpha(i\,;\,1)\cdots\alpha(i\,;\,l-1) = 0\}, \tag{2.40}$$

and

$$E^l_{\text{cup}} = \bigcup_{0 \le i \le l} E^l_i, \quad E^l_{\text{cap}} = \bigcap_{0 \le i \le l} E^l_i, \tag{2.41}$$

where

$$\alpha(i\,;\,k) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k}, \quad i,\,k \geq 0. \tag{2.42}$$

The following Lemma 2.6 will be proved in [2].

Lemma 2.6. *We have*

(i) Z_ξ *is a* (*not necessarily irreducible*) *complex analytic subset with* $\dim(Z_\xi) = 1$.

(ii) *If* $\alpha \in P \backslash E^l_{\mathrm{cap}}$, *then* $H_{\hat{t}}$ *and* Z_ξ *properly intersect for any* $\hat{t} \in \mathbb{C} \cup \{\infty\}$.

(iii) *If* $\alpha \in P \backslash E^l_{\mathrm{cup}}$, *then* Z_ξ *is connected and smooth.*

Taking Lemma 2.6 into account, we obtain Lemma 2.7 whose proof will be given in Section 4.

Lemma 2.7. *If* $\alpha \in P \backslash E^l_{\mathrm{cup}}$, *then* $\{\hat{D}(\xi) = 0\} \cap Z_\xi$ *is a finite set.*

Thanks to the above lemmas, we can apply the Bézout theorem [3] to our situation if $\alpha \in P \backslash E^l_{\mathrm{cup}}$, and we can obtain the number of zero points of $\hat{D}(\xi)$ in Z_ξ. Those points, however, contain many irrelevant ones which come from the compactification of $\mathbb{C}_t \times \mathbb{C}^{l+1}_\xi$. As a matter of fact, to obtain a formula for the number of turning points of the second kind, we should exclude those points contained in H_∞ (resp. $H_\infty \cup H_0$) if l is even (resp. odd).

§ 3. Irrelevant zero points of $\hat{D}|_{Z_\xi}$

In this section, we obtain the number of zero points of $\hat{D}|_{Z_\xi}$ in $H_\infty \cup H_0$ (for an odd l) or in H_∞ (for an even l) that are considered to be irrelevant for our formulas. We first study the odd case. Suppose that l is an odd integer. We need to know, in this case, a precise estimate of the each number of the zero points contained in the following locally closed analytic subsets in $\mathbb{P}^{l+1}_\xi$:

$$(Z_\xi \cap H_\infty) \backslash H_0, \qquad Z_\xi \cap H_\infty \cap H_0 \quad \text{and} \quad (Z_\xi \backslash H_\infty) \cap H_0.$$

Note that, to give the similar formula for the even case, we need to calculate the number of zero points only in $(Z_\xi \cap H_\infty) \setminus H_0$ because $t = 0$ is not a singular point of NY_{2m} and because, as we show below, $Z_\xi \cap H_\infty \cap H_0$ is empty if l is even.

Before we are going to study zero points of $\hat{D}|_{Z_\xi}$, we need some preparations. We first recall the definition of intersection multiplicity number. Let X be a complex manifold, and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be coherent $\mathcal{O}_X$-modules. We set $W_1 = \mathrm{Supp}(\mathcal{M}_1)$ and

$W_2 = \mathrm{Supp}(\mathcal{M}_2)$, and we suppose $\dim_p(W_1 \cap W_2) = 0$ for any point $p \in W_1 \cap W_2$. Then we define *an intersection multiplicity number* of $\mathcal{M}_1$ and $\mathcal{M}_2$ at $p \in W_1 \cap W_2$ by

$$\mathrm{mult}_p(\mathcal{M}_1,\, \mathcal{M}_2) = \sum_{k \geq 0} (-1)^k \dim_{\mathbb{C}} \left(\mathrm{Tor}^k_{\mathcal{O}_X}(\mathcal{M}_1,\, \mathcal{M}_2)_p \right). \tag{3.1}$$

Note that, by virtue of the fact that $\dim_p(W_1 \cap W_2) = 0$, the $\mathcal{O}_{X,p}$-module $\mathrm{Tor}^k_{\mathcal{O}_X}(\mathcal{M}_1,\, \mathcal{M}_2)_p$ is a finite dimensional $\mathbb{C}$-vector space. For any analytic subsets Y_1 and Y_2 which have a perfect intersection at p, we set

$$\mathrm{mult}_p(Y_1,\, Y_2) = \mathrm{mult}_p \left(\frac{\mathcal{O}_X}{\mathcal{I}_{Y_1}},\, \frac{\mathcal{O}_X}{\mathcal{I}_{Y_2}} \right), \tag{3.2}$$

where $\mathcal{I}_{Y_i}$ denotes the defining ideal of Y_i $(i = 1,\, 2)$. In the same way, for an analytic subset Y with $\dim(Y) = 1$ and a divisor D in X that is not identically zero on any irreducible component of Y, we also set

$$\mathrm{mult}_p(Y,\, D) = \mathrm{mult}_p \left(\frac{\mathcal{O}_X}{\mathcal{I}_Y},\, \frac{\mathcal{O}_X}{(f_D)} \right), \tag{3.3}$$

where f_D is a local defining function of D near p. Moreover, if Y is smooth near p with a local coordinate(s) (where the point p corresponds to $s = 0$), then we have

$$\mathrm{mult}_p(Y,\, D) = \nu_{\{s=0\}}(f_D(s)). \tag{3.4}$$

Here $f_D(s)$ is the restriction of a local defining function f_D of D to Y that is regarded as a holomorphic function on Y, and $\nu_{\{s=0\}}(g(s))$ denotes the order of a zero point of a holomorphic function $g(s)$ at $s = 0$. Thus the intersection multiplicity number defined above is a natural extension of the classical one.

Let l be an integer greater than 1, i.e., $l = 2m$ or $l = 2m + 1$ for $m \geq 1$. We can easily see that, by putting $\eta = 0$ into (2.35), any point $p \in Z_\xi \cap H_\infty$ can be expressed as

$$p = (\sigma_0,\, \sigma_1,\, \ldots,\, \sigma_l;\, 0) \in \mathbb{P}^{l+1}_\xi, \quad \sigma_0 = 1, \quad \sigma_i = \pm 1, \quad i = 1,\, \ldots,\, l. \tag{3.5}$$

Then we set

$$\begin{aligned} l_+(p) &= \#\{i;\, \sigma_i = 1\}, \\ l_-(p) &= \#\{i;\, \sigma_i = -1\}, \end{aligned} \tag{3.6}$$

where $\#G$ denotes the number of elements in a set G, and we define the number $l(p)$ by

$$l(p) = \min\{l_-(p),\, l_+(p)\}. \tag{3.7}$$

Note that a point in $Z_\xi \cap H_\infty \cap H_0$ is characterized, in terms of $l_\pm(p)$, as

$$p \in Z_\xi \cap H_\infty \cap H_0 \iff l_+(p) = l_-(p). \tag{3.8}$$

Therefore, if l is even, then $Z_\xi \cap H_\infty \cap H_0$ is empty because $l_+(p) + l_-(p) = l + 1$ holds.

Let p be a point in $Z_\xi \cap H_\infty$. Then we can take

$$(1, \xi_1, \xi_2, \ldots, \xi_l; \eta) \in \mathbb{P}_\xi^{l+1} \tag{3.9}$$

as a system of local coordinates of $\mathbb{P}_\xi^{l+1}$ in a neighborhood U_p of p. Now let us recall the definition of $\hat{C}(\lambda, \xi)$ given in (2.26), and let $\lambda_i(\xi)$ $(i = 1, \ldots, m)$ be a root of $\hat{C}(\lambda, \xi) = 0$ where we fix $\xi_0 = 1$. Then $\lambda_i(\xi)$ is a multi-valued holomorphic function on U_p and we can easily see

$$\hat{D}|_{U_p} = \prod_{i<j} (\lambda_i(\xi) - \lambda_j(\xi))^2 \qquad \text{for } \xi = (1, \xi_1, \ldots, \xi_l). \tag{3.10}$$

The following lemma is the most fundamental result for a root of $\hat{C}(\lambda, \xi) = 0$ at $\xi = p$.

Lemma 3.1. *Let p be a point in $Z_\xi \cap H_\infty$. We assume $p \notin H_0$ (note that this assumption is always satisfied if l is even as we have already mentioned it). Then we have*

(i) *If $l(p) > 1$, then $l(p)$-roots of the equation $\hat{C}(\lambda, p) = 0$ coincide and the other roots are simple.*

(ii) *If either $l(p) = 0$ or $l(p) = 1$, all the roots of $\hat{C}(\lambda, p) = 0$ are simple.*

Proof. Since the proof for the even case is the same as that for the odd case, we give the proof only for the odd case. By the assumption $p \notin H_0$, we may consider the equation $G(\lambda, p) = 0$ instead of $\hat{C}(\lambda, p) = 0$ where $G(\lambda, \xi)$ was defined by (2.32).

Assume the point p is given by coordinates (3.5). Then, as $p \notin H_0$ implies $\sigma_0 + \sigma_1 + \cdots + \sigma_l \neq 0$, we have $G(0, p) = \sigma_0 \ldots \sigma_l \times (\sigma_0 + \sigma_1 + \cdots + \sigma_l) \neq 0$. Hence $\lambda = 0$ is not a root of $G(\lambda, p) = 0$, and we also suppose $\lambda \neq 0$ in what follows.

Set $d = 2m + 2 - 2l(p)$, which is always positive by the assumption. Then $G(\lambda, p)$ is written in the form

$$G(\lambda, p) = \begin{cases} \dfrac{1}{2\sqrt{\lambda}}(\lambda - 1)^{l(p)}\{(\sqrt{\lambda}+1)^d - (\sqrt{\lambda}-1)^d\}, & \text{if} \quad l(p) = l_-(p), \\ -\dfrac{1}{2\sqrt{\lambda}}(\lambda - 1)^{l(p)}\{(\sqrt{\lambda}+1)^d - (\sqrt{\lambda}-1)^d\}, & \text{if} \quad l(p) = l_+(p). \end{cases} \tag{3.11}$$

To show the claim of the lemma, it suffices to investigate a common root of $G(\lambda, p) = 0$ and $\dfrac{\partial G}{\partial \lambda}(\lambda, p) = 0$. We set

$$g(\lambda) = \frac{1}{2\sqrt{\lambda}}\{(\sqrt{\lambda}+1)^d - (\sqrt{\lambda}-1)^d\}, \quad h(\lambda) = (\lambda-1)^{l(p)}. \tag{3.12}$$

Since $g(\lambda) = 0$ and $h(\lambda) = 0$ do not have any common root, if a common root λ_0 exists, then λ_0 satisfies either Case 1 or Case 2 below:

Case 1. $h(\lambda_0) = 0$ and $\dfrac{\partial h}{\partial \lambda}(\lambda_0) = 0$.

Case 2. $g(\lambda_0) = 0$ and $\dfrac{\partial g}{\partial \lambda}(\lambda_0) = 0$.

It is clear that Case 1 occurs if and only if $l(p) > 1$. In this situation, $h(\lambda) = 0$ and $\dfrac{\partial h}{\partial \lambda}(\lambda) = 0$ share exactly $(l(p) - 1)$-common roots. While we can easily observe that Case 2 never occurs. Indeed, as we have

$$\frac{\partial g}{\partial \lambda}(\lambda) = -\frac{1}{2\lambda}g(\lambda) + \frac{d}{4\lambda}\{(\sqrt{\lambda}+1)^{d-1} - (\sqrt{\lambda}-1)^{d-1}\}, \tag{3.13}$$

Case 2 is equivalent to

$$(\sqrt{\lambda_0}+1)^{d-1} - (\sqrt{\lambda_0}-1)^{d-1} = 0, \quad (\sqrt{\lambda_0}+1)^d - (\sqrt{\lambda_0}-1)^d = 0. \tag{3.14}$$

Since

$$x^d - y^d = x^{d-1} - y^{d-1} = 0 \tag{3.15}$$

have a unique solution $(x, y) = (0, 0)$, such a λ_0 never exists. This completes the proof. □

§ 3.1. Zero points in $(Z_\xi \cap H_0)\backslash H_\infty$

In this subsection 3.1 and in the following subsection 3.2, we assume that l is an odd integer: $l = 2m + 1$.

Let p be a point in $(Z_\xi \cap H_0)\backslash H_\infty$.

Theorem 3.2. *Assume $\alpha \in P\backslash E^l_{\mathrm{cup}}$. Then we have*

$$\mathrm{mult}_p(Z_\xi, \hat{D}) \geq 2\,\mathrm{mult}_p(Z_\xi, H_0)(m-1)^2. \tag{3.16}$$

For a generic parameter α, we obtain

$$\mathrm{mult}_p(Z_\xi, \hat{D}) = 2(m-1)^2. \tag{3.17}$$

Proof. Let $p = (\xi_0, \xi_1, \ldots, \xi_{2m+1}; 1) \in (Z_\xi \cap H_0) \setminus H_\infty$. It follows from Lemma 2.6 that Z_ξ is a smooth variety. Let (s) be a local coordinate system of Z_ξ near the point p. We denote by $\xi(s)$ (resp. $\xi_{\mathrm{total}}(s)$) the restriction of ξ (resp. ξ_{total}) to Z_ξ which can be regarded as a holomorphic function of the variable s. By (ii) in Lemma 2.6, there exist a positive integer τ and a non-zero constant c_τ for which $\xi_{\mathrm{total}}(s)$ has an expansion at $s = 0$ in the form

$$\xi_{\mathrm{total}}(s) = c_\tau s^\tau + c_{\tau+1} s^{\tau+1} + \cdots . \tag{3.18}$$

Then we obtain

$$\hat{C}(\lambda; \xi(s)) = \lambda^m + \sum_{k=1}^{m} \xi_{\mathrm{total}}(s)^{2k-1} S_{2k+1}(\xi(s)) \lambda^{m-k}, \tag{3.19}$$

where $S_j(\xi)$ denotes the j-th symmetric polynomial of the variable ξ, i.e.,

$$S_j(\xi) = \sum_{0 \le i_1 < \cdots < i_j \le 2m+1} \xi_{i_1} \cdots \xi_{i_j}, \quad 1 \le j \le 2m+2. \tag{3.20}$$

Since we have

$$\nu_{\{s=0\}}(\xi_{\mathrm{total}}^{2k-1}(s) S_{2k+1}(\xi(s))) \ge (2k-1)\tau, \quad 1 \le k \le m, \tag{3.21}$$

the Newton polygon of $\hat{C}(\lambda; \xi(s))$ is contained in the region described in Fig. 3.1 below. Then a root $\lambda_i(s)$ of $\hat{C}(\lambda; \xi(s)) = 0$ has a Puiseux expansion at $s = 0$ as

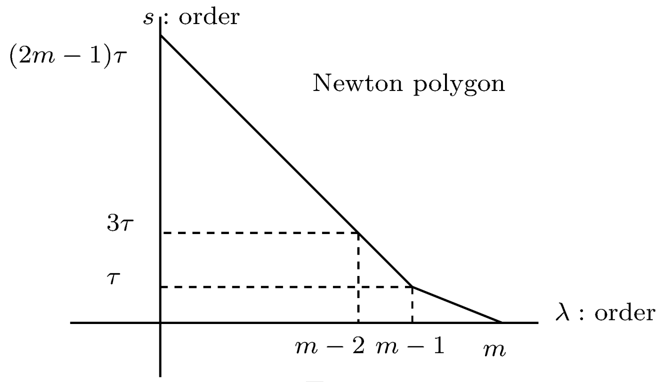

Figure. 3.1

$$\lambda_i(s) = c_i s^{d_i} + \cdots , \quad c_i \neq 0, \quad d_i \in \mathbb{Q}_+, \quad 1 \le i \le m. \tag{3.22}$$

Here d_j's satisfy the conditions

$$\begin{aligned} &1 \le d_1 \le d_2 \le \cdots \le d_m, \\ &d_1 + \cdots + d_k \ge (2k-1)\tau, \quad 1 \le k \le m. \end{aligned} \tag{3.23}$$

Noticing

(3.24)
$$\hat{D}(\xi(s)) = \prod_{i<j}(\lambda_i - \lambda_j)^2 \quad = \prod_{1<j}(\lambda_1 - \lambda_j)^2 \prod_{2<j}(\lambda_2 - \lambda_j)^2 \cdots \prod_{m-1<j}(\lambda_{m-1} - \lambda_j)^2,$$

we have

$$\begin{aligned} \nu_{\{s=0\}}(\hat{D}(\xi(s))) &\geq 2\big((m-1)d_1 + (m-2)d_2 + \cdots + d_{m-1}\big) \\ &= 2\big(d_1 + (d_1 + d_2) + \cdots + (d_1 + \cdots + d_{m-1})\big) \\ &\geq 2(1 + 3 + \cdots + 2m - 3)\tau = 2\tau(m-1)^2. \end{aligned} \tag{3.25}$$

Hence we have obtained the estimate (3.16).

We shall prove that (3.17) holds for a generic parameter. To prove this, it is enough to show that the conditions (i), (ii) and (iii) below are satisfied for a generic parameter.

(i) $S_{2m+1}(\xi(0)) \neq 0$,

(ii) $S_3(\xi(0)) \neq 0$,

(iii) The algebraic equation of the variable X

$$S_3(\xi(0))X^{m-1} + S_5(\xi(0))X^{m-2} + \cdots + S_{2m+1}(\xi(0)) = 0 \tag{3.26}$$

has mutually distinct $(m-1)$-roots.

Since we have

$$S_{2m+1}(\xi) = \sum_{k=0}^{2m+1} \xi_0 \cdots \check{\xi}_k \cdots \xi_{2m+1} = 2^{2m+1} J_\xi(g_0, \ldots, g_{l-1}, \xi_{\mathrm{total}}), \tag{3.27}$$

where J_ξ denotes the Jacobian matrix of functions $(g_0, \ldots, g_{l-1}, \xi_{\mathrm{total}})$ with respect to the variables ξ_j's and since Z_ξ is smooth, i.e., $J_\xi \neq 0$ on Z_ξ, the condition (i) is clearly satisfied for a parameter $\alpha \notin E^l_{\mathrm{cup}}$.

Let us show that the conditions (ii) and (iii) are also satisfied for a generic parameter. For this purpose, we prepare a few lemmas. Let T denote the closed analytic subset defined by

$$T = \left\{(\xi, \eta; \alpha) \in \mathbb{P}^{l+1}_\xi \times P;\, g_0 = 0,\, g_1 = 0,\, \ldots,\, g_l = 0,\, \xi_{\mathrm{total}} = 0\right\}. \tag{3.28}$$

We set

$$\overline{T} = \overline{T \setminus (H_\infty \times P)} \subset \mathbb{P}^{l+1}_\xi \times P \tag{3.29}$$

and

$$E^{(1),l} = \bigcup_{1+\sigma_1+\sigma_2+\cdots+\sigma_l=0,\sigma_i=\pm 1} \{(\alpha_0, \ldots, \alpha_l) \in P;\ \sigma_1\beta_1+\sigma_2\beta_2+\cdots+\sigma_l\beta_l = 0\}, \tag{3.30}$$

where β_i denotes

$$\beta_i = 4\alpha(0;\, i-1) = 4(\alpha_0 + \alpha_1 + \cdots + \alpha_{i-1}), \quad 1 \le i \le l. \tag{3.31}$$

Note that $E^{(1),l}$ is a closed analytic subset with $\dim(E^{(1),l}) = l-1$. Let $\pi : \mathbb{P}^{l+1} \times P \to P$ denote the canonical projection.

Lemma 3.3. *For any $\alpha \notin E^{(1),l}$ we have*

$$\left(\overline{T} \cap (H_\infty \times P)\right) \cap \pi^{-1}(\alpha) = \phi. \tag{3.32}$$

Proof. First, as we already observed at the beginning of this section, a point $p \in \overline{T} \cap (H_\infty \times P)$ has the coordinates

$$(1, \sigma_1, \ldots, \sigma_l;\, 0;\, \alpha) \tag{3.33}$$

for some α and $\sigma_i = \pm 1$ with $1 + \sigma_1 + \cdots + \sigma_l = 0$, and Z_ξ near p is described by a local coordinate(s) in the following way:

$$\eta = s, \qquad \xi_0 = 1, \quad \xi_i = \sigma_i\sqrt{1 - \beta_i s^2}, \quad 1 \le i \le l. \tag{3.34}$$

Note that $\xi_{\mathrm{total}}(s)$ has also an expansion with respect to s in the form

$$\xi_{\mathrm{total}}(s) = -\frac{1}{2}(\sigma_1\beta_1 + \cdots + \sigma_l\beta_l)s^2 - \frac{1}{4}(\sigma_1\beta_1^2 + \cdots + \sigma_l\beta_l^2)s^4 - \cdots. \tag{3.35}$$

Now suppose that (3.32) were false. Then we can find sequences $s^{(k)} \in \mathbb{C}\backslash\{0\}$ and $\alpha^{(k)} \in P$ $(k = 1, 2, \ldots)$ that satisfy

$$(\xi_0^{(k)}, \ldots, \xi_l^{(k)};\, s^{(k)};\, \alpha^{(k)}) \in Z_\xi \times P \xrightarrow[k\to\infty]{} (1, \sigma_1, \ldots, \sigma_l;\, 0;\, \alpha)$$

where $\xi_i^{(k)}$ are given by

$$\xi_0^{(k)} = 1, \quad \xi_i^{(k)} = \sigma_i\sqrt{1 - \beta_i^{(k)}(s^{(k)})^2}, \quad 1 \le i \le l. \tag{3.36}$$

Taking (3.35) and (3.36) into account, we have for any $k = 1, 2, \ldots$

$$0 = (s_{(k)})^2 \left\{ \left(\sigma_1\beta_1^{(k)} + \cdots + \sigma_l\beta_l^{(k)}\right) + \frac{1}{2}\left(\sigma_1(\beta_1^{(k)})^2 + \cdots + \sigma_l(\beta_l^{(k)})^2\right)\left(s^{(k)}\right)^2 + \cdots \right\}. \tag{3.37}$$

Hence, by letting $k \to \infty$, we obtain $\sigma_1\beta_1 + \sigma_2\beta_2 + \cdots + \sigma_l\beta_l = 0$. This contradicts to the assumption $\alpha \notin E^{(1),l}$. The proof has been completed. $\square$

Let us consider an algebraic equation of the variable X with coefficients $a_0, \ldots, a_{m-1}$

$$a_0X^{m-1} + a_1X^{m-2} + \cdots + a_{m-1} = 0, \tag{3.38}$$

and let $D_{m-1}(a_0, \ldots, a_{m-1})$ be a polynomial of $a_0, \ldots, a_{m-1}$ defined by

$$D_{m-1}(a_0, \ldots, a_{m-1}) = a_{m-1} \times \text{ the discriminant of (3.38)}\ . \tag{3.39}$$

Note that, since $D_{m-1}(S_3(\xi), S_5(\xi), \ldots, S_{2m+1}(\xi))$ is a homogeneous polynomial of ξ, we can regard it as a divisor on $\mathbb{P}_\xi^{2m+2} \times P$. Let F denote the analytic set defined by

$$F = \{(\xi, \eta; \alpha) \in \overline{T};\ D_{m-1}(S_3(\xi), S_5(\xi), \ldots, S_{2m+1}(\xi)) = 0\} \subset \mathbb{P}_\xi^{2m+2} \times P. \tag{3.40}$$

Then we have

Lemma 3.4. $\dim(F \cap (\mathbb{C}_\xi^{2m+2} \times P)) = 2m.$

Proof. We may assume $\eta = 1$. Since $\alpha_i = -\dfrac{1}{4}(\xi_{i+1}^2 - \xi_i^2)$, $i = 0, 1, \ldots, l$ holds on $\overline{T} \cap (\mathbb{C}^{2m+2} \times P)$, we obtain an isomorphism

$$F \cap (\mathbb{C}_\xi^{2m+2} \times P) \simeq \{\xi \in \mathbb{C}_\xi^{2m+2};\ \xi_{\mathrm{total}} = 0,\ D_{m-1}(S_3(\xi), S_5(\xi), \ldots, S_{2m+1}(\xi)) = 0\}. \tag{3.41}$$

The morphism $\rho : \mathbb{C}_\xi^{2m+2} \to \mathbb{C}_w^{2m+2}$ defined by

$$w_i = S_i(\xi), \qquad i = 1, 2, \ldots, 2m+2 \tag{3.42}$$

is clearly proper, surjective and has a finite fiber. Let K be an analytic subset in $\mathbb{C}_w^{2m+2}$ defined by

$$K = \{(w_1, w_2, \ldots, w_{2m+2});\ w_1 = 0, \quad D_{m-1}(w_3, w_5, \ldots, w_{2m+1}) = 0\}\,. \tag{3.43}$$

Then, as $\dim(K) = 2m$ and ρ has a finite fiber, we have $\dim(\rho^{-1}(K)) = 2m$. Hence we have the result. $\square$

We back to the proof for Theorem 3.2. If $\alpha \notin \pi(F)$, then

$$D_{m-1}(S_3(\xi), S_5(\xi), \ldots, S_{2m+1}(\xi)) \neq 0$$

holds for any point ξ in $(Z_\xi \cap H_0) \setminus H_\infty$, that is, the conditions (ii) and (iii) are satisfied for such a parameter. Hence, to finish the proof, it suffices to show

$$\dim(\pi(F)) \leq 2m. \tag{3.44}$$

This fact can be proved in the following way: By Lemma 3.3 we have

$$\pi(\overline{T} \cap (H_\infty \times P)) \subset E^{(1),l}. \tag{3.45}$$

Since $\pi|_{\overline{T}} : \overline{T} \to P$ is a proper map with a finite fiber and $\dim(E^{(1),l}) \leq 2m$, we have $\dim(\overline{T} \cap (H_\infty \times P)) \leq 2m$. In particular, noticing $F \subset \overline{T}$, we get

$$\dim(F \cap (H_\infty \times P)) \leq 2m. \tag{3.46}$$

Then, as $\dim(\pi(F))$ is less than or equal to $\dim(F)$, (3.44) follows from Lemma 3.4 and (3.46). This completes the proof. □

§ 3.2. Zero points in $Z_\xi \cap H_\infty \cap H_0$

Throughout this subsection we suppose that $l = 2m + 1$ is odd. Let p be a point in $Z_\xi \cap H_\infty \cap H_0$.

Theorem 3.5. *Assume $\alpha \in P \backslash E^l_{\mathrm{cup}}$. Then we have*

$$\mathrm{mult}_p(Z_\xi,\, \hat{D}) \geq 6m(m-1) + 2(2m^2 - 4m + 1 + d(p))d(p), \tag{3.47}$$

where $d(p) = \dfrac{1}{2}\,\mathrm{mult}_p(Z_\xi,\, H_0) - 1$. For a generic parameter α we obtain

$$\mathrm{mult}_p(Z_\xi,\, \hat{D}) = 6m(m-1). \tag{3.48}$$

Proof. Since Z_ξ is smooth by the assumption, Z_ξ is described by a local coordinate system as follows.

$$\eta = s, \quad \xi_0 = 1, \quad \xi_i = \sigma_i \sqrt{1 - \beta_i s^2}, \quad 1 \leq i \leq l. \tag{3.49}$$

Here β_i was given in (3.31). Since $\nu_{\{s=0\}}(\xi_{\mathrm{total}}(s)) = 2d(p) + 2$ holds by the definition of an intersection multiplicity number (see (3.4)), the expansion of $\xi_{\mathrm{total}}(s)$ at $s = 0$ can be written in the form

$$\xi_{\mathrm{total}}(s) = \kappa s^{2+2d(p)} + \kappa_1 s^{4+2d(p)} + \cdots, \qquad \kappa \neq 0. \tag{3.50}$$

Let us recall the definition of $G(\lambda,\, \xi)$ given in (2.32) and we consider its expansion at $\lambda = 1$:

$$G(\lambda,\, \xi) = \tilde{S}_1(\xi)(\lambda-1)^m + \tilde{S}_3(\xi)(\lambda-1)^{m-1} + \cdots + \tilde{S}_{2m-1}(\xi)(\lambda-1) + \tilde{S}_{2m+1}(\xi). \tag{3.51}$$

Lemma 3.6. *We have*

$$\nu_{\{s=0\}}(\tilde{S}_{2k+1}(\xi(s))) \geq \max\{2(k+1),\, 2(1 + d(p))\}, \quad k = 0,\, 1,\, 2,\, \ldots,\, m. \tag{3.52}$$

The proof for Lemma 3.6 will be given at the end of the subsection. Let us continue the proof for Theorem 3.5. Note that the above lemma particularly implies $\nu_{\{s=0\}}(\tilde{S}_{2k+1}(\xi(s))) \geq 2$ for any $k = 0, 1, \ldots, m$, and thus, the function

$$\hat{G}(\lambda, s) = \frac{1}{s^2} G(\lambda, \xi(s)) \tag{3.53}$$

is holomorphic at $s = 0$. Then, by the lemma, we can easily observe that the Newton polygon of $\hat{G}(\lambda, s)$ is contained in the region described in the Fig.3.2 below:

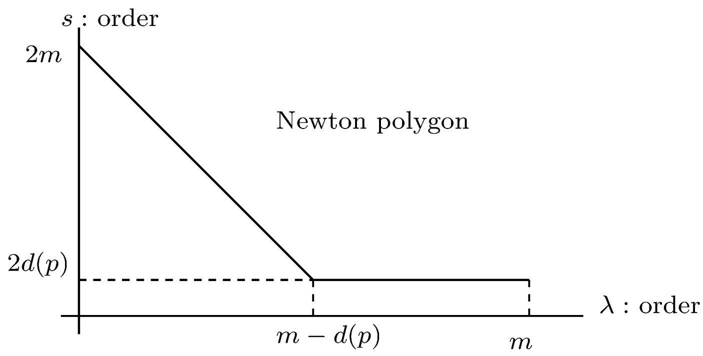

Figure.3.2

Hence a root $\hat{\lambda}_i(s)$ of the equation $\hat{G}(\lambda, s) = 0$ has a Puiseux expansion at $s = 0$ in the form

$$\hat{\lambda}_i(s) = 1 + c_i s^{d_i} + \cdots, \quad c_i \neq 0, \quad d_i \in \mathbb{Q}_+, \quad 1 \leq j \leq m. \tag{3.54}$$

Here d_j's satisfy the conditions

$$0 \leq d_1 \leq d_2 \leq \cdots \leq d_m, \tag{3.55}$$

and

$$\begin{aligned} &d_1 + d_2 + \cdots + d_j \geq 0, \qquad \text{if } j \leq d(p), \\ &d_1 + d_2 + \cdots + d_j \geq 2(j - d(p)), \qquad \text{if } j \geq d(p) + 1. \end{aligned} \tag{3.56}$$

Let $\lambda_j(s)$ (resp. $\hat{\lambda}_j(s)$) denote a root of $\hat{C}(\lambda, \xi(s)) = 0$ (resp. $\hat{G}(\lambda, s) = 0$). Then, by noticing $\lambda_j(s) = s^2 \hat{\lambda}_j(s)$, we have

$$\begin{aligned} \hat{D}(\xi(s)) &= \prod_{0 \leq i < j < m} (\lambda_i - \lambda_j)^2 \\ &= s^{8(1+d(p))\frac{m(m-1)}{2}} \prod_{0 \leq i < j < m} (\hat{\lambda}_i - \hat{\lambda}_j)^2 g_{ij}(s) \\ &= s^{4(1+d(p))m(m-1)} \prod_{0 \leq i < j < m} ((\hat{\lambda}_i - 1) - (\hat{\lambda}_j - 1))^2 g_{ij}(s), \end{aligned} \tag{3.57}$$

where $g_{ij}(s)$ is a multi-valued holomorphic function in a neighborhood of $s = 0$. This concludes the estimate (3.47). In fact, we have

(3.58)

$$\begin{aligned}
&\nu_{\{s=0\}}(\hat{D}(\xi(s))) \\
&\qquad \geq 4(1+d(p))m(m-1) + 2((m-1)d_1 + (m-2)d_2 + \cdots + d_{m-1}) \\
&\qquad = 4(1+d(p))m(m-1) + 2(d_1 + (d_1+d_2) + \cdots + (d_1+d_2+\cdots+d_{m-1})) \\
&\qquad \geq 4(1+d(p))m(m-1) + 4(1+2+\cdots+m-1-d(p)) \\
&\qquad = 4(1+d(p))m(m-1) + 2(m-d(p))(m-d(p)-1) \\
&\qquad = 6m(m-1) + (4m^2 - 8m + 2 + 2d(p))d(p).
\end{aligned}$$

Next we prove (3.48) for a generic parameter. We may assume

$$p = (1, \ldots, 1, -1, \ldots, -1; 0) \in Z_\xi \cap H_\infty \cap H_0,$$

because the proof for any point in $Z_\xi \cap H_\infty \cap H_0$ goes in the same way as that for p. We take $\alpha \notin E^{(1),l}$ as a parameter where $E^{(1),l}$ was defined by (3.30). Since $d(p) = 0$ follows from the condition $\alpha \notin E^{(1),l}$, we have

$$\nu_{\{s=0\}}(\hat{D}(\xi(s))) \geq 6m(m-1), \tag{3.59}$$

and hence, a Puiseux expansion of a root $\lambda_i(s)$ of $\hat{C}(\lambda, \xi(s)) = 0$ can be written in the form

$$\lambda_i(s) = \xi_{\mathrm{total}}^2(1 + c_i s^2 + \cdots), \quad 1 \leq i \leq m. \tag{3.60}$$

Here the coefficient c_i satisfies the algebraic equation of the variable X:

(3.61)

$$b_0(\beta)X^m + b_1(\beta)X^{m-1} + \cdots + b_{m-1}(\beta)X + b_m(\beta) = 0, \qquad b_k(\beta) = \left.\frac{\tilde{S}_{2k+1}(\xi(s))}{s^{2(k+1)}}\right|_{s=0}.$$

Note that, in virtue of $\alpha \notin E^{(1),l}$, we have

$$b_0(\beta) = \left.\frac{\xi_{\mathrm{total}}(s)}{s^2}\right|_{s=0} \neq 0.$$

For the coefficient $b_k(\beta)$, we have the following lemma whose proof is given after we finish the proof for the theorem.

Lemma 3.7. *Assume $p = (1, \ldots, 1, -1, \ldots, -1; 0) \in Z_\xi \cap H_\infty \cap H_0$. Then we have*

$$b_{k-1}(\beta) = (-2)^k \left(\sum_{m+1 \leq i_1 < i_2 \cdots < i_k \leq 2m+1} \beta_{i_1} \cdots \beta_{i_k} - \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq m} \beta_{i_1} \cdots \beta_{i_k} \right) \tag{3.62}$$

for any $1 \leq k \leq m+1$. *In particular,* $b_0(\beta), \dots, b_m(\beta)$ *are algebraically independent as a polynomial of variables* $(\beta_1, \beta_2, \dots, \beta_{2m+1})$.

If the equation (3.61) has mutually distinct m-roots, then equalities in (3.58) hold, and from which (3.48) follows. Hence it suffices to show that (3.61) has mutually distinct roots for a generic parameter.

Let $D_m(b_0, \dots, b_m)$ denote the discriminant of the equation (3.61). Since $b_0(\beta), \dots, b_m(\beta)$ are algebraically independent, $D_m(b_0(\beta), \dots, b_m(\beta))$ is a non-zero polynomial of the variables $(\beta_1, \dots, \beta_{2m+1})$. Set

$$F = E^{(1),l} \cup \{D_m(b_0(\beta), \dots, b_m(\beta)) = 0\}. \tag{3.63}$$

Then we have $\dim(F) < \dim(P) = 2m+1$ and (3.48) are satisfied if $\alpha \notin F$. This completes the proof. □

Now we give the proofs of Lemmas 3.6 and 3.7 in the rest of this subsection. We first give the proof for Lemma 3.6.

Proof of Lemma 3.6. We set

$$\tilde{\beta} = (\beta_0, \beta) = (\beta_0, \beta_1, \dots, \beta_{2m+1}) \tag{3.64}$$

and

$$g(\lambda, s; \tilde{\beta}) = \frac{1}{\sqrt{\lambda}} \left(\prod_{k=0}^{2m+1} \left(\sqrt{\lambda} + \sigma_k \sqrt{1 + \beta_k s}\right) - \prod_{k=0}^{2m+1} \left(\sqrt{\lambda} - \sigma_k \sqrt{1 + \beta_k s}\right) \right) \tag{3.65}$$

Note that for $\tilde{\beta} = (0, \beta)$ we have

$$\begin{aligned} g(\lambda, s^2; \tilde{\beta}) &= G(\lambda, \xi(s)) \\ &= S_1(\xi(s))(\lambda - 1)^m + \tilde{S}_3(\xi(s))(\lambda - 1)^{m-1} + \cdots + \tilde{S}_{2m+1}(\xi(s)), \end{aligned} \tag{3.66}$$

where $\tilde{S}_{2k-1}(\xi)$'s are given in (3.51). To prove the lemma, we may assume $p = (1, \dots, 1, -1, \dots, -1; 0)$, i.e.,

$$\sigma_0 = \cdots = \sigma_m = 1 \text{ and } \sigma_{m+1} = \cdots = \sigma_{2m+1} = -1. \tag{3.67}$$

Then $g(\lambda, s; \tilde{\beta})$ becomes

$$\begin{aligned} g(\lambda, s; \tilde{\beta}) = \frac{1}{\sqrt{\lambda}} \Bigg(&\prod_{k=0}^{m} \left(\sqrt{\lambda} + \sqrt{1 + \beta_k s}\right) \prod_{k=m+1}^{2m+1} \left(\sqrt{\lambda} - \sqrt{1 + \beta_k s}\right) \\ &- \prod_{k=0}^{m} \left(\sqrt{\lambda} - \sqrt{1 + \beta_k s}\right) \prod_{k=m+1}^{2m+1} \left(\sqrt{\lambda} + \sqrt{1 + \beta_k s}\right) \Bigg). \end{aligned} \tag{3.68}$$

In this situation, we first show

$$\nu_{\{s=0\}}(\tilde{S}_{2k+1}(\xi(s))) \geq 2(1+d(p)). \tag{3.69}$$

Set $\tilde{\beta} = (\tilde{\beta}', \tilde{\beta}'')$ with $\tilde{\beta}' = (\beta_0, \ldots, \beta_m)$ and $\tilde{\beta}'' = (\beta_{m+1}, \ldots, \beta_{2m+1})$. We can easily observe the following properties (i) and (ii):

(i) $g(\lambda, s; \tilde{\beta})$ is symmetric with respect to an exchange of the variables in $\{\tilde{\beta}'\}$ (resp. $\{\tilde{\beta}''\}$), that is, we have for $0 \leq i < j \leq m$

$$\begin{aligned} &g(\lambda, s; \beta_0, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_m, \tilde{\beta}'') \\ &\qquad = g(\lambda, s; \beta_0, \ldots, \beta_j, \ldots, \beta_i, \ldots, \beta_m, \tilde{\beta}''), \end{aligned} \tag{3.70}$$

and for $m+1 \leq i < j \leq 2m+1$

$$\begin{aligned} &g(\lambda, s; \tilde{\beta}', \beta_{m+1}, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_{2m+1}) \\ &\qquad = g(\lambda, s; \tilde{\beta}', \beta_{m+1}, \ldots \beta_j, \ldots, \beta_i, \ldots, \beta_{2m+1}). \end{aligned} \tag{3.71}$$

(ii) $g(\lambda, s; \beta)$ is anti-symmetric with respect to exchange of $\tilde{\beta}'$ and $\tilde{\beta}''$:

$$g(\lambda, s; \tilde{\beta}', \tilde{\beta}'') = -g(\lambda, s; \tilde{\beta}'', \tilde{\beta}') \tag{3.72}$$

We set for non-negative integers k_1 and k_2:

$$g^{(k_1, k_2)}(\tilde{\beta}) = \frac{\partial^{k_1+k_2} g}{\partial \lambda^{k_1} \partial s^{k_2}}(1, 0; \tilde{\beta}). \tag{3.73}$$

Note that $g^{(k_1,k_2)}(\tilde{\beta})$ has the same properties (i) and (ii) above and it is a polynomial of $\tilde{\beta}$ with degree k_2.

Set

$$\gamma_k = \beta_0^k + \beta_1^k + \cdots + \beta_m^k, \quad \hat{\gamma}_k = \beta_{m+1}^k + \beta_{m+2}^k + \cdots + \beta_{2m+1}^k, \quad k = 1, 2, \ldots. \tag{3.74}$$

Then the property (i) of $g^{(k_1, k_2)}(\tilde{\beta})$ implies that $g^{(k_1, k_2)}(\tilde{\beta})$ is also a function of the variables $\gamma_1, \ldots, \gamma_{k_2}$ and $\hat{\gamma}_1, \ldots, \hat{\gamma}_{k_2}$, that is, there exists a polynomial φ_{k_1,k_2} of the variables $\gamma_1, \ldots, \gamma_{k_2}$ and $\hat{\gamma}_1, \ldots, \hat{\gamma}_{k_2}$ which satisfies

$$g^{(k_1, k_2)}(\tilde{\beta}) = \varphi_{k_1,k_2}(\gamma_1, \ldots, \gamma_{k_2}, \hat{\gamma}_1, \ldots, \hat{\gamma}_{k_2}). \tag{3.75}$$

It follows from the property (ii) of $g^{(k_1, k_2)}(\tilde{\beta})$ that φ_{k_1,k_2} also satisfies

$$\varphi_{k_1,k_2}(\gamma_1, \ldots, \gamma_{k_2}, \hat{\gamma}_1, \ldots, \hat{\gamma}_{k_2}) = -\varphi_{k_1,k_2}(\hat{\gamma}_1, \ldots, \hat{\gamma}_{k_2}, \gamma_1, \ldots, \gamma_{\kappa_2}). \tag{3.76}$$

Especially, by taking $\hat{\gamma}_j = \gamma_j$ $(1 \leq j \leq k_2)$ in (3.76), we have

$$\varphi_{k_1,k_2}(\gamma_1, \ldots, \gamma_{k_2}, \gamma_1, \ldots, \gamma_{k_2}) = 0. \tag{3.77}$$

On the other hand, since the expansion of $\xi_{\mathrm{total}}(s)$ at $s=0$ is in the form

$$\xi_{\mathrm{total}}(s) = (1+\sigma_1+\cdots+\sigma_{2m+1}) - \sum_{j>0} \frac{1}{2^j}(\sigma_1\beta_1^j + \cdots + \sigma_{2m+1}\beta_{2m+1}^j)s^{2j}, \tag{3.78}$$

the equalities

$$\gamma_j = \hat{\gamma}_j, \qquad \text{for } 1 \le j < 1+d(p) \text{ and } \tilde{\beta}=(0,\,\beta) \tag{3.79}$$

follow from the definition of $d(p)$ (see also (3.50)). As a consequence of (3.77) and (3.79), we get

$$g^{(k_1,\,k_2)}(\tilde{\beta}) = \varphi_{k_1,k_2}(\gamma_0,\,\ldots,\,\gamma_{k_2},\,\gamma_0,\,\ldots,\,\gamma_{k_2}) = 0, \qquad k_2 < 1+d(p),\ \tilde{\beta}=(0,\,\beta). \tag{3.80}$$

This completes the proof for (3.69).

Finally we show

$$\nu_{\{s=0\}}(\tilde{S}_{2k+1}(\xi(s))) \ge 2(k+1). \tag{3.81}$$

This is equivalent to saying that

$$k_1+k_2 < m+1 \Rightarrow g^{(k_1,\,k_2)}(\tilde{\beta}) = 0. \tag{3.82}$$

We set

$$g_+(\lambda,\,s) = \prod_{0\le i\le m} f_i^+ \prod_{m+1\le i\le 2m+1} f_i^-, \quad g_-(\lambda,\,s) = \prod_{0\le i\le m} f_i^- \prod_{m+1\le i\le 2m+1} f_i^+, \tag{3.83}$$

$$f_i^+ = \left(\sqrt{\lambda}+\sqrt{1+\beta_i s}\right), \quad f_i^- = \left(\sqrt{\lambda}-\sqrt{1+\beta_i s}\right). \tag{3.84}$$

Then (3.82) follows from the fact:

$$k_1+k_2 < m+1 \Rightarrow \partial^{(k_1,\,k_2)} g_\pm(1,0) = 0, \tag{3.85}$$

which can be easily shown. Indeed, we have

$$\begin{aligned}\partial^{(k_1,\,k_2)}&g_+(\lambda,\,s)\\ &= \sum_{|\kappa_1|=k_1}\sum_{|\kappa_2|=k_2}\binom{k_1}{\kappa_1}\binom{k_2}{\kappa_2}\prod_{0\le i\le m}\partial^{(\kappa_{1i},\,\kappa_{2i})}f_i^+ \prod_{m+1\le j\le 2m+1}\partial^{(\kappa_{1j},\,\kappa_{2j})}f_j^-\end{aligned} \tag{3.86}$$

where κ_1 (resp. κ_2) denotes a multi-indices $(\kappa_{10},\,\kappa_{11},\,\ldots,\,\kappa_{12m+1})$ (resp. $(\kappa_{20},\,\kappa_{21},\,\ldots,\,\kappa_{22m+1})$). It follows from $|\kappa_1|+|\kappa_2| = k_1+k_2 < m+1$ that there exist indices $0\le i\le m$ and $m+1\le j\le 2m+1$ which satisfy

$$\kappa_{1i} = \kappa_{2i} = 0 \text{ and } \kappa_{1j} = \kappa_{2j} = 0. \tag{3.87}$$

For such indices we have

$$\partial^{(\kappa_{1i},\,\kappa_{2i})} f_i^+(1,0)\partial^{(\kappa_{1j},\,\kappa_{2j})} f_j^-(1,0) = \partial^{(\kappa_{1i},\,\kappa_{2i})} f_i^-(1,0)\partial^{(\kappa_{1j},\,\kappa_{2j})} f_j^+(1,0) = 0.$$

This entails (3.85). The proof has been completed.

Now we give the proof for Lemma 3.7 where we continue to use the same notations as those in the proof for Lemma 3.6.

Proof of Lemma 3.7. Taking (3.66) into account, we have

$$b_{k-1}(\beta) = \frac{1}{(m+1-k)!k!}\frac{\partial^{m+1} g}{\partial\lambda^{m+1-k}\partial s^k}(1,\,0;\,\tilde{\beta}), \quad 1 \le k \le m+1, \tag{3.88}$$

where we set $\tilde{\beta} = (0,\,\beta)$. Then, by the same argument as in the proof for Lemma 3.6, we have

$$\begin{aligned} &\partial^{(m+1-k,\,k)} g_+(1,\,0) \\ &= \left(\prod_{i=0}^{m} f_i^+\right)(1,\,0) \times \left(\partial^{(m+1-k,\,k)} \prod_{j=m+1}^{2m+1} f_j^-\right)(1,\,0) \\ &= 2^{m+k+1} \times (m+1-k)!k! \\ &\quad \times \sum_{m+1\le i_1<i_2<\cdots<i_k\le 2m+1} \beta_{i_1}\beta_{i_2}\cdots\beta_{i_k} \times \big(-\frac{1}{2}\big)^k \times \big(\frac{1}{2}\big)^{m+1-k} \\ &= (-2)^k(m+1-k)!k! \sum_{m+1\le i_1<i_2<\cdots<i_k\le 2m+1} \beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}. \end{aligned} \tag{3.89}$$

We also get

$$\partial^{(m+1-k,\,k)} g_-(1,\,0) = (-2)^k(m+1-k)!k! \sum_{0\le i_1<i_2<\cdots<i_k\le m} \beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}. \tag{3.90}$$

Hence we have obtained (3.62). The last assertion in Lemma 3.7 follows from the easy fact that the morphism

$$\begin{array}{ccc} \mathbb{C}^{2m+1} & \longrightarrow & \mathbb{C}^{m+1} \\ \beta = (\beta_1,\,\ldots,\,\beta_{2m+1}) & \mapsto & (b_0(\beta),\,\ldots,\,b_m(\beta)) \end{array} \tag{3.91}$$

is surjective. The proof has been completed.

§ 3.3. Zero points in $(Z_\xi \cap H_\infty)\backslash H_0$

In this subsection, the index l denotes either $2m$ or $2m+1$. Let p be a point in $(Z_\xi \cap H_\infty) \setminus H_0$ with $l(p) > 1$.

Theorem 3.8. *Assume $\alpha \in P\backslash E^l_{\mathrm{cup}}$. Then we have*

$$\mathrm{mult}_p(Z_\xi,\, \hat{D}) \geq 4 \times {}_{l(p)}C_2. \tag{3.92}$$

The equality holds for a generic parameter α.

Proof. We shall prove the odd case. Since $\alpha \in P\backslash E^l_{\mathrm{cup}}$, the manifold Z_ξ is expressed by a system of local coordinates

$$\eta = s, \quad \xi_0 = 1, \quad \xi_i = \sigma_i \sqrt{1-\beta_i s^2}, \quad 1 \leq i \leq l, \tag{3.93}$$

where β_i was given in (3.31). Now let us recall the definition of $G(\lambda,\, \xi)$ given in (2.32). Note that we may consider $G(\lambda,\, \xi)$ instead of $\hat{C}(\lambda,\, \xi)$ outside $t = 0$. Hence, as $p \notin H_0$, it suffices to consider the discriminant of $G(\lambda,\, \xi(s))$ instead of that of $\hat{C}(\lambda,\, \xi(s))$. We set for non-negative integers k_1 and k_2:

$$G^{(k_1,\, k_2)}(\lambda,\, s) = \frac{\partial^{k_1+k_2} G(\lambda,\, \xi(s))}{\partial \lambda^{k_1} \partial s^{k_2}}. \tag{3.94}$$

Taking Lemma 3.1 into account, we have

$$k_1 + k_2 < l(p) \Rightarrow G^{(k_1,\, k_2)}(1,\, 0) = 0, \tag{3.95}$$

and

$$G^{(l(p),\, 0)}(1,\, 0) \neq 0. \tag{3.96}$$

By the Weierstrass preparation theorem, there is a unique representation of $G(\lambda,\, \xi(s))$ locally near $(1,\, \xi(0))$ as

$$G(\lambda,\, \xi(s)) = b(\lambda,\, s)\big((\lambda-1)^{l(p)} + a_1(s)(\lambda-1)^{l(p)-1} + \cdots + a_{l(p)}(s)\big), \tag{3.97}$$

where $b(\lambda,\, s)$ is a polynomial which does not vanish at $(1,\, 0)$ and the coefficient $a_j(s)$ is a holomorphic function of s with degree $2j$. We denote the roots of $G(\lambda,\, \xi(s)) = 0$ by

$$\lambda_1(s),\, \ldots,\, \lambda_{l(p)}(s),\, \lambda_{l(p)+1},\, \ldots,\, \lambda_m(s), \tag{3.98}$$

where roots λ_j's satisfy the conditions

$$\lambda_1(0) = \cdots = \lambda_{l(p)}(0) = 1, \quad \lambda_i(0) \neq 1, \quad l(p)+1 \leq i \leq m. \tag{3.99}$$

Since the first $l(p)$-roots of $G(\lambda,\, \xi(s)) = 0$ have Puiseux expansions of the form

$$\lambda_i(s) = 1 + \theta_i s^{d_i} + \cdots, \quad \theta_i \neq 0, \quad 2 \leq d_i \in \mathbb{Q}_+, \quad 1 \leq i \leq l(p), \tag{3.100}$$

we have

$$\begin{aligned}\hat{D}(\xi(s)) &= \prod_{i<j,\lambda_i(0)=\lambda_j(0)} (\lambda_i(s)-\lambda_j(s))^2 \prod_{i<j,\lambda_i(0)\neq\lambda_j(0)} (\lambda_i(s)-\lambda_j(s))^2 \\ &= \prod_{1\le i<j\le l(p)} (\lambda_i(s)-\lambda_j(s))^2 \prod_{i<j,\lambda_i(0)\neq\lambda_j(0)} (\lambda_i(s)-\lambda_j(s))^2 \\ &= s^{4\times {}_{l(p)}C_2} \prod_{1\le i<j\le l(p)} g_{ij}(s) \prod_{i<j,\lambda_i(0)\neq\lambda_j(0)} (\lambda_i(s)-\lambda_j(s))^2,\end{aligned} \tag{3.101}$$

where $g_{ij}(s)$ is a multi-valued holomorphic function in a neighborhood of $s=0$. This concludes

$$\nu_{\{s=0\}}(\hat{D}(\xi(s))) \ge 4\times {}_{l(p)}C_2. \tag{3.102}$$

Hence we have the first assertion.

Finally we prove that the equality in (3.92) holds for a generic parameter, which can be shown by the same argument as that for Theorem 3.5. In fact, for a Puiseux expansion of a root $\lambda_j(s)$ of $G(\lambda,\,\xi(s))=0$ in the form

$$\lambda_j(s) = 1+\theta_j s^2+\cdots, \quad 1\le j\le l(p), \tag{3.103}$$

its coefficient θ_j satisfies the algebraic equation of the variable X:

$$X^{l(p)}+b_1(\beta)X^{l(p)-1}+b_2(\beta)X^{l(p)-2}+\cdots+b_{l(p)}(\beta)=0, \quad b_k(\beta)=\left.\frac{a_k(s)}{s^{2k}}\right|_{s=0}, \tag{3.104}$$

where $a_k(s)$'s are given in (3.97). Then we have the following lemma by the similar argument as in the proof for Lemma 3.7.

Lemma 3.9. *Assume* $p=(1,\,-1,\,\ldots,\,-1,\,1,\,\ldots\ldots,\,1;\,0)\in(Z_\xi\cap H_\infty)\backslash H_0$ *with* $l(p)=l_-(p)$. *Then we have*

$$b_k(\beta)=\frac{(-2)^k 2^{2(m+1-l(p))}}{b(1,\,0)}\left(\sum_{1\le i_1<i_2\cdots<i_k\le l(p)} \beta_{i_1}\cdots\beta_{i_k}\right) \tag{3.105}$$

for any $1\le k\le l(p)$. *In particular* $b_1(\beta),\,\ldots,\,b_{l(p)}(\beta)$ *are algebraically independent as a polynomial of variables* $(\beta_1,\,\ldots,\,\beta_{l(p)})$.

Therefore the rest of the proof is the same as that for Theorem 3.5. □

§ 4. Formulas for the number of the turning points of the second kind

In the last section, we give formulas for the number of the turning points of the second kind. We first prove Lemma 2.7 that makes sure of proper intersections between

Z_ξ and $\hat{D}(\xi)$.

Proof of Lemma 2.7. Assume $\alpha \in P \backslash E^l_{\mathrm{cup}}$. Since Z_ξ is connected by (iii) in Lemma 2.6, to prove the assertion of Lemma 2.7, it is enough to show that the discriminant $\hat{D}$ is not zero at some point $p \in Z_\xi$.

For example, we take a point $p = (\xi; \eta) = (1, 1, \ldots, 1; 0) \in Z_\xi \cap H_\infty$, i.e., $l(p) = 0$. Then roots of the equation $\hat{C}(\lambda, \xi) = 0$ are all simple by Lemma 3.1. Then, as (3.10) holds, we have $\hat{D}(p) \neq 0$. Hence we obtain the assertion. □

Let NT_l denote the number of the turning points of the second kind of NY_l defined by

$$\mathrm{NT}_l = \sum_{p \in G} \mathrm{mult}_p(Z_\xi, \hat{D}), \tag{4.1}$$

where

$$G = \begin{cases} \big(Z_\xi \cap \{\hat{D} = 0\}\big) \cap \mathbb{C}^{l+1}_\xi, & l : \text{even}, \\ \big((Z_\xi \backslash H_0) \cap \{\hat{D} = 0\}\big) \cap \mathbb{C}^{l+1}_\xi, & l : \text{odd}. \end{cases} \tag{4.2}$$

§ 4.1. A formula for NY_{2m+1}

We assume that $l = 2m + 1$ is odd.

Theorem 4.1. *Assume $\alpha \in P \backslash E^{2m+1}_{\mathrm{cup}}$. Then we have the estimate*

$$\mathrm{NT}_l \leq 2\Big((2m^2+3m+3)\binom{2m+1}{m} - 3(m+1)2^{2m}\Big) - 2 \sum_{p \in Z_\xi \cap H_\infty \cap H_0} (d(p)-1)d(p), \tag{4.3}$$

where $d(p) = \dfrac{1}{2}\mathrm{mult}_p(Z_\xi, H_0) - 1$. There exists, in particular, a hypersurface E in P such that

$$\mathrm{NT}_l = 2\left((2m^2 + 3m + 3)\binom{2m+1}{m} - 3(m+1)2^{2m}\right) \tag{4.4}$$

holds for $\alpha \notin E$.

Proof. Note that the degree of Z_ξ is 2^{2m+1}. It follows from Lemma 2.5 that the discriminant $\hat{D}(\xi)$ of $\iota^* \hat{C}(\lambda, \xi)|_{W_{t,\xi}}$ is a homogeneous polynomial of ξ with degree $4m(m-1)$ where the morphism ι was defined by (2.38). By the Bézout theorem [3], we have

$$\sum_{p \in Z_\xi} \mathrm{mult}_p(Z_\xi, \hat{D}) = 4m(m-1) \times 2^{2m+1}. \tag{4.5}$$

Now we calculate the total number of the irrelevant zero points of $\hat{D}|_{Z_\xi}$.

Case 1. The number of zero points of $\hat{D}(\xi)$ in $(Z_\xi \cap H_\infty)\backslash H_0$.

We have the following estimate that follows from Theorem 3.8:

(4.6)
$$\begin{aligned}
\sum_{p\in(Z_\xi\cap H_\infty)\backslash H_0, l(p)>1} \operatorname{mult}_p(Z_\xi,\, \hat{D}) &\geq \sum_{p\in(Z_\xi\cap H_\infty)\backslash H_0, l(p)>1} 4 \times {}_{l(p)}C_2 \\
&= 4 \sum_{2\leq k\leq m} {}_kC_2 \times {}_{2m+2}C_k \\
&= 4(m+1)(2m+1) \sum_{0\leq k\leq m-2} {}_{2m}C_k \\
&= 2(m+1)(2m+1)\{2^{2m} - 2 \times {}_{2m}C_{m-1} - {}_{2m}C_m\}.
\end{aligned}$$

Case 2. The number of zero points of $\hat{D}(\xi)$ in $Z_\xi \cap H_\infty \cap H_0$.

Since a point $p \in Z_\xi \cap H_\infty \cap H_0$ is given by coordinates (3.5), we can easily see that the number of the elements in $Z_\xi \cap H_\infty \cap H_0$ is ${}_{2m+1}C_m$. Hence, by Theorem 3.5, we have

$$\begin{aligned}
\sum_{p\in Z_\xi\cap H_\infty\cap H_0} \operatorname{mult}_p(Z_\xi,\, \hat{D}) &\geq 6m(m-1)_{2m+1}C_m \\
&\quad + \sum_{p\in Z_\xi\cap H_\infty\cap H_0} (4m^2 - 8m + 2 + 2d(p))d(p),
\end{aligned} \tag{4.7}$$

where

$$d(p) = \frac{1}{2} \operatorname{mult}_p(Z_\xi,\, H_0) - 1. \tag{4.8}$$

Case 3. The number of zero points of $\hat{D}(\xi)$ in $(Z_\xi \cap H_0)\backslash H_\infty$.

Set $\tau(p) = \operatorname{mult}_p(Z_\xi,\, H_0)$ for a point $p \in (Z_\xi \cap H_0)\backslash H_\infty$. Since Z_ξ and H_0 properly intersect by the second statement in Theorem 2.6, we can apply the Bézout theorem [3] to a pair $(Z_\xi,\, H_0)$ and we obtain

$$\sum_{p\in Z_\xi\cap H_0} \operatorname{mul}_p(Z_\xi,\, H_0) = 2^{2m+1}. \tag{4.9}$$

Hence we get

$$\begin{aligned}
\sum_{p\in(Z_\xi\cap H_0)\backslash H_\infty} \tau(p) &= 2^{2m+1} - \sum_{p\in Z_\xi\cap H_\infty\cap H_0} (2 + 2d(p)) \\
&= 2^{2m+1} - 2_{2m+1}C_m - 2 \sum_{p\in Z_\xi\cap H_\infty\cap H_0} d(p).
\end{aligned} \tag{4.10}$$

Then it follows from Theorem 3.5 and (4.10) that we have the estimate

$$
\begin{aligned}
\sum_{p\in(Z_\xi\cap H_0)\setminus H_\infty} \mathrm{mult}_p(Z_\xi,\,\hat{D}) &\geq \sum_{p\in(Z_\xi\cap H_0)\setminus H_\infty} 2\tau(p)(m-1)^2 \\
&= 2(m-1)^2 \sum_{p\in(Z_\xi\cap H_0)\setminus H_\infty} \tau(p) \\
&= 2(m-1)^2\left(2^{2m+1} - {}_{2m+1}C_m\right) \\
&\quad - 4(m-1)^2 \sum_{p\in Z_\xi\cap H_\infty\cap H_0} d(p).
\end{aligned}
\tag{4.11}
$$

Summing up, we have

(4.12)

$$
\begin{aligned}
\mathrm{NT}_l &\leq 4m(m-1)2^{2m+1} - 2(m+1)(2m+1)(2^{2m} - {}_{2m}C_{m-1} - {}_{2m}C_m) \\
&\quad - 6m(m-1)_{2m+1}C_m - \sum_{p\in Z_\xi\cap H_\infty\cap H_0} (4m^2-8m+2+2d(p))d(p) \\
&\quad - 2(m-1)^2\left(2^{2m+1} - {}_{2m+1}C_m\right) + (4m^2-8m+4)\sum_{p\in Z_\xi\cap H_\infty\cap H_0} d(p) \\
&= 2\left((3m^2+5m+4)_{2m}C_{m-1} + (m^2+2m+3)_{2m}C_m - 3(m+1)2^{2m}\right) \\
&\quad - 2\sum_{p\in Z_\xi\cap H_\infty\cap H_0} (d(p)-1)d(p) \\
&= 2\Big((2m^2+3m+3)_{2m+1}C_m - 3(m+1)2^{2m}\Big) - 2\sum_{p\in Z_\xi\cap H_\infty\cap H_0} (d(p)-1)d(p).
\end{aligned}
$$

Therefore we have obtained the first assertion and we can easily see that the second assertion follows from Theorems 3.2, 3.5 and 3.8. $\square$

§ 4.2. A formula for NY_{2m}

We assume that $l = 2m$ is even.

Theorem 4.2. *Assume $\alpha \in P\setminus E^{2m}_{\mathrm{cup}}$. Then we have the estimate*

$$
\mathrm{NT}_l \leq \left(2m(2m+1)\binom{2m}{m} - 3m2^{2m}\right). \tag{4.13}
$$

There exists, in particular, a hypersurface E in P such that the equality in (4.11) *holds for $\alpha \notin E$.*

Proof. Note that, in this case, the degree of Z_ξ is 2^{2m} and that of the discriminant $\hat{D}(\xi)$ is $2m(m-1)$ by Lemma 2.5. Hence we have

$$
\sum_{p\in Z_\xi} \mathrm{mult}_p(Z_\xi,\,\hat{D}) = 2m(m-1)\times 2^{2m}. \tag{4.14}
$$

Let us calculate the number of the irrelevant zero points of $\hat{D}|_{Z_\xi}$. For the even case, the following estimate follows from Theorem 3.8,

$$
\begin{aligned}
\sum_{p\in Z_\xi\cap H_\infty, l(p)>1} \mathrm{mult}_p(Z_\xi,\, \hat{D}) &\geq \sum_{p\in Z_\xi\cap H_\infty, l(p)>1} 4 \times {}_{l(p)}C_2 \\
&= 4 \sum_{2\leq k\leq m} {}_kC_2 \times {}_{2m+1}C_k \\
&= 4m(2m+1) \sum_{0<k\leq m-2} {}_{2m-1}C_k \\
&= 4m(2m+1)(2^{2m-2} - {}_{2m-1}C_{m-1}).
\end{aligned}
\tag{4.15}
$$

Therefore we have

$$
\begin{aligned}
\mathrm{NT}_l &\leq 2m(m-1)2^{2m} - 4m(2m+1)(2^{2m-2} - {}_{2m-1}C_{m-1}) \\
&= 4m(2m+1)_{2m-1}C_{m-1} - 3m2^{2m} \\
&= 2m(2m+1)_{2m}C_m - 3m2^{2m}.
\end{aligned}
\tag{4.16}
$$

We have obtained the first assertion and we can easily see that the second assertion holds by Theorem 3.8. □

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