RIMS Kôkyûroku Bessatsu
B38 (2013), 119–137

Tangent varieties and openings of map-germs

Dedicated to the memory of Vladimir Zakalyukin

By

Goo Ishikawa*

Abstract

By taking embedded tangent spaces to a submanifold in an affine space, we obtain a ruled variety, which is called the tangent variety to the submanifold and has non-isolated singularities in general. We explain a method of modifications of map-germs, which we call openings of map-germs, and study the local classification problem of tangent varieties in terms of the opening construction. In particular, we present the general stable classification result of tangent varieties to generic submanifolds of sufficiently high codimension.

§ 1. Introduction

Embedded tangent spaces to a submanifold in an affine space draw a variety, which is called the tangent variety to the submanifold.

Let N be an n-dimensional C^∞ manifold. We denote by TN the tangent bundle of N. Let $f : N^n \to \mathbf{R}^m$ be an immersion. Then the **tangent mapping** $\mathrm{Tan}(f) : TN \to \mathbf{R}^m$ of f is defined by

$$\mathrm{Tan}(f)(x, v) := f(x) + f_*(v), \quad (x, v) \in TN,$$

using the affine structure of $\mathbf{R}^m$.

Then we define the **tangent variety** of f as the parametrised variety which is defined by the right equivalence class of $\mathrm{Tan}(f)$. If $(x_1, \dots, x_n)$ is a system of local

Received June 30, 2012. Revised December 03, 2012.
2000 Mathematics Subject Classification(s): Primary 58K40; Secondly 57R45, 53A20.
Key Words: tangent mapping, opening, generalised frontal mapping, cuspidal-conical edge.
Supported by KAKENHI No. 2234003002 and No. 2365405801.
*Department of Mathematics, Hokkaido University, Japan
e-mail: ishikawa@math.sci.hokudai.ac.jp

coordinates of N, and $(x_1, \ldots, x_n, t_1, \ldots, t_n)$ the induced system of local coordinates of TN, then $\mathrm{Tan}(f)$ is given by

$$\mathrm{Tan}(f)(x, t) = f(x) + \sum_{j=1}^{n} t_j \frac{\partial f}{\partial x_j}(x).$$

Also note that we can define similarly the tangent varieties of mappings to a projective space. Tangent varieties appear in various geometric problems and applications naturally ([1][3][8][16][17][23][18][27]). See [14], for the geometric exposition on the local classification problem of tangent varieties.

It is known, in the three dimensional Euclidean space, that the tangent variety (tangent developable) to a generic space curve has singularities each of which is locally diffeomorphic, i.e. right-left equivalent, to the cuspidal edge or to the folded umbrella, as is found by Cayley and Cleave ([6][7]).

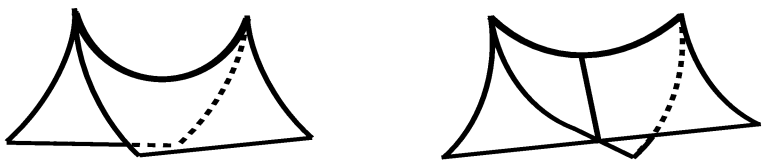

Figure 1. cuspidal edge and folded umbrella.

The **cuspidal edge** is parametrised by the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ defined by

$$(w, x) \mapsto (w,\ x^2,\ x^3).$$

Note that it is diffeomorphic to the germ

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^3 + 3st^2)$$

of a parametrisation of tangent variety.

The **folded umbrella** is parametrised by the germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ defined by

$$(t, s) \mapsto (t + s, t^2 + 2st, t^4 + 4st^3),$$

which is diffeomorphic to

$$(w, x) \mapsto (w,\ x^2 + ux,\ \frac{1}{2}x^4 + \frac{1}{3}ux^3).$$

The folded umbrella is often called the **cuspidal cross cap**.

Cuspidal edge singularities appear along ordinary points of a curve in $\mathbf{R}^3$, while the folded umbrella appears at an isolated point of zero torsion([8][23]).

For more degenerate curves in $\mathbf{R}^3$, the singularities of tangent varieties were classified by Mond [19][20] and Scherbak [26][4]. See also the survey [13].

Tangent varieties are defined also for higher codimensional curves. Then, it is known that the tangent variety to a generic curve in $\mathbf{R}^m$ with $m \geq 4$ has singularities each of which is locally diffeomorphic to the higher codimensional cuspidal edge in $\mathbf{R}^m$ (Theorem 2.6 [14]).

The **higher codimensional cuspidal edge** is parametrised by the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^m, 0)$ defined by

$$(w, x) \mapsto (w,\ x^2,\ x^3,\ 0, \dots,\ 0),$$

which is diffeomorphic to the germ

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^3 + 3st^2,\ \dots,\ t^m + mst^{m-1}),$$

and also to

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^3 + 3st^2,\ 0, \dots,\ 0).$$

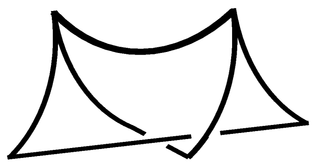

Figure 2. The higher codimensional cuspidal edge

Thus we understand that the local diffeomorphism class of tangent varieties to generic curves of sufficiently high codimension is determined uniquely. Moreover the tangent variety to any immersed curve in $\mathbf{R}^m$ is obtained locally by a projection of a higher codimensional cuspidal edge in $\mathbf{R}^{m'}$ for some $m' \geq m$.

Tangent varieties are defined also for higher dimensional submanifolds. In [14], we observe several results of tangent varieties to surfaces. For instance, let us consider a surface in $\mathbf{R}^5$. Then the tangent variety to a generic surface becomes a hypersurface in $\mathbf{R}^5$ and it has conical singularities along the original surface itself, together with several self-intersection loci. The local classification problem for singularities of tangent varieties of generic surfaces in $\mathbf{R}^5$ is still open, as far as the author knows. Note that, in [14], it is treated the classification problem for singularities of tangent varieties to Legendre surfaces in $\mathbf{R}^5$. In particular we show that the tangent variety of an elliptic or a hyperbolic Legendre surface in $\mathbf{R}^5$ has D_4-singularity along the Legendre surface itself using the criterion in [24] (Theorem 9.5 of [14]).

Instead, in this paper, we show a simple observation that the local diffeomorphism class of tangent varieties to generic surface in $\mathbf{R}^m$ is unique if m is sufficiently large:

Theorem 1.1. *Suppose $m \geq 11$. Then the tangent map-germ*

$$\mathrm{Tan}(f) : (TN, (p, 0)) \to \mathbf{R}^m$$

of a proper immersion $f : N^2 \to \mathbf{R}^m$ of a two dimensional manifold N, which is generic in Whitney C^∞-topology, has the unique right-left equivalence class for any $(p, 0) \in N \times \{0\}$. Moreover $\mathrm{Tan}(f) : TN \setminus N \times \{0\} \longrightarrow \mathbf{R}^m$ *is an immersion.*

We have that the tangent variety to any surface in $\mathbf{R}^m$ is obtained by a projection of the universal singularity in $\mathbf{R}^{m'}$ for some $m' \geq m$. Further, we show this is true for any dimension of submanifolds. (See Theorem 6.3, Corollary 6.4).

The singularities of tangent varieties are obtained in general by so called the **opening construction**. In general, given a C^∞ map-germ $g : (N, a) \to (M, b)$ with $\dim N \leq \dim M$, we associate a sub-algebra $\mathcal{R}_g$ in the $\mathbf{R}$-algebra of C^∞ function-germs on (N, a) such that, for any element $h \in \mathcal{R}_g$, the map-germ $(g, h) : (N, a) \to (M \times \mathbf{R}, (b, h(a)))$ has the same singular locus with g in (N, a) and the same kernels of the differential $(g, h)_* : (TN, a \times T_aN) \to T(M \times \mathbf{R})$ with $g_* : (TN, a \times T_aN) \to TM$. By adding a finite number of elements in $\mathcal{R}_g$ as components, we obtain an "opening" of g.

The tangent variety to a curve in $\mathbf{R}^m, m \geq 3$ projects locally to the tangent variety to a space curve in the osculating 3-space, and to a plane curve in the osculating 2-plane. In [14] we observed that the tangent variety in $\mathbf{R}^m$ can be regarded as an "opening" of a tangent variety to a space curve and to a plane curve.

Though name "opening" is firstly used in [14], the notion of opening is, for instance, used in [11][12] intrinsically. In fact, openings of map-germs appear naturally as typical singularities in several problems of geometry and its applications. For example, the open swallowtail, which is an opening of the swallowtail as a singular Lagrangian variety [2], and as a singular solution to certain partial differential equation [10]. The open folded umbrella appears as a "frontal-symplectic singularity" ([15]). In this paper, we show one example of this fact that openings appear naturally in geometry.

In §2, we introduce the notion of openings of map-germs and prepare necessary results to show the classification results of tangent varieties in this paper. In §3, we introduce the generalised notion of frontal mappings to connect the singularities of tangent varieties and opening constructions. In §4, we recall the genericity results of immersions into higher dimensional space. In §5, we give the proof of Theorem 1.1. In §6, we show the stable classification for singularities on tangent varieties of generic submanifold of arbitrary dimension. We introduce the singularity, **cuspidal-conical edge** for our classification problem, which is a generalisation of the cuspidal edge in the three space.

In [25], it is studied the cuspidal edges in the three space from the differential geometric point of view. It would be an interesting problem to investigate the differential geometry also on higher codimensional cuspidal edges and higher dimensional cuspidal-conical edges.

In this paper all manifolds and mappings are assumed of class C^∞ unless otherwise stated.

The author would like to thank the referee for helpful comments to improve the original manuscript.

§ 2. Opening construction of differentiable map-germs

Let (N, a) be a germ of n-dimensional manifold at a point $a \in N$. Let $g : (N, a) \to (\mathbf{R}^m, b), g = (g_1, \dots, g_m), n \leq m$ be a map-germ.

We define the **Jacobi module** $\mathcal{J}_g$ of g by

$$\mathcal{J}_g = \{ \sum_{j=1}^{m} p_j \ dg_j \ \mid \ p_j \in \mathcal{E}_{N,a} \ (1 \leq j \leq m) \ \} \subset \Omega^1_{N,a}$$

in the space $\Omega^1_{N,a}$ of 1-form-germs on (N, a). Then define the **ramification module** $\mathcal{R}_g$ of g by

$$\mathcal{R}_g = \{h \in \mathcal{E}_{N,a} \mid dh \in \mathcal{J}_g\},$$

in the space $\mathcal{E}_{N,a}$ of function-germs on (N, a). See [11][12]. The ramification module is regarded as the set of generating function in symplectic geometry. Note that a related notion was introduced firstly in [21]. See [28] for the related notion of "generating ideal".

For $g : (N, a) \to (\mathbf{R}^m, b), g' : (N, a) \to (\mathbf{R}^{m'}, b')$, easily we see that $\mathcal{J}_{g'} \subseteq \mathcal{J}_g$ if and only if $\mathcal{R}_{g'} \subseteq \mathcal{R}_g$, and therefore that $\mathcal{J}_{g'} = \mathcal{J}_g$ if and only if $\mathcal{R}_{g'} = \mathcal{R}_g$.

Definition 2.1. Let $g : (N, a) \to (\mathbf{R}^m, b), g = (g_1, \dots, g_m),\ n \leq m$ be a map-germ. A map-germ $G : (N, a) \to \mathbf{R}^m \times \mathbf{R}^r = \mathbf{R}^{m+r}$ defined by

$$G = (g_1, \dots, g_m, h_1, \dots, h_r)$$

is called an **opening** of g if $h_1, \dots, h_r \in \mathcal{R}_g$. Then g is called a **closing** of G.

For any opening G of g, we have $\mathcal{R}_G = \mathcal{R}_g$ and $\mathcal{J}_G = \mathcal{J}_g$.

Note that an opening of an opening of g is an opening of g.

Definition 2.2. An opening $G = (g, h_1, \dots, h_r)$ of g is called a **versal opening** (resp. a **mini-versal opening**) of $g : (N, a) \to (\mathbf{R}^m, b)$, if $1, h_1, \dots, h_r$ form a (minimal) system of generators of $\mathcal{R}_g$ as an $\mathcal{E}_{\mathbf{R}^m,b}$-module via $g^* : \mathcal{E}_{\mathbf{R}^m,b} \to \mathcal{E}_{N,a}$.

Note that a versal opening of an opening of g is a versal opening of g. An opening of a versal opening of g is a versal opening of g.

Example 2.3. (1) Let $h : (\mathbf{R}, 0) \to (\mathbf{R}, 0),\ h(x) = x^2$. Then $\mathcal{R}_h = \langle 1,\ x^3 \rangle_{h^*(\mathcal{E}_1)}$. The map-germ $H : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0), H(x) := (x^2,\ x^3)$, the simple cusp map, is the mini-versal opening of h.
(2) Let $g : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0),\ g(x, t) = (x, t^2)$. $\mathcal{R}_g = \langle 1,\ t^3 \rangle_{g^*(\mathcal{E}_2)}$. The map-germ $G : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0), G(x) := (x,\ t^2,\ t^3)$, the cuspidal edge, is the mini-versal opening of g.
In this example, we set $\mathcal{E}_n = \mathcal{E}_{\mathbf{R}^n,0}$.

As examples in a more degenerate case, the swallowtail is an opening of the Whitney's cusp. The open swallowtail is an opening of the swallowtail and of the Whitney's cusp ([14]).

In many cases, versal openings do exist. For the general results on the existence of versal openings, consult with [14].

In this paper we are concerned with only the uniqueness:

Proposition 2.4. (Proposition 6.9 of [14]) *Let $g : (N^n, a) \to (\mathbf{R}^m, b)$ be a C^∞ map-germ $(n \leq m)$. Then the mini-versal opening $G : (N, a) \to \mathbf{R}^{m+r}$ of g is, if it exists, unique up to left-equivalence and any versal opening $G : (N, a) \to \mathbf{R}^{m+s}$ of g is left-equivalent to a mini-versal opening composed with an immersion $(\mathbf{R}^n, a) \to \mathbf{R}^{m+r} \hookrightarrow \mathbf{R}^{m+s}$.*

To make assure ourselves, we give a proof of Proposition 2.4 briefly.

Lemma 2.5. *Assume that there exists a versal opening of g. Then an opening $G = (g, h_1, \ldots, h_r)$ of g is a mini-versal opening if and only if $1, h_1, \ldots, h_r$ form a basis of $\mathbf{R}$-vector space $\mathcal{R}_g / g^*(\mathfrak{m}_{\mathbf{R}^m,b})\mathcal{R}_g$.*

Proof: By the assumption, we have that $\mathcal{R}_g$ is a finite $\mathcal{E}_{\mathbf{R}^m,b}$-module via g^*. Thus by Nakayama's lemma (see for instance [5]), we have that $1, h_1, \ldots, h_r$ generate $\mathcal{R}_g$ as $\mathcal{E}_{\mathbf{R}^m,b}$-module via g^* if and only if they form a generator of $\mathcal{R}_g / g^*(\mathfrak{m}_{\mathbf{R}^m,b})\mathcal{R}_g$ over $\mathbf{R}$. Therefore $1, h_1, \ldots, h_r$ form a minimal system of generators of $\mathcal{R}_g$ as $\mathcal{E}_{\mathbf{R}^m,b}$-module via g^* if and only if they form a basis of $\mathcal{R}_g / g^*(\mathfrak{m}_{\mathbf{R}^m,b})\mathcal{R}_g$ over $\mathbf{R}$. □

Proof of Proposition 2.4: Let $G = (g, h_1, \ldots, h_r)$ and $G' = (g, k_1, \ldots, k_s)$ be mini-versal openings of g. Then, by Lemma 2.5, we have $r = s$ and $(h_1, \ldots, h_r)$ (resp. $(k_1, \ldots, k_r)$) form a basis of $\mathcal{R}_g / g^*(\mathfrak{m}_{\mathbf{R}^m,b})\mathcal{R}_g$. We may assume $h_i(a) = 0, k_j(a) = 0, 1 \leq i, j \leq r$. Since $k_j \in \mathcal{R}_g$, there exist ${c_j}^0, {c_j}^1, \ldots, {c_j}^r \in \mathcal{E}_{\mathbf{R}^m,b}$ such that $k_j = {c_j}^0 \circ g + ({c_j}^1 \circ g)h_1 + \cdots + ({c_j}^r \circ g)h_r, (1 \leq j \leq r)$. Then we see the $r \times r$-matrix $({c_j}^i(b))$ is regular. We set $\Psi(y, z) = (y, ({c_j}^0(y) + {c_j}^1(y)z_1 + \cdots + {c_j}^r(y)z_r)_{1 \leq j \leq s})$. Then $\Psi : (\mathbf{R}^{m+r}, (b, 0)) \to (\mathbf{R}^{m+r}, (b, 0))$ is a diffeomorphism germ and $G' = \Psi \circ G$. Now let G'' be a versal opening of g. Then similarly as above, $G'' = \Psi \circ G$ and the matrix $({c_j}^i(b))$ is of rank r. Then Ψ is an immersion-germ. $\square$

Actually we need more general result to show the main Theorem 1.1.

Definition 2.6. Let $g : (N, a) \to (\mathbf{R}^m, b), g' : (N, a) \to (\mathbf{R}^{m'}, b')$ be map-germs. The map-germs g and g' are called $\mathcal{J}$-**equivalent** if their Jacobi modules coincide: $\mathcal{J}_g = \mathcal{J}_{g'}$.

If g and g' are left equivalent, then $\mathcal{R}_g = \mathcal{R}_{g'}$ and therefore, they are $\mathcal{J}$-equivalent. However the converse does not hold.

Example 2.7. Define $g, g' : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$, by $g(s,t) = (s^2, st, t^2), g'(s,t) = (s^2 + s^3, st, t^2)$. Then g, g' are $\mathcal{J}$-equivalent. However g, g' are not left equivalent. In fact $g|_{\mathbf{R}^2 \setminus 0}$ is 2 to 1, while $g'|_{\mathbf{R}^2 \setminus 0}$ is injective on $\mathbf{R}^2 \setminus \{t = 0\}$.

Definition 2.8. A map-germ $g : (N, a) \to (\mathbf{R}^m, b)$ is called $\mathcal{J}$-**minimal** if $dg_1, \dots, dg_m$ form a minimal system of generators of $\mathcal{J}_g$ as an $\mathcal{E}_{N,a}$-module.

Lemma 2.9. *Suppose two map-germs $g : (N, a) \to (\mathbf{R}^m, b)$ and $g' : (N, a) \to (\mathbf{R}^{m'}, b')$ are $\mathcal{J}$-equivalent and they are both $\mathcal{J}$-minimal. Then $m = m'$.*

In fact $m = m' = \dim_{\mathbf{R}} \mathcal{J}_g / \mathfrak{m}_{N,a} \mathcal{J}_g$.

The following generalisation of Proposition 2.4 is the key of the proof on our main result.

Proposition 2.10. *Suppose $g : (N, a) \to (\mathbf{R}^m, b)$ and $g' : (N, a) \to (\mathbf{R}^m, b')$ are $\mathcal{J}$-equivalent and they are both $\mathcal{J}$-minimal. Moreover suppose G is a mini-versal openings of g, and G' is a mini-versal opening of g', Then G and G' are left equivalent.*

Proof: We may suppose $b = b' = 0$ and $G(a) = G'(a) = 0$. Let

$$G = (g_1, \dots, g_m, h_1, \dots, h_r), \quad G' = (g'_1, \dots, g'_m, h'_1, \dots, h'_r).$$

Let

$$dh_k = \sum_{\ell=1}^{m} a_{k\ell}\, dg_\ell, \quad (1 \le k \le r).$$

Then set $\widetilde{h}_k := h_k - \sum_{\ell=1}^m a_{k\ell}(a)\, g_\ell, (1 \le k \le r)$, and set

$$\widetilde{G} := (g_1, \dots, g_m, \widetilde{h}_1, \dots, \widetilde{h}_r).$$

Then we have that $\widetilde{G}$ is left-equivalent to G and that $\widetilde{G}$ is a mini-versal opening of g. Note that $d\widetilde{h}_k \in \mathfrak{m}_{N,a}\mathcal{J}_g, (1 \leq k \leq r)$. Now set

$$\widetilde{G}' := (g_1', \dots, g_m', \widetilde{h}_1, \dots, \widetilde{h}_r).$$

Since g' is $\mathcal{J}$-equivalent to g, we see that $\widetilde{G}'$ is a mini-versal opening of g'. Then, by Proposition 2.4, we have that $\widetilde{G}'$ is left-equivalent to G'. Therefore it suffices to show that $\widetilde{G}'$ and $\widetilde{G}$ are left equivalent. Since g' is $\mathcal{J}$-equivalent to g and $\widetilde{G}$ is a versal opening of g, we have that $g' = \Phi \circ \widetilde{G}$ for some $\Phi : (\mathbf{R}^{m+r}, 0) \to (\mathbf{R}^m, 0)$. Then $\widetilde{G}' = (\Phi \times \mathrm{id}) \circ \widetilde{G}$. Then we have

$$dg_i' = d(\Phi_i \circ \widetilde{G}) \equiv \sum_{j=1}^{m} \frac{\partial \Phi_i}{\partial y_j}(0)\ dg_j, \ \text{mod.}\ \mathfrak{m}_{N,a}\mathcal{J}_g, \ (1 \leq i \leq m).$$

Since g' and g are $\mathcal{J}$-minimal, we have that the matrix $\left(\frac{\partial \Phi_i}{\partial y_j}(0)\right)_{1 \leq i,j \leq m}$ is regular and therefore Φ is of rank m at 0. Therefore $\Phi \times \mathrm{id}$ is a diffeomorphism-germ, and we have that $\widetilde{G}'$ is left equivalent to $\widetilde{G}$. $\square$

§ 3. Generalised frontal mappings

In this paper we introduce a key notion that connects the study on tangent varieties and opening procedures of map-germs.

Let M be an m-dimensional manifold and $0 \leq \ell \leq m$. Let

$$\mathrm{Gr}(\ell, TM) = \{(y, V) \mid y \in M, V \subset T_yM, \dim V = \ell\}$$

denote the Grassmannian bundle over M consisting of ℓ-dimensional linear tangential subspaces to M, and $\pi : \mathrm{Gr}(\ell, TM) \to M$ the natural projection. The canonical differential system $\mathcal{C} \subset T\mathrm{Gr}(\ell, TM)$ on $\mathrm{Gr}(\ell, TM)$ is defined by, for $(y, V) \in \mathrm{Gr}(\ell, TM)$,

$$\mathcal{C}_{(y,V)} := \{v \in T_{(y,V)}\mathrm{Gr}(\ell, TM) \mid \pi_* v \in V (\subset T_yM)\}.$$

Note that $\dim \mathrm{Gr}(\ell, TM) = m + \ell(m - \ell)$ and $\mathrm{rank}\, \mathcal{C} = \ell + \ell(m - \ell)$.

Let N be an n-dimensional manifold with $n \leq m$. A C^∞ mapping $f : N^n \to M^m$ is called **frontal** if

(I) the locus of regular points $\mathrm{Reg}(f) = \{x \in N \mid f : (N, x) \to (M, f(x)) \text{ is immersive}\}$ of f is dense in N, and

(II) there exists a C^∞ mapping $\widetilde{f} : N \to \mathrm{Gr}(n, TM)$ satisfying $\pi \circ \widetilde{f} = f$ and $\widetilde{f}(x) = f_*(T_xN)$ for any $x \in \mathrm{Reg}(f)$.

The mapping $\widetilde{f}$ is uniquely determined if it exists, which we call the **Grassmann lifting** of f.

The Grassmann lifting $\widetilde{f}$ of a frontal mapping $f : N \to M$ is an integral mapping to the canonical differential system $\mathcal{C}$ on $\mathrm{Gr}(n, TM)$.

If f is an immersion, then f is frontal. For example, the tangent mapping of a curve of finite type is a non-immersive frontal mapping ([14]). The name "frontal" was firstly introduced by Vladimir Zakalyukin.

If f is frontal and $\widetilde{f}$ is an immersion, then f is called a **front**. Note that the above notions of frontals and fronts are applied usually in the case $m = n + 1$.

The notion of frontal mappings is generalised naturally to the following: Let M be a manifold of dimension m and ℓ a natural number with $0 \leq \ell \leq m$. Let N be a manifold of dimension n with $n \leq \ell$.

A mapping $f : N \to M$ is called ℓ-**frontal** if
(I) the locus of regular points $\mathrm{Reg}(f) \subset N$ is dense in N and
(II) there exists a $\mathcal{C}$-integral map $\widetilde{f} : N \to \mathrm{Gr}(\ell, TM)$ such that $\pi \circ \widetilde{f} = f$ and $\widetilde{f}(x) \supset f_*(T_x N)$ for any $x \in N$.

Note that an n-frontal mapping $f : N^n \to M$ is frontal. Any mapping $f : N^n \to M^m$ is m-frontal, because $\mathrm{Gr}(m, TM^m) \cong M$.

A germ $f : (N, p) \to M$ is called ℓ-**frontal** if some representative of f is ℓ-frontal in the above sense. For $n \leq \ell \leq \ell' \leq m$, if a germ f is ℓ-frontal, then it is ℓ'-frontal.

Remark 3.1. A mapping $f : N \to M$ is called **sub-frontal** if there exist a frontal mapping $g : L \to M$ and an embedding $i : N \to L$ such that $f = g \circ i$. Then f is ℓ-frontal with $\ell = \dim L$.

Remark 3.2. Let ℓ, m be a natural number with $0 \leq \ell \leq m$. Consider the Grassmannian bundle $\mathrm{Gr}(\ell, TP(\mathbf{R}^{m+1}))$ over m-dimensional projective space $P(\mathbf{R}^{m+1})$. Let $V \subset T_y P(\mathbf{R}^{m+1})$ be an ℓ-dimensional linear subspace for a $y \in P(\mathbf{R}^{m+1})$. Then there exists uniquely an $(\ell+1)$-dimensional linear subspace $\widetilde{V} \subset \mathbf{R}^{m+1}$ such that the projective subspace $P(\widetilde{V}) \subset P(\mathbf{R}^{m+1})$ has the tangent space $V = T_y P(\widetilde{V}) \subset T_y P(\mathbf{R}^{m+1})$ at y. Then $\mathrm{Gr}(\ell, TP(\mathbf{R}^{m+1}))$ is identified with the flag manifold

$$\mathcal{F}_{1,\ell+1}(\mathbf{R}^{m+1}) = \{(V_1, V_{\ell+1}) \mid V_1 \subset V_{\ell+1} \subset \mathbf{R}^{m+1}, \dim(V_1) = 1, \dim(V_{\ell+1}) = \ell + 1\},$$

by mapping $(y, V) \mapsto (y, \widetilde{V})$ ([14]). The canonical differential system $\mathcal{C} = \mathcal{C}_{1,\ell+1}(\mathbf{R}^{m+1})$ is defined by, for each $(V_1, V_{\ell+1}) \in \mathcal{F}_{1,\ell+1}(\mathbf{R}^{m+1})$,

$$\mathcal{C}_{(V_1, V_{\ell+1})} = \{v \in T_{(V_1, V_{\ell+1})}\mathcal{F}_{1,\ell+1}(\mathbf{R}^{m+1}) \mid \pi_{1*}(v) \in TP(V_{\ell+1})(\subset TP(\mathbf{R}^{m+1}))\},$$

where $\pi_1 : \mathcal{F}_{1,\ell+1}(\mathbf{R}^{m+1}) \to P(\mathbf{R}^{m+1})$ is the natural projection.

The following lemma gives a description of the canonical system $\mathcal{C}$ in terms of local coordinates:

Lemma 3.3. (Remark 3.7 of [14]) *The canonical system $\mathcal{C}$ on $\mathcal{F}_{1,\ell+1}(\mathbf{R}^{m+1})$ is locally given by*

$$dx_{i+1}^{\,0} - \sum_{j=1}^{\ell} x_{i+1}^{\,j} dx_j^0 = 0, \quad (\ell \le i \le m-1),$$

for a system of local coordinates $x_{i+1}^{\,0}, (0 \le i \le m-1), x_{i+1}^{\,j}, (1 \le j \le \ell, \ell \le i \le m-1)$. The projection $\pi_1 : \mathcal{F}_{1,\ell+1}(\mathbf{R}^{m+1}) \to P(\mathbf{R}^{m+1})$ is represented by $(x_1^{\,0}, \dots, x_m^{\,0})$. If we write $y_j = x_j^{\,0}$ $(1 \le j \le \ell), z_i = x_{n+i}^{\,0}$ $(1 \le i \le m-\ell)$ and $p_{ij} = x_{n+i}^{\,j}$ $(1 \le i \le m-\ell, 1 \le j \le \ell)$, then we have

$$dz_i - \sum_{j=1}^{n} p_{ij} dy_j = 0, \quad 1 \le i \le m-\ell.$$

Therefore the condition that a map $F : N^n \to \mathrm{Gr}(\ell, TP(\mathbf{R}^{m+1}))$ is $\mathcal{C}$-integral is expressed by

$$d(z_i \circ F) - \sum_{j=1}^{\ell} (p_{ij} \circ F)\, d(y_j \circ F) = 0, \quad 1 \le i \le m-\ell.$$

The following lemma characterises ℓ-frontal map-germs.

Lemma 3.4. *Let $1 \le n \le \ell \le m$. A map-germ $f : (N^n, p) \to M^m$ is ℓ-frontal if and only if f is left equivalent to an opening of a map-germ $g : (N, p) \to \mathbf{R}^\ell$ with dense* $\mathrm{Reg}(g)$.

Proof: Suppose f is ℓ-frontal. Let $\widetilde{f} : (N, p) \to \mathrm{Gr}(\ell, TM)$ be an $\mathcal{C}$-integral map such that $\pi \circ \widetilde{f} = f$ and $\widetilde{f}(x) \supset f_*(T_x N)$ for any $x \in (N, p)$. We take a system of local coordinates $y_1, \dots, y_\ell, z_1, \dots, z_{m-\ell}; p_{ij}$ of $\mathrm{Gr}(\ell, TM)$ around $\widetilde{f}(p)$ such that the ℓ-dimensional subspace $\widetilde{f}(p)$ of $T_p M$ is given by $dz_1(p) = \cdots = dz_{m-\ell}(p) = 0$, and that

$$z_i \circ f = \sum_{j=1}^{\ell} (p_{ij} \circ \widetilde{f}) d(y_j \circ f), \quad (1 \le i \le m-\ell) \quad (*).$$

We set $g = (y_1, \dots, y_\ell) \circ f$. Then $\mathrm{Reg}(g)$ is dense in (N, p) and $z_i \circ f \in \mathcal{R}_g$ by (*), $1 \le i \le m-\ell$. Therefore f is left-equivalent to an opening of g.

Conversely let $g : (N, p) \to \mathbf{R}^\ell$ be a map-germ with dense $\mathrm{Reg}(g)$ and $G = (g, h_1, \dots, h_{m-\ell})$ be an opening of g. Then $\mathrm{Reg}(G) = \mathrm{Reg}(g)$ is dense in (N, p). Since $h_i \in \mathcal{R}_g$, there exist a_{ij} on (N, p) satisfying

$$dh_i = \sum_{j=1}^{\ell} a_{ij} dg_j, \quad (1 \le i \le m-\ell).$$

Now we define a map-germ $H : (N, p) \to \mathrm{Gr}(\ell, TM)$ by $H(x) = (G(x), a_{ij}(x)) = (g(x), h(x), a_{ij}(x))$. Then H is a $\mathcal{C}$-integral map and satisfies $\pi \circ H = G$. Moreover $H(x) \supset G_*(T_x N)$ for any $x \in N$. Therefore G is ℓ-frontal and hence f is ℓ-frontal.

□

Corollary 3.5. *Openings of an ℓ-frontal map-germ are ℓ-frontal.*

Proof: Let f be ℓ-frontal and $F = (f, h)$ be an opening of f. By Lemma 3.4, f is left-equivalent to an opening $G = (g, k)$ of a map-germ $g : (N, a) \to \mathbf{R}^\ell$. Then $f = \Psi \circ G$ for a diffeomorphism-germ Ψ. Then $\mathcal{R}_f = \mathcal{R}_G = \mathcal{R}_g$. Therefore (G, h) is an opening of g. Moreover F is left-equivalent to (G, h). Hence we see that F is ℓ-frontal again by Lemma 3.4. □

§ 4. Higher order non-degenerate immersions

Let $f : N^n \to \mathbf{R}^m$ be a mapping and $f(x) = (f_1(x), \dots, f_m(x))$ be a local expression of f via an affine system of local coordinates on $\mathbf{R}^m$. We define the matrix $W_i(f)(x)$, for $i = 1, 2, 3, \dots$, by

$$W_i(f)(x) := \left(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \;\middle|\; \alpha \in \mathbf{N}^n \setminus \{\mathbf{0}\}, 1 \le |\alpha| \le i \right).$$

Here we regard each $\dfrac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)$ as a column vector. Let k be a positive integer. Then f is called k-**non-degenerate** if $\operatorname{rank} W_k(x) = \sum_{j=1}^k {}_nC_j$, the number of columns, (cf. [22][9]). Here ${}_nC_j = \frac{n!}{(n-j)!j!}$. Note that, in this case, we must have $m \ge \sum_{j=1}^k {}_nC_j$.

Lemma 4.1. *Let N be an n-dimensional manifold and $m \ge n + \sum_{j=1}^k {}_nC_j$. Then, in the space of proper immersions $N^n \to \mathbf{R}^m$ with Whitney C^∞ topology, the set of k-non-degenerate immersions form a residual set.*

Proof: In the k-jet space $J^k(N, \mathbf{R}^m)$, the condition that the rank of W_k is less than $\sum_{j=1}^k {}_nC_j$ defines an algebraic subset of codimension $m - (\sum_{j=1}^k {}_nC_j) + 1$. Since $n < m - (\sum_{j=1}^k {}_nC_j) + 1$, we see that k-non-degenerate immersions form a residual set in the space of proper immersions $N \to \mathbf{R}^m$, by the transversality theorem. □

Corollary 4.2. *Let N be an n-dimensional manifold and $m \ge 2n + \frac{1}{2}n(n+1)$. Then, in the space of proper immersions $N^n \to \mathbf{R}^m$, the set of 2-non-degenerate immersions form a residual set.*

Corollary 4.3. *Let N be an n-dimensional manifold and $m \ge 2n + \frac{1}{2}n(n+1) + \frac{1}{6}n(n+1)(n+2)$. Then, in the space of proper immersions $N^n \to \mathbf{R}^m$, the set of 3-non-degenerate immersions form a residual set.*

For to make clear the behaviour of tangent mapping outside of zero-section, we need the following:

Lemma 4.4. *Let $f : N \to \mathbf{R}^m$ be a 2-non-degenerate immersion. Then* $\mathrm{Tan}(f) : TN \setminus N \times \{0\} \to \mathbf{R}^m$ *is an immersion.*

Proof: Recall that $F = \mathrm{Tan}(f)$ is defined locally by

$$F(x,t) = \mathrm{Tan}(f)(x,t) = f(x) + \sum_{j=1}^{n} t_j \frac{\partial f}{\partial x_j}(x).$$

Then we have

$$\frac{\partial F}{\partial t_i} = \frac{\partial f}{\partial x_i}, \quad \frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{n} t_j \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Then, for the rank of Jacobi matrix of F, we have

$$\mathrm{rank}\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}\right) = \mathrm{rank}\left(\frac{\partial f}{\partial x_1}, \ \dots, \ \frac{\partial f}{\partial x_n}, \ \sum_{j=1}^{n} t_j \frac{\partial^2 f}{\partial x_1 \partial x_j}, \ \dots, \ \sum_{j=1}^{n} t_j \frac{\partial^2 f}{\partial x_n \partial x_j}\right).$$

Since f is 2-non-degenerate, we see that $\mathrm{rank}\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}\right) = 2n$, for any $(t_1, \dots, t_n) \neq (0, \dots, 0)$. In fact suppose

$$\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} b_i \left(\sum_{j=1}^{n} t_j \frac{\partial^2 f}{\partial x_n \partial x_j}\right) = 0,$$

for some $a_i, b_i \in \mathbf{R}$. Then we have

$$\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} + \sum_{1 \leq i \leq j \leq n} (b_i t_j + b_j t_i) \frac{\partial^2 f}{\partial x_i \partial x_j} = 0.$$

Therefore $a_i = 0, (1 \leq i \leq n)$ and $b_i t_j + b_j t_i = 0, (1 \leq i \leq j \leq n)$. Suppose $t_1 \neq 0$. Then, since $2b_1 t_1 = 0$, we have $b_1 = 0$. Then, since $b_1 t_j + b_j t_1 = 0$, we have $b_j = 0, 2 \leq j \leq n$. Suppose, in general, $t_i \neq 0$ for some i. Then $b_i = 0$. For $j < i$, we have $b_j t_i + b_i t_j = 0$, hence $b_j = 0$. For $j > i$, we have $b_i t_j + b_j t_i = 0$, hence $b_j = 0$. Thus $b_j = 0, 1 \leq j \leq n$. $\square$

Remark 4.5. A mapping $f : N^n \to \mathbf{R}^m$, $n \leq m$ is called **of finite type** at $p \in N$ if the $m \times \infty$-matrix

$$W_\infty(f)(x) = \left(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \;\middle|\; \alpha \in \mathbf{N}^n \setminus \{\mathbf{0}\}\right)$$

is of rank m. Moreover, f is called **of finite type** if it is if finite type at every point in N.

Using the transversality theorem, we can show easily:

Lemma 4.5. *Let N be an n-dimensional manifold and $n \leq m$. Then mappings of non-finite type $N \to \mathbf{R}^m$ form an infinite codimensional subset of $C^\infty(N, \mathbf{R}^m)$. Namely, for any ℓ, any ℓ-dimensional family of mapping $F : N \times \mathbf{R}^\ell \to \mathbf{R}^m$ is approximated by $\widetilde{F}$ in C^∞-topology such that any $\widetilde{F}_\lambda, (\lambda \in \mathbf{R}^\ell)$ is of finite type.*

§ 5. Tangent varieties of surfaces with large codimension

First we consider the classification problem of tangent varieties of generic immersions $N^2 \to \mathbf{R}^m$ for a sufficiently large m and prove Theorem 1.1.

Lemma 5.1. *Let $g : (\mathbf{R}^4, 0) \to (\mathbf{R}^5, 0)$ be the map-germ defined by*

$$g(u, v, s, t) = (u,\ v,\ s^2,\ st,\ t^2).$$

Then we have

$$\begin{aligned}\mathcal{R}_g &= \{h \in \mathcal{E}_{u,v,s,t} \mid h_s(u, v, 0, 0) = 0,\ h_t(u, v, 0, 0) = 0\} \\ &= \mathbf{R} + \mathfrak{m}^2_{s,t}\mathcal{E}_{u,v,s,t}\end{aligned}$$

Moreover $\mathcal{R}_g$ is minimally generated by $1, s^3, s^2t, st^2, t^3$ as a $g^*\mathcal{E}_5$-module.

Proof: Set $A = \{h \in \mathcal{E}_{u,v,s,t} \mid h_s(u, v, 0, 0) = 0,\ h_t(u, v, 0, 0) = 0\}, B = \mathbf{R} + \mathfrak{m}^2_{s,t}\mathcal{E}_{u,v,s,t}$ and $C = \langle 1, s^3, s^2t, st^2, t^3\rangle_{g^*\mathcal{E}_5}$. It is clear that $\mathcal{R}_g \subseteq A$. By Haramard's lemma we see $A \subseteq B$. Let $h \in B$. Using the preparation theorem, we write $h = g^*K + g^*L \cdot s + g^*M \cdot t$, for some $K, L, M \in \mathcal{E}_5$. Then $L(u, v, 0, 0, 0) = M(u, v, 0, 0, 0) = 0$. Therefore $g^*L = (g^*L_1)s^2 + (g^*L_2)st + (g^*L_2)t^2, g^*M = (g^*M_1)s^2 + (g^*M_2)st + (g^*M_3)t^2$ for some $L_1, L_2, L_3, M_1, M_2, M_3 \in \mathcal{E}_5$. Thus we see $h \in C$. Hence we have $B \subseteq C$. Since $s^3, s^2t, st^2, t^3 \in \mathcal{R}_g$, we have $C \subseteq \mathcal{R}_g$. Thus we have $\mathcal{R}_g = A = B = C$. The minimality is clear. □

Corollary 5.2. *The mini-versal opening of $g : (\mathbf{R}^4, 0) \to (\mathbf{R}^5, 0)$ in Lemma 5.1 is given by $G : (\mathbf{R}^4, 0) \to (\mathbf{R}^5 \times \mathbf{R}^4, 0) = (\mathbf{R}^9, 0)$,*

$$G(u, v, s, t) = (u,\ v,\ s^2,\ st,\ t^2,\ s^3,\ s^2t,\ st^2,\ t^3).$$

A map-germ from a 4-dimensional manifold to a 9-dimensional manifold is called the 4-**dimensional cuspidal conical edge** if it is diffeomorphic to the above map-germ G.

Proposition 5.3. *Let $f : N^2 \to \mathbf{R}^m$ be an immersion with $m \geq 9$. Suppose f is 3-non-degenerate at a point $p \in N$. Then the germ $\mathrm{Tan}(f) : (TN, (p, 0)) \to \mathbf{R}^m$ is diffeomorphic to the 4-dimensional cuspidal-conical edge composed with an immersion.*

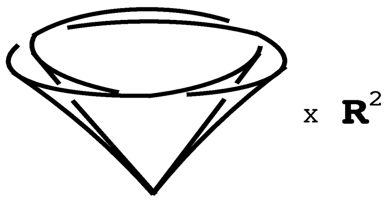

Figure 3. Four-dimensional cuspidal-conical edge

Proof: Let f is 3-non-degenerate at $p \in N$. Then, for a system of affine local coordinates of $\mathbf{R}^m$ and a system of local coordinates of N centred at p, we have a local representation $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^m, 0)$ in the form

$$\begin{aligned} f(x_1, x_2) = \big(x_1,\ x_2,\ x_1^2 + \varphi_1(x),\ x_1x_2 + \varphi_2(x),\ x_2^2 + \varphi_3(x), \\ x_1^3 + \psi_1(x),\ x_1^2x_2 + \psi_2(x),\ x_1x_2^2 + \psi_3(x),\ x_2^3 + \psi_4(x), \\ \rho_1(x), \dots, \rho_{m-9}(x)\big)\,, \end{aligned}$$

with $\varphi_i \in \mathfrak{m}_2^3, \psi_j \in \mathfrak{m}_2^4, \rho_k \in \mathfrak{m}_2^4, i = 1,2,3, j = 1,2,3,4, k = 1,\dots,m-9,$ where $\mathfrak{m}_2 = \mathfrak{m}_{\mathbf{R}^2,0}$.

Then the tangent mapping $F = \mathrm{Tan}(f) : (\mathbf{R}^4, 0) \to \mathbf{R}^m$ of f is given by

$$F(x_1, x_2, s, t) = f(x_1, x_2) + s\,\frac{\partial f}{\partial x_1}(x_1, x_2) + t\,\frac{\partial f}{\partial x_2}(x_1, x_2),$$

namely by

$$\begin{cases} F_1 = x_1 + s \\ F_2 = x_2 + t \\ F_3 = x_1^2 + \varphi_1 + s(2x_1 + \dfrac{\partial \varphi_1}{\partial x_1}) + t\dfrac{\partial \varphi_1}{\partial x_2} \\ F_4 = x_1x_2 + \varphi_2 + s(x_2 + \dfrac{\partial \varphi_2}{\partial x_1}) + t(x_1 + \dfrac{\partial \varphi_2}{\partial x_2}) \\ F_5 = x_2^2 + \varphi_3 + s\dfrac{\partial \varphi_3}{\partial x_1} + t(2x_2 + \dfrac{\partial \varphi_3}{\partial x_2}) \\ F_6 = x_1^3 + \psi_1 + s(3x_1^2 + \dfrac{\partial \psi_1}{\partial x_1}) + t\dfrac{\partial \psi_1}{\partial x_2} \\ F_7 = x_1^2x_2 + \psi_2 + s(2x_1x_2 + \dfrac{\partial \psi_2}{\partial x_1}) + t(x_1^2 + \dfrac{\partial \psi_2}{\partial x_2}) \\ F_8 = x_1x_2^2 + \psi_3 + s(x_2^2 + \dfrac{\partial \psi_3}{\partial x_1}) + t(2x_1x_2 + \dfrac{\partial \psi_3}{\partial x_2}) \\ F_9 = x_2^3 + \psi_4 + s\dfrac{\partial \psi_4}{\partial x_1} + t(3x_2^2 + \dfrac{\partial \psi_4}{\partial x_2}) \\ F_{9+k} = \rho_k + s\dfrac{\partial \rho_k}{\partial x_1} + t\dfrac{\partial \rho_k}{\partial x_2}, (1 \le k \le m-9) \end{cases}$$

We set $u = x_1 + s, v = x_2 + t$. Then, in terms of coordinates u, v, s, t of $(\mathbf{R}^4, 0)$, we have

$$\begin{cases} F_1 = u \\ F_2 = v \\ F_3 = -s^2 + u^2 + \Phi_1 \\ F_4 = -st + uv + \Phi_2 \\ F_5 = -t^2 + v^2 + \Phi_3 \\ F_6 = 2s^3 - 3us^2 + u^3 + \Psi_1 \\ F_7 = 2s^2t - 2ust - vs^2 + u^2v + \Psi_2 \\ F_8 = 2st^2 - ut^2 - 2vst + uv^2 + \Psi_3 \\ F_9 = 2t^3 - 3vt^2 + v^3 + \Psi_4 \\ F_{9+k} = R_k \end{cases}$$

with $\Phi_i \in \mathfrak{m}_4^3, \Psi_j \in \mathfrak{m}_4^4, R_k \in \mathfrak{m}_4^4, i = 1,2,3, j = 1,2,3,4, k = 1, \dots, m-9$, where $\mathfrak{m}_4 = \mathfrak{m}_{\mathbf{R}^4,0}$.

We set $g, g' : (\mathbf{R}^4, 0) \to (\mathbf{R}^5, 0)$ by

$$g(u,v,s,t) = (u,\ v,\ s^2,\ st,\ t^2), \quad g'(u,v,s,t) = (F_1, F_2, F_3, F_4, F_5).$$

Then we see $\mathcal{R}_{g'} = \mathcal{R}_g$ by Lemma 5.1. Therefore we have $\mathcal{J}_{g'} = \mathcal{J}_g$ and that g, g' are $\mathcal{J}$-equivalent. Moreover both g and g' are $\mathcal{J}$-minimal.

We set $G, G' : (\mathbf{R}^4, 0) \to (\mathbf{R}^9, 0)$ by

$$G(u,v,s,t) = (g;\ s^3,\ s^2t,\ st^2,\ t^3), \qquad G'(u,v,s,t) = (g'; F_6, F_7, F_8, F_9).$$

Then G is a mini-versal opening of g. Moreover G' is a mini-versal opening of g'. Therefore by Proposition 2.10, G and G' are left equivalent. By Lemma 5.1, F is an opening of G'. Thus we have, by Proposition 2.4, F is left equivalent to $(G', 0)$, which is left equivalent to $(G, 0)$, namely to the 4-dimensional cuspidal-conical edge composed with an immersion. □

Now to show Theorem 1.1, it is enough to prove the following:

Theorem 5.4. *Let $m \geq 11$. Then for a generic proper immersion $f : N^2 \to \mathbf{R}^m$, in Whitney C^∞ topology, from a 2-dimensional manifold N, we have, at any point $p \in N$,* $\mathrm{Tan}(f) : (TN, (p,0)) \to \mathbf{R}^m$ *is diffeomorphic to the 4-dimensional cuspidal-conical edge composed with an immersion. In particular* $\mathrm{Tan}(f)$ *is 5-frontal. Moreover* $\mathrm{Tan}(f)$ *is an immersion on $TN \setminus N \times \{0\}$.*

Proof: By Corollary 4.3 ($n = 2$), we may suppose f is 3-non-degenerate at p. Then by Proposition 5.3, we have the first-half. The second-half follows from Lemma 4.4 ($n = 2$). □

§ 6. Stable classification of tangential singularities

Let $v_n : (\mathbf{R}^n, 0) \to (\mathbf{R}^{\frac{1}{2}n(n+1)}, 0)$ be the Veronese map defined by

$$v_n(t_1, t_2, \dots, t_n) := \left(t_1^2,\ t_1t_2,\ \dots,\ t_1t_n,\ \dots,\ t_n^2\right),$$

all monomials of second order appearing. Then consider the trivial ℓ-parameter unfolding $v_{\ell,n} : (\mathbf{R}^{\ell+n}, 0) \to (\mathbf{R}^{\ell+\frac{1}{2}n(n+1)}, 0)$ of v_n,

$$v_{\ell,n}(u_1, \dots, u_\ell, t_1, t_2, \dots, t_n) = (u_1, \dots, u_\ell, v_n(t_1, t_2, \dots, t_n)).$$

Lemma 6.1. *Let $m = \ell + \frac{1}{2}n(n+1)$ and $g = v_{\ell,n} : (\mathbf{R}^{\ell+n}, 0) \to (\mathbf{R}^m, 0)$ be the trivial unfolding of Veronese map-germ. Then*

$$\begin{aligned}\mathcal{R}_g &= \{h \in \mathcal{E}_{\mathbf{R}^{\ell+n},0} \mid \frac{\partial h}{\partial t_i}|_{\mathbf{R}^\ell \times 0} = 0, 1 \le i \le n\} \\ &= \mathbf{R} + \mathfrak{m}^2_{\mathbf{R}^n,0}\mathcal{E}_{\mathbf{R}^{\ell+n},0}\end{aligned}$$

Moreover $\mathcal{R}_g$ is generated by 1 and all cubic monomials on $t_1, \dots, t_n$ as a $g^*\mathcal{E}_{\mathbf{R}^m,0}$-module.

Proof: Set $A = \{h \in \mathcal{E}_{\mathbf{R}^{\ell+n},0} \mid \frac{\partial h}{\partial t_i}|_{\mathbf{R}^\ell \times 0} = 0, 1 \le i \le n\}, B = \mathbf{R} + \mathfrak{m}^2_{\mathbf{R}^n,0}\mathcal{E}_{\mathbf{R}^{\ell+n},0}$ and denote by C the $g^*\mathcal{E}_{\mathbf{R}^n,0}$-module generated by 1 and all cubic monomials on $t_1, \dots, t_n$. It is clear that $\mathcal{R}_g \subseteq A$. By Haramard's lemma we see $A \subseteq B$. Let $h \in B$. Using the preparation theorem, we write $h = g^*K + \sum_{i=1}^n (g^*L_i)t_i$, for some $K, L_1, \dots, L_n \in \mathcal{E}_{\mathbf{R}^m,0}$. Then $L_i(u, 0) = 0, 1 \le i \le n$. Therefore $h \in C$. Hence we have $B \subseteq C$. Since any cubic monomial belongs to $\mathcal{R}_g$, we have $C \subseteq \mathcal{R}_g$. Thus we have $\mathcal{R}_g = A = B = C$. □

The versal opening of $v_{1,1}$ is the cuspidal edge. The versal opening of $v_{2,2}$ is the 4-dimensional cuspidal-conical edge.

Similarly we get the mini-versal opening of $v_{n,n}$ by just putting all monomials of degree 3 of $t_1, \dots, t_n$ to $v_{n,n}$. We define $w_n : (\mathbf{R}^n, 0) \to (\mathbf{R}^{\frac{1}{6}n(n+1)(n+2)}, 0)$ by all cubic monomials on $t_1, \dots, t_n$.

A map-germ is called $2n$**-dimensional cuspidal-conical edge** if it is diffeomorphic to the map-germ $(\mathbf{R}^{2n}, 0) \to (\mathbf{R}^m, 0)$ defined by

$$(u, t) = (u_1, \dots, u_n, t_1, \dots, t_n) \mapsto (u, v_n(t), w_n(t), 0) =$$

$$(u_1, \dots, u_n, t_1^2,\ t_1t_2,\ \dots,\ t_1t_n,\ \dots,\ t_n^2,$$

$$t_1^3,\ t_1^2t_2, \dots, t_1^2t_n,\ t_1t_2^2,\ t_1t_2t_3, \dots\dots, t_n^3,\ 0, \dots, 0).$$

Proposition 6.2. *Let $f : N^n \to \mathbf{R}^m$ be an immersion with $m \ge 2n + \dfrac{1}{2}n(n+1)$. Suppose f is 3-non-degenerate at a point $p \in N$. Then the germ* $\mathrm{Tan}(f) : (TN, (p, 0)) \to$

$\mathbf{R}^m$ is diffeomorphic to the $2n$-dimensional cuspidal-conical edge composed with an immersion.

Recall that f is 3-non-degenerate at $p \in N$ if

$$\operatorname{rank}\left(\frac{\partial^\alpha f}{\partial x^\alpha}(p) \;\middle|\; |\alpha| = 1, 2, 3\right) = n + \frac{1}{2}n(n+1) + \frac{1}{6}n(n+1)(n+2).$$

Proof: The proof of Proposition 6.2 is similar to that of Proposition 5.3 in the previous section. In fact f is affine equivalent to the form

$$f(x) = (x, v_n(x) + \varphi(x), w_n(x) + \psi(x), \rho(x)),$$

such that each component of φ (resp. ψ, ρ) belongs to $\mathfrak{m}_n^3$, (resp. $\mathfrak{m}_n^4$), where $\mathfrak{m}_n = \mathfrak{m}_{\mathbf{R}^n,0}$. Then $F = \operatorname{Tan}(f)$ is given by

$$\begin{aligned} F(x,t) &= f(x) + \textstyle\sum_{i=1}^n t_i \displaystyle\frac{\partial f}{\partial x_i}(x) \\ &= \begin{pmatrix} x + t \\ v_n(x) + \sum_{i=1}^n t_i \dfrac{\partial v_n(x)}{\partial x_i}(x) + \varphi(x) + \sum_{i=1}^n t_i \dfrac{\partial \varphi}{\partial x_i}(x) \\ w_n(x) + \sum_{i=1}^n t_i \dfrac{\partial w_n(x)}{\partial x_i}(x) + \psi(x) + \sum_{i=1}^n t_i \dfrac{\partial \psi}{\partial x_i}(x) \\ \rho(x) + \sum_{i=1}^n t_i \dfrac{\partial \rho}{\partial x_i}(x) \end{pmatrix} \end{aligned}$$

We set $u = x + t$. Then $x = u - t$. Put $g(u,t) = (u, v_n(t))$ and

$$g'(u,t) = (u, v_n(u-t) + \sum_{i=1}^n t_i \frac{\partial v_n(x)}{\partial x_i}(u-t) + \varphi(u-t) + \sum_{i=1}^n t_i \frac{\partial \varphi}{\partial x_i}(u-t)).$$

Then g and g' are $\mathcal{J}$-equivalent and both g and g' are $\mathcal{J}$-minimal. Moreover, we set $G(u,t) = (u, v_n(t), w_n(t))$ and

$$G'(u,t) = (g'(u,t), w_n(u-t) + \sum_{i=1}^n t_i \frac{\partial w_n(x)}{\partial x_i}(u-t) + \psi(u-t) + \sum_{i=1}^n t_i \frac{\partial \psi}{\partial x_i}(u-t)).$$

Then G is a mini-versal opening of g and G' is a mini-versal opening of g'. Thus we see G and G' are left equivalent by Proposition 2.10. Then F is left equivalent to $(G', 0)$ and therefore to $(G, 0)$. $\square$

The normal form for singularity of tangent variety of a generic n-dimensional submanifold in $\mathbf{R}^m$ for sufficiently large m is given by $2n$-dimensional cuspidal-conical edge:

Theorem 6.3. *For a generic immersion $f : N^n \to \mathbf{R}^m$ with $m \geq 2n + \frac{1}{2}n(n+1) + \frac{1}{6}n(n+1)(n+2)$, the germ of tangent variety* $\operatorname{Tan}(f) : (TN, (p,0)) \to \mathbf{R}^m$ *has unique* *local diffeomorphism class for any $p \in N$, that is the $2n$-dimensional cuspidal-conical edge composed with an immersion. In particular* $\mathrm{Tan}(f)$ *is* $\{n + \frac{1}{2}n(n+1)\}$*-frontal. Moreover* $\mathrm{Tan}(f)$ *is an immersion on* $TN \setminus N \times \{0\}$.

The genericity condition of Theorem 6.3 is given by that f is 3-non-degenerate. Also we have

Corollary 6.4. *Any tangent variety* $\mathrm{Tan}(f) : (TN, (p, 0)) \to \mathbf{R}^m$ *of any immersion* $N^n \to \mathbf{R}^m$ *is obtained locally by some projection of the $2n$-dimensional cuspidal-conical edge, for any $p \in N$.*

Proof: Let $f : N \to \mathbf{R}^m$ be any immersion and $p \in N$. Then by a right equivalence and a linear coordinate change of the target we have

$$f = (x_1, \dots, x_n, \varphi_1, \dots, \varphi_{m-n}),$$

with $\varphi_i \in \mathfrak{m}_n^2, 1 \leq i \leq m-n$. Consider the map

$$f' = (x_1, \dots, x_n, \varphi_1, \dots, \varphi_{m-n}, v_n, w_n),$$

where v_n (resp. w_n) is the mapping with components which consist of all quadratic (resp. cubic) monomials in $x_1, \dots, x_n$ as above. Then $\mathrm{Tan}(f')$ is $2n$-dimensional cuspidal-conical edge by Theorem 6.3 and $\mathrm{Tan}(f) = \Pi \circ \mathrm{Tan}(f')$ by the projection Π to the first m-components. $\square$

References

[1] M.A. Akivis, V.V. Goldberg, *Differential geometry of varieties with degenerate Gauss maps,* CMS Books in Mathematematics, **18**, Springer-Verlag, New York, (2004).

[2] V.I. Arnol'd, *Lagrangian manifolds with singularities, asymptotic rays, and the open swallowtail,* Funkt. Anal. i Prilozhen., **15–4** (1981), 1–14. English transl. Funct. Anal. and its Appl., **15–4** (1981), 235–246.

[3] V.I. Arnol'd, *Singularities of caustics and wave fronts,* Math. its Appl. (Soviet Series), **62**. Kluwer Academic Publishers, (1990).

[4] V.I. Arnol'd, *Catastrophe theory,* 3rd edition, Springer-Verlag, (1992).

[5] T. Bröcker, *Differentiable Germs and Catastrophes,* London Math. Soc. Lecture Note Series **17**, Cambridge Univ. Press (1975).

[6] A. Cayley, *On certain developable surfaces,* Quarterly J. Pure Appl. Math., **6** (1864), 108–126.

[7] J.P. Cleave, *The form of the tangent-developable at points of zero torsion on space curves,* Math. Proc. Camb. Philos. Soc., **88–3** (1980), 403–407.

[8] J.W. Bruce, P.J. Giblin, *Curves and singularities, A geometrical introduction to singularity theory,* 2nd ed., Cambridge Univ. Press, (1992).

[9] E.F. Feldman, *The geometry of immersions I,* Trans. Amer. Math. Soc., **120** (1965), 185–224.
[10] A.B. Givental', *Whitney singularities of solutions of partial differential equations,* J. Geom. Phys. **15–4** (1995) 353–368.
[11] G. Ishikawa, *Parametrization of a singular Lagrangian variety,* Trans. Amer. Math. Soc., **331–2** (1992), 787–798.
[12] G. Ishikawa, *Symplectic and Lagrange stabilities of open Whitney umbrellas,* Invent. math., **126** (1996), 215–234.
[13] G. Ishikawa, *Singularities of developable surfaces,* London Math. Soc. Lect. Notes Series, **263** (1999), 403–418.
[14] G. Ishikawa, *Singularities of tangent varieties to curves and surfaces,* arXiv 1201. 2778, to appear in Journal of Singularities.
[15] G. Ishikawa, S. Janeczko, *Symplectic bifurcations of plane curves and isotropic liftings,* Quarterly J. Math. Oxford, **54** (2003), 73–102.
[16] G. Ishikawa, Y. Machida, M. Takahashi, *Asymmetry in singularities of tangent surfaces in contact-cone Legendre-null duality,* Journal of Singularities, **3** (2011), 126–143.
[17] S. Izumiya, K. Saji, *The mandala of Legendrian dualities for pseudo-spheres in Lorenz-Minkowski space and "flat" spacelike surfaces.* Journal of Singularities **2** (2010), 92–127.
[18] J.M. Landsberg, J. Weyman, *On tangential varieties of rational homogeneous varieties,* J. London Math. Soc., **76–2** (2007), 513–530.
[19] D. Mond, *On the tangent developable of a space curve,* Math. Proc. Cambridge Philos. Soc., **91–3** (1982), 351–355.
[20] D. Mond, *Singularities of the tangent developable surface of a space curve,* Quart. J. Math. Oxford, **40–157** (1989), 79–91.
[21] D. Mond, *Deformations which preserve the non-immersive locus of a map-germ,* Math. Scand., **66** (1990), 21–32.
[22] W.F. Pohl, *Differential geometry of higher order,* Topology **1** (1962), 169–211.
[23] I.R. Porteous, *Geometric Differentiation: For the intelligence of curves and surfaces,* Camb. Univ. Press., (1994).
[24] K. Saji, *Criteria for D_4 singularities of wave fronts,* Tohoku Math. J., **63** (2011), 137–147.
[25] K. Saji, M. Umehara, K. Yamada, *The geometry of fronts,* Annals of Math., **169** (2009), 491–520.
[26] O.P. Scherbak, *Projective dual space curves and Legendre singularities,* Trudy Tbiliss Univ., 232–233 (1982), 280–336.
[27] F.L. Zak, *Tangents and Secants of Algebraic Varieties,* Transl. of Math. Monographs **127**, Amer. Math. Soc., (1993).
[28] V. Zakalyukin, *Generating ideals of Lagrangian varieties,* Theory of Singularities and its Applications, Advances in Soviet Math. **1**, Amer. Math. Soc., (1990), pp. 201–210.