<table>
<thead>
<tr>
<th>Title</th>
<th>On line degenerated torus curves and weak Zariski pairs (Singularity theory, geometry and topology)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kawashima, Masayuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2013), B38: 107-118</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/207805">http://hdl.handle.net/2433/207805</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
On line degenerated torus curves and weak Zariski pairs

By

Kawashima Masayuki *

Abstract

Let $C = \{f = 0\}$ be an affine plane curve. We are interested in a form of the defining polynomial $f$. In this paper, we study line degenerations of torus curves. Line degenerations of torus type are divided into two types which are called visible or invisible degenerations. We construct a pair of plane curves of degree $2p - 2$ such that they have the same configuration of singularities. If $p$ is even, their complements in $\mathbb{P}^2$ have different topologies. Thus they give a weak Zariski pair.

§1. Introduction

Let $\mathbb{P}^2$ be a complex projective space of dimension 2 with homogeneous coordinates $[X_0, X_1, X_2]$ and let $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{X_2 = 0\}$ be the affine space with coordinates $(x, y) = (X_0/X_2, X_1/X_2)$. We study reduced plane curves in $\mathbb{P}^2$ and $\mathbb{C}^2$. Let $\mathcal{M}(d)$ and $\mathcal{M}^a(d)$ be the set of projective and affine plane curves of degree $d$ respectively. For a given curve $C \in \mathcal{M}(d)$ or $\mathcal{M}^a(d)$, we are interested in forms of the defining polynomial of $C$. Let $p$ and $q$ be positive integers such that $p > q \geq 2$. We say that $C = \{f = 0\} \in \mathcal{M}^a(d)$ is a torus curve of type $(p, q)$ if $f$ is written as $f = f_a^p + f_b^q$ where $f_j$ is a polynomial in $\mathbb{C}[x, y]$ of degree $j$. Put $\mathcal{T}(p, q; d)$ as the set of curves of $(p, q)$ torus type of degree $d$.

We also consider another class of plane curves which are called quasi torus curves of type $(p, q)$ (c.f [7], [2]). We say that $C = \{f = 0\} \in \mathcal{M}^a(d)$ quasi torus curve of type $(p, q)$ if there exist three polynomials $f_a$, $f_b$ and $f_c$ such that they do not have same
components and they satisfy the following relation:

\[ f_{c}^{pq}f = f_{a}^{p} + f_{b}^{q} \quad \text{in} \quad \mathbb{C}[x, y] \quad \deg f_{j} = j \]

where \( \deg f_{j} \) is the degree of \( f_{j} \). Put \( \mathcal{Q}\mathcal{T}(p, q; d) \) as the set of curves of \((p, q)\) quasi torus type of degree \( d \).

For a given curve \( C \in \mathcal{M}^{a}(d) \), we say that \( C \) has a torus decomposition (resp. quasi torus decomposition) if \( C \) is in \( \mathcal{T}(p, q; d) \) (resp. \( \mathcal{Q}\mathcal{T}(p, q; d) \)) for some \((p, q)\).

**Example 1.1.** The following example is the motivation of this work. Let \( Q = \{f = 0\} \in \mathcal{M}^{a}(4) \) be a 3-cuspidal quartic. Then \( Q \) has at least two torus and one quasi torus decompositions ([6]):

\[
(1.1) \quad f = f_{3}^{1} + f_{2}^{2}, \quad f = g_{2}^{3} + g_{3}^{2}, \quad h_{1}^{6}f = h_{3}^{3} + h_{5}^{2}
\]

where \( \deg f_{i} = i \), \( \deg g_{i} = i \) and \( \deg h_{i} = i \).

To construct these torus decompositions, we used line degenerated torus curves. Now we recall line degeneration of torus curves which are defined by M. Oka in [8].

**Definition 1.2.** Let \( C = \{F = F_{p}^{q} + F_{q}^{p} = 0\} \in \mathcal{M}(pq) \) be a projective \((p, q)\) torus curve. Suppose that \( F \) has the following form:

\[
(1.2) \quad F(X_{0}, X_{1}, X_{2}) = X_{2}^{j}G(X_{0}, X_{1}, X_{2})
\]

where \( G(X, Y, Z) \) is a reduced homogeneous polynomial of degree \( pq - j \). We call a curve \( D = \{G = 0\} \) a line degenerated torus curve of type \((p, q)\) of order \( j \) and the line \( L_{\infty} = \{X_{2} = 0\} \) the limit line of the degeneration ([8]).

Put \( \mathcal{L}\mathcal{T}_{j}(p, q; d) \) as the set of line degenerated torus curves of type \((p, q)\) of order \( j \). and \( \mathcal{L}\mathcal{T}(p, q) \) is the union of \( \mathcal{L}\mathcal{T}_{j}(p, q; d) \) with respect to \( j \).

We divide the situations (1.2) into two cases which are called visible degenerations and invisible degenerations. Put the integer \( r_{k} := \max\{r \in \mathbb{Z} \mid X_{2}^{r} \mid F_{k}\} \) for \( k = p, q \).

**Visible case.** Suppose that \( r_{p} \cdot r_{q} \neq 0 \) and \( qr_{p} \neq pr_{q} \). Then \( F_{q} \) and \( F_{p} \) are written as follows:

\[
F_{q}(X_{0}, X_{1}, X_{2}) = F'_{q-r_{q}}(X_{0}, X_{1}, X_{2})X_{2}^{r_{q}}, \quad F_{p}(X_{0}, X_{1}, X_{2}) = F'_{p-r_{p}}(X_{0}, X_{1}, X_{2})X_{2}^{r_{p}}.
\]

Putting \( j := \min\{qr_{p}, pr_{q}\} \), we can factor \( F \) as \( F(X_{0}, X_{1}, X_{2}) = X_{2}^{j}G(X_{0}, X_{1}, X_{2}) \). Then \( G \) is written using \( F'_{p-r_{p}} \) and \( F'_{q-r_{q}} \) as

\[
(1.3) \quad G(X_{0}, X_{1}, X_{2}) = \begin{cases} 
F'_{q-r_{q}}(X_{0}, X_{1}, X_{2})^{p} + F'_{p-r_{p}}(X_{0}, X_{1}, X_{2})^qX_{2}^{qr_{p} - pr_{q}} & \text{if } j = pr_{q}, \\
F'_{q-r_{q}}(X_{0}, X_{1}, X_{2})^{p}X_{2}^{pr_{p} - qr_{q}} + F'_{p-r_{p}}(X_{0}, X_{1}, X_{2})^q & \text{if } j = qr_{p}.
\end{cases}
\]
Such a factorization is called a visible factorization and $D$ is called a visible degeneration of $(p, q)$ torus curves. We denote the set of visible degenerations of order $j$ by $\mathcal{LT}^V_j(p, q; d)$.

**Invisible case.** Either $r_p = 0$ or $r_q = 0$ but $F$ can be written as (1.2). Then $D$ is called an invisible degeneration of $(p, q)$ torus curves. In this case, write $F_q^p + F_p^q = \sum_{i=0}^{pq} A_i(X_0, X_1)X_2^i$. Then $A_j(X_0, X_1) = 0$ for $i \leq j - 1$ and therefore $X_2^j \mid F$. We denote the set of invisible degenerations of order $j$ by $\mathcal{LT}^I_j(p, q; d)$.

Using these terminologies, we will show that torus decompositions (1.1) satisfy:

$$\{f_1^3 + f_2^2 = 0\} \in \mathcal{LT}^V_2(3, 2; 4), \{g_2^3 + g_3^2 = 0\} \in \mathcal{LT}^I_2(3, 2; 4).$$

Thus $Q = \{f = 0\}$ is in $\mathcal{LT}^V_2(3, 2; 4) \cap \mathcal{LT}^I_2(3, 2; 4)$.

We consider whether such phenomena occur or not for other curves. Before we consider this problem, we study line degenerated torus curves. More precisely, we look for a pair of curves $\{C, D\}$ such that $C \in \mathcal{LT}^V_j(p, q; d)$ and $D \in \mathcal{LT}^I_j(p, q; d)$ such that $\text{Sing} C = \text{Sing} D$. Here $\text{Sing} C$ is the configuration of the singularities. If there exists such a pair $(C, D)$, then we discuss if the topologies of $C$ and $D$ are the same or not.

**Definition 1.3.** A pair of plane curves $(C_1, C_2)$ is called a weak Zariski pair if they have the same degree and configuration of singularities, while the complements $\mathbb{P}^2 \setminus C_1$ and $\mathbb{P}^2 \setminus C_2$ are not homeomorphic to each other ([9, 5]).

To express singularities of curves, we use an important class of singularities which is called Brieskorn-Pham singularities:

$$B_{n,m} : x^n + y^m = 0, \quad n, m \geq 2.$$

**Theorem 1.4.** For each $p \geq 3$, there is a pair of plane curves $(C, D) \in \mathcal{LT}^V_2(p, 2; 2p - 2) \times \mathcal{LT}^I_2(p, 2; 2p - 2)$ with

$$\text{Sing} C = \text{Sing} D = \{pA_{p-1}, A_{p-3}, B_{p-2,2(p-2)}\}.$$

If $p$ is even, then $(C, D)$ is a weak Zariski pair.

§ 2. Preliminaries

In section 2, we follow the terminologies in [3] and [4].

Let $p : \Sigma_d \to \mathbb{P}^1$ be a Hirzebruch surface of degree $d$ and let $\Delta_{\infty,d}$ be the exceptional section with the self-intersection multiplicity $\Delta_{\infty,d}^2 = -d$. Let $(X_0, X_1, X_2)$ and $(Y_0, Y_1)$ be homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively. Using these coordinates, $\Sigma_d$ is defined as

$$\Sigma_d := \{((X_0, X_1, X_2), (Y_0, Y_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid X_1Y_1^d = X_2Y_0^d\}$$
and \( p : \Sigma_d \to \mathbb{P}^1 \) is the canonical projection. There are four affine coordinates which cover \( \Sigma_d \). We use two affine spaces \( W_d^1, W_d^2 \subset \Sigma_d \) with coordinates \((y_d, \tau_d)\) and \((z_d, \tau_d)\) respectively where
\[
y_d = X_2/X_0, \quad z_d = X_0/X_2, \quad \tau_d = Y_0/Y_1
\]
and they are glued by the relation \( y_dz_d = 1 \). Putting \( V_1 = \{(Y_0, Y_1) \in \mathbb{P}^1 \mid Y_1 \neq 0\} \), they satisfy \( p^{-1}(V_1) = W_d^1 \cup W_d^2 \).

We denote the fiber over \( \tau_d = 0 \) in \( \Sigma_d \) by \( F_{\infty}^\circ \) and the origin of the affine space \( W_d^i \) by \( O_{i,d} := (0,0) \in W_d^i \). We put the affine line \( F_{\infty}^o := F_{\infty} \setminus \triangle_{\infty,d} = F_{\infty} \cap W_d^2 \).

§ 2.1. \( p \)-gonal curves

Let \( B \subset \Sigma_d \) be a reduced curve such that \( B \) does not contain the exceptional section \( \Delta_{\infty,d} \). If \( B \) intersects with a generic fiber at \( p \) points, then we call \( B \) a generalized \( p \)-gonal curve. A generalized \( p \)-gonal curve \( B \) is called a \( p \)-gonal curve if \( B \) disjoint from the exceptional section \( \Delta_{\infty,d} \).

Let \( f_i \) be a defining equation of \( B \) on \( W_d^i \) and then we have the equality \( f_1(y_d, \tau_d) = y_d^pf_2(z_d, \tau_d) \) on \( W_d^1 \cap W_d^2 \). Using affine coordinates \((z_d, \tau_d) \in W_d^2 \), the local equation \( f_2(z_d, \tau_d) \) is written as
\[
f_2(z_d, \tau_d) = \sum_{i=0}^{p} b_i(\tau_d)z_d^i, \quad \deg b_i(\tau_d) \leq d(p - i).
\]
The exceptional section \( \Delta_{\infty,d} \) is defined as \( \{y_d = 0\} \) in the affine coordinates \((y_d, \tau_d) \in W_d^1 \).

§ 2.2. Nagata transformations

Let \( P \) be a fixed point in \( \Sigma_2 \setminus \Delta_{\infty,2} \) and let \( F \) be the fiber which passes through \( P \). A Nagata transformation \( N : \Sigma_2 \dashrightarrow \Sigma_1 \) is a birational transformation which consists of the blowing-up at \( P \notin \Delta_{\infty,2} \) and the blowing-down the strict transform \( F^* \) of \( F \). We observe that the exceptional section \( \Delta_{\infty,1} \) of \( \Sigma_1 \) is the image \( N(\Delta_{\infty,2}) \).

We express a Nagata transformation using local coordinates \((z_2, \tau_2)\) and \((z_1, \tau_1)\) assuming \( P = O_{2,2} \in W_2^2 \). Let \( \mu_1 : \tilde{W}_2^2 \to W_2^2 \) and \( \mu_2 : \tilde{W}_1^1 \to W_1^1 \) be blowing-ups centered at \( O_{2,2} \) and \( O_{1,1} \) respectively. There is an affine coordinate \( \tilde{W} \) with coordinates \((s, t)\) such that \( \mu_1(s, t) = (t, ts) \) and \( \mu_2(s, t) = (s, st) \). Note that \( \{t = 0\} \) defines the exceptional curve of \( \mu_1 \) and \( \{s = 0\} \) defines the exceptional curve of \( \mu_2 \). Then we have:
\[
N(z_2, \tau_2) = (z_1, \tau_1) = \left( \frac{z_2}{\tau_2}, \tau_2 \right).
\]

Let \( B \) be a \( p \)-gonal curve in \( \Sigma_2 \) which is defined by \( \{f_2(z_2, \tau_2) = 0\} \) in \( W_2^2 \). We consider the defining equation of the image of a \( p \)-gonal curve by a Nagata transformation. By the definition of a Nagata transformation, \( B' := N(B) \subset \Sigma_1 \) is defined as
(2.1) \[ B' : f'_2(z_1, \tau_1) = \frac{1}{\tau_1^M} f_2(z_1 \tau_1, \tau_1) = 0 \]

where \( M \) is the multiplicity of \( B \) at \( P \). As \( B \) is assumed to be \( p \)-gonal, \( B' \cap \Delta_{\infty,1} \) is \( \{O_{1,1}\} \). Thus \( B' \) is a generalized \( p \)-gonal curve.

§2.3. Contraction of \( p \)-gonal curves from \( \Sigma_2 \) to \( \mathbb{P}^2 \)

We recall that a Hirzebruch surface \( \Sigma_1 \) is obtained as a blowing-up at any point in \( \mathbb{P}^2 \). In this section, we consider the defining polynomial of a plane curve which is obtained as the image of the composition of a Nagata transformation and a blowing-up.

Let \( B = \{f_2(z_2, \tau_2) = 0\} \) be a \( p \)-gonal curve in \( W_2^2 \) and let \( B' = \{f'_2(z_1, \tau_1) = 0\} \subset W_1^2 \) be the image of \( B \) by a Nagata transformation \( N : \Sigma_2 \to \Sigma_1 \) at \( O_{2,2} \). Put \( m \) the intersection multiplicity of \( B' \) and \( \triangle_{\infty,1} \) at \( O_{1,1} \). Let \( U_1 \) be the affine coordinate chart \( \mathbb{P}^2 \setminus \{X_1 = 0\} \) with the coordinate \( (x_0, x_2) = (X_0/X_1, X_2/X_1) \). Let \( \pi : \tilde{U}_1 \to U_1 \) be a blowing-up at \((0,0) \in U_1 \). We naturally identify \( \tilde{U}_1 \) with \( \Sigma_1 \) as follows: Let \( \tilde{U}_{10} \) and \( \tilde{U}_{11} \) be two affine coordinates of \( \tilde{U}_1 \) and let \( (s, t) \) be the affine coordinate of \( \tilde{U}_{11} \). Then \( \pi \) is defined as \( \pi(s, t) = (x_0, x_2) = (s, st) \) on \( \tilde{U}_{11} \). We identify \( \tilde{U}_{11} \) with \( W_1^1 \) as \( (s, t) \mapsto (y_1, \tau_1) \).

By the definition of \( \pi : \Sigma_1 \to U_1 \) and the equality (2.1), the defining polynomial \( f \) of \( C := (\pi \circ N)(B) \subset U_1 \) as

\[
(2.2) \quad f(x_0, x_2) = \frac{x_0^{M+m+p}}{x_2^M} f_2 \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right).
\]

Indeed, let \( f'_1(y_1, \tau_1) \) be the defining equation of \( B' \) in \( W_1^1 \) which is written as

\[
f'_1(y_1, \tau_1) = y_1^p f'_2(1/y_1, \tau_1) = \frac{y_1^p}{\tau_1^M} f_2(\tau_1/y_1, \tau_1).
\]

where we use (2.1) for the second equality. And \( f \) must satisfy \( f(y_1, y_1 \tau_1) = y_1^m f'_1(y_1, \tau_1) \). Using these equalities and \( \pi(y_1, \tau_1) = (x_0, x_2) = (y_1, y_1 \tau_1) \), we have the equality (2.2).

Next we consider singularities of \( B' \) and \( C \). Assume that \( B \) satisfies the following conditions:

- \( B \) has an \( A_{\ell-1} = B_{\ell,2} \) singularity at \( O_{2,2} \in W_2^2 \) and its tangent cone is transverse to the fiber \( F_\infty = \{\tau_2 = 0\} \).

- \( B \) intersects transversely at \( p - 2 \) distinct points with \( F_\infty \) outside of \( O_{2,2} \in W_2^2 \).

Under the above conditions, the intersection \( B \cap (F_\infty \setminus \{O_{2,2}\}) \) consists of distinct \( p - 2 \) points and \( B' \) intersects with \( F_\infty^\circ \) so that
If $\ell = 2$, then $B'$ intersects transversely with $F_\infty$ at two points.

- If $\ell = 3$, then $B'$ is tangent to $F_\infty$ with the intersection multiplicity 2.
- If $\ell > 3$, then $B'$ has $A_{\ell - 3} = B_{\ell - 2, 2}$ singularity.

**Observation.** If $B$ is a trigonal curve ($p = 3$), then $B'$ is smooth and intersects transversely with $\Delta_{\infty, 1}$ at $O_{1, 1}$. If $p$ is greater than 3, then $B'$ has $B_{p-2, p-2}$ singularity at $(0, 0) \in U_1$.

**Proof.** The first assertion is obvious. Assume $p > 3$. The defining equation $f_1(y_1, \tau_1)$ of $B'$ in $W^1_1$ is written as:

$$f_1(y_1, \tau_1) = c \prod_{i=1}^{p-2} (y_1 - \alpha_i \tau_1) + \text{(higher terms)}, \quad c \neq 0, \quad \alpha_i \neq \alpha_j \ (i \neq j).$$

Now we use the equality $f(x_0, x_2) = x_0^{p-2} f_1(x_0, x_2/x_0)$ which is obtained from $(2.2)$. Then we have

$$f(x_0, x_2) = x_0^{p-2} f_1(x_0, x_2/x_0) = \prod_{i=1}^{p-2} (x_0^2 - \alpha_i x_2) + \text{(higher terms)}.$$ 

Thus $C$ has $B_{p-2, 2(p-2)}$ singularity at $(0, 0) \in U_1$.

§ 3. $p$-gonal curves of $(p, 2)$ torus type

Let $B$ be a $p$-gonal curve in $\Sigma_2$. We say that $B$ is a *torus curve of type $(p, 2)$* if the defining equation $f_2$ of $B$ in the affine space $(W^2_2, (z_2, \tau_2))$ is written as

$$f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2.$$

We assume further that

$$\begin{aligned}
k(z_2, \tau_2) &= z_2 + b_2(\tau_2), \\
h(z_2, \tau_2) &= b_{p-2}(\tau_2)z_2 + b_p(\tau_2),
\end{aligned} \quad \text{deg } b_i(\tau_2) = i.$$

§ 3.1. Singularities of $(p, 2)$ torus type

We consider curves $K := \{k = 0\}$ and $H := \{h = 0\}$ in $W^2_2$ where $h$ and $k$ are as above. Let $P \in B$ be a singular point. If $P \in K \cap H$, we call $P$ an *inner singularity*. Otherwise $P$ is called an *outer singularity*. We put $\Delta_1(\tau_2) := h(-b_2(\tau_2), \tau_2) = b_p(\tau_2) - b_{p-2}(\tau_2)b_2(\tau_2)$ and take an inner singular point $P \in K \cap H$. Then $P$ is written as $(-b_2(s), s)$ for some $s \in \mathbb{C}$ with $\Delta_1(s) = 0$ and the multiplicity of $\Delta_1(\tau_2)$ at $s$, say $i$, is equal to the intersection multiplicity of $K$ and $H$ at $P$.

By a similar argument as that in Lemma 1 in [1], we have the following.
Lemma 3.1. Let $B$ be the $p$-gonal curve as above in $\Sigma_2$. Suppose that $s$ is a root of $\Delta_1(\tau)$ and let $P = (-b_q(s), s) \in B$ be an inner singular point with the intersection multiplicity $\iota$. If $\Delta_2(s) \neq 0$, then $B$ has $B_{p,2} = A_{p^{\frac{\iota}{2}}}$ singularity at $P$.

§ 4. Proof of Theorem 1.4

Let $B \subset \Sigma_2$ be a $p$-gonal curve of $(p, 2)$ torus type. As the degree of $\Delta_1(\tau_2)$ is $p$, $B$ has $pA_{p-1}$ inner singularities by Lemma 3.1. We may assume that $B$ has an outer $A_{p-1}$ singularity. For example, we take $b_2(\tau_2), b_{p-2}(\tau_2)$ and $b_p(\tau_2)$ as

$$b_2(\tau_2) = 1 + \tau_2^2, \quad b_p(\tau_2) = 1 + \frac{p}{2}\tau_2^2 + \tau_2^p, \quad b_{p-2}(\tau_2) = \frac{p}{2} + p\tau_2^{p-2}.$$ 

Then $f_2 = k^p - h^2$ has an outer $A_{p-1}$ singularity at $O_{2,2}$ and its tangent cone does not contain $\{\tau_2 = 0\}$. As $\Delta_1(\tau_2) = 1 - \frac{p}{2} - p\tau_2^{p-2} + (1 - p)\tau_2^p$ and $p \geq 3$, $K$ and $H$ intersect transversely at distinct $p$ points and $K \cap H \cap F_\infty = \emptyset$.

Let $P$ be an inner $A_{p-1}$ singular point and let $Q$ be an outer $A_{p-1}$ singular point of $B$. Let $N_1$ and $N_2$ be the Nagata transformations from $\Sigma_2$ to $\Sigma_1$ at $P$ and $Q$ respectively. We consider the defining polynomial of $C := (\pi \circ N_1)(B)$ and $D := (\pi \circ N_2)(B)$ where $\pi : \Sigma_1 \to U_1$ is the blowing-up at $(0,0) \in U_1$.

§ 4.1. Construction of a visible degeneration

Hereafter we assume that $K$ and $H$ intersect transversely at $p$ points. Assume that $P = O_{2,2}$ in the affine space $W_2^2$. Let $f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2$ be the defining
equation of $B$ where
\[ k(z_2, \tau_2) = z_2 + b_2(\tau_2), \quad h(z_2, \tau_2) = b_{p-2}(\tau_2)z_2 + b_p(\tau_2), \quad \deg b_i(\tau_2) = i. \]
As $k(0, 0) = h(0, 0) = 0$, we can write $b_2(\tau_2)$ and $b_p(\tau_2)$ as
\[ b_2(\tau_2) = \tau_2 b_1(\tau_2), \quad b_p(\tau_2) = \tau_2 b_{p-1}(\tau_2), \quad \deg b_i = i. \]
Let $f$ be the defining polynomial of $C$ and using (2.2), we have
\[ f(x_0, x_2) = \frac{x_0^{2p}}{x_2^2} f_2 \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right). \]
We calculate the above equation as the following:
\[
x_2^2 f(x_0, x_2) = x_0^{2p} \left( k \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right)^p - h \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right)^2 \right)
= x_0^{2p} \left( \left( \frac{x_2}{x_0^2} + \frac{x_2}{x_0} b_1(\frac{x_2}{x_0}) \right)^p - \left( \frac{x_2}{x_0} + \frac{x_2}{x_0} b_{p-2}(\frac{x_2}{x_0}) \right)^2 \right)
= x_0^{2p} \left( 1 + x_0 b_1(\frac{x_2}{x_0}) \right)^p - x_0^{2p} \left( x_0^{p-2} b_{p-2}(\frac{x_2}{x_0}) + x_0^{p-1} b_{p-1}(\frac{x_2}{x_0}) \right)^2
= f_1(x_0, x_2)^p x_2^p - f_{p-1}(x_0, x_2)^2 x_2^2.
\]
and then where
\[ f_1(x_0, x_2) := 1 + c_1(x_0, x_2), \quad f_{p-1}(x_0, x_2) := c_{p-2}(x_0, x_2) + c_{p-1}(x_0, x_2). \]
Note that $c_i(x_0, x_2) := x_0^i b_i(x_2/x_0)$ is a polynomial for $i = 1, p - 2$ and $p - 1$. Hence we have
\[ x_2^2 f(x_0, x_2) = (f_1(x_0, x_2)x_2)^p - (f_{p-1}(x_0, x_2)x_2)^2. \]
Thus the above equation shows that $C := \{ f = 0 \}$ is a visible line degeneration of order 2 of $(p, 2)$ torus type.

§ 4.2. Construction of an invisible degeneration

Assume that $Q = O_{2, 2}$ in the affine space $W_2^2$. Let $f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2$ be the defining equation of $B$ where
\[ k(z_2, \tau_2) = z_2 + b_2(\tau_2), \quad h(z_2, \tau_2) = b_{p-2}(\tau_2)z_2 + b_p(\tau_2), \quad \deg b_i(\tau_2) = i. \]
Let $g$ be the defining polynomial of $D$ and using (2.2) in §2.3, we have
\[ g(x_0, x_2) = \frac{x_0^{2p}}{x_2^2} f_2 \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right). \]
We calculate the above equation:

\[
x_2^2 g(x_0, x_2) = x_0^{2p} \left( k \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right)^p - h \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right)^2 \right)
\]

\[
= x_0^{2p} \left( \left( \frac{x_2}{x_0^2} + b_2 \left( \frac{x_2}{x_0} \right) \right)^p - \left( b_{p-2} \left( \frac{x_2}{x_0} \right) \frac{x_2}{x_0} + b_p \left( \frac{x_2}{x_0} \right) \right)^2 \right)
\]

\[
= \left( x_2 + x_0^2 b_2 \left( \frac{x_2}{x_0} \right) \right)^p - \left( x_0^{p-2} b_{p-2} \left( \frac{x_2}{x_0} \right) x_2 + x_0^p b_p \left( \frac{x_2}{x_0} \right) \right)^2
\]

\[
= g_2(x_0, x_2)^p - g_p(x_0, x_2)^2
\]

where the polynomials \( g_2 \) and \( g_p \) are defined as

\[
g_2(x_0, x_2) := x_2 + d_2(x_0, x_2) \quad g_p(x_0, x_2) := d_{p-2}(x_0, x_2) x_2 + d_p(x_0, x_2)
\]

where \( d_i(x_0, x_2) := x_0^i b_i \left( \frac{x_2}{x_0} \right) \) for \( i = 2, p-2 \) and \( p \). Thus the above equation shows that \( D := \{ g = 0 \} \) is an invisible line degeneration of order \( 2 \) of \( (p, 2) \) torus type:

\[
x_2^2 g(x_0, x_2) = g_2(x_0, x_2)^p - g_p(x_0, x_2)^2.
\]

§ 4.3. Singularities of constructed curves

We consider singularities of \( C \) and \( D \). By our constructions and the argument in §2.3, we have the following:

- Sing \( C \) and Sing \( D \) are the same:

  \[
  \text{Sing } C = \text{Sing } D = \{ pA_{p-1}, A_{p-3}, B_{p-2,2(p-2)} \}.
  \]

- \( C \) has \( (p-1)A_{p-1} \) and \( A_{p-3} \) singularities as inner and one \( A_{p-1} \) singularity as outer.
- \( D \) has \( pA_{p-1} \) singularities as are inner and \( A_{p-3} \) singularity as outer.

Thus we have a pair \((C, D)\) which satisfy the statement of the first part of Theorem 1.4.

§ 4.4. The case \( p \) is even

In this section, we suppose that \( p \) is even. We will show that the pair \((C, D)\) is a weak Zariski pair. Recall that the defining polynomials \( f \) and \( g \) of \( C \) and \( D \) satisfy

\[
f(x_0, x_2) = f_1(x_0, x_2)^p x_2^{p-2} - f_{p-1}(x_0, x_2)^2
\]

\[
x_2^2 g(x_0, x_2) = g_2(x_0, x_2)^p - g_p(x_0, x_2)^2.
\]

As \( p \) is even, \( C \) is decomposed as \( C = C_1 \cup C_2 \) where \( \deg C_1 = \deg C_2 = p - 1 \).
Lemma 4.1. $D$ is decomposed as $D = D_{p-2} \cup D_{p}$ where $\deg D_{p-2} = p - 2$ and $D_{p} = p$.

Proof. Put $p = 2s$. Let $B = \{f_{2} = k^{2s} - h^{2} = 0 \}$ be a $2s$-gonal curve in $\Sigma_{2}$ and let $\pi \circ N_{2} : \Sigma_{2} \rightarrow \mathbb{P}^{2}$ be a birational map which are considered in the proof of Theorem 1.4. Then we can factorize $f_{2}(z_{2}, \tau_{2})$ as

$$f_{2}(z_{2}, \tau_{2}) = (k(z_{2}, \tau_{2})^{s} - h(z_{2}, \tau_{2}))(k(z_{2}, \tau_{2})^{s} + h(z_{2}, \tau_{2})) = k_{1}(z_{2}, \tau_{2})k_{2}(z_{2}, \tau_{2})$$

where $k_{1}(z_{2}, \tau_{2}) = k(z_{2}, \tau_{2})^{s} - h(z_{2}, \tau_{2})$, $k_{2}(z_{2}, \tau_{2}) = k(z_{2}, \tau_{2})^{s} + h(z_{2}, \tau_{2})$.

As we assumed that $O_{2,2}$ is an outer singular point of $B$, we may assume that $O_{2,2}$ is in $\{k_{1} = 0\} \setminus \{k_{2} = 0\}$. Then, using (2.2) in §2.3, the defining polynomial $w_{1}$ of $\pi \circ N_{2}(\{k_{1} = 0\})$ is given by

$$w_{1}(x_{0}, x_{2}) = \frac{x_{0}^{2s}}{x_{0}^{2}}k_{1}(x_{0}, x_{2}) = \frac{1}{x_{0}^{2}}(g_{2}(x_{0}, x_{2})^{s} - g_{p}(x_{0}, x_{2})) = k_{1}(x_{0}, x_{2}).$$

As $w_{1}, g_{2}$ and $g_{p}$ are polynomials and $\deg g_{2} = 2$ and $\deg g_{p} = p$, the degree $w_{1}$ must be $p - 2$. Note that $\{w_{1} = 0\}$ has $A_{p-3}$ singularity. As $g$ is obtained as

$$g(x_{0}, x_{2}) = \frac{x_{0}^{4s}}{x_{2}^{2}}f_{2}(x_{0}, x_{2}) = w_{1}(x_{0}, x_{2})w_{2}(x_{0}, x_{2})$$

where $w_{2} := x_{0}^{2s}k_{2}$. As $\deg g = 2p - 2$ and $\deg w_{1} = p - 2$, the degree $w_{2}$ must be $p$. \square

Now we consider the irreducibility of $C_{1}$ and $C_{2}$. Let $P_{1}, \ldots, P_{p-1}, Q, R$ and $O^{*}$ be the singular points of $C$ such that

$$(C, P_{i}) \sim A_{p-1}, \quad i = 1, \ldots, p - 1,$$

$$(C, Q) \sim A_{p-3}, \quad (C, R) \sim A_{p-1}, \quad (C, O^{*}) \sim B_{p-2,2(p-2)}$$

and $P_{i}$ and $Q$ are inner singularities and $R$ is an outer singular point of $C$. As $P_{i}$ and $Q$ are inner, they are in $\{f_{1} = 0\} \cap \{f_{p-1} = 0\}$. Hence $P_{i}$ and $Q$ are also in $C_{1} \cap C_{2}$. Note that $C_{1}$ and $C_{2}$ are smooth at $P_{i}$ and $Q$. As $R$ is the outer singular point, we may assume that $R \in C_{1} \setminus C_{2}$.

By the form of the defining polynomials of $C_{1}$ and $C_{2}$, both curves have $B_{\frac{p-2}{2},p-2}$ singularity at $O^{*}$. Note that $C_{1}$ and $C_{2}$ have no other singularities.

Now we assume that $C_{1}$ is reducible as $C_{1} = E_{a} \cup E_{b}$ where $\deg E_{i} = i$ and $a \leq b$. Assume that $p > 4$. As $O^{*}$ and $R$ are singular points of $C_{1}$, the intersection $E_{a} \cap E_{b}$ is one of the following:

$$\{O^{*}\}, \quad \{R\}, \quad \{O^{*}, R\}.$$
We consider the cases $E_a \cap E_b = \{O^*\}$ or $\{O^*, R\}$. Let $n$ and $m$ be positive integers such that $(E_a, O^*) \sim B_{n,2n}$ and $(E_b, O^*) \sim B_{m,2m}$. Positive integers $(a, b, n, m)$ must satisfy the following equations:

1. $a + b = p - 1$.
2. $2m + 2n = p - 2$.
3. $a \geq 2n$, $b \geq 2m$.
4. If $E_a \cap E_b = \{O^*\}$, then $ab = 2mn$.
5. If $E_a \cap E_b = \{O^*, R\}$, then $ab = \frac{p}{2} + 2mn$.

Equalities (4) and (5) are obtained by Bézout theorem. By simple calculations, there are no positive integers $(a, b, n, m)$ which satisfy the above equations. Hence if $O^* \in E_a \cap E_b$, then $C_1$ is irreducible. By the same argument, we can show that $C_2$ is irreducible because $C_2$ has only a $B_{\frac{p-2}{2},p-2}$ singularity.

Now we consider the case $E_a \cap E_b = \{R\}$. Then $E_a$ and $E_b$ are smooth at $R$ with $I(E_a, E_b; R) = \frac{p}{2}$. As $E_a \cap E_b = \{R\}$, we have $ab = \frac{p}{2}$ by Bézout theorem. The equations $a + b = p - 1$ and $ab = \frac{p}{2}$ are satisfied for the case $(p, a, b) = (4, 1, 2)$ only. Hence if $p > 4$, then $C_1$ and $C_2$ are irreducible. Therefore the pair $(C, D)$ is a weak Zariski pair.

§ 4.5. The case $p = 4$

We suppose that $p = 4$. Then $\deg C = \deg D = 6$ and their singularities are

\[ \text{Sing } C = \text{Sing } D = \{5A_3, A_1\}. \]

By the above argument, $C$ is decomposed as $E_1 \cup E_2 \cup C_2$ and $C_2$ is a smooth cubic. Their intersection points and intersection multiplicities of these curves are the following:

\[ E_1 \cap E_2 = \{R\}, \quad E_2 \cap C_3 = \{P_1, P_2, P_3\}, \quad E_1 \cap C_3 = \{P_4, Q\} \]

\[ I(E_1, E_2; R) = 2, \quad I(E_1, C_3; Q) = 1, \quad I(E_i, C_3; P_k) = 2, \quad k = 1, \ldots, 4. \]
On the other hand, $D$ is also decomposed as $D_4 \cup D_1 \cup D_1'$ where $\deg D_4 = 4$ and $\deg D_1 = \deg D_1' = 1$. Indeed, outer $A_1$ singularity must be in $D_2$. Hence $D_2$ consists of two distinct lines. Thus $D$ is decomposed as $D_4 \cup D_1 \cup D_1'$. Note that $D_1$ and $D_1'$ are bitangent lines of $D_4$.

\[ D_4 \ni D_1 \ni D_1' \ni L_{\infty} \ni D_1' \ni D_1 \ni D_4 \]

Thus $C$ and $D$ have different irreducible decompositions. Hence the pair $(C, D)$ is a weak Zariski pair.

\textbf{§4.6. Observation for the case $p = 3$}

By our construction, $C$ and $D$ are 3-cuspidal quartics. As we mentioned in the introduction, each curve has both torus decompositions. Moreover it is known that the moduli space of 3-cuspidal quartic is irreducible and hence $C$ and $D$ are in the same moduli space.

\textbf{Acknowledgment.} The author expresses his deepest gratitude to Professor Mutsuo Oka and the referee for their various important advices during the preparation of this paper.

\textbf{References}