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<td>Kawashima, Masayuki</td>
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Kyoto University
On line degenerated torus curves and weak Zariski pairs

By

Kawashima Masayuki *

Abstract

Let $C = \{ f = 0 \}$ be an affine plane curve. We are interested in a form of the defining polynomial $f$. In this paper, we study line degenerations of torus curves. Line degenerations of torus type are divided into two types which are called visible or invisible degenerations. We construct a pair of plane curves of degree $2p-2$ such that they have the same configuration of singularities. If $p$ is even, their complements in $\mathbb{P}^2$ have different topologies. Thus they give a weak Zariski pair.

§1. Introduction

Let $\mathbb{P}^2$ be a complex projective space of dimension 2 with homogeneous coordinates $[X_0, X_1, X_2]$ and let $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{ X_2 = 0 \}$ be the affine space with coordinates $(x, y) = (X_0/X_2, X_1/X_2)$. We study reduced plane curves in $\mathbb{P}^2$ and $\mathbb{C}^2$. Let $\mathcal{M}(d)$ and $\mathcal{M}^a(d)$ be the set of projective and affine plane curves of degree $d$ respectively. For a given curve $C \in \mathcal{M}(d)$ or $\mathcal{M}^a(d)$, we are interested in forms of the defining polynomial of $C$.

Let $p$ and $q$ be positive integers such that $p > q \geq 2$. We say that $C = \{ f = 0 \} \in \mathcal{M}^a(d)$ is a torus curve of type $(p, q)$ if $f$ is written as $f = f_a^p + f_b^q$ where $f_j$ is a polynomial in $\mathbb{C}[x, y]$ of degree $j$. Put $\mathcal{T}(p, q; d)$ as the set of curves of $(p, q)$ torus type of degree $d$.

We also consider another class of plane curves which are called quasi torus curves of type $(p, q)$ (c.f [7], [2]). We say that $C = \{ f = 0 \} \in \mathcal{M}^a(d)$ quasi torus curve of type $(p, q)$ if there exist three polynomials $f_a$, $f_b$ and $f_c$ such that they do not have same
components and they satisfy the following relation:

\[ f_{c}^{pq}f = f_{a}^{p} + f_{b}^{q} \quad \text{in} \quad \mathbb{C}[x, y] \quad \deg f_j = j \]

where \( \deg f_j \) is the degree of \( f_j \). Put \( \mathcal{QT}(p, q; d) \) as the set of curves of \((p, q)\) quasi torus type of degree \( d \).

For a given curve \( C \in \mathcal{M}^{a}(d) \), we say that \( C \) has a torus decomposition (resp. quasi torus decomposition) if \( C \) is in \( \mathcal{T}(p, q; d) \) (resp. \( \mathcal{QT}(p, q; d) \)) for some \((p, q)\).

**Example 1.1.** The following example is the motivation of this work. Let \( Q = \{f = 0\} \in \mathcal{M}^{a}(4) \) be a 3-cuspidal quartic. Then \( Q \) has at least two torus and one quasi torus decompositions ([6]):

\[
\begin{align*}
    f &= f_{1}^{3} + f_{2}^{2}, \\
    f &= g_{2}^{3} + g_{3}^{2}, \\
    h_{1}^{6}f &= h_{3}^{3} + h_{5}^{2}
\end{align*}
\]

where \( \deg f_{i} = i \), \( \deg g_{i} = i \) and \( \deg h_{i} = i \).

To construct these torus decompositions, we used line degenerated torus curves. Now we recall line degeneration of torus curves which are defined by M. Oka in [8].

**Definition 1.2.** Let \( C = \{F = F_{q}^{p} + F_{p}^{q} = 0\} \in \mathcal{M}(pq) \) be a projective \((p, q)\) torus curve. Suppose that \( F \) has the following form:

\[
F(X_{0}, X_{1}, X_{2}) = X_{2}^{j}G(X_{0}, X_{1}, X_{2})
\]

where \( G(X, Y, Z) \) is a reduced homogeneous polynomial of degree \( pq - j \). We call a curve \( D = \{G = 0\} \) a line degenerated torus curve of type \((p, q)\) of order \( j \) and the line \( L_{\infty} = \{X_{2} = 0\} \) the limit line of the degeneration ([8]).

Put \( \mathcal{LT}_{j}(p, q; d) \) as the set of line degenerated torus curves of type \((p, q)\) of order \( j \). and \( \mathcal{LT}(p, q) \) is the union of \( \mathcal{LT}_{j}(p, q; d) \) with respect to \( j \).

We divide the situations (1.2) into two cases which are called visible degenerations and invisible degenerations. Put the integer \( r_{k} := \max\{r \in \mathbb{Z} \mid X_{2}^{r} \mid F_{k}\} \) for \( k = p, q \).

**Visible case.** Suppose that \( r_{p} \cdot r_{q} \neq 0 \) and \( qr_{p} \neq pr_{q} \). Then \( F_{q} \) and \( F_{p} \) are written as follows:

\[
F_{q}(X_{0}, X_{1}, X_{2}) = F_{q-r_{q}}'(X_{0}, X_{1}, X_{2})X_{2}^{r_{q}}, \quad F_{p}(X_{0}, X_{1}, X_{2}) = F_{p-r_{p}}'(X_{0}, X_{1}, X_{2})X_{2}^{r_{p}}.
\]

Putting \( j := \min\{qr_{p}, pr_{q}\} \), we can factor \( F \) as \( F(X_{0}, X_{1}, X_{2}) = X_{2}^{j}G(X_{0}, X_{1}, X_{2}) \). Then \( G \) is written using \( F_{p-r_{p}}' \) and \( F_{q-r_{q}}' \) as

\[
G(X_{0}, X_{1}, X_{2}) = \begin{cases} 
    F_{q-r_{q}}'(X_{0}, X_{1}, X_{2})^{p} + F_{p-r_{p}}'(X_{0}, X_{1}, X_{2})^{q}X_{2}^{qr_{p}-pr_{q}} & \text{if } j = pr_{q}, \\
    F_{q-r_{q}}'(X_{0}, X_{1}, X_{2})^{p}X_{2}^{pr_{q}-qr_{p}} + F_{p-r_{p}}'(X_{0}, X_{1}, X_{2})^{q} & \text{if } j = qr_{p}.
\end{cases}
\]
Such a factorization is called a \textit{visible factorization} and $D$ is called a \textit{visible degeneration} of $(p, q)$ torus curves. We denote the set of visible degenerations of order $j$ by $\mathcal{L}T^V_j(p, q; d)$.

**Invisible case.** Either $r_p = 0$ or $r_q = 0$ but $F$ can be written as (1.2). Then $D$ is called an \textit{invisible degeneration} of $(p, q)$ torus curves. In this case, write $F_p^q + F_q^p = \sum_{i=0}^{pq} A_i(X_0, X_1)X_2^i$. Then $A_j(X_0, X_1) = 0$ for $i \leq j - 1$ and therefore $X_2^j \mid F$. We denote the set of invisible degenerations of order $j$ by $\mathcal{L}T^I_j(p, q; d)$.

Using these terminologies, we will show that torus decompositions (1.1) satisfy:

\[
\{f_1^3 + f_2^2 = 0\} \in \mathcal{L}T^V_2(3, 2; 4), \quad \{g_2^3 + g_3^2 = 0\} \in \mathcal{L}T^I_2(3, 2; 4).
\]

Thus $Q = \{f = 0\}$ is in $\mathcal{L}T^V_2(3, 2; 4) \cap \mathcal{L}T^I_2(3, 2; 4)$.

We consider whether such phenomena occur or not for other curves. Before we consider this problem, we study line degenerated torus curves. More precisely, we look for a pair of curves $\{C, D\}$ such that $C \in \mathcal{L}T^V_j(p, q; d)$ and $D \in \mathcal{L}T^I_j(p, q; d)$ such that $\text{Sing} \ C = \text{Sing} \ D$. Here $\text{Sing} \ C$ is the configuration of the singularities. If there exists such a pair $(C, D)$, then we discuss if the topologies of $C$ and $D$ are the same or not.

**Definition 1.3.** A pair of plane curves $(C_1, C_2)$ is called a \textit{weak Zariski pair} if they have the same degree and configuration of singularities, while the complements $\mathbb{P}^2 \setminus C_1$ and $\mathbb{P}^2 \setminus C_2$ are not homeomorphic to each other ([9, 5]).

To express singularities of curves, we use an important class of singularities which is called Brieskorn-Pham singularities:

\[B_{n,m} : x^n + y^m = 0, \quad n, m \geq 2.\]

**Theorem 1.4.** For each $p \geq 3$, there is a pair of plane curves $(C, D) \in \mathcal{L}T^V_2(p, 2; 2p - 2) \times \mathcal{L}T^I_2(p, 2; 2p - 2)$ with

\[\text{Sing} \ C = \text{Sing} \ D = \{pA_{p-1}, A_{p-3}, B_{p-2,2(p-2)}\}.\]

If $p$ is even, then $(C, D)$ is a weak Zariski pair.

§ 2. Preliminaries

In section 2, we follow the terminologies in [3] and [4].

Let $p : \Sigma_d \to \mathbb{P}^1$ be a Hirzebruch surface of degree $d$ and let $\Delta_{\infty,d}$ be the exceptional section with the self-intersection multiplicity $\Delta_{\infty,d}^2 = -d$. Let $(X_0, X_1, X_2)$ and $(Y_0, Y_1)$ be homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively. Using these coordinates, $\Sigma_d$ is defined as

\[
\Sigma_d := \{(X_0, X_1, X_2), (Y_0, Y_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid X_1Y_1^d = X_2Y_0^d\}
\]
and $p : \Sigma_{d} \to \mathbb{P}^{1}$ is the canonical projection. There are four affine coordinates which cover $\Sigma_{d}$. We use two affine spaces $W_{d}^{1}, W_{d}^{2} \subset \Sigma_{d}$ with coordinates $(y_{d}, \tau_{d})$ and $(z_{d}, \tau_{d})$ respectively where
\[
y_{d} = X_{2}/X_{0}, \ z_{d} = X_{0}/X_{2}, \ \tau_{d} = Y_{0}/Y_{1}
\]
and they are glued by the relation $y_{d}z_{d} = 1$. Putting $V_{1} = \{(Y_{0}, Y_{1}) \in \mathbb{P}^{1} \mid Y_{1} \neq 0\}$, they satisfy $p^{-1}(V_{1}) = W_{d}^{1} \cup W_{d}^{2}$.

We denote the fiber over $\tau_{d} = 0$ in $\Sigma_{d}$ by $F_{\infty}$ and the origin of the affine space $W_{d}^{i}$ by $O_{i,d} := (0, 0) \in W_{d}^{i}$. We put the affine line $F_{\infty}^{\circ} := F_{\infty} \setminus \Delta_{\infty,d} = F_{\infty} \cap W_{d}^{2}$.

\section{2.1. $p$-gonal curves}

Let $B \subset \Sigma_{d}$ be a reduced curve such that $B$ does not contain the exceptional section $\Delta_{\infty,d}$. If $B$ intersects with a generic fiber at $p$ points, then we call $B$ a generalized $p$-gonal curve. A generalized $p$-gonal curve $B$ is called a $p$-gonal curve if $B$ disjoint from the exceptional section $\Delta_{\infty,d}$.

Let $f_{i}$ be a defining equation of $B$ on $W_{d}^{i}$ and then we have the equality $f_{1}(y_{d}, \tau_{d}) = y_{d}^{p}f_{2}(z_{d}, \tau_{d})$ on $W_{d}^{1} \cap W_{d}^{2}$. Using affine coordinates $(z_{d}, \tau_{d}) \in W_{d}^{2}$, the local equation $f_{2}(z_{d}, \tau_{d})$ is written as
\[
f_{2}(z_{d}, \tau_{d}) = \sum_{i=0}^{p} b_{i}(\tau_{d})z_{d}^{i}, \quad \deg b_{i}(\tau_{d}) \leq d(p - i).
\]
The exceptional section $\Delta_{\infty,d}$ is defined as $\{y_{d} = 0\}$ in the affine coordinates $(y_{d}, \tau_{d}) \in W_{d}^{1}$.

\section{2.2. Nagata transformations}

Let $P$ be a fixed point in $\Sigma_{2} \setminus \Delta_{\infty,2}$ and let $F$ be the fiber which passes through $P$. A Nagata transformation $N : \Sigma_{2} \dashrightarrow \Sigma_{1}$ is a birational transformation which consists of the blowing-up at $P \notin \Delta_{\infty,2}$ and the blowing-down the strict transform $F^{*}$ of $F$. We observe that the exceptional section $\Delta_{\infty,1}$ of $\Sigma_{1}$ is the image $N(\Delta_{\infty,2})$.

We express a Nagata transformation using local coordinates $(z_{2}, \tau_{2})$ and $(z_{1}, \tau_{1})$ assuming $P = O_{2,2} \in W_{2}^{2}$. Let $\mu_{1} : \tilde{W}_{2}^{2} \to W_{2}^{2}$ and $\mu_{2} : \tilde{W}_{1}^{1} \to W_{1}^{1}$ be blowing-ups centered at $O_{2,2}$ and $O_{1,1}$ respectively. There is an affine coordinate $\tilde{W}$ with coordinates $(s, t)$ such that $\mu_{1}(s, t) = (t, ts)$ and $\mu_{2}(s, t) = (s, st)$. Note that $\{t = 0\}$ defines the exceptional curve of $\mu_{1}$ and $\{s = 0\}$ defines the exceptional curve of $\mu_{2}$. Then we have:
\[
N(z_{2}, \tau_{2}) = (z_{1}, \tau_{1}) = \left(\frac{z_{2}}{\tau_{2}}, \tau_{2}\right).
\]

Let $B$ be a $p$-gonal curve in $\Sigma_{2}$ which is defined by $\{f_{2}(z_{2}, \tau_{2}) = 0\}$ in $W_{2}^{2}$. We consider the defining equation of the image of a $p$-gonal curve by a Nagata transformation. By the definition of a Nagata transformation, $B' := N(B) \subset \Sigma_{1}$ is defined as
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(2.1) \[ B' : f'_2(z_1, \tau_1) = \frac{1}{\tau_1^M} f_2(z_1 \tau_1, \tau_1) = 0 \]

where \( M \) is the multiplicity of \( B \) at \( P \). As \( B \) is assumed to be \( p \)-gonal, \( B' \cap \Delta_{\infty,1} \) is \( \{O_{1,1}\} \). Thus \( B' \) is a generalized \( p \)-gonal curve.

§ 2.3. Contraction of \( p \)-gonal curves from \( \Sigma_2 \) to \( \mathbb{P}^2 \)

We recall that a Hirzebruch surface \( \Sigma_1 \) is obtained as a blowing-up at any point in \( \mathbb{P}^2 \). In this section, we consider the defining polynomial of a plane curve which is obtained as the image of the composition of a Nagata transformation and a blowing-up.

Let \( B = \{f_2(z_2, \tau_2) = 0\} \) be a \( p \)-gonal curve in \( W_2^2 \) and let \( B' = \{f'_2(z_1, \tau_1) = 0\} \subset W_1^2 \) be the image of \( B \) by a Nagata transformation \( N : \Sigma_2 \dashrightarrow \Sigma_1 \) at \( O_{2,2} \). Put \( m \) the intersection multiplicity of \( B' \) and \( \Delta_{\infty,1} \) at \( O_{1,1} \). Let \( U_1 \) be the affine coordinate chart \( \mathbb{P}^2 \setminus \{X_1 = 0\} \) with the coordinate \((x_0, x_2) = (X_0/X_1, X_2/X_1)\). Let \( \pi : \tilde{U}_1 \rightarrow U_1 \) be a blowing-up at \((0,0) \in U_1 \). We naturally identify \( \tilde{U}_1 \) with \( \Sigma_1 \) as follows: Let \( \tilde{U}_{10} \) and \( \tilde{U}_{11} \) be two affine coordinates of \( \tilde{U} \) and let \((s, t) \) be the affine coordinate of \( \tilde{U}_{11} \). Then \( \pi \) is defined as \( \pi(s, t) = (x_0, x_2) = (s, st) \) on \( \tilde{U}_{11} \). We identify \( \tilde{U}_{11} \) with \( W_1^1 \) as \((s, t) \mapsto (y_1, \tau_1)\).

By the definition of \( \pi : \Sigma_1 \rightarrow U_1 \) and the equality (2.1), the defining polynomial \( f \) of \( C : = (\pi \circ N)(B) \subset U_1 \) as

\[ f(x_0, x_2) = \frac{x_0^{M+m+p}}{x_2^M} f_2 \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right). \]

Indeed, let \( f'_{1}(y_1, \tau_1) \) be the defining equation of \( B' \) in \( W_1^1 \) which is written as

\[ f'_{1}(y_1, \tau_1) = y_1^p f'_2(1/y_1, \tau_1) = y_1^p \frac{\tau_1^M}{f_2(t_1/y_1, \tau_1)}. \]

where we use (2.1) for the second equality. And \( f \) must satisfy \( f(y_1, y_1 \tau_1) = y_1^p f'_{1}(y_1, \tau_1) \). Using these equalities and \( \pi(y_1, \tau_1) = (x_0, x_2) = (y_1, y_1 \tau_1) \), we have the equality (2.2).

Next we consider singularities of \( B' \) and \( C \). Assume that \( B \) satisfies the following conditions:

- \( B \) has an \( A_{\ell-1} = B_{\ell,2} \) singularity at \( O_{2,2} \in W_2^2 \) and its tangent cone is transverse to the fiber \( F_\infty = \{\tau_2 = 0\} \).

- \( B \) intersects transversely at \( p - 2 \) distinct points with \( F_\infty \) outside of \( O_{2,2} \in W_2^2 \).

Under the above conditions, the intersection \( B \cap (F_\infty \setminus \{O_{2,2}\}) \) consists of distinct \( p - 2 \) points and \( B' \) intersects with \( F_\infty \) so that
• If \( \ell = 2 \), then \( B' \) intersects transversely with \( F_{\infty}^{\omega} \) at two points.

• If \( \ell = 3 \), then \( B' \) is tangent to \( F_{\infty}^{\omega} \) with the intersection multiplicity 2.

• If \( \ell > 3 \), then \( B' \) has \( A_{\ell-3} = B_{\ell-2,2} \) singularity.

**Observation.** If \( B \) is a trigonal curve \( (p = 3) \), then \( B' \) is smooth and intersects transversely with \( \Delta_{\infty,1} \) at \( O_{1,1} \). If \( p \) is greater than 3, then \( B' \) has \( B_{p-2,p-2} \) singularity at \( (0,0) \in U_{1} \).

**Proof.** The first assertion is obvious. Assume \( p > 3 \). The defining equation \( f_{1}'(y_{1}, \tau_{1}) \) of \( B' \) in \( W_{1}^{1} \) is written as:

\[
f_{1}'(y_{1}, \tau_{1}) = c \prod_{i=1}^{p-2}(y_{1} - \alpha_{i}\tau_{1}) + \text{(higher terms)}, \quad c \neq 0, \quad \alpha_{i} \neq \alpha_{j} (i \neq j).
\]

Now we use the equality \( f(x_{0}, x_{2}) = x_{0}^{p-2}f_{1}'(x_{0}, x_{2}/x_{0}) \) which is obtained from (2.2). Then we have

\[
f(x_{0}, x_{2}) = x_{0}^{p-2}f_{1}'(x_{0}, x_{2}/x_{0}) = \prod_{i=1}^{p-2}(x_{0}^{2} - \alpha_{i}x_{2}) + \text{(higher terms)}.
\]

Thus \( C \) has \( B_{p-2,2(p-2)} \) singularity at \( (0,0) \in U_{1} \). \( \square \)

§ 3. \( p \)-gonal curves of \((p, 2)\) torus type

Let \( B \) be a \( p \)-gonal curve in \( \Sigma_{2} \). We say that \( B \) is **torus curve of type** \((p, 2)\) if the defining equation \( f_{2} \) of \( B \) in the affine space \( (W_{2}^{2}, (z_{2}, \tau_{2})) \) is written as

\[
f_{2}(z_{2}, \tau_{2}) = k(z_{2}, \tau_{2})^{p} - h(z_{2}, \tau_{2})^{2}.
\]

We assume further that

\[
\begin{align*}
k(z_{2}, \tau_{2}) &= z_{2} + b_{2}(\tau_{2}), \\
h(z_{2}, \tau_{2}) &= b_{p-2}(\tau_{2})z_{2} + b_{p}(\tau_{2}), \\
\deg b_{i}(\tau_{2}) &= i.
\end{align*}
\]

§ 3.1. **Singularities of** \((p, 2)\) **torus type**

We consider curves \( K := \{k = 0\} \) and \( H := \{h = 0\} \) in \( W_{2}^{2} \) where \( h \) and \( k \) are as above. Let \( P \in B \) be a singular point. If \( P \in K \cap H \), we call \( P \) an **inner singularity**. Otherwise \( P \) is called an **outer singularity**. We put \( \Delta_{1}(\tau_{2}) := h(-b_{2}(\tau_{2}), \tau_{2}) - b_{p-2}(\tau_{2})b_{2}(\tau_{2}) \) and take an inner singular point \( P \in K \cap H \). Then \( P \) is written as \((-b_{2}(s), s)\) for some \( s \in \mathbb{C} \) with \( \Delta_{1}(s) = 0 \) and the multiplicity of \( \Delta_{1}(\tau_{2}) \) at \( s \), say \( \iota \), is equal to the intersection multiplicity of \( K \) and \( H \) at \( P \).

By a similar argument as that in Lemma 1 in [1], we have the following.
Lemma 3.1. Let $B$ be the $p$-gonal curve as above in $\Sigma_2$. Suppose that $s$ is a root of $\Delta_1(\tau)$ and let $P = (-b_q(s), s) \in B$ be an inner singular point with the intersection multiplicity $t$. If $\Delta_2(s) \neq 0$, then $B$ has $B_{p,2} = A_{p-1}$ singularity at $P$.

§ 4. Proof of Theorem 1.4

Let $B \subset \Sigma_2$ be a $p$-gonal curve of $(p, 2)$ torus type. As the degree of $\Delta_1(\tau_2)$ is $p$, $B$ has $pA_{p-1}$ inner singularities by Lemma 3.1. We may assume that $B$ has an outer $A_{p-1}$ singularity. For example, we take $b_2(\tau_2), b_{p-2}(\tau_2)$ and $b_p(\tau_2)$ as

$$ b_2(\tau_2) = 1 + \tau_2^2, \quad b_p(\tau_2) = 1 + \frac{p}{2} \tau_2^2 + \tau_2^p, \quad b_{p-2}(\tau_2) = \frac{p}{2} + p \tau_2^{p-2}. $$

Then $f_2 = k^p - h^2$ has an outer $A_{p-1}$ singularity at $O_{2,2}$ and its tangent cone does not contain $\{\tau_2 = 0\}$. As $\Delta_1(\tau_2) = 1 - \frac{p}{2} - p \tau_2^{p-2} + (1 - p) \tau_2^p$ and $p \geq 3$, $K$ and $H$ intersect transversely at distinct $p$ points and $K \cap H \cap F_{\infty} = \emptyset$.

Let $P$ be an inner $A_{p-1}$ singular point and let $Q$ be an outer $A_{p-1}$ singular point of $B$. Let $N_1$ and $N_2$ be the Nagata transformations from $\Sigma_2$ to $\Sigma_1$ at $P$ and $Q$ respectively. We consider the defining polynomial of $C := (\pi \circ N_1)(B)$ and $D := (\pi \circ N_2)(B)$ where $\pi : \Sigma_1 \to U_1$ is the blowing-up at $(0,0) \in U_1$.

§ 4.1. Construction of a visible degeneration

Hereafter we assume that $K$ and $H$ intersect transversely at $p$ points. Assume that $P = O_{2,2}$ in the affine space $W_2$. Let $f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2$ be the defining
equation of $B$ where
\[ k(z_2, \tau_2) = z_2 + b_2(\tau_2), \quad h(z_2, \tau_2) = b_{p-2}(\tau_2)z_2 + b_p(\tau_2), \quad \deg b_i(\tau_2) = i. \]
As $k(0,0) = h(0,0) = 0$, we can write $b_2(\tau_2)$ and $b_p(\tau_2)$ as
\[ b_2(\tau_2) = \tau_2 b_1(\tau_2), \quad b_p(\tau_2) = \tau_2 b_{p-1}(\tau_2), \quad \deg b_i = i. \]
Let $f$ be the defining polynomial of $C$ and using (2.2), we have
\[ f(x_0, x_2) = \frac{x_0^{2p}}{x_2^2} f_2 \left( \frac{x_2}{x_0}, \frac{x_2}{x_0} \right). \]
We calculate the above equation as the following:
\[
x_2^2 f(x_0, x_2) = x_0^{2p} \left( k \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right)^p - h \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right)^2 \right)
\]
\[
= x_0^{2p} \left( \left( \frac{x_2}{x_0^2} + \frac{x_2}{x_0} b_1 \left( \frac{x_2}{x_0} \right) \right)^p - \left( b_{p-2} \left( \frac{x_2}{x_0} \right) \frac{x_2}{x_0^2} + \frac{x_2}{x_0} b_{p-1} \left( \frac{x_2}{x_0} \right) \right)^2 \right)
\]
\[
= x_0^{2p} \left( 1 + x_0 b_1 \left( \frac{x_2}{x_0} \right) \right)^p - x_0^2 \left( x_0^{p-2} b_{p-2} \left( \frac{x_2}{x_0} \right) + x_0^{p-1} b_{p-1} \left( \frac{x_2}{x_0} \right) \right)^2
\]
\[
= f_1(x_0, x_2)^p x_2^p - f_{p-1}(x_0, x_2)^2 x_2^2.
\]
and then where
\[
f_1(x_0, x_2) := 1 + c_1(x_0, x_2), \quad f_{p-1}(x_0, x_2) := c_{p-2}(x_0, x_2) + c_{p-1}(x_0, x_2).
\]
Note that $c_i(x_0, x_2) := x_0^i b_i(x_2/x_0)$ is a polynomial for $i = 1, p - 2$ and $p - 1$. Hence we have
\[
x_2^2 f(x_0, x_2) = (f_1(x_0, x_2)x_2)^p - (f_{p-1}(x_0, x_2)x_2)^2.
\]
Thus the above equation shows that $C := \{f = 0\}$ is a visible line degeneration of order 2 of $(p, 2)$ torus type.

\section*{§ 4.2. Construction of an invisible degeneration}
Assume that $Q = O_{2,2}$ in the affine space $W^2_2$. Let $f_2(z_2, \tau_2) = k(z_2, \tau_2)^p - h(z_2, \tau_2)^2$ be the defining equation of $B$ where
\[ k(z_2, \tau_2) = z_2 + b_2(\tau_2), \quad h(z_2, \tau_2) = b_{p-2}(\tau_2)z_2 + b_p(\tau_2), \quad \deg b_i(\tau_2) = i. \]
Let $g$ be the defining polynomial of $D$ and using (2.2) in §2.3, we have
\[ g(x_0, x_2) = \frac{x_0^{2p}}{x_2^2} f_2 \left( \frac{x_2}{x_0}, \frac{x_2}{x_0} \right). \]
We calculate the above equation:
\[ x_2^2 g(x_0, x_2) = x_0^{2p} \left( k \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right)^p - h \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right)^2 \right) \]
\[ = x_0^{2p} \left( \left( \frac{x_2^2}{x_0^4} + 2 \frac{x_2}{x_0} \right)^p - \left( b_{p-2} \frac{x_2}{x_0} \right)^2 \right) \]
\[ = \left( x_2 + x_0^2 b_2 \frac{x_2}{x_0} \right)^p - \left( x_0^{p-2} b_{p-2} \frac{x_2}{x_0} + x_0^p b_p \left( \frac{x_2}{x_0} \right)^2 \right) \]
\[ = g_2(x_0, x_2)^p - g_p(x_0, x_2)^2 \]

where the polynomials \( g_2 \) and \( g_p \) are defined as
\[ g_2(x_0, x_2) := x_2 + d_2(x_0, x_2) \quad g_p(x_0, x_2) := d_{p-2}(x_0, x_2) x_2^2 + d_p(x_0, x_2) \]

where \( d_i(x_0, x_2) := x_0^i b_i \frac{x_2}{x_0} \) for \( i = 2, p - 2 \) and \( p \). Thus the above equation shows that \( D := \{ g = 0 \} \) is an invisible line degeneration of order 2 of \( (p, 2) \) torus type:
\[ x_2^2 g(x_0, x_2) = g_2(x_0, x_2)^p - g_p(x_0, x_2)^2. \]

§ 4.3. Singularities of constructed curves

We consider singularities of \( C \) and \( D \). By our constructions and the argument in §2.3, we have the following:

- **Sing** \( C \) and **Sing** \( D \) are the same:
  \[ \text{Sing} \ C = \text{Sing} \ D = \{ pA_{p-1}, A_{p-3}, B_{p-2,2(p-2)} \}. \]

- **C** has \((p-1)A_{p-1}\) and \( A_{p-3} \) singularities as inner and one \( A_{p-1} \) singularity as outer.
- **D** has \( pA_{p-1} \) singularities as are inner and \( A_{p-3} \) singularity as outer.

Thus we have a pair \((C, D)\) which satisfy the statement of the first part of Theorem 1.4.

§ 4.4. The case \( p \) is even

In this section, we suppose that \( p \) is even. We will show that the pair \((C, D)\) is a weak Zariski pair. Recall that the defining polynomials \( f \) and \( g \) of \( C \) and \( D \) satisfy
\[ f(x_0, x_2) = f_1(x_0, x_2)^p x_2^{p-2} - f_{p-1}(x_0, x_2)^2 \]
\[ x_2^2 g(x_0, x_2) = g_2(x_0, x_2)^p - g_p(x_0, x_2)^2. \]

As \( p \) is even, \( C \) is decomposed as \( C = C_1 \cup C_2 \) where \( \deg C_1 = \deg C_2 = p - 1 \).
Lemma 4.1. \( D \) is decomposed as \( D = D_{p-2} \cup D_p \) where \( \deg D_{p-2} = p - 2 \) and \( D_p = p \).

Proof. Put \( p = 2s \). Let \( B = \{ f_2 = k^{2s} - h^2 = 0 \} \) be a \( 2s \)-gonal curve in \( \Sigma_2 \) and let \( \pi \circ N_2 : \Sigma_2 \to \mathbb{P}^2 \) be a birational map which are considered in the proof of Theorem 1.4. Then we can factorize \( f_2(z_2, \tau_2) \) as

\[
    f_2(z_2, \tau_2) = (k(z_2, \tau_2)^s - h(z_2, \tau_2))(k(z_2, \tau_2)^s + h(z_2, \tau_2))
\]

where

\[
    k_1(z_2, \tau_2) = k(z_2, \tau_2)^s - h(z_2, \tau_2), \quad k_2(z_2, \tau_2) = k(z_2, \tau_2)^s + h(z_2, \tau_2).
\]

As we assumed that \( O_{2,2} \) is an outer singular point of \( B \), we may assume that \( O_{2,2} \) is in \( \{ k_1 = 0 \} \setminus \{ k_2 = 0 \} \). Then, using (2.2) in \( \S 2.3 \), the defining polynomial \( w_1 \) of \( \pi \circ N_2(\{ k_1 = 0 \}) \) is given by

\[
    w_1(x_0, x_2) = \frac{x_0^{2s}}{x_0^2} k_1 \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right) = \frac{1}{x_0^2} (g_2(x_0, x_2)^s - g_p(x_0, x_2)).
\]

As \( w_1, g_2 \) and \( g_p \) are polynomials and \( \deg g_2 = 2 \) and \( \deg g_p = p \), the degree \( w_1 \) must be \( p - 2 \). Note that \( \{ w_1 = 0 \} \) has \( A_{p-3} \) singularity. We can obtain \( g(x_0, x_2) = \frac{x_0^{4s}}{x_2^2} f_2 \left( \frac{x_2}{x_0^2}, \frac{x_2}{x_0} \right) = w_1(x_0, x_2) w_2(x_0, x_2) \)

where \( w_2 := x_0^{2s} k_2 \). As \( \deg g = 2p - 2 \) and \( \deg w_1 = p - 2 \), the degree \( w_2 \) must be \( p \). \( \square \)

Now we consider the irreducibility of \( C_1 \) and \( C_2 \). Let \( P_1, \ldots, P_{p-1}, Q, R \) and \( O^* \) be the singular points of \( C \) such that

\[
(C, P_i) \sim A_{p-1}, \quad i = 1, \ldots, p - 1,
\]

\[
(C, Q) \sim A_{p-3}, \quad (C, R) \sim A_{p-1}, \quad (C, O^*) \sim B_{p-2, 2(p-2)}
\]

and \( P_i \) and \( Q \) are inner singularities and \( R \) is an outer singular point of \( C \). As \( P_i \) and \( Q \) are inner, they are in \( \{ f_1 = 0 \} \cap \{ f_{p-1} = 0 \} \). Hence \( P_i \) and \( Q \) are also in \( C_1 \cap C_2 \). Note that \( C_1 \) and \( C_2 \) are smooth at \( P_i \) and \( Q \). As \( R \) is the outer singular point, we may assume that \( R \in C_1 \setminus C_2 \).

By the form of the defining polynomials of \( C_1 \) and \( C_2 \), both curves have \( B_{\frac{p-2}{2}, p-2} \) singularity at \( O^* \). Note that \( C_1 \) and \( C_2 \) have no other singularities.

Now we assume that \( C_1 \) is reducible as \( C_1 = E_a \cup E_b \) where \( \deg E_i = i \) and \( a \leq b \). Assume that \( p > 4 \). As \( O^* \) and \( R \) are singular points of \( C_1 \), the intersection \( E_a \cap E_b \) is one of the following:

\[
\{ O^* \}, \quad \{ R \}, \quad \{ O^*, R \}.
\]
We consider the cases $E_a \cap E_b = \{O^*\}$ or $\{O^*, R\}$. Let $n$ and $m$ be positive integers such that $(E_a, O^*) \sim B_{n,2n}$ and $(E_b, O^*) \sim B_{m,2m}$. Positive integers $(a, b, n, m)$ must satisfy the following equations:

1. $a + b = p - 1$.
2. $2m + 2n = p - 2$.
3. $a \geq 2n, \ b \geq 2m$.
4. If $E_a \cap E_b = \{O^*\}$, then $ab = 2mn$.
5. If $E_a \cap E_b = \{O^*, R\}$, then $ab = \frac{p}{2} + 2mn$.

Equalities (4) and (5) are obtained by Bézout theorem. By simple calculations, there are no positive integers $(a, b, n, m)$ which satisfy the above equations. Hence if $O^* \in E_a \cap E_b$, then $C_1$ is irreducible. By the same argument, we can show that $C_2$ is irreducible because $C_2$ has only a $B_{\frac{p-2}{2}, p-2}$ singularity.

Now we consider the case $E_a \cap E_b = \{R\}$. Then $E_a$ and $E_b$ are smooth at $R$ with $I(E_a, E_b; R) = \frac{p}{2}$. As $E_a \cap E_b = \{R\}$, we have $ab = \frac{p}{2}$ by Bézout theorem. The equations $a + b = p - 1$ and $ab = \frac{p}{2}$ are satisfied for the case $(p, a, b) = (4, 1, 2)$ only. Hence if $p > 4$, then $C_1$ and $C_2$ are irreducible. Therefore the pair $(C, D)$ is a weak Zariski pair.

§ 4.5. The case $p = 4$

We suppose that $p = 4$. Then $\deg C = \deg D = 6$ and their singularities are

$$\text{Sing} \ C = \text{Sing} \ D = \{5A_3, A_1\}.$$  

By the above argument, $C$ is decomposed as $E_1 \cup E_2 \cup C_2$ and $C_2$ is a smooth cubic. Their intersection points and intersection multiplicities of these curves are the following:

$E_1 \cap E_2 = \{R\}, \ E_2 \cap C_3 = \{P_1, P_2, P_3\}, \ E_1 \cap C_3 = \{P_4, Q\}$

$I(E_1, E_2; R) = 2, \ I(E_1, C_3; Q) = 1, \ I(E_i, C_3; P_k) = 2, \ k = 1, \ldots, 4.$
On the other hand, $D$ is also decomposed as $D_4 \cup D_1 \cup D'_1$ where $\deg D_4 = 4$ and $\deg D_1 = \deg D'_1 = 1$. Indeed, outer $A_1$ singularity must be in $D_2$. Hence $D_2$ consists of two distinct lines. Thus $D$ is decomposed as $D_4 \cup D_1 \cup D'_1$. Note that $D_1$ and $D'_1$ are bitangent lines of $D_4$.

Thus $C$ and $D$ have different irreducible decompositions. Hence the pair $(C, D)$ is a weak Zariski pair.

§ 4.6. Observation for the case $p = 3$

By our construction, $C$ and $D$ are 3-cuspidal quartics. As we mentioned in the introduction, each curve has both torus decompositions. Moreover it is known that the moduli space of 3-cuspidal quartic is irreducible and hence $C$ and $D$ are in the same moduli space.

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References