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On the Stokes geometry for the Pearcey system and the (1,4) hypergeometric system

By

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Abstract

In this paper, we study the exact WKB analysis for two concrete holonomic systems, that is, the Pearcey system and the (1,4) hypergeometric system, and investigate the structure of their Stokes geometry. In particular, we discuss the relationship between the Stokes geometry for these two holonomic systems and the Stokes geometry for third-order ordinary differential equations obtained by restricting these systems. We show that the Stokes surfaces of the systems contain the new Stokes curves relevant to Stokes phenomena for WKB solutions of the ordinary differential equations obtained by restricting them. Furthermore, it is also shown that new Stokes curves irrelevant to Stokes phenomena are included in the Stokes surface as well. We also discuss the relationship between the Stokes surface for the (1,4) hypergeometric system and the structure of virtual turning points of the ordinary differential equation obtained by restricting it.

§ 1. Introduction

The exact WKB analysis was first developed for second-order ordinary differential equations and then has been extended to higher-order ordinary differential equations and non-linear ordinary differential equations. However, the exact WKB analysis for holonomic systems, although being initiated in [1], is not well developed yet. In this paper, as the first step toward the generalization of the exact WKB analysis to holonomic systems, we study two concrete holonomic systems from the viewpoint of the exact WKB analysis and, in particular, investigate their Stokes geometry.

The first holonomic system we study is

$$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{1}{2}x_2\eta^2\dfrac{\partial}{\partial x_1} + \dfrac{1}{4}x_1\eta^3\right)\psi = 0, \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0. \end{cases} \tag{1.1}$$

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Since this system has the following Pearcey integral as a particular solution:

$$\psi = \int \exp\left\{\eta\left(t^4 + x_2 t^2 + x_1 t\right)\right\} dt, \tag{1.2}$$

we call (1.1) the Pearcey system. In [1] Aoki constructed WKB solutions and defined turning points and Stokes surfaces for the Pearcey system. On the other hand, for fixed $x_2 = c_2$, the Pearcey integral satisfies the following third-order ordinary differential equation:

$$\left(\frac{d^3}{dx_1^3} + \frac{1}{2}c_2\eta^2\frac{d}{dx_1} + \frac{1}{4}x_1\eta^3\right)\psi = 0. \tag{1.3}$$

Since (1.3) is the equation which Berk-Nevins-Roberts discussed in considering the extension of the WKB analysis to higher order equations, it is often called the BNR equation. As was pointed out by Berk et al, there exist crossing points of Stokes curves for the BNR equation and Stokes phenomena for WKB solutions of the BNR equation occur not only on ordinary Stokes curves but also on the so-called "new Stokes curves", that is, Stokes curves passing through such crossing points. After the pioneering work of [3], Aoki-Kawai-Takei showed in [2] that a new Stokes curve can be interpreted as a Stokes curve emanating from the so-called "virtual turning point". The existence of new Stokes curves and virtual turning points shows the difficulty of the exact WKB analysis for higher-order ordinary differential equations. The first purpose of this paper is to develop the exact WKB analysis for the Pearcey system and to investigate the relationship between the Stokes geometry for the Pearcey system and the Stokes geometry for the BNR equation. In particular, we will show that the new Stokes curves for the BNR equation are included in the restriction of the Stokes surface for the Pearcey system.

The second holonomic system we consider is

$$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{2}{3}x_2\eta\dfrac{\partial^2}{\partial x_1^2} + \dfrac{1}{3}x_1\eta^2\dfrac{\partial}{\partial x_1} - \dfrac{\alpha}{3}\eta^3\right)\psi = 0, \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0, \end{cases} \tag{1.4}$$

($\alpha \in \mathbb{C}$ is a complex constant), which belongs to the class of hypergeometric systems of two variables studied in [5]. Since (1.4) has the following solution

$$\psi = \int \exp\left\{\eta\left(t^3 + x_2 t^2 + x_1 t\right)\right\} t^{-\eta\alpha - 1} dt \tag{1.5}$$

and it is determined by the partition "$(1, 4)$" of the natural number 5, we call (1.5) the (1,4) hypergeometric function and (1.4) the (1,4) hypergeometric system. For fixed

$x_2 = c_2$, the (1,4) hypergeometric function satisfies the following third-order ordinary differential equation:

$$\left(\frac{d^3}{dx_1^3} + \frac{2}{3}c_2\eta\frac{d^2}{dx_1^2} + \frac{1}{3}x_1\eta^2\frac{d}{dx_1} - \frac{\alpha}{3}\eta^3\right)\psi = 0. \tag{1.6}$$

Since (1.6) is the equation which Aoki-Kawai-Takei discussed in considering the exact WKB analysis for higher order equations, we call it the AKT equation in this paper. Just like the BNR equation the AKT equation has new Stokes curves relevant to Stokes phenomena for its WKB solutions. As a matter of fact, it was shown in [2] that there exist three new Stokes curves relevant to Stokes phenomena for the AKT equation. Furthermore, [2] also showed that the AKT equation has infinitely many virtual turning points and new Stokes curves emanating from them, which are considered to be irrelevant to Stokes phenomena. The second purpose of this paper is to develop the exact WKB analysis for the (1,4) hypergeometric system and to investigate the relationship between its Stokes geometry and the Stokes geometry for the AKT equation, particularly the relationship between the Stokes surface for the (1,4) hypergeometric system and the new Stokes curves for the AKT equation. Our conclusion is that, in parallel to the result for the Pearcey system, the (three) new Stokes curves for the AKT equation relevant to the Stokes phenomena are included in the restriction of the Stokes surface for the (1,4) hypergeometric system. With the aid of a computer we will further show that the Stokes surface for the (1,4) hypergeometric system also contains (at least some of) new Stokes curves for the AKT equation irrelevant to Stokes phenomena. Inspired by these observations, we also study the behavior of infinitely many virtual turning points of the AKT equation when the c_2-variable varies. These results lead to an expectation that all of the infinitely many new Stokes curves for the AKT equation are included in the Stokes surface for the (1,4) hypergeometric system.

This paper is constructed as follows: In §2 we study the Pearcey system and its Stokes geometry. We review the Stokes geometry for the BNR equation in §2.1 and recall some basic definitions for the exact WKB analysis for the Pearcey system given by [1], i.e., WKB solutions, turning points and Stokes surfaces in §2.2. Then the relationship between the Stokes geometry for the Pearcey system and the Stokes geometry for the BNR equation is discussed in §2.3. In §3 we study the (1,4) hypergeometric system and its Stokes geometry. In §3.1 we review the Stokes geometry for the AKT equation studied in [2]. Then, in parallel to the case of the Pearcey system, we give some basic definitions for the exact WKB analysis for the (1,4) hypergeometric system in §3.2. In §3.3, §3.4 and §3.5, we investigate the relationship between the Stokes geometry for the (1,4) hypergeometric system and new Stokes curves for the AKT equation. Finally in §3.6 we discuss the behavior of virtual turning points of the AKT equation when c_2 varies and its relationship with the Stokes geometry for the (1,4) hypergeometric system.

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§ 2. The Stokes geometry for the Pearcey system

§ 2.1. The Stokes geometry for the BNR equation

In this section we study the Pearcey system

$$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{1}{2}x_2\eta^2\dfrac{\partial}{\partial x_1} + \dfrac{1}{4}x_1\eta^3\right)\psi = 0, \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0, \end{cases} \tag{2.1}$$

and the structure of its Stokes geometry. Before discussing the Stokes geometry for the Pearcey system, we first consider the following ordinary differential equation obtained by restricting the Pearcey system to $x_2 = c_2$:

$$\left(\frac{d^3}{dx_1^3} + \frac{1}{2}c_2\eta^2\frac{d}{dx_1} + \frac{1}{4}x_1\eta^3\right)\psi = 0. \tag{2.2}$$

As this equation was discussed by Berk-Nevins-Roberts in detail ([3]), we call it the BNR_{c_2} equation.

A turning point of (2.2) is, by definition, a point where the algebraic equation

$$\xi^3 + \frac{1}{2}c_2\xi + \frac{1}{4}x_1 = 0 \tag{2.3}$$

has a multiple root, that is, a zero of the discriminant of (2.3). An ordinary Stokes curve emanating from a turning point $x_1 = a_1$ is defined by

$$\Im\int_{a_1}^{x_1}(\xi_i - \xi_{i'})\,dx_1 = 0, \tag{2.4}$$

where $\xi_i, \xi_{i'}$ are two roots of (2.3) satisfying $\xi_i(a_1) = \xi_{i'}(a_1)$. The configuration of the turning points and the ordinary Stokes curves of the BNR_{c_2} equation for $c_2 = 1 + \sqrt{-1}$ is shown in Figure 1. Stokes phenomena for WKB solutions occur on ordinary Stokes curves. Furthermore, Berk et al showed that Stokes phenomena for WKB solutions also occur on the so-called "new Stokes curves". In [2] Aoki-Kawai-Takei showed that a new Stokes curve can be interpreted also as a Stokes curve emanating from a virtual turning point. Figure 2 shows the complete Stokes geometry of the BNR_{c_2} equation for $c_2 = 1 + \sqrt{-1}$, that is, the configuration of turning points and ordinary Stokes curves with the virtual turning point, i.e., $x_1 = 0$, and new Stokes curves also being added.

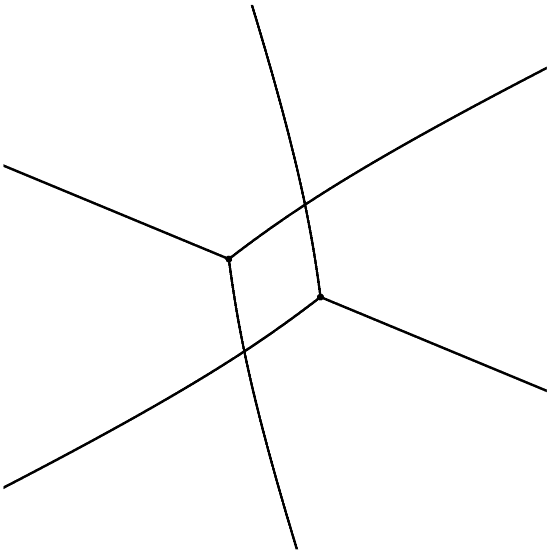

Figure 1. The turning points and the ordinary Stokes curves of the $\mathrm{BNR}_{1+\sqrt{-1}}$ equation.

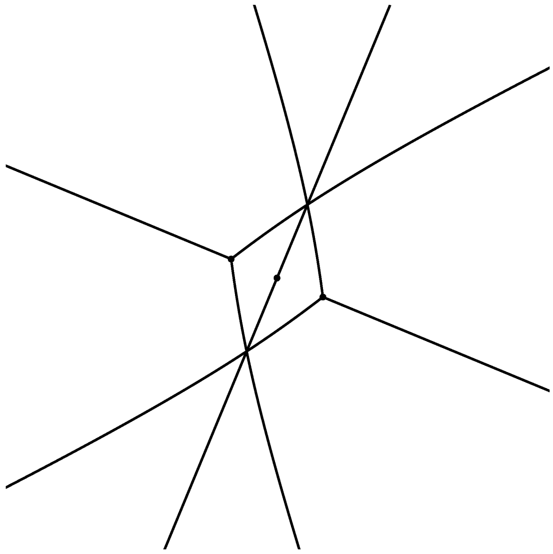

Figure 2. The complete Stokes geometry of the $\mathrm{BNR}_{1+\sqrt{-1}}$ equation.

The purpose of this section is to investigate the relationship between the Stokes geometry for the BNR_{c_2} equation and the Stokes geometry for the Pearcey system. In particular, we consider the relationship between the new Stokes curve of the BNR_{c_2} equation and the Stokes surface of the Pearcey system which is the corresponding notion of the Stokes curve for a holonomic system.

§ 2.2. Some definitions for the exact WKB analysis for the Pearcey system

In this subsection we review some basic definitions for the Pearcey system, i.e., a WKB solution, a turning point and a Stokes surface. These definitions are given in [1] with explanations for their relevance to the analysis on the Borel plane.

Substituting

$$S^{(1)} = \frac{\partial u}{\partial x_1}\Big/ u, \quad S^{(2)} = \frac{\partial u}{\partial x_2}\Big/ u \tag{2.5}$$

into the Pearcey system, we have

$$\left(S^{(1)}\right)^3 + 3S^{(1)}\frac{\partial S^{(1)}}{\partial x_1} + \frac{\partial^2 S^{(1)}}{\partial x_1^2} + \frac{1}{2}x_2\eta^2 S^{(1)} + \frac{1}{4}x_1\eta^3 = 0, \tag{2.6}$$

$$\eta S^{(2)} - \left(S^{(1)}\right)^2 - \frac{\partial S^{(1)}}{\partial x_1} = 0. \tag{2.7}$$

Assuming $S^{(1)}$ and $S^{(2)}$ have the following form

$$S^{(1)}(x,\eta)=\sum_{j\geq -1} S_j^{(1)}(x)\eta^{-j},\quad S^{(2)}(x,\eta)=\sum_{j\geq -1} S_j^{(2)}(x)\eta^{-j}, \tag{2.8}$$

we then obtain the following relations:

$$\begin{cases} \left(S_{-1}^{(1)}\right)^3+\frac{1}{2}x_2S_{-1}^{(1)}+\frac{1}{4}x_1=0, \\ \left\{3\left(S_{-1}^{(1)}\right)^2+\frac{1}{2}x_2\right\}S_0^{(1)}+3S_{-1}^{(1)}\frac{\partial S_{-1}^{(1)}}{\partial x_1}=0, \\ \left\{3\left(S_{-1}^{(1)}\right)^2+\frac{1}{2}x_2\right\}S_j^{(1)}+\sum_{\substack{k+l+m=j+1\\ j\geq k,l,m\geq 0}} S_{k-1}^{(1)}S_{l-1}^{(1)}S_{m-1}^{(1)} \\ \qquad +3\sum_{\substack{k+l=j\\ j\geq k,l\geq 0}} S_{k-1}^{(1)}\frac{\partial S_{l-1}^{(1)}}{\partial x_1}+\frac{\partial^2 S_{j-2}^{(1)}}{\partial x_1^2}=0 \quad (j\geq 1). \end{cases} \tag{2.9}$$

$$\begin{cases} S_{-1}^{(2)}=\left(S_{-1}^{(1)}\right)^2, \\ S_j^{(2)}=\sum_{\substack{k+l=j+1\\ j+1\geq k,l\geq 0}} S_{k-1}^{(1)}S_{l-1}^{(1)}+\frac{\partial S_{j-1}^{(1)}}{\partial x_1} \quad (j\geq 0). \end{cases} \tag{2.10}$$

Hence $S_{-1}^{(1)}$ is a root of the algebraic equation

$$p(x,\xi)=\xi^3+\frac{1}{2}x_2\xi+\frac{1}{4}x_1=0 \tag{2.11}$$

and $S_{-1}^{(2)}=\left(S_{-1}^{(1)}\right)^2$. The other coefficients $S_j^{(k)}$ $(j\geq 0, k=1,2)$ are uniquely determined in a recursive manner. As there exist three roots ξ_i $(i=1,2,3)$ of (2.11), we thus obtain three solutions $\left(S^{(1),i},S^{(2),i}\right)$ of (2.6) and (2.7) satisfying $\left(S_{-1}^{(1),i},S_{-1}^{(2),i}\right)=\left(\xi_i,\xi_i^2\right)$.

For $S^{(k),i}$ we have

Proposition 2.1. *$\omega^i=S^{(1),i}dx_1+S^{(2),i}dx_2$ is a closed one form.*

Proof. For simplicity, $S^{(k),i}$ is denoted by $S^{(k)}$ in this proof. We show the proposition near a point $x\in\mathbb{C}^2$ which satisfies $3\left(S_{-1}^{(1)}\right)^2+x_2/2\neq 0$, that is, x is outside the set of the turning points. (The definition of a turning point is given by Definition 2.3 below.)

Taking the partial derivative of (2.6) with respect to the variable x_2, we obtain

$$3\left(S^{(1)}\right)^2\frac{\partial S^{(1)}}{\partial x_2}+3\frac{\partial S^{(1)}}{\partial x_1}\frac{\partial S^{(1)}}{\partial x_2}+3S^{(1)}\frac{\partial^2 S^{(1)}}{\partial x_1\partial x_2}+\frac{\partial^3 S^{(1)}}{\partial x_1^2\partial x_2}+\frac{1}{2}\eta^2S^{(1)}+\frac{1}{2}x_2\eta^2\frac{\partial S^{(1)}}{\partial x_2}=0. \tag{2.12}$$

Thus it suffices to prove that $\partial S^{(2)}/\partial x_1$ is the unique solution of the following equation

$$3\left(S^{(1)}\right)^2 T + 3\frac{\partial S^{(1)}}{\partial x_1}T + 3S^{(1)}\frac{\partial T}{\partial x_1} + \frac{\partial^2 T}{\partial x_1^2} + \frac{1}{2}\eta^2 S^{(1)} + \frac{1}{2}x_2\eta^2 T = 0. \tag{2.13}$$

First let us prove the uniqueness of the solution of (2.13). Let $\widetilde{T} = \sum_{j\geq -1}\widetilde{T}_j\eta^{-j}$ be a solution of the following equation:

$$3\left(S^{(1)}\right)^2 \widetilde{T} + 3\frac{\partial S^{(1)}}{\partial x_1}\widetilde{T} + 3S^{(1)}\frac{\partial \widetilde{T}}{\partial x_1} + \frac{\partial^2 \widetilde{T}}{\partial x_1^2} + \frac{1}{2}x_2\eta^2 \widetilde{T} = 0. \tag{2.14}$$

Comparing the terms of degree j with respect to η of both sides of (2.14), we obtain

$$\left\{3\left(S_{-1}^{(1)}\right)^2 + \frac{1}{2}x_2\right\}\widetilde{T}_j + F_j = 0 \quad (j \geq -1), \tag{2.15}$$

where

(2.16)

$$F_j = \begin{cases} 0 \quad (j=-1), \\ 6S_{-1}^{(1)}S_0^{(1)}\widetilde{T}_{-1} + 3\dfrac{\partial}{\partial x_1}\left(S_{-1}^{(1)}\widetilde{T}_{-1}\right) \quad (j=0), \\ 3\displaystyle\sum_{\substack{k+l+m=j+1\\ j+1\geq k,l\geq 0\\ j\geq m\geq 0}} S_{k-1}^{(1)}S_{l-1}^{(1)}\widetilde{T}_{m-1} + 3\frac{\partial}{\partial x_1}\left(\sum_{\substack{k+m=j\\ j\geq k,m\geq 0}} S_{k-1}^{(1)}\widetilde{T}_{m-1}\right) + \frac{\partial^2 \widetilde{T}_{j-2}}{\partial x_1^2} \quad (j\geq 1). \end{cases}$$

Since $3\left(S_{-1}^{(1)}\right)^2 + x_2/2 \neq 0$ and $F_{-1} = 0$, we get $\widetilde{T}_{-1} = 0$. Hence we have $F_0 = 0$. Then we get $\widetilde{T}_0 = 0$ from the relation (2.15) for $j = 0$. In a similar way, we get $\widetilde{T}_j = 0$ in a recursive manner. Therefore the uniqueness of solutions of (2.13) holds.

Next we prove that $\partial S^{(2)}/\partial x_1$ satisfies (2.13). Using (2.6) and (2.7), we obtain

(2.17)

$$\begin{aligned} &3\left(S^{(1)}\right)^2\frac{\partial S^{(2)}}{\partial x_1} + 3\frac{\partial S^{(1)}}{\partial x_1}\frac{\partial S^{(2)}}{\partial x_1} + 3S^{(1)}\frac{\partial^2 S^{(2)}}{\partial x_1^2} + \frac{\partial^3 S^{(2)}}{\partial x_1^3} + \frac{1}{2}\eta^2 S^{(1)} + \frac{1}{2}x_2\eta^2\frac{\partial S^{(2)}}{\partial x_1} \\ =&6\eta^{-1}\left(S^{(1)}\right)^3\frac{\partial S^{(1)}}{\partial x_1} + 9\eta^{-1}\left(S^{(1)}\right)^2\frac{\partial^2 S^{(1)}}{\partial x_1^2} + 12\eta^{-1}S^{(1)}\left(\frac{\partial S^{(1)}}{\partial x_1}\right)^2 + 9\eta^{-1}\frac{\partial S^{(1)}}{\partial x_1}\frac{\partial^2 S^{(1)}}{\partial x_1^2} \\ &+ 5\eta^{-1}S^{(1)}\frac{\partial^3 S^{(1)}}{\partial x_1^3} + \eta^{-1}\frac{\partial^4 S^{(1)}}{\partial x_1^4} + \frac{1}{2}\eta^2 S^{(1)} + x_2\eta S^{(1)}\frac{\partial S^{(1)}}{\partial x_1} + \frac{1}{2}x_2\eta\frac{\partial^2 S^{(1)}}{\partial x_1^2} \\ =&6\eta^{-1}\left(S^{(1)}\right)^3\frac{\partial S^{(1)}}{\partial x_1} + 6\eta^{-1}\left(S^{(1)}\right)^2\frac{\partial^2 S^{(1)}}{\partial x_1^2} + 6\eta^{-1}S^{(1)}\left(\frac{\partial S^{(1)}}{\partial x_1}\right)^2 \\ &+ 2\eta^{-1}S^{(1)}\frac{\partial^3 S^{(1)}}{\partial x_1^3} + \frac{1}{2}\eta^2 S^{(1)} + x_2\eta S^{(1)}\frac{\partial S^{(1)}}{\partial x_1} \end{aligned}$$

$$=2\eta^{-1}S^{(1)}\frac{\partial}{\partial x_1}\left\{\left(S^{(1)}\right)^3+3S^{(1)}\frac{\partial S^{(1)}}{\partial x_1}+\frac{\partial^2 S^{(1)}}{\partial x_1^2}+\frac{1}{2}x_2\eta^2S^{(1)}+\frac{1}{4}x_1\eta^3\right\}$$
$$=0.$$

□

Proposition 2.1 implies the following integral is well-defined as a formal power series in η^{-1}

$$\int^x \omega^i. \tag{2.18}$$

Definition 2.2.

$$\psi_i=\eta^{-1/2}\exp\left(\int^x \omega^i\right) \tag{2.19}$$

is called a WKB solution for the Pearcey system.

In the exact WKB analysis for ordinary differential equations, turning points and ordinary Stokes curves are important notions. We now define turning points and Stokes surfaces for the Pearcey system, which are the notions corresponding to turning points and ordinary Stokes curves for ordinary differential equations, respectively.

Definition 2.3. A point $a\in\mathbb{C}^2$ is called a turning point for the Pearcey system if there exist $i,i'\in\{1,2,3\}$ $(i\neq i')$ for which

$$\omega^i_{-1}(a)=\omega^{i'}_{-1}(a) \tag{2.20}$$

holds, where ω^i_{-1} is the coefficient of η of ω^i.

Definition 2.4. Let $a\in\mathbb{C}^2$ be a turning point. A Stokes surface for the Pearcey system emanating from $x=a$ is the real 3-dimensional surface defined by

$$\Im\int_a^x\left(\omega^i_{-1}-\omega^{i'}_{-1}\right)=0. \tag{2.21}$$

§ 2.3. The structure of the Stokes geometry for the Pearcey system

In this subsection we investigate the relationship between the Stokes geometry for the Pearcey system and the Stokes geometry for the BNR_{c_2} equation.

First we have

Proposition 2.5. *The set of the turning points for the Pearcey system restricted to $x_2=c_2$ coincides with the set of the turning points for the BNR_{c_2} equation.*

Proof. Let $a \in \mathbb{C}^2$ be a turning point for the Pearcey system, that is, there exist $i, i' \in \{1,2,3\}$ $(i \neq i')$ such that $\omega^i(a) = \omega^{i'}(a)$. Since $S_{-1}^{(2)} = \left(S_{-1}^{(1)}\right)^2$, this condition is equivalent to

$$\xi_i(a) = \xi_{i'}(a), \tag{2.22}$$

where $\xi_i, \xi_{i'}$ are two roots of the algebraic equation $p(x,\xi) = 0$. Hence a turning point of the Pearcey system is a point where the algebraic equation $p(x,\xi) = 0$ has a multiple root, that is, a zero of the discriminant of $p(x,\xi) = 0$. This completes the proof of Proposition 2.5. □

The set of the turning points for the Pearcey system is explicitly given by

$$\left\{(x_1,x_2) \in \mathbb{C}^2 \ ; \ 27x_1^2 + 8x_2^3 = 0\right\}. \tag{2.23}$$

In view of Proposition 2.5, we readily find that ordinary Stokes curves for the BNR_{c_2} equation are contained in the Stokes surface for the Pearcey system restricted to $x_2 = c_2$.

Furthermore, we can prove

Proposition 2.6. *New Stokes curves for the* BNR_{c_2} *equation are also contained in the Stokes surface for the Pearcey system restricted to* $x_2 = c_2$.

Before we prove Proposition 2.6, we give some notations related to turning points. A turning point for the BNR_{c_2} equation is a zero of

$$27x_1^2 + 8c_2^3 = 0. \tag{2.24}$$

Let H be the c_2-plane equipped with cut lines as is shown in Figure 3. Then the turning points for the BNR_{c_2} equation are single-valued functions in this cut plane. We denote a turning point (viewed as a single-valued function in H) by $a_j(c_2)$ $(j = 1,2)$. For the sake of convenience, we also use the notation $a_3(c_2)$ which is given by $a_3(c_2) = a_1(c_2)$. Since $a_j(1)$ $(j = 1,2)$ are turning points for the BNR_1 equation, we may assume

$$\xi_i\Big|_{x=(a_i(1),1)} = \xi_{i+1}\Big|_{x=(a_i(1),1)} \qquad (i = 1,2), \tag{2.25}$$

where $\xi_i(x) = \xi_i(x_1,x_2)$ $(i = 1,2,3)$ are the roots of $p(x,\xi) = 0$.

In this notation we prove

Proposition 2.7. *For* $i = 1,2$ *we have*

$$\int_{a_i(c_2)}^{a_{i+1}(c_2)} (\xi_i(x_1,c_2) - \xi_{i+1}(x_1,c_2))\,dx_1 = \int_0^{c_2} t_i(c_2)dc_2, \tag{2.26}$$

where t_i $(i = 1,2)$ *are single-valued functions on* H *and satisfies the following two conditions*

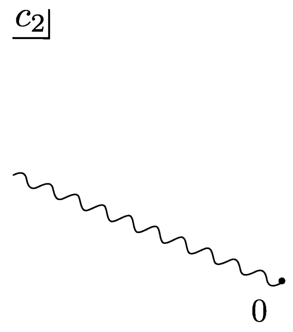

Figure 3. The cut plane H. (A wiggly line designates a cut.)

- t_i *is the root of* $t^2 - 9c_2^2/4 = 0$,
- $t_i(1) = (-1)^i 3c_2/2$.

Proof. For simplicity, $\xi_i(a_j(c_2), c_2)$ are denoted by $\xi_{i,j}(c_2)$. Then $\xi_{i,j}$ $(i = 1, 2, 3)$ satisfy

$$p\,(a_j(c_2), c_2, \xi) = 0. \tag{2.27}$$

On the other hand, by the definition of a_j, the discriminant in ξ of (2.27) is equal to zero, that is, (2.27) has a multiple root, which is given by

$$\xi_{j,j} = \xi_{j+1,j} \quad (j = 1, 2) \tag{2.28}$$

in view of (2.25). Since $\xi_1 + \xi_2 + \xi_3 = 0$ and $\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 = c_2/2$, we have the following relations for $j = 1, 2$

$$2\xi_{j,j} + \xi_{j+2,j} = 0, \quad \xi_{j,j}^2 + 2\xi_{j,j}\xi_{j+2,j} = \frac{1}{2}c_2, \tag{2.29}$$

where $\xi_{4,2}$ is defined by $\xi_{4,2} = \xi_{1,2}$. By using these relations, we find that $\xi_{j,j}$ $(j = 1, 2)$ satisfy the following algebraic equation

$$6\xi^2 + c_2 = 0. \tag{2.30}$$

In particular, $\xi_{i,j}$ $(i, j = 1, 2, 3)$ are single-valued functions on H.

Thanks to Proposition 2.1, $\xi_i = \xi_i(x_1, c_2)$ satisfies

$$\frac{\partial}{\partial c_2}\xi_i = \frac{\partial}{\partial x_1}\xi_i^2. \tag{2.31}$$

It follows from this relation and (2.28) that

$$\begin{aligned}
&\frac{d}{dc_2}\int_{a_i(c_2)}^{a_{i+1}(c_2)} (\xi_i - \xi_{i+1})\, dx_1 \\
&= \int_{a_i(c_2)}^{a_{i+1}(c_2)} \frac{\partial}{\partial c_2}(\xi_i - \xi_{i+1})\, dx_1 + (\xi_i - \xi_{i+1})\Big|_{x_1=a_{i+1}(c_2)}\frac{da_{i+1}}{dc_2} - (\xi_i - \xi_{i+1})\Big|_{x_1=a_i(c_2)}\frac{da_i}{dc_2} \\
&= \int_{a_i(c_2)}^{a_{i+1}(c_2)} \frac{\partial}{\partial x_1}\left(\xi_i^2 - \xi_{i+1}^2\right) dx_1 + (\xi_i - \xi_{i+1})\Big|_{x_1=a_{i+1}(c_2)}\frac{da_{i+1}}{dc_2} \\
&= \left(\xi_i^2 - \xi_{i+1}^2\right)\Big|_{x_1=a_{i+1}(c_2)} - \left(\xi_i^2 - \xi_{i+1}^2\right)\Big|_{x_1=a_i(c_2)} + (\xi_i - \xi_{i+1})\Big|_{x_1=a_{i+1}(c_2)}\frac{da_{i+1}}{dc_2} \\
&= \left(\xi_{i,i+1}^2 + \frac{da_{i+1}}{dc_2}\xi_{i,i+1}\right) - \left(\xi_{i+1,i+1}^2 + \frac{da_{i+1}}{dc_2}\xi_{i+1,i+1}\right).
\end{aligned} \tag{2.32}$$

Since $\xi_{i,i}$ $(i = 1, 2)$ satisfy

$$\begin{cases} p\left(a_i(c_2), c_2, \xi_{i,i}(c_2)\right) = 0, \\ \dfrac{\partial p}{\partial \xi}\left(a_i(c_2), c_2, \xi_{i,i}(c_2)\right) = 0, \end{cases} \tag{2.33}$$

we have

$$\begin{aligned}
0 &= \frac{d}{dc_2} p\left(a_i(c_2), c_2, \xi_{i,i}(c_2)\right) \\
&= \frac{\partial p}{\partial x_1}\left(a_i(c_2), c_2, \xi_{i,i}(c_2)\right)\frac{da_i}{dc_2} + \frac{\partial p}{\partial c_2}\left(a_i(c_2), c_2, \xi_{i,i}(c_2)\right) + \frac{\partial p}{\partial \xi}\left(a_i(c_2), c_2, \xi_{i,i}(c_2)\right)\frac{d\xi_{i,i}}{dc_2} \\
&= \frac{1}{4}\frac{da_i}{dc_2} + \frac{1}{2}\xi_{i,i},
\end{aligned} \tag{2.34}$$

that is,

$$\frac{da_i}{dc_2} = -2\xi_{i,i} \tag{2.35}$$

holds. Hence, by (2.28), (2.29) and (2.30), we have

$$\begin{aligned}
\frac{d}{dc_2}\int_{a_1(c_2)}^{a_2(c_2)} (\xi_1 - \xi_2)\, dx_1 &= (\xi_{1,2} - \xi_{2,2})\left(\xi_{1,2} + \xi_{2,2} + \frac{da_2}{dc_2}\right) \\
&= (\xi_{1,2} - \xi_{2,2})^2
\end{aligned} \tag{2.36}$$

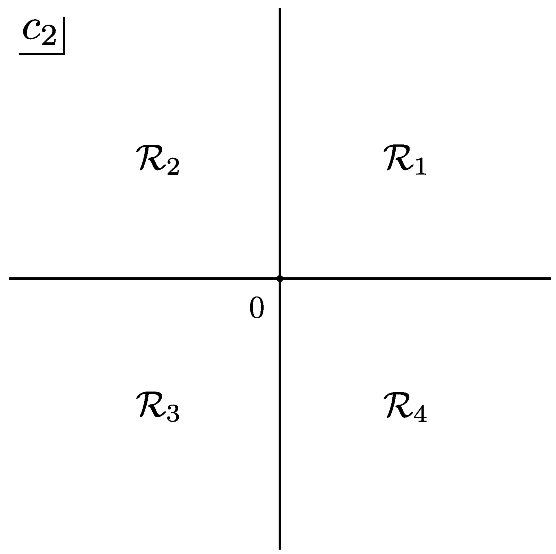

Figure 4. The curves defined by (2.38).

$$\begin{aligned} &= 9\xi_{2,2}^2 \\ &= -\frac{3}{2}c_2 \\ &= t_1 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dc_2}\int_{a_2(c_2)}^{a_3(c_2)} (\xi_2 - \xi_3)\, dx_1 &= (\xi_{2,3} - \xi_{3,3})\left(\xi_{2,3} + \xi_{3,3} + \frac{da_3}{dc_2}\right) \\ &= (\xi_{2,1} - \xi_{3,1})\left(\xi_{2,1} + \xi_{3,1} + \frac{da_1}{dc_2}\right) \\ &= -9\xi_{1,1}^2 \\ &= \frac{3}{2}c_2 \\ &= t_2. \end{aligned} \tag{2.37}$$

Since $a_i(0) = a_{i+1}(0)$ $(i = 1, 2)$, we thus obtain (2.26). $\square$

Using Proposition 2.7, we can easily draw the figure of the curves defined by

$$\Im \int_{a_i(c_2)}^{a_{i+1}(c_2)} (\xi_i - \xi_{i+1})\, dx_1 = 0 \quad (i = 1, 2), \tag{2.38}$$

which is shown in Figure 4.

Note that, if $c_2 \in \mathbb{C}_{c_2}$ is on the curve defined by (2.38), then an ordinary Stokes curve of the BNR_{c_2} equation emanating from a turning point passes through another turning point.

By using Proposition 2.7, we prove Proposition 2.6.

Proof of Proposition 2.6. Let $\mathcal{R}_i$ $(i = 1, 2, 3, 4)$ be domains bounded by the curves defined by (2.38) (cf. Figure 4):

$$\begin{aligned} &\mathcal{R}_1 = \{\Re c_2 > 0, \Im c_2 > 0\}, \quad \mathcal{R}_2 = \{\Re c_2 < 0, \Im c_2 > 0\}, \\ &\mathcal{R}_3 = \{\Re c_2 < 0, \Im c_2 < 0\}, \quad \mathcal{R}_4 = \{\Re c_2 > 0, \Im c_2 < 0\}. \end{aligned} \tag{2.39}$$

By Proposition 2.7, the Stokes geometry for the BNR_{c_2} equation for a point c_2 in $\mathcal{R}_i$ is topologically equivalent to that for any other point c_2' in the same domain $\mathcal{R}_i$. On the other hand, the Stokes geometry for the BNR_{c_2} equation in $\mathcal{R}_i$ is topologically different from that in a different domain $\mathcal{R}_{i'}$. For example, let us pick up the following points

$$\mu_1 = 1 + \sqrt{-1}, \quad \mu_2 = -1 + \sqrt{-1}, \quad \mu_3 = -1 - \sqrt{-1}, \quad \mu_4 = 1 - \sqrt{-1} \tag{2.40}$$

from each domain $\mathcal{R}_i$. The Stokes geometry for the BNR_{c_2} equation for $c_2 = \mu_1, \mu_2, \mu_3$ and μ_4 is shown in Figure 5, Figure 6, Figure 7 and Figure 8, respectively. Comparing these figures, we find that a new Stokes curve for the BNR_{c_2} equation for $c_2 \in \mathcal{R}_i$ is changed to an ordinary Stokes curve in an adjacent domain $\mathcal{R}_{i'}$. Using this property, we prove that a new Stokes curve for the BNR_{c_2} equation for $c_2 \in \mathcal{R}_i$ is contained in the Stokes surface for the Pearcey system in what follows.

Since the proof is similar for any domain $\mathcal{R}_i$, we consider only $\mathcal{R}_1$. In particular, taking $\lambda_1 = 1 + \sqrt{-1}/4$ in $\mathcal{R}_1$ and a point p on the new Stokes curve for the BNR_{λ_1} equation (cf. Figure 9), we show that (p, λ_1) is contained in the Stokes surface for the Pearcey system. That is, we show that there exist a turning point $a \in \mathbb{C}^2$ for the Pearcey system, two one forms $\omega^i_{-1}, \omega^{i'}_{-1}$ and a path γ connecting a and (p, λ_1) that satisfy the following conditions.

$$\omega^i_{-1}(a) = \omega^{i'}_{-1}(a), \quad \Im \int_a^q \left(\omega^i_{-1} - \omega^{i'}_{-1}\right) = 0 \quad (\text{for any } q \in \gamma). \tag{2.41}$$

Let $a_1(\lambda_1), a_2(\lambda_1)$ be turning points for the BNR_{λ_1} equation, and b_1 be a crossing point of two ordinary Stokes curves of the BNR_{λ_1} equation emanating from $a_1(\lambda_1), a_2(\lambda_1)$ in Figure 9. We take a point in an adjacent domain, for example, $\lambda_2 = 1 - \sqrt{-1}/4 \in \mathcal{R}_4$, and a path $l_2(t)$ connecting λ_1 and λ_2 defined by

$$l_2(t) = (1 - t)\lambda_1 + t\lambda_2 \quad (0 \le t \le 1). \tag{2.42}$$

Note that $l_2(t)$ crosses the curve defined by (2.38) at $c_2 = 1$, where the Stokes geometry of the BNR_1 equation is degenerate (cf. Figure 10). Let $a_1(\lambda_2), a_2(\lambda_2)$ be turning points

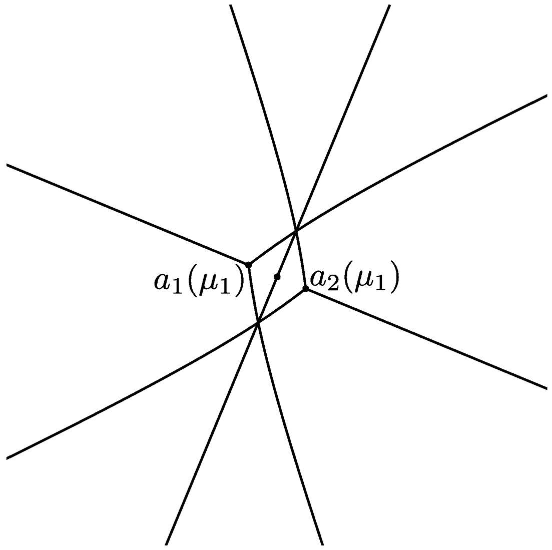

Figure 5. The Stokes geometry for the BNR_{μ_1} equation.

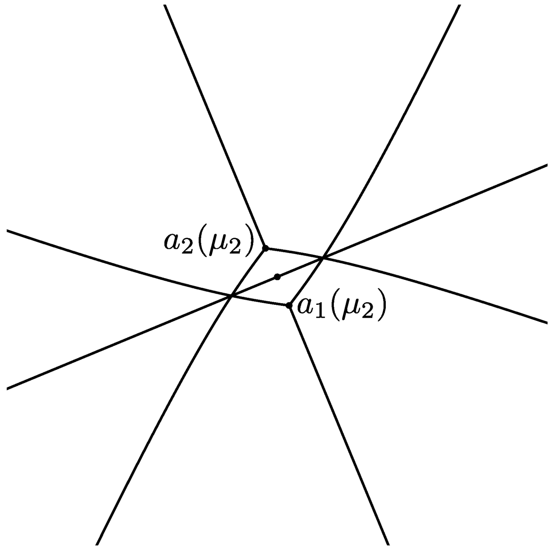

Figure 6. The Stokes geometry for the BNR_{μ_2} equation.

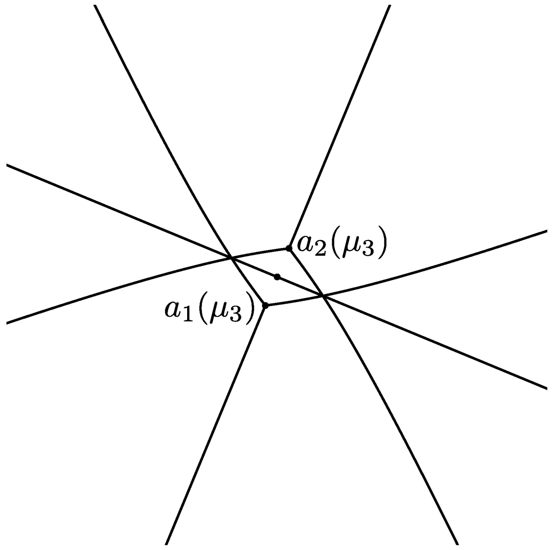

Figure 7. The Stokes geometry for the BNR_{μ_3} equation.

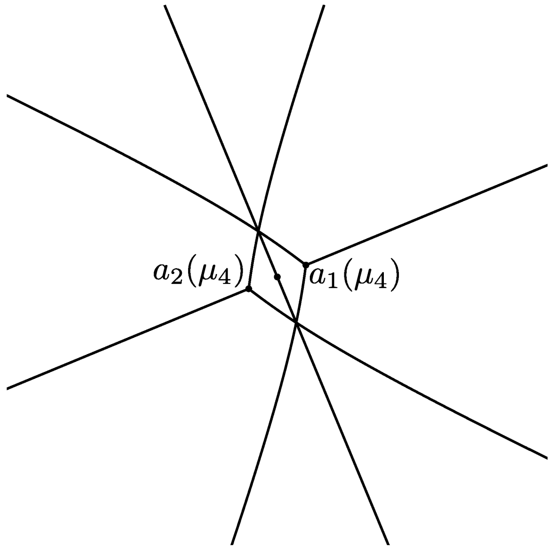

Figure 8. The Stokes geometry for the BNR_{μ_4} equation.

for the BNR_{λ_2} equation, and b_2 be a crossing point of two ordinary Stokes curves for the BNR_{λ_2} equation emanating from $a_1(\lambda_2), a_2(\lambda_2)$ in Figure 11. Furthermore, $a_1(1), a_2(1)$ are two turning points for the BNR_1 equation in Figure 10. Let $l_1(t)$ be a path defined by the following

- if $t \neq 1/2$, then $l_1(t)$ is a crossing point in the upper half plane of two ordinary Stokes curves for the $\mathrm{BNR}_{l_2(t)}$ equation emanating from two turning points,
- if $t = 1/2$, then $l_1(1/2)$ is a turning point for the $\mathrm{BNR}_{l_2(1/2)}$ equation in the upper half plane.

Then $l(t) = (l_1(t), l_2(t))$ $(0 \leq t \leq 1)$ is a path in $\mathbb{C}^2$ connecting $l(0) = (b_1, \lambda_1)$ and $l(1) = (b_2, \lambda_2)$. By using l, we define a path γ connecting $(a_2(\lambda_2), \lambda_2)$ and (p, λ_1) as the composition of the following three paths:

- a path from $(a_2(\lambda_2), \lambda_2)$ to (b_2, λ_2) along an ordinary Stokes curve for the BNR_{λ_2} equation (this portion is contained completely in $\{c_2 = \lambda_2\}$),
- l^{-1}, that is, a path connecting (b_2, λ_2) and (b_1, λ_1) through crossing points of ordinary Stokes curves for the $\mathrm{BNR}_{l_2(t)}$ equation $(0 \leq t \leq 1)$,
- a path from (b_1, λ_1) to (p, λ_1) along a new Stokes curve for the BNR_{λ_1} equation (this portion is contained completely in $\{c_2 = \lambda_1\}$).

We first prove

$$\Im \int_{(a_2(\lambda_2),\lambda_2)}^{(p,\lambda_1)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) = 0, \tag{2.43}$$

where the integration is taken along γ and the branch ω_{-1}^i $(i = 1, 2, 3)$ of ω_{-1} is assumed to be fixed at the endpoint (p, λ_1) (that is, ω_{-1}^i expresses the branch in the x_1-plane (given by $c_2 = \lambda_1$) with cuts; cf. Figure 9). Note that after the analytic continuation along γ^{-1} ω_{-1}^1 and ω_{-1}^3 are changed to ω_{-1}^2 and ω_{-1}^3, respectively, and hence the integrand vanishes at the starting point $(a_2(\lambda_2), \lambda_2)$. To prove (2.43), we decompose the integral as

$$\begin{aligned}
&\int_{(a_2(\lambda_2),\lambda_2)}^{(p,\lambda_1)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) \\
&= \int_{(a_2(\lambda_2),\lambda_2)}^{(b_2,\lambda_2)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{(b_2,\lambda_2)}^{l(1/2)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) \\
&\quad + \int_{l(1/2)}^{(b_1,\lambda_1)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{(b_1,\lambda_1)}^{(p,\lambda_1)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.44}$$

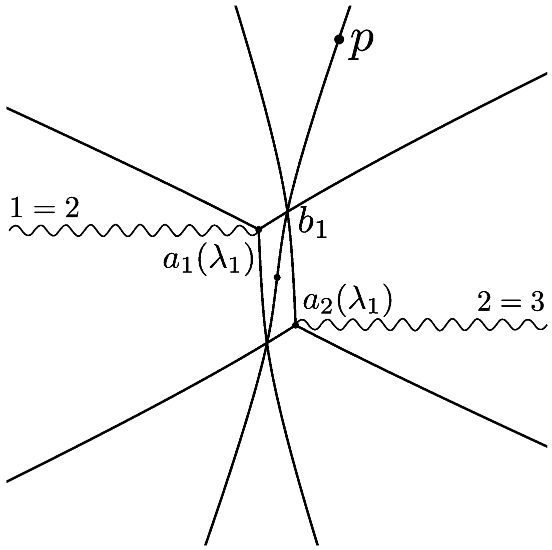

Figure 9. The Stokes geometry for the BNR_{λ_1} equation. (A wiggly line designates a cut to define $\xi_i(x_1, \lambda_1)$. For example, on a cut with the symbol "$1 = 2$" the numbering of ξ_1 and ξ_2 should be interchanged.)

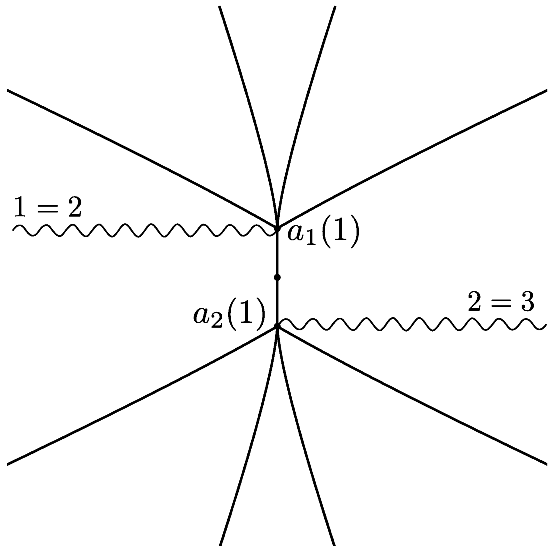

Figure 10. The Stokes geometry for the BNR_1 equation.

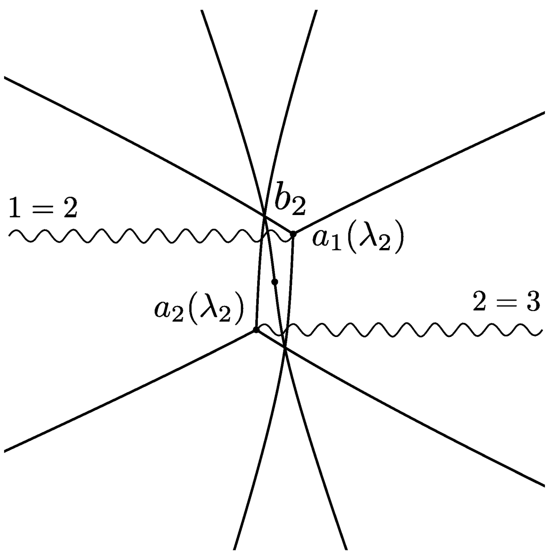

Figure 11. The Stokes geometry for the BNR_{λ_2} equation.

First, since p lies on a new Stokes curve (of type (1,3)) for the BNR_{λ_1} equation, we have

$$\Im I_4 = 0. \tag{2.45}$$

We next consider I_3, which is the sum of two integrals:

$$\begin{aligned} I_3 &= \int_{l(1/2)}^{(b_1,\lambda_1)} \left(\omega^1_{-1} - \omega^2_{-1}\right) + \int_{l(1/2)}^{(b_1,\lambda_1)} \left(\omega^2_{-1} - \omega^3_{-1}\right) \\ &=: I_{31} + I_{32}. \end{aligned} \tag{2.46}$$

By deforming the path of integration, we can express I_{31} as

$$I_{31} = \int_{l(1/2)}^{(a_1(\lambda_1),\lambda_1)} \left(\omega^1_{-1} - \omega^2_{-1}\right) + \int_{(a_1(\lambda_1),\lambda_1)}^{(b_1,\lambda_1)} \left(\omega^1_{-1} - \omega^2_{-1}\right), \tag{2.47}$$

where the first integral is taken along the set of turning points for the Pearcey system and the second integral is done along an ordinary Stokes curve for the BNR_{λ_1} equation emanating from $a_1(\lambda_1)$. Since $\omega^1_{-1} = \omega^2_{-1}$ holds at turning points $(a_1(l_2(t)), l_2(t))$ $(0 \le t \le 1/2)$, the first integral vanishes. Furthermore, as the second integral is taken along an ordinary Stokes curve, its imaginary part is zero. Hence

$$I_{31} \in \mathbb{R}. \tag{2.48}$$

On the other hand, I_{32} can be expressed as

$$I_{32} = \int_{l(1/2)}^{(a_2(1),1)} \left(\omega^2_{-1} - \omega^3_{-1}\right) + \int_{(a_2(1),1)}^{(a_2(\lambda_1),\lambda_1)} \left(\omega^2_{-1} - \omega^3_{-1}\right) + \int_{(a_2(\lambda_1),\lambda_1)}^{(b_1,\lambda_1)} \left(\omega^2_{-1} - \omega^3_{-1}\right), \tag{2.49}$$

where the first and the third integrals are done along ordinary Stokes curves for the BNR_1 and the BNR_{λ_1} equation, respectively, and the second integral is taken along the set of turning points for the Pearcey system. Hence, similarly to I_{31}, we have

$$I_{32} = \int_{l(1/2)}^{(a_2(1),1)} \left(\omega^2_{-1} - \omega^3_{-1}\right) + \int_{(a_2(\lambda_1),\lambda_1)}^{(b_1,\lambda_1)} \left(\omega^2_{-1} - \omega^3_{-1}\right) \in \mathbb{R}. \tag{2.50}$$

Thus we obtain

$$\Im I_3 = 0. \tag{2.51}$$

The integral I_2 can be discussed in a similar manner: By expressing I_2 as

$$I_2 = \int_{(b_2,\lambda_2)}^{(a_2(\lambda_2),\lambda_2)} \left(\omega^2_{-1} - \omega^3_{-1}\right) + \int_{(a_2(\lambda_2),\lambda_2)}^{(a_2(1),1)} \left(\omega^2_{-1} - \omega^3_{-1}\right) + \int_{(a_2(1),1)}^{l(1/2)} \left(\omega^2_{-1} - \omega^3_{-1}\right), \tag{2.52}$$

we can confirm

$$I_2 = \int_{(b_2,\lambda_2)}^{(a_2(\lambda_2),\lambda_2)} \left(\omega_{-1}^2 - \omega_{-1}^3\right) + \int_{(a_2(1),1)}^{l(1/2)} \left(\omega_{-1}^2 - \omega_{-1}^3\right) \in \mathbb{R}. \tag{2.53}$$

Finally, since b_2 lies on an ordinary Stokes curve for the BNR_{λ_2} equation, we have

$$\Im I_1 = 0. \tag{2.54}$$

(Note that the path of integration for I_1 crosses a cut and consequently the integrand of I_1 vanishes at $(a_2(\lambda_2), \lambda_2)$.) Thus we have verified (2.43).

Using the above reasoning, we can also prove that

$$\Im \int_{(a_2(\lambda_2),\lambda_2)}^{q} \left(\omega_{-1}^1 - \omega_{-1}^3\right) = 0 \tag{2.55}$$

holds for any point q on γ. This completes the proof of Proposition 2.6. □

Thus, in the case of the BNR_{c_2} equation, not only the ordinary Stokes curves but also the new Stokes curves are included in the restriction of the Stokes surface for the Pearcey system. In other words, the Stokes surface for the Pearcey system describes the whole Stokes curves for the BNR equation.

§ 3. The Stokes geometry for the (1,4) hypergeometric system

§ 3.1. The Stokes geometry for the ordinary differential equation obtained by restricting the (1,4) hypergeometric system to some line

In this section we study the (1,4) hypergeometric system

$$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{2}{3}x_2\eta\dfrac{\partial^2}{\partial x_1^2} + \dfrac{1}{3}x_1\eta^2\dfrac{\partial}{\partial x_1} - \dfrac{\alpha}{3}\eta^3\right)\psi = 0, \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0 \end{cases} \tag{3.1}$$

(where $\alpha \in \mathbb{C}^2$ is a complex constant) and the structure of its Stokes geometry. In parallel to §2, we first consider the following ordinary differential equation obtained by restricting the (1,4) hypergeometric system to $x_2 = c_2$:

$$\left(\frac{d^3}{dx_1^3} + \frac{2}{3}c_2\eta\frac{d^2}{dx_1^2} + \frac{1}{3}x_1\eta^2\frac{d}{dx_1} - \frac{\alpha}{3}\eta^3\right)\psi = 0. \tag{3.2}$$

As this equation was discussed by Aoki-Kawai-Takei in [2], we call it the AKT_{c_2} equation. The configuration of the turning points and the ordinary Stokes curves of the

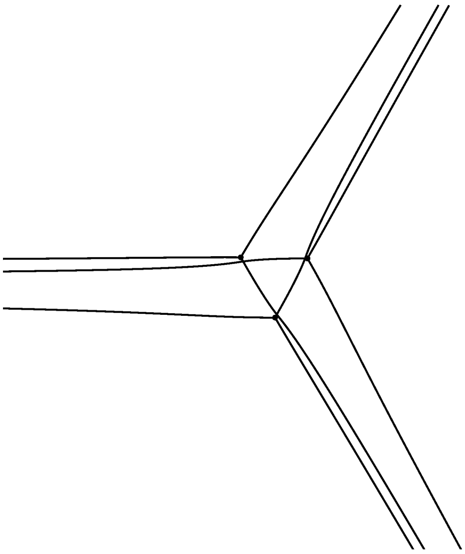

Figure 12. The turning points and the ordinary Stokes curves of the AKT_{c_2} equation for $c_2 = \sqrt{-1}/20$ and $\alpha = 1/2 - \sqrt{-1}/2$.

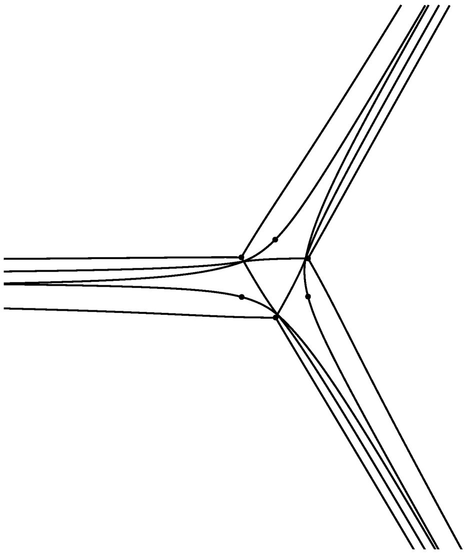

Figure 13. The Stokes geometry shown in Figure 12 with the new Stokes curves relevant to Stokes phenomena being added; the dot on a new Stokes curve designates the virtual turning point from which it emanates.

AKT_{c_2} equation for $c_2 = \sqrt{-1}/20$ and $\alpha = 1/2 - \sqrt{-1}/2$ is shown in Figure 12. (The choice of $\alpha = 1/2 - \sqrt{-1}/2$ has no special meaning. We can take a generic value for α. But, for the sake of definiteness, we fix $\alpha = 1/2 - \sqrt{-1}/2$ in what follows.)

Similarly to the case of the BNR_{c_2} equation, new Stokes curves appear also for the AKT_{c_2} equation. In [2] Aoki-Kawai-Takei investigated the new Stokes curves of such an equation. (Although the equation they considered does not have exactly the same form as (3.2), it can be transformed to (3.2) by a simple gauge transformation of the unknown function and a translation and a scaling of the independent variable.) As explained in Introduction, they showed that the equation (equivalent to the AKT_{c_2} equation in the above sense) has three new Stokes curves where Stokes phenomena for the WKB solutions occur and that it has also infinitely many virtual turning points. The latter result implies that, in addition to the three new Stokes curves relevant to Stokes phenomena for WKB solutions, the AKT_{c_2} equation has infinitely many new Stokes curves which are irrelevant to Stokes phenomena. The figure where the three new Stokes curves relevant to Stokes phenomena and the virtual turning points corresponding to them are added to Figure 12 is shown in Figure 13.

In the previous section we have shown that the new Stokes curve relevant to Stokes phenomena for the BNR_{c_2} equation is contained in the Stokes surface for the Pearcey system. Thus the following questions (A) and (B) naturally arise for the (1,4) hypergeometric system (3.1) and its restriction (3.2) to $x_2 = c_2$:

(A) Are the (three) new Stokes curves for the AKT_{c_2} equation relevant to Stokes phenomena contained in the Stokes surface for (3.1)?

(B) Are infinitely many new Stokes curves for the AKT_{c_2} equation irrelevant to Stokes phenomena also contained in the Stokes surface for (3.1)?

The purpose of this section is to investigate these two questions (A) and (B). As it will be clarified below, our answer to the question (A) is 'Yes'. We also find that the answer to the question (B) is positive as far as the examples we have checked are concerned; this strongly supports our expectation that the answer to (B) is also 'Yes'. Our expectation will be further fortified by our observations in §3.6 below.

§ 3.2. Some definitions for the exact WKB analysis for the (1,4) hypergeometric system

Before discussing the questions (A) and (B), in this subsection we review some basic definitions for the (1,4) hypergeometric system. These definitions are given in parallel to the case of the Pearcey system.

Substituting

$$S^{(1)} = \frac{\partial u}{\partial x_1}\Big/ u, \quad S^{(2)} = \frac{\partial u}{\partial x_2}\Big/ u \tag{3.3}$$

into the (1,4) hypergeometric system (3.1), we have

(3.4)
$$\left(S^{(1)}\right)^3 + 3S^{(1)}\frac{\partial S^{(1)}}{\partial x_1} + \frac{\partial^2 S^{(1)}}{\partial {x_1}^2} + \frac{2}{3}x_2\eta\left\{\left(S^{(1)}\right)^2 + \frac{\partial S^{(1)}}{\partial x_1}\right\} + \frac{1}{3}x_1\eta^2 S^{(1)} - \frac{\alpha}{3}\eta^3 = 0,$$

$$\eta S^{(2)} - \left(S^{(1)}\right)^2 - \frac{\partial S^{(1)}}{\partial x_1} = 0. \tag{3.5}$$

Assuming $S^{(1)}$ and $S^{(2)}$ have the following form

$$S^{(1)}(x,\eta) = \sum_{j\geq -1} S_j^{(1)}(x)\eta^{-j}, \quad S^{(2)}(x,\eta) = \sum_{j\geq -1} S_j^{(2)}(x)\eta^{-j}, \tag{3.6}$$

we then find that $S_{-1}^{(1)}$ is a root of the algebraic equation

$$\widetilde{p}\,(x,\xi) = \xi^3 + \frac{2}{3}x_2\xi^2 + \frac{1}{3}x_1\xi - \frac{\alpha}{3} = 0 \tag{3.7}$$

and $S^{(2)}_{-1} = \left(S^{(1)}_{-1}\right)^2$. The other coefficients $S^{(k)}_j$ $(j \geq 0,\ k = 1, 2)$ are uniquely determined in a recursive manner. As there exist three roots ξ_i $(i = 1, 2, 3)$ of (3.7), we thus obtain three solutions $\left(S^{(1),i}, S^{(2),i}\right)$ of (3.4) and (3.5) satisfying $\left(S^{(1,i)}_{-1}, S^{(2),i}_{-1}\right) = \left(\xi_i, \xi_i^2\right)$.

Similarly to Proposition 2.1, we can prove

Proposition 3.1. *$\omega^i = S^{(1),i}dx_1 + S^{(2),i}dx_2$ is a closed one form.*

Proposition 3.1 implies the following integral is well-defined as a formal power series in η^{-1}

$$\int^x \omega^i. \tag{3.8}$$

Definition 3.2.

$$\psi_i = \eta^{-1/2} \exp\left(\int^x \omega^i\right) \tag{3.9}$$

is called a WKB solution for the (1,4) hypergeometric system.

Definition 3.3. A point $a \in \mathbb{C}^2$ is called a turning point for the (1,4) hypergeometric system if there exist $i, i' \in \{1, 2, 3\}$ $(i \neq i')$ for which

$$\omega^i_{-1}(a) = \omega^{i'}_{-1}(a) \tag{3.10}$$

holds, where ω^i_{-1} is the coefficient of η of ω^i.

Definition 3.4. Let $a \in \mathbb{C}^2$ be a turning point. A Stokes surface for the (1,4) hypergeometric system emanating from $x = a$ is the real 3-dimensional surface defined by

$$\Im \int_a^x \left(\omega^i_{-1} - \omega^{i'}_{-1}\right) = 0. \tag{3.11}$$

§ 3.3. The structure of the Stokes geometry for the (1,4) hypergeometric system, I

In the subsequent subsections we investigate the structure of the Stokes geometry for the (1,4) hypergeometric system and answer the questions (A) and (B) raised in §3.1.

We first give some notations related to turning points. A turning point for the AKT_{c_2} equation is given by a zero of

$$12x_1^3 - 4c_2^2x_1^2 + 108\alpha c_2 x_1 + 243\alpha^2 - 32\alpha c_2^3 = 0. \tag{3.12}$$

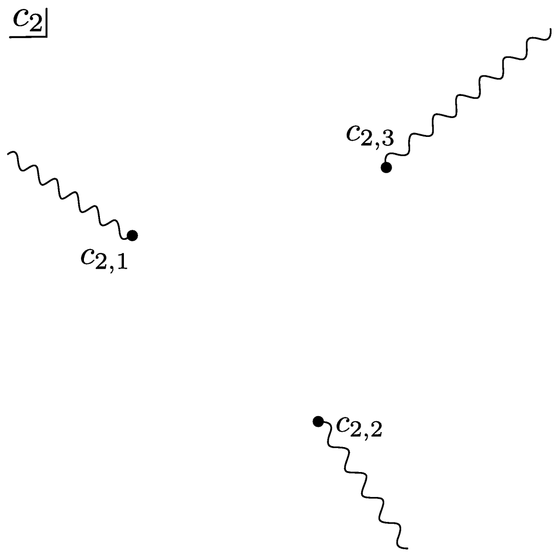

Figure 14. The cut plane H.

The discriminant of this algebraic equation is $-16\alpha\left(8c_2^3+243\alpha\right)^3$. In what follows we denote the zeros of this discriminant by $c_{2,j}$ $(j=1,2,3)$, that is,

$$c_{2,j}=-\frac{3^{5/3}}{2}\alpha^{1/3}e^{2\pi\sqrt{-1}(j-1)/3}\quad(j=1,2,3). \tag{3.13}$$

Let H be the c_2-plane equipped with cut lines emanating from $c_{2,j}$ $(j=1,2,3)$ as is shown in Figure 14. In H, turning points for the AKT_{c_2} equation are single-valued functions, and denoted by $a_j(c_2)$ $(j=1,2,3)$. By the definition of $c_{2,j}$, we assume

$$a_j(c_{2,j})=a_{j+1}(c_{2,j})\quad(j=1,2,3). \tag{3.14}$$

Here and in what follows, we consider all indices as modulo 3. Since $a_j(0)$ $(j=1,2,3)$ are turning points for the AKT_{c_2} equation, we may assume the following conditions

$$\xi_i\Big|_{x=(a_i(0),0)}=\xi_{i+1}\Big|_{x=(a_i(0),0)}\quad(i=1,2,3), \tag{3.15}$$

where ξ_i $(i=1,2,3)$ are the roots of $\widetilde{p}(x,\xi)=0$.

In the discussion of §2 Proposition 2.7 plays an important role. As a counterpart of Proposition 2.7 for the (1,4) hypergeometric system, we prove

Proposition 3.5. *For $i=1,2,3$ we have*

$$\int_{a_i(c_2)}^{a_{i+1}(c_2)}\left(\xi_i(x_1,c_2)-\xi_{i+1}(x_1,c_2)\right)dx_1=\int_{c_{2,i}}^{c_2}t_i(c_2)dc_2, \tag{3.16}$$

$$\int_{a_{i+1}(c_2)}^{a_i(c_2)}\left(\xi_{i+1}(x_1,c_2)-\xi_{i+2}(x_1,c_2)\right)dx_1=-\int_{c_{2,i}}^{c_2}t_{i-1}(c_2)dc_2, \tag{3.17}$$

where t_i $(i = 1, 2, 3)$ *are single-valued functions on* H*, and satisfies the following conditions*

- t_i *is the root of the following algebraic equation*

$$t^3 - c_2^2 t^2 + \frac{1}{27} c_2 (8c_2^3 + 243\alpha) t - \frac{1}{2916} (8c_2^3 + 243\alpha)^2 = 0, \tag{3.18}$$

- $t_{i-1}(c_{2,i}) = t_i(c_{2,i}) \quad (i = 1, 2, 3)$.

Proof. Since the proof is similar for both relations, we only prove (3.16). For simplicity, $\xi_i(a_j(c_2), c_2)$ are denoted by $\xi_{i,j}(c_2)$. Then $\xi_{i,j}$ $(i = 1, 2, 3)$ satisfy

$$\widetilde{p}\,(a_j(c_2), c_2, \xi) = 0. \tag{3.19}$$

On the other hand, by the definition of a_j, the discriminant in ξ of (3.19) is equal to zero, that is, (3.19) has a multiple root, which is given by

$$\xi_{j,j} = \xi_{j+1,j} \quad (j = 1, 2, 3) \tag{3.20}$$

in view of (3.15). Thus we have the following relations for $j = 1, 2, 3$

$$2\xi_{j,j} + \xi_{j+2,j} = -\frac{2}{3} c_2, \quad \xi_{j,j}^2 \xi_{j+2,j} = \frac{\alpha}{3}. \tag{3.21}$$

By using these relations, we find that $\xi_{j,j}$ $(j = 1, 2, 3)$ satisfy the following algebraic equation

$$6\xi^3 + 2c_2\xi^2 + \alpha = 0. \tag{3.22}$$

Since the discriminant of this algebraic equation is $-4\alpha\left(8c_2^3 + 243\alpha\right)$, $\xi_{i,j}$ $(i, j = 1, 2, 3)$ are single-valued functions on H. By (3.14) and (3.20), we have

$$\xi_{i,i}(c_{2,i}) = \xi_{i+1,i}(c_{2,i}) = \xi_{i+1}\Big|_{x=(a_i(c_{2,i}), c_{2,i})} = \xi_{i+1}\Big|_{x=(a_{i+1}(c_{2,i}), c_{2,i})} = \xi_{i+1,i+1}(c_{2,i}). \tag{3.23}$$

Thanks to Proposition 3.1, $\xi_i = \xi_i(x_1, c_2)$ satisfies

$$\frac{\partial}{\partial c_2}\xi_i = \frac{\partial}{\partial x_1}\xi_i^2. \tag{3.24}$$

It follows from this relation and (3.20) that

$$\frac{d}{dc_2}\int_{a_i(c_2)}^{a_{i+1}(c_2)} (\xi_i - \xi_{i+1})\, dx_1 \tag{3.25}$$

$$= \int_{a_i(c_2)}^{a_{i+1}(c_2)} \frac{\partial}{\partial c_2}(\xi_i - \xi_{i+1})\,dx_1 + (\xi_i - \xi_{i+1})\Big|_{x_1=a_{i+1}(c_2)} \frac{da_{i+1}}{dc_2} - (\xi_i - \xi_{i+1})\Big|_{x_1=a_i(c_2)} \frac{da_i}{dc_2}$$
$$= \int_{a_i(c_2)}^{a_{i+1}(c_2)} \frac{\partial}{\partial x_1}\left(\xi_i^2 - \xi_{i+1}^2\right)dx_1 + (\xi_i - \xi_{i+1})\Big|_{x_1=a_{i+1}(c_2)} \frac{da_{i+1}}{dc_2}$$
$$= \left(\xi_i^2 - \xi_{i+1}^2\right)\Big|_{x_1=a_{i+1}(c_2)} - \left(\xi_i^2 - \xi_{i+1}^2\right)\Big|_{x_1=a_i(c_2)} + (\xi_i - \xi_{i+1})\Big|_{x_1=a_{i+1}(c_2)} \frac{da_{i+1}}{dc_2}$$
$$= \left(\xi_{i,i+1}^2 + \frac{da_{i+1}}{dc_2}\xi_{i,i+1}\right) - \left(\xi_{i+1,i+1}^2 + \frac{da_{i+1}}{dc_2}\xi_{i+1,i+1}\right).$$

Since $\xi_{i,i}$ $(i = 1, 2, 3)$ satisfy

$$\begin{cases} \widetilde{p}\,(a_i(c_2), c_2, \xi_{i,i}(c_2)) = 0, \\ \dfrac{\partial \widetilde{p}}{\partial \xi}\,(a_i(c_2), c_2, \xi_{i,i}(c_2)) = 0, \end{cases} \tag{3.26}$$

we have

(3.27)
$$0 = \frac{d}{dc_2}\widetilde{p}\,(a_i(c_2), c_2, \xi_{i,i}(c_2))$$
$$= \frac{\partial \widetilde{p}}{\partial x_1}(a_i(c_2), c_2, \xi_{i,i}(c_2))\frac{da_i}{dc_2} + \frac{\partial \widetilde{p}}{\partial c_2}(a_i(c_2), c_2, \xi_{i,i}(c_2)) + \frac{\partial \widetilde{p}}{\partial \xi}(a_i(c_2), c_2, \xi_{i,i}(c_2))\frac{d\xi_{i,i}}{dc_2}$$
$$= \frac{1}{3}\xi_{i,i}\frac{da_i}{dc_2} + \frac{2}{3}\xi_{i,i}^2$$
$$= \frac{1}{3}\xi_{i,i}\left(\frac{da_i}{dc_2} + 2\xi_{i,i}\right),$$

that is,

$$\frac{da_i}{dc_2} = -2\xi_{i,i} \tag{3.28}$$

holds. Hence, by (3.21), we have

$$\begin{aligned} \frac{d}{dc_2}\int_{a_i(c_2)}^{a_{i+1}(c_2)} (\xi_i - \xi_{i+1})\,dx_1 &= (\xi_{i,i+1} - \xi_{i+1,i+1})\left(\xi_{i,i+1} + \xi_{i+1,i+1} + \frac{da_{i+1}}{dc_2}\right) \\ &= (\xi_{i,i+1} - \xi_{i+1,i+1})^2 \\ &= \left(3\xi_{i+1,i+1} + \frac{2}{3}c_2\right)^2. \end{aligned} \tag{3.29}$$

We now set

$$t_i = \left(3\xi_{i+1,i+1} + \frac{2}{3}c_2\right)^2 \quad (i = 1, 2, 3), \tag{3.30}$$

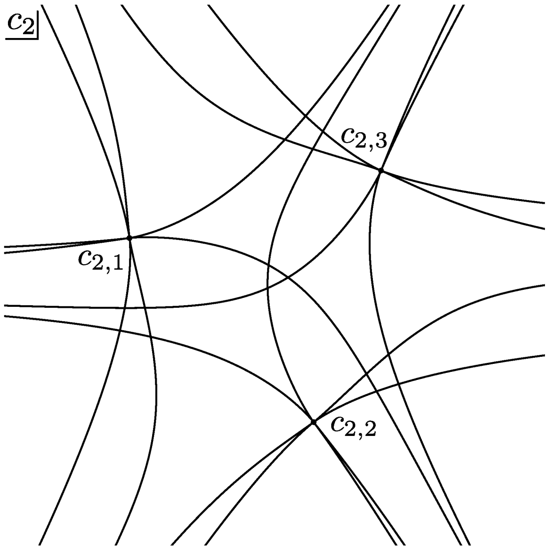

Figure 15. The curves defined by (3.31).

then t_i are single-valued functions on H and satisfy the two conditions in the claim of Proposition 3.5. In view of (3.14), we thus obtain (3.16). $\square$

Figure 15 shows the configuration of the curves defined by

(3.31)
$$\Im \int_{a_i(c_2)}^{a_{i+1}(c_2)} (\xi_i - \xi_{i+1})\, dx_1 = 0 \ \text{ and } \ \Im \int_{a_{i+1}(c_2)}^{a_i(c_2)} (\xi_{i+1} - \xi_{i+2})\, dx_1 = 0 \quad (i = 1, 2, 3)$$

in Proposition 3.5. Note that, if c_2 lies on a curve defined by (3.31), then the Stokes geometry for the AKT_{c_2} equation is degenerate in the sense that a Stokes curve of the AKT_{c_2} equation emanating from a turning point hits another turning point.

Remark. As is clear from Figure 15, the curves defined by (3.31) may cross. These crossing points are dissolved if we draw the figure of the curves defined by (3.31) on the Riemann surface determined by (3.18).

By using Figure 15, we now confirm that the new Stokes curves for the AKT_{c_2} equation relevant to Stokes phenomena in Figure 13 are contained in the Stokes surface for the (1,4) hypergeometric system as follows: Among the three new Stokes curves in Figure 13 relevant to Stokes phenomena we pick up a particular one, which is specified in Figure 16. We first show that this new Stokes curve specified in Figure 16 is contained in the Stokes surface for the (1,4) hypergeometric system. Recall that Figure 13 and Figure 16 are figures of (a part of) the Stokes geometry of the AKT_{c_2} equation for $c_2 = \sqrt{-1}/20$, which is denoted by λ_1 in what follows. Let γ_1 be a path in the c_2-plane emanating

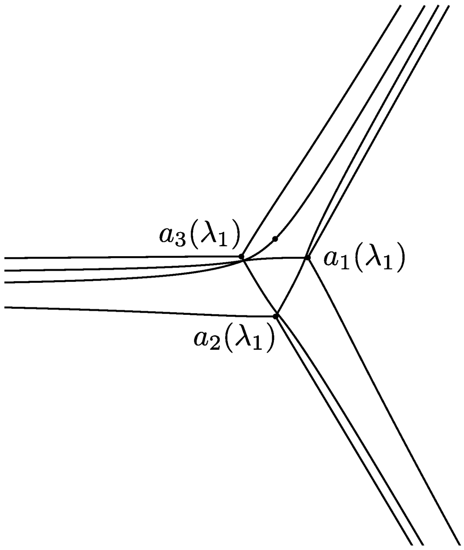

Figure 16. The ordinary Stokes curves and one particular new Stokes curve of the AKT_{λ_1} equation; here and in what follows the dot on a new Stokes curve designates the virtual turning point from which it emanates.

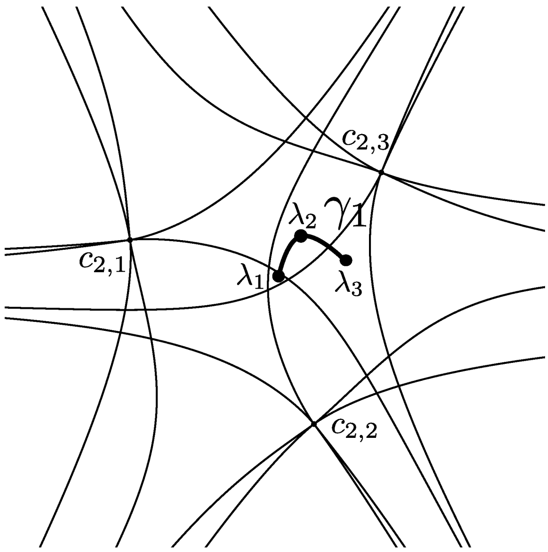

Figure 17. The path γ_1 and points λ_j ($j = 1, 2, 3$) on it.

from λ_1 and let us take points λ_2 and λ_3 on γ_1, as is shown in Figure 17. To observe how the configuration of the Stokes geometry for the AKT_{c_2} equation is deformed when c_2 varies along γ_1, we give figures of the Stokes geometry (i.e., ordinary Stokes curves and a new Stokes curve in question) for the AKT_{c_2} equation with $c_2 = \lambda_1, \lambda_2$ and λ_3 in Figure 18, 19 and 20, respectively. We examine, in particular, the difference between Figure 19 and Figure 20.

Let $l_{1,2}, l_{2,2}$ and nl_2 be Stokes curves in Figure 19 defined as follows:

- $l_{1,2}$: an ordinary Stokes curve emanating from a turning point $a_1(\lambda_2)$,
- $l_{2,2}$: an ordinary Stokes curve emanating from a turning point $a_2(\lambda_2)$,
- nl_2 : a new Stokes curve passing through a crossing point of two ordinary Stokes curves emanating from two turning points $a_1(\lambda_2)$ and $a_3(\lambda_2)$.

Let $l_{1,3}, l_{2,3}$ and nl_3 be Stokes curves in Figure 20 defined as follows:

- $l_{1,3}$: an ordinary Stokes curve emanating from a turning point $a_1(\lambda_3)$,
- $l_{2,3}$: an ordinary Stokes curve emanating from a turning point $a_2(\lambda_3)$,

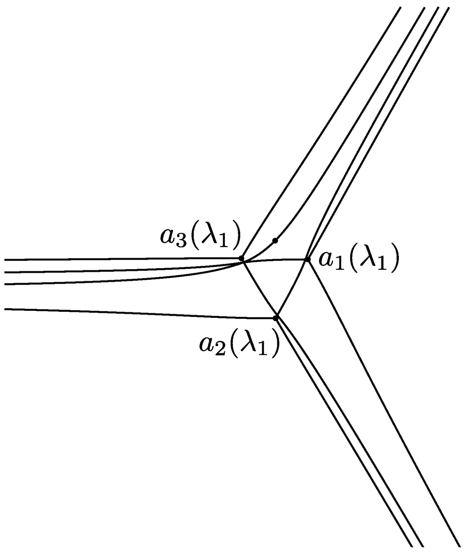

Figure 18. The Stokes geometry in question of the AKT_{λ_1} equation.

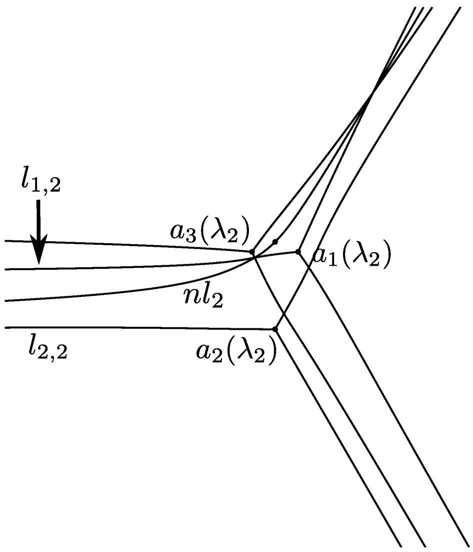

Figure 19. The Stokes geometry in question of the AKT_{λ_2} equation.

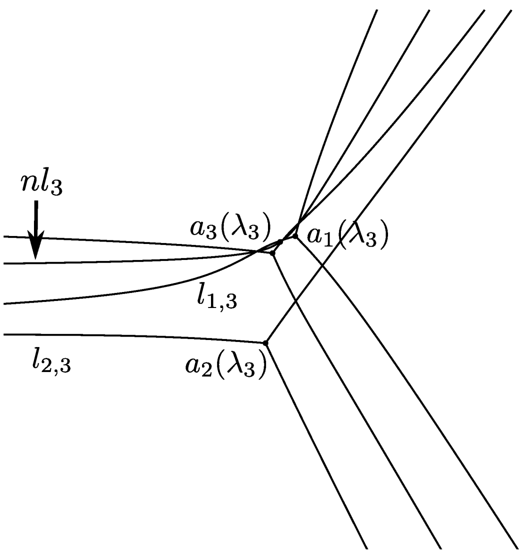

Figure 20. The Stokes geometry in question of the AKT_{λ_3} equation.

- nl_3 : a new Stokes curve passing through a crossing point of two ordinary Stokes curves emanating from two turning points $a_1(\lambda_3)$ and $a_3(\lambda_3)$.

We can observe that in Figure 19 nl_2 is asymptotically approaching to $l_{2,2}$, while $l_{1,2}$ is not approaching to $l_{2,2}$ when they go to the left toward ∞. On the other hand, in Figure 20 nl_3 is not approaching to $l_{2,3}$, while $l_{1,3}$ is asymptotically approaching to $l_{2,3}$ when they go to the left toward ∞. Then, by the same argument as in Proposition 2.6, we can verify that the new Stokes curve nl_2 in Figure 19 is contained in the Stokes surface for the (1,4) hypergeometric system. This immediately implies that the new Stokes curve in Figure 16 is contained in the Stokes surface for the (1,4) hypergeometric system.

In a similar manner, we can verify that the other two new Stokes curves in Figure 13 are also contained in the Stokes surface for the (1,4) hypergeometric system. In fact, by taking a path in Figure 21 (resp. Figure 23), we can confirm that the new Stokes curve specified in Figure 22 (resp. Figure 24) is contained in the Stokes surface for the (1,4) hypergeometric system.

Thus we have confirmed that the new Stokes curves in Figure 13 relevant to Stokes phenomena are all contained in the Stokes surface for the (1,4) hypergeometric system, that is, the answer to the question (A) in §3.1 is 'Yes'.

§ 3.4. The structure of the Stokes geometry for the (1,4) hypergeometric system, II

In the preceding subsection we showed that the new Stokes curves for the AKT_{c_2} equation relevant to Stokes phenomena are contained in the Stokes surface for the (1,4) hypergeometric system. In that argument the curves defined by (3.31) and their configuration (Figure 15) played a crucially important role. In the case of the Pearcey system, the degeneracy of the Stokes geometry for the BNR_{c_2} equation only occurs on the curve defined by (2.38). However, in the case of the (1,4) hypergeometric system, this is not true. The degeneracy of the Stokes geometry for the AKT_{c_2} equation occurs not only on the curve defined by (3.31) but also on other curves. In this subsection, we discuss this problem and, as its consequence, we show that a new Stokes curve irrelevant to Stokes phenomena is also contained in the Stokes surface for the (1,4) hypergeometric system.

Let γ_2 be a path in Figure 25 and let λ_j $(j = 1, \cdots, 7)$ be points on γ_2. (The first three points λ_j $(j = 1, 2, 3)$ are the same points as in the previous subsection.) In what follows we consider ordinary Stokes curves and new Stokes curves shown in Figure 26. Note that in Figure 26 there is a new Stokes curve irrelevant to Stokes phenomena, which in newly added to Figure 13. Figure 27, Figure28, Figure 29, Figure 30, Figure 31, Figure 32 and Figure 33 show the Stokes geometry for the AKT_{c_2} equation with

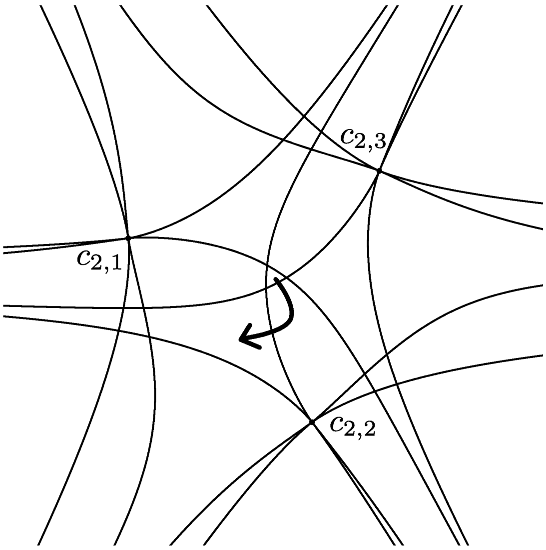

Figure 21.

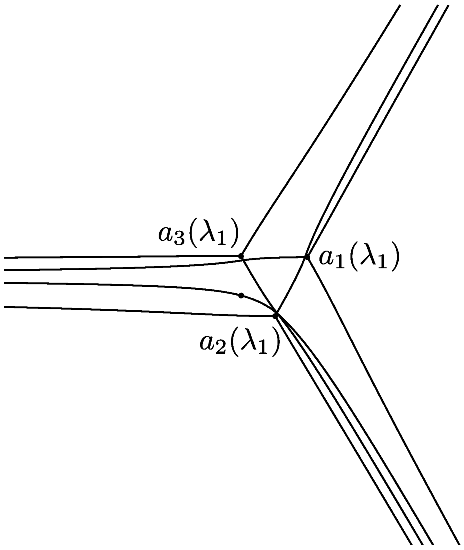

Figure 22. The ordinary Stokes curves and the second new Stokes curve of the AKT_{λ_1} equation shown in Figure 13.

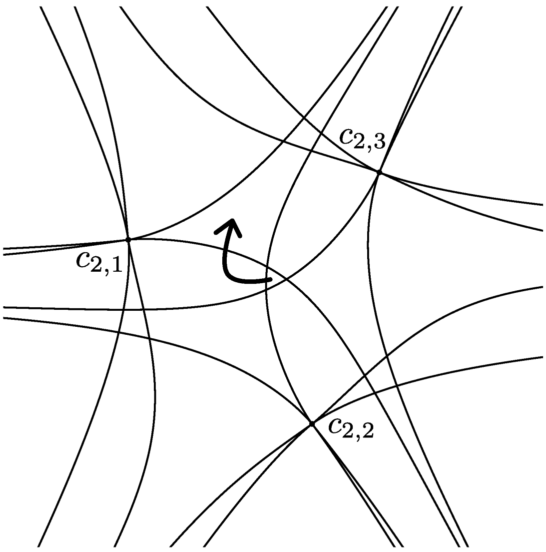

Figure 23.

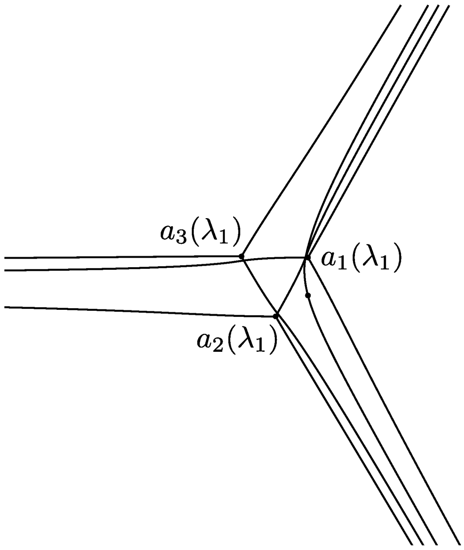

Figure 24. The ordinary Stokes curves and the third new Stokes curve of the AKT_{λ_1} equation shown in Figure 13.

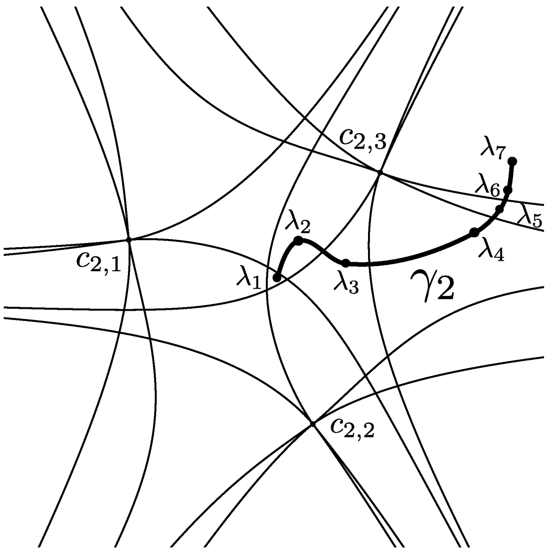

Figure 25. The path γ_2 and points λ_j ($j = 1, \cdots, 7$) on it.

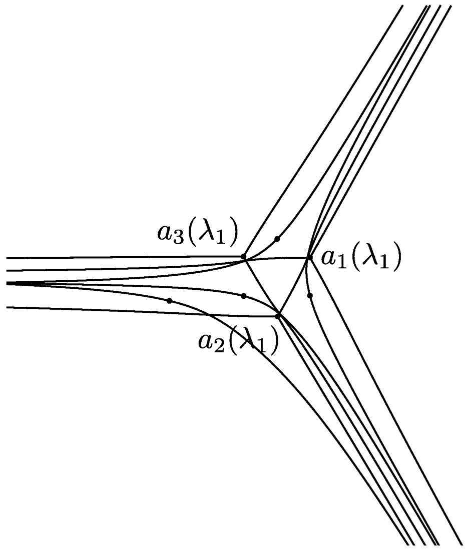

Figure 26. A new Stokes curve irrelevant to Stokes phenomena being added to Figure 13.

$c_2 = \lambda_j$ ($j = 1, \cdots, 7$), respectively. We examine, in particular, the difference between Figure 32 and Figure 33.

Let $l_6, nl_{1,6}, nl_{2,6}$ and $nl_{3,6}$ be Stokes curves in Figure 32 defined as follows:

- l_6 : an ordinary Stokes curve emanating from a turning point $a_3(\lambda_3)$,
- $nl_{1,6}$: a new Stokes curve passing through a crossing point of two ordinary Stokes curves emanating from two turning points $a_1(\lambda_6)$ and $a_3(\lambda_6)$,
- $nl_{2,6}$: a new Stokes curve passing through a crossing point of two ordinary Stokes curves emanating from two turning points $a_2(\lambda_6)$ and $a_3(\lambda_6)$,
- $nl_{3,6}$: a new Stokes curve passing through a crossing point of an ordinary Stokes curve emanating from $a_1(\lambda_6)$ and the new Stokes curve $nl_{2,6}$.

Let $l_7, nl_{1,7}, nl_{2,7}$ and $nl_{3,7}$ be Stokes curves in Figure 33 defined as follows:

- l_7 : an ordinary Stokes curve emanating from a turning point $a_3(\lambda_7)$,
- $nl_{1,7}$: a new Stokes curve passing through a crossing point of two ordinary Stokes curves emanating from two turning points $a_1(\lambda_7)$ and $a_3(\lambda_7)$,
- $nl_{2,7}$: a new Stokes curve passing through a crossing point of two ordinary Stokes curves emanating from two turning points $a_2(\lambda_7)$ and $a_3(\lambda_7)$,

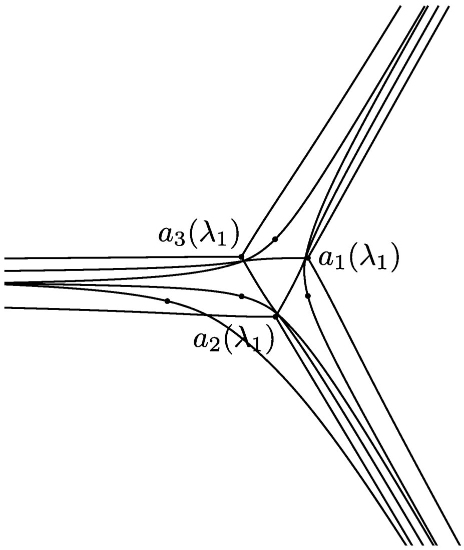

Figure 27. The Stokes geometry in question of the AKT_{λ_1} equation.

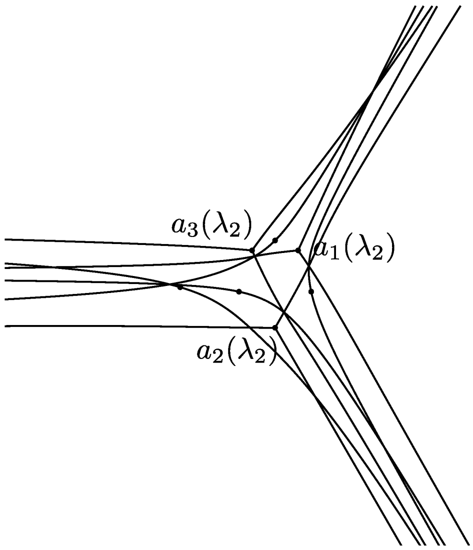

Figure 28. The Stokes geometry in question of the AKT_{λ_2} equation.

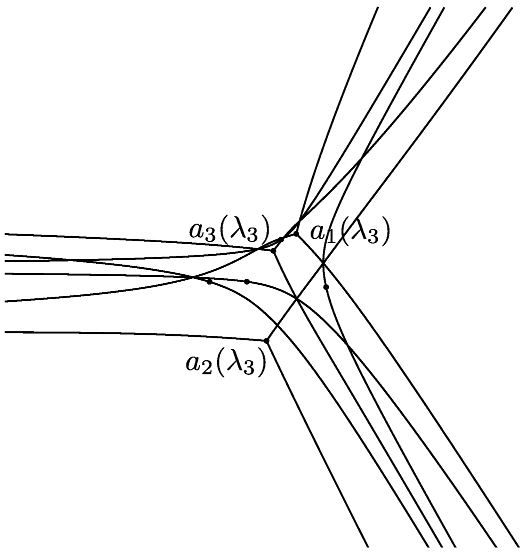

Figure 29. The Stokes geometry in question of the AKT_{λ_3} equation.

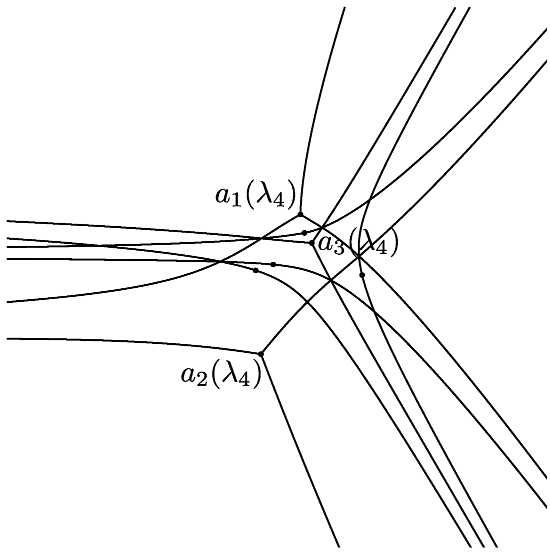

Figure 30. The Stokes geometry in question of the AKT_{λ_4} equation.

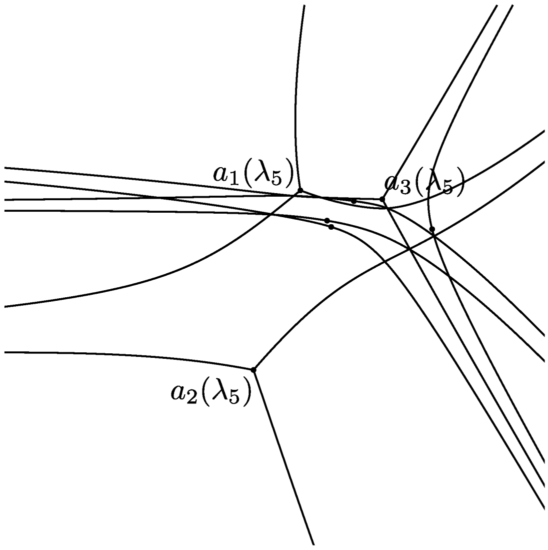

Figure 31. The Stokes geometry in question of the AKT_{λ_5} equation.

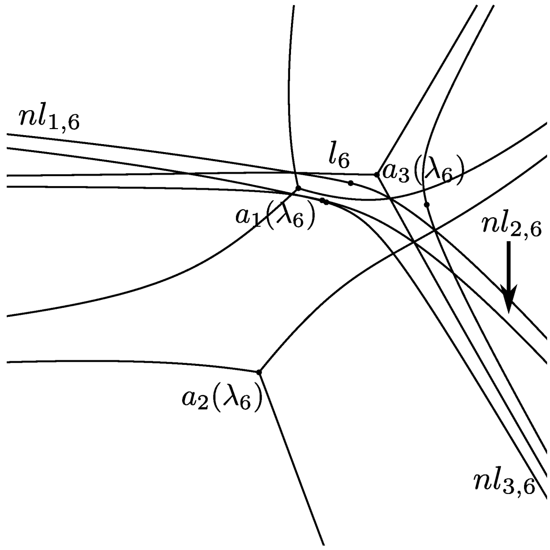

Figure 32. The Stokes geometry in question of the AKT_{λ_6} equation.

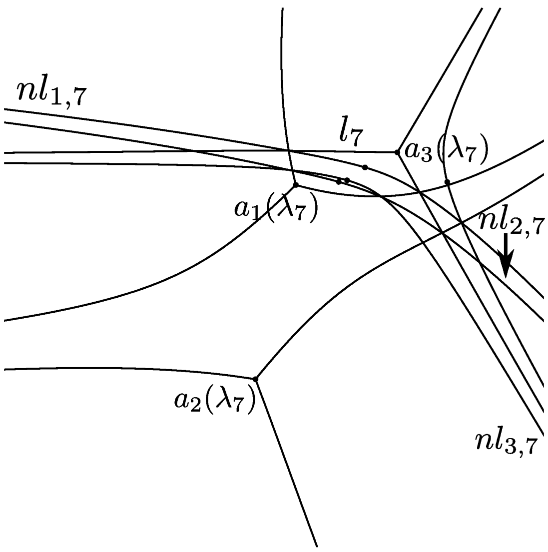

Figure 33. The Stokes geometry in question of the AKT_{λ_7} equation.

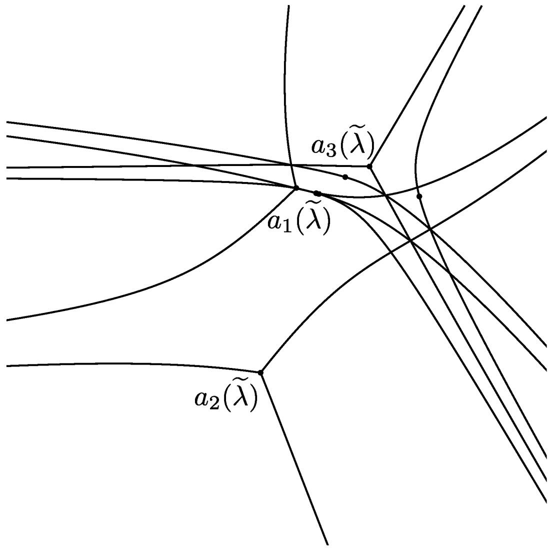

Figure 34. New type of degeneracy for the Stokes geometry of the AKT_{c_2} equation. (An enlarged figure is given in Figure 44.)

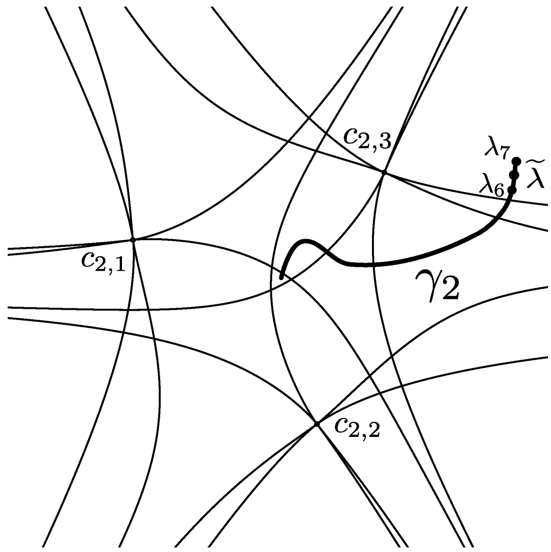

Figure 35. The point $\widetilde{\lambda}$ on γ_2 where the degenerate configuration Figure 34 is observed.

- $nl_{3,7}$: a new Stokes curve passing through a crossing point of an ordinary Stokes curve emanating from $a_1(\lambda_7)$ and the new Stokes curve $nl_{2,7}$.

We then observe that in Figure 32 $nl_{2,6}$ is asymptotically approaching to l_6 and $nl_{3,6}$ is asymptotically approaching to $nl_{1,6}$ when they go to the upper-left toward ∞. On the other hand, in Figure 33 $nl_{2,7}$ is asymptotically approaching to $nl_{1,7}$ and $nl_{3,7}$ is asymptotically approaching to l_7 when they go to the upper-left toward ∞. Comparing these two figures, we find that at some point between λ_6 and λ_7 on γ_2 we should encounter a new type of degeneracy of the Stokes geometry shown in Figure 34, that is, there exist two new Stokes curves emanating from two different virtual turning points hit an (ordinary) turning point. (With the aid of a computer we observe in Figure 34 the new type of degeneracy at $\widetilde{\lambda}$ on the path γ_2 in Figure 35.) Thus the degeneracy of the Stokes geometry for the AKT_{c_2} equation occurs also outside the curve defined by (3.31) (as a matter of fact, occurs at $\widetilde{\lambda}$ in the current situation).

In the preceding section we observed that at, say, $c_2 = 1$ an ordinary Stokes curve and a new Stokes curve of the BNR_{c_2} equation are switched and, using this fact, we proved that a new Stokes curve of the BNR_{c_2} equation is contained in the Stokes surface for the Pearcey system (cf. Proposition 2.6). In the current situation the new Stokes curve $nl_{2,*}(* = 6, 7)$ of the AKT_{c_2} equation plays a role similar to that the ordinary Stokes curve of the BNR_{c_2} equation played in the preceding section: That is, at $c_2 = \widetilde{\lambda}$ the turning point $a_1(\widetilde{\lambda})$ of the AKT_{c_2} equation switches the new Stokes curves $nl_{2,*}$ and

$nl_{3,*}(* = 6, 7)$ and, as was shown in the previous subsection, $nl_{2,6}$ is contained in the Stokes surface for the (1,4) hypergeometric system. Hence, by the reasoning similar to that used in the proof of Proposition 2.6, we expect that the new Stokes curve $nl_{3,7}$ is also contained in the Stokes surface for the (1,4) hypergeometric system. In fact, we can prove

Proposition 3.6. *The new Stokes curve $nl_{3,7}$ in Figure 33 is contained in the Stokes surface for the (1,4) hypergeometric system.*

Proof. Taking a point p on the new Stokes curve $nl_{3,7}$, we show that (p, λ_7) is contained in the Stokes surface for the (1,4) hypergeometric system. That is, we show that there exist a turning point $a \in \mathbb{C}^2$ for the (1,4) hypergeometric system, two branches of the one form $\omega^i_{-1}, \omega^{i'}_{-1}$ and a path γ connecting a and (p, λ_7) that satisfy the following conditions:

$$\omega^i_{-1}(a) = \omega^{i'}_{-1}(a), \quad \Im \int_a^q \left(\omega^i_{-1} - \omega^{i'}_{-1}\right) = 0 \quad \text{(for any } q \in \gamma). \tag{3.32}$$

Let $a_1(\lambda_6), a_2(\lambda_6)$ and $a_3(\lambda_6)$ be turning points for the AKT_{λ_6} equation, let b_2 be the crossing point of the new Stokes curve $nl_{2,6}$ and an ordinary Stokes curve emanating from $a_1(\lambda_6)$, and let b_4 be the crossing point of two ordinary Stokes curves of the AKT_{λ_6} equation emanating from $a_2(\lambda_6)$ and $a_3(\lambda_6)$ in Figure 36. Similarly, let $a_1(\lambda_7), a_2(\lambda_7)$ and $a_3(\lambda_7)$ be turning points for the AKT_{λ_7} equation, and let b_1 be the crossing point of an ordinary Stokes curve emanating from $a_1(\lambda_7)$ and the new Stokes curve $nl_{2,7}$, and let b_3 be the crossing point of two ordinary Stokes curves of the AKT_{λ_7} equation emanating from $a_2(\lambda_7)$ and $a_3(\lambda_7)$ in Figure 38. We further let $a_1(\widetilde{\lambda}), a_2(\widetilde{\lambda})$ and $a_3(\widetilde{\lambda})$ be three turning points for the $\mathrm{AKT}_{\widetilde{\lambda}}$ equation in Figure 37. As a path connecting λ_6 and λ_7 we take the restriction $\gamma_2\big|_{[\lambda_6,\lambda_7]}$ of the path γ_2. Let $l_{4\to3,1}(c_2)$ $(c_2 \in \gamma_2\big|_{[\lambda_6,\lambda_7]})$ be a crossing point of two ordinary Stokes curves for the AKT_{c_2} equation emanating from the two turning points $a_2(c_2)$ and $a_3(c_2)$, and let us define $l_{2\to1,1}(c_2)$ $(c_2 \in \gamma_2\big|_{[\lambda_6,\lambda_7]})$ by the following

- if $c_2 \neq \widetilde{\lambda}$, then $l_{2\to1,1}(c_2)$ is a crossing point of an ordinary Stokes curve for the AKT_{c_2} equation emanating from $a_1(c_2)$ and a new Stokes curve for the AKT_{c_2} equation passing through $l_{4\to3,1}(c_2)$, (that is, $l_{2\to1,1}(c_2)$ coincides with b_2 and b_1 when $c_2 = \lambda_6$ and $c_2 = \lambda_7$, respectively,)
- if $c_2 = \widetilde{\lambda}$, then $l_{2\to1,1}(\widetilde{\lambda})$ is $a_1(\widetilde{\lambda})$.

Then $l_{2\to1}(c_2) = (l_{2\to1,1}(c_2), c_2)$(resp. $l_{4\to3}(c_2) = (l_{4\to3,1}(c_2), c_2)$) $(c_2 \in \gamma_2\big|_{[\lambda_6,\lambda_7]})$ is a path in $\mathbb{C}^2$ connecting $l_{2\to1}(\lambda_6) = (b_2, \lambda_6)$ and $l_{2\to1}(\lambda_7) = (b_1, \lambda_7)$ (resp. $l_{4\to3}(\lambda_6) = (b_4, \lambda_6)$ and $l_{4\to3}(\lambda_7) = (b_3, \lambda_7)$). By using $l_{2\to1}$, we define a path $\widetilde{\gamma}_1$ connecting (b_2, λ_6) and (p, λ_7) as the composition of the following two paths:

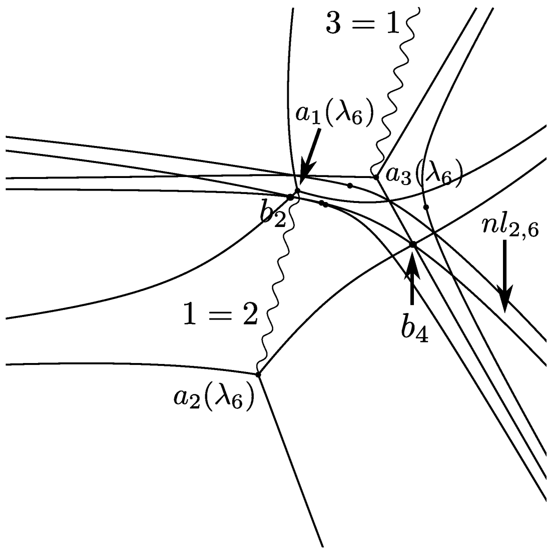

Figure 36. The Stokes geometry for the AKT_{λ_6} equation.

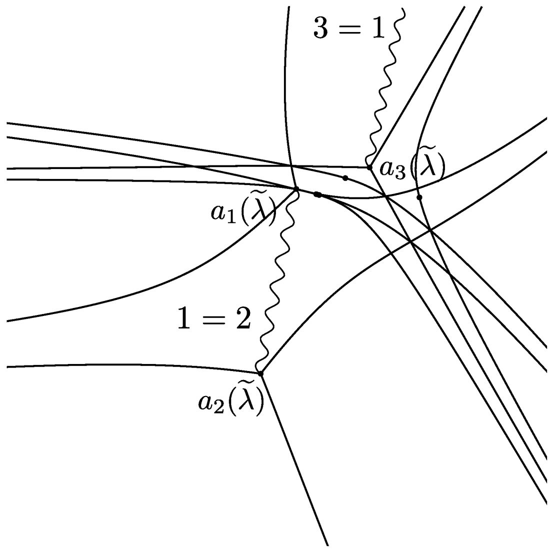

Figure 37. The Stokes geometry for the $\mathrm{AKT}_{\widetilde{\lambda}}$ equation.

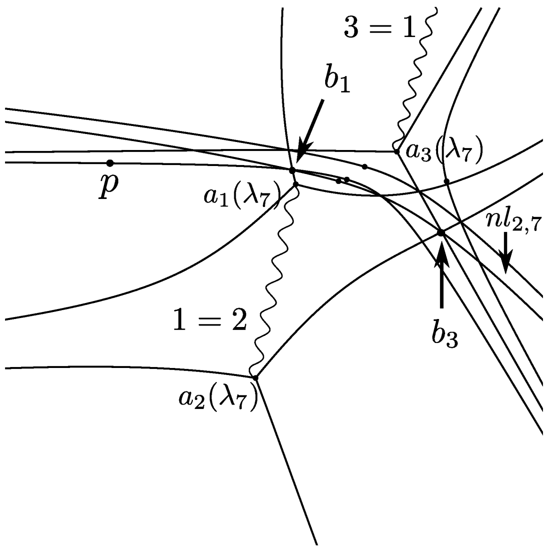

Figure 38. The Stokes geometry for the AKT_{λ_7} equation.

- a path from (b_2, λ_6) to (b_1, λ_7) along the path $l_{2\to 1}$,
- a path from (b_1, λ_7) to (p, λ_7) along the new Stokes curve $nl_{3,7}$ (this portion is contained completely in $\{c_2 = \lambda_7\}$).

We first prove

$$\Im \int_{(b_2,\lambda_6)}^{(p,\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) = 0, \tag{3.33}$$

where the integration is taken along $\widetilde{\gamma}_1$ and the branch ω_{-1}^i $(i = 1, 2, 3)$ of ω_{-1} is assumed to be fixed at the starting point (b_2, λ_6). Note that the wiggly lines in Figure 36, 37 and 38 designate cuts to define the branch ω_{-1}^i of ω_{-1}. To prove (3.33), we decompose the integral as

$$\begin{aligned} &\int_{(b_2,\lambda_6)}^{(p,\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) \\ &= \int_{(b_2,\lambda_6)}^{l_{2\to 1}(\widetilde{\lambda})} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{l_{2\to 1}(\widetilde{\lambda})}^{(b_1,\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{(b_1,\lambda_7)}^{(p,\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.34}$$

First, since p lies on a new Stokes curve (of type (1,3)) for the AKT_{λ_7} equation, we have

$$\Im I_3 = 0. \tag{3.35}$$

We next consider I_2, which is the sum of two integrals:

$$\begin{aligned} I_2 &= \int_{l_{2\to 1}(\widetilde{\lambda})}^{(b_1,\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^2\right) + \int_{l_{2\to 1}(\widetilde{\lambda})}^{(b_1,\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^3\right) \\ &=: I_{21} + I_{22}. \end{aligned} \tag{3.36}$$

By deforming the path of integration, we can express I_{21} as

$$I_{21} = \int_{l_{2\to 1}(\widetilde{\lambda})}^{(a_1(\lambda_7),\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^2\right) + \int_{(a_1(\lambda_7),\lambda_7)}^{(b_1,\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^2\right), \tag{3.37}$$

where the first integral is taken along the set of turning points for the (1,4) hypergeometric system and the second integral is done along an ordinary Stokes curve for the AKT_{λ_7} equation emanating from $a_1(\lambda_7)$. Since $\omega_{-1}^1 = \omega_{-1}^2$ holds at turning points $(a_1(c_2), c_2)$ $(c_2 \in \gamma_2\big|_{[\lambda_6,\lambda_7]})$, the first integral vanishes. Furthermore, as the second integral is taken along an ordinary Stokes curve, its imaginary part is zero. Hence

$$I_{21} \in \mathbb{R}. \tag{3.38}$$

On the other hand, I_{22} can be expressed as

$$I_{22} = \int_{(a_1(\widetilde{\lambda}),\widetilde{\lambda})}^{l_{4\to 3}(\widetilde{\lambda})} \left(\omega_{-1}^2 - \omega_{-1}^3\right) + \int_{l_{4\to 3}(\widetilde{\lambda})}^{l_{4\to 3}(\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^3\right) + \int_{l_{4\to 3}(\lambda_7)}^{(b_1,\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^3\right), \tag{3.39}$$

where the first and the third integrals are done along new Stokes curves for the $\mathrm{AKT}_{\widetilde{\lambda}}$ and the AKT_{λ_7} equation, respectively, and the second integral is taken along $l_{4\to 3}$. Furthermore, we have

$$\begin{aligned}
&\int_{l_{4\to 3}(\widetilde{\lambda})}^{l_{4\to 3}(\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^3\right) \\
&= \int_{l_{4\to 3}(\widetilde{\lambda})}^{l_{4\to 3}(\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^1\right) + \int_{l_{4\to 3}(\widetilde{\lambda})}^{l_{4\to 3}(\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) \\
&= \int_{l_{4\to 3}(\widetilde{\lambda})}^{(a_2(\widetilde{\lambda}),\widetilde{\lambda})} \left(\omega_{-1}^2 - \omega_{-1}^1\right) + \int_{(a_2(\widetilde{\lambda}),\widetilde{\lambda})}^{(a_2(\lambda_7),\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^1\right) + \int_{(a_2(\lambda_7),\lambda_7)}^{l_{4\to 3}(\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^1\right) \\
&\qquad + \int_{l_{4\to 3}(\widetilde{\lambda})}^{(a_3(\widetilde{\lambda}),\widetilde{\lambda})} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{(a_3(\widetilde{\lambda}),\widetilde{\lambda})}^{(a_3(\lambda_7),\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{(a_3(\lambda_7),\lambda_7)}^{l_{4\to 3}(\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^3\right),
\end{aligned} \tag{3.40}$$

where the first and the fourth integrals are done along ordinary Stokes curves for the $\mathrm{AKT}_{\widetilde{\lambda}}$ equation, the third and the sixth integrals are done along ordinary Stokes curves for the AKT_{λ_7} equation, and the second and the fifth integrals are taken along the set of turning points for the (1,4) hypergeometric system, respectively. Hence, similarly to I_{21}, we have

$$\begin{aligned}
I_{22} &= \int_{(a_1(\widetilde{\lambda}),\widetilde{\lambda})}^{l_{4\to 3}(\widetilde{\lambda})} \left(\omega_{-1}^2 - \omega_{-1}^3\right) + \int_{l_{4\to 3}(\widetilde{\lambda})}^{(a_2(\widetilde{\lambda}),\widetilde{\lambda})} \left(\omega_{-1}^2 - \omega_{-1}^1\right) + \int_{(a_2(\lambda_7),\lambda_7)}^{l_{4\to 3}(\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^1\right) \\
&\quad + \int_{l_{4\to 3}(\widetilde{\lambda})}^{(a_3(\widetilde{\lambda}),\widetilde{\lambda})} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{(a_3(\lambda_7),\lambda_7)}^{l_{4\to 3}(\lambda_7)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{l_{4\to 3}(\lambda_7)}^{(b_1,\lambda_7)} \left(\omega_{-1}^2 - \omega_{-1}^3\right) \\
&\in \mathbb{R}.
\end{aligned} \tag{3.41}$$

Thus we obtain

$$\Im I_2 = 0. \tag{3.42}$$

Finally, the integral I_1 can be discussed in a similar manner: By expressing I_1 as

$$I_1 = \int_{(b_2,\lambda_6)}^{l_{4\to 3}(\lambda_6)} \left(\omega_{-1}^1 - \omega_{-1}^3\right) + \int_{l_{4\to 3}(\lambda_6)}^{l_{4\to 3}(\widetilde{\lambda})} \left(\omega_{-1}^2 - \omega_{-1}^3\right) + \int_{l_{4\to 3}(\widetilde{\lambda})}^{l_{2\to 1}(\widetilde{\lambda})} \left(\omega_{-1}^2 - \omega_{-1}^3\right) \tag{3.43}$$

(here the branch ω_{-1}^{i} of ω_{-1} is fixed at the lower endpoint of each integral. Note that ω_{-1}^{1} is changed to ω_{-1}^{2} on the way from (b_2, λ_6) to $l_{4\to 3}(\lambda_6) = (b_4, \lambda_6)$ as this portion of the integration path crosses a cut "1 = 2"), we can confirm

$$
\begin{aligned}
(3.44) \qquad I_1 = {} & \int_{(b_2,\lambda_6)}^{l_{4\to 3}(\lambda_6)} \left(\omega_{-1}^{1} - \omega_{-1}^{3}\right) + \int_{l_{4\to 3}(\lambda_6)}^{(a_2(\lambda_6),\lambda_6)} \left(\omega_{-1}^{2} - \omega_{-1}^{1}\right) \\
& + \int_{(a_2(\lambda_6),\lambda_6)}^{(a_2(\widetilde{\lambda}),\widetilde{\lambda})} \left(\omega_{-1}^{2} - \omega_{-1}^{1}\right) + \int_{(a_2(\widetilde{\lambda}),\widetilde{\lambda})}^{l_{4\to 3}(\widetilde{\lambda})} \left(\omega_{-1}^{2} - \omega_{-1}^{1}\right) \\
& + \int_{l_{4\to 3}(\lambda_6)}^{(a_3(\lambda_6),\lambda_6)} \left(\omega_{-1}^{1} - \omega_{-1}^{3}\right) + \int_{(a_3(\lambda_6),\lambda_6)}^{(a_3(\widetilde{\lambda}),\widetilde{\lambda})} \left(\omega_{-1}^{1} - \omega_{-1}^{3}\right) \\
& + \int_{(a_3(\widetilde{\lambda}),\widetilde{\lambda})}^{l_{4\to 3}(\widetilde{\lambda})} \left(\omega_{-1}^{1} - \omega_{-1}^{3}\right) + \int_{l_{4\to 3}(\widetilde{\lambda})}^{l_{2\to 1}(\widetilde{\lambda})} \left(\omega_{-1}^{2} - \omega_{-1}^{3}\right) \\
\in {} & \mathbb{R}.
\end{aligned}
$$

Thus we have verified (3.33).

Using the above reasoning, we can also prove that

$$
(3.45) \qquad \Im \int_{(b_2,\lambda_6)}^{q} \left(\omega_{-1}^{1} - \omega_{-1}^{3}\right) = 0
$$

holds for any point q on $\widetilde{\gamma}_1$.

Since $nl_{2,6}$ is contained in the Stokes surface for the (1,4) hypergeometric system, there exist a turning point $a \in \mathbb{C}^2$ for the (1,4) hypergeometric system, two branches of the one form $\omega_{-1}^{i}, \omega_{-1}^{i'}$ and a path $\widetilde{\gamma}_2$ connecting a and (b_2, λ_6) that satisfy the following conditions:

$$
(3.46) \qquad \omega_{-1}^{i}(a) = \omega_{-1}^{i'}(a), \quad \Im \int_{a}^{q} \left(\omega_{-1}^{i} - \omega_{-1}^{i'}\right) = 0 \quad (\text{for any } q \in \widetilde{\gamma}_2).
$$

Note that, after the analytic continuation along $\widetilde{\gamma}_2$, ω_{-1}^{i} and $\omega_{-1}^{i'}$ are changed to ω_{-1}^{1} and ω_{-1}^{3}, respectively.

We define a path γ connecting a and (p, λ_7) as the composition of two paths $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. Then, by (3.33) and (3.46), we have

$$
(3.47) \qquad \Im \int_{a}^{q} \left(\omega_{-1}^{i} - \omega_{-1}^{i'}\right) = 0 \quad (q \in \gamma).
$$

□

Remark. In proving (3.42) (resp. (3.44)), we need to deform the path of integration for I_{22} (resp. I_1) so that it may pass the two turning points $a_2(\lambda_7)$ and $a_3(\lambda_7)$. Such a deformation was essentially used in the previous subsection to confirm that the new Stokes curve $nl_{2,7}$ is contained in the Stokes surface for the (1,4) hypergeometric system.

As is clear from the proof of Proposition 3.6, in order to prove that the new Stokes curve $nl_{3,7}$ is contained in the Stokes surface for the (1,4) hypergeometric system, we further need to take another turning point $a_1(\lambda_7)$ into account.

In this manner, to confirm that a new Stokes curve of the AKT_{c_2} equation is contained in the Stokes surface for the (1,4) hypergeometric system, we need to consider several ordinary turning points and some portions of (ordinary and/or new) Stokes curves connecting them. These ordinary turning points and portions of Stokes curves are nothing but those used in [7] (cf. [4] also) to define a virtual turning point corresponding to the new Stokes curve in question through the so-called "integral relation".

Thus we have confirmed that all of the new Stokes curves in Figure 26 are contained in the Stokes surface for the (1,4) hypergeometric system. This implies that at least one new Stokes curve of AKT_{c_2} equation which is irrelevant to Stoke phenomena is contained in the Stokes surface for the (1,4) hypergeometric system.

§ 3.5. The structure of the Stokes geometry for the (1,4) hypergeometric system, III

In the previous subsection it was shown that the degeneracy of the Stokes geometry for AKT_{c_2} equation occurs also outside the curve defined by (3.31). Then, where does the degeneracy of the Stokes geometry for the AKT_{c_2} equation occur? Is it possible to describe explicitly the set in the c_2-plane where such a degeneracy occurs? In this subsection we discuss this problem. Furthermore, we consider the relationship between this new type of degeneracy and virtual turning points of the AKT_{c_2} equation.

Let $\mathcal{P}$ be the set of "periods" for the (1,4) hypergeometric system, that is,

$$\mathcal{P} = \left\{ \int_\ell \omega_{-1} \in \mathbb{C} \;;\; \ell \text{ is a closed path in } \mathbb{C}^2 \right\}. \tag{3.48}$$

In the case of the (1,4) hypergeometric system it is known that $\mathcal{P}$ is generated by one element so that

$$\mathcal{P} = \left\{ 2\pi\sqrt{-1}\alpha k \;;\; k \in \mathbb{Z} \right\}, \tag{3.49}$$

where α is the parameter that the (1,4) hypergeometric system (3.1) contains and it is also given by

$$\alpha = \frac{1}{2\pi\sqrt{-1}} \int_{\ell_0} \omega_{-1} \tag{3.50}$$

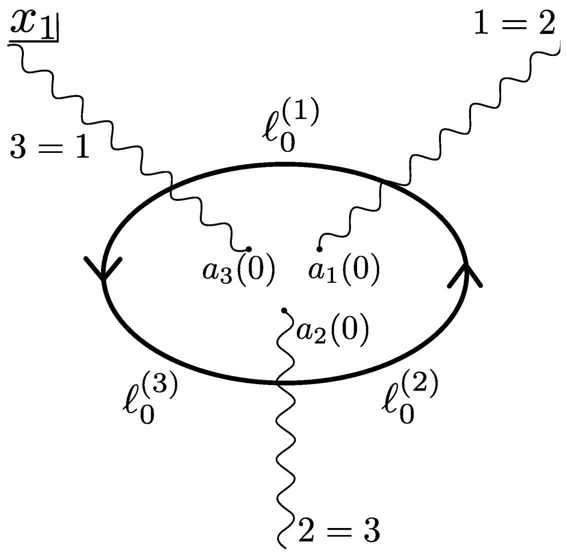

Figure 39. The closed path ℓ_0, which is contained in $\{c_2=0\}$. (Note that in (3.50) ω_{-1} is assumed to take a branch ω_{-1}^i on each portion $\ell_0^{(i)}$.)

with, for example, the closed path ℓ_0 shown in Figure 39. Using the period $\mathcal{P} = \left\{2\pi\sqrt{-1}\alpha k\right\}$, we consider the curves in the c_2-plane defined by

(3.51)
$$\Im\left(\int_{c_{2,i}}^{c_2} t_i(c_2)dc_2 - 2\pi\sqrt{-1}\alpha k\right) = 0 \text{ and } \Im\left(-\int_{c_{2,i}}^{c_2} t_{i-1}(c_2)dc_2 - 2\pi\sqrt{-1}\alpha k\right) = 0$$

for $i=1,2,3,\ k\in\mathbb{Z}$. Their configuration for $k=0,\pm1,\pm2$ is shown in Figure 40. By numerical computations we have checked that the degeneracy of the Stokes geometry for the AKT$_{c_2}$ equation is occurring at several points on these curves. For example, the point $\widetilde{\lambda}$ predicted in §3.4 lies on the following curve

$$\Im\left(\int_{c_{2,3}}^{c_2} t_3(c_2)dc_2 - 2\pi\sqrt{-1}\alpha\right) = 0, \tag{3.52}$$

as is shown in Figure 41.

Remark. In the case of the Pearcey system we have $\mathcal{P}=\{0\}$, that is, there exist no non-trivial periods. This fact can be considered as the main reason why the degeneracy for the Stokes geometry for the BNR$_{c_2}$ equation occurs only on the curve defined by (2.38).

It was observed in §3.4 that two new Stokes curves emanating from two virtual turning points hit an ordinary turning point at $\widetilde{\lambda}$ on the curve (3.52). In what follows

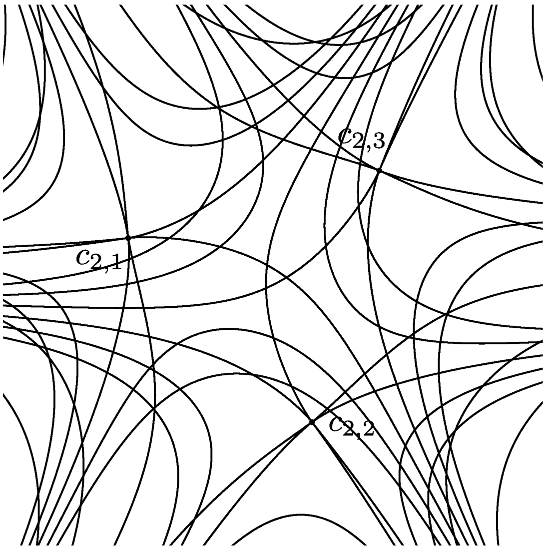

Figure 40. The curves defined by (3.51) for $k = 0, \pm 1, \pm 2$.

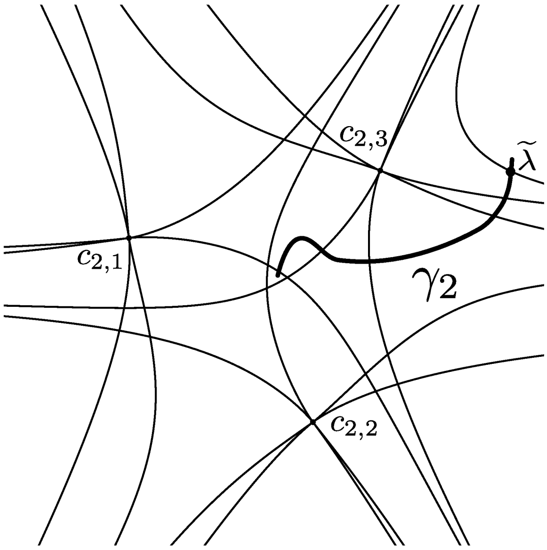

Figure 41. A curve (3.51) passing through $\widetilde{\lambda}$ being added to Figure 15.

we investigate the behavior of these two virtual turning points of the AKT_{c_2} equation along the curve (3.52) in more details. First, let us take a (unique) zero of

$$\int_{c_{2,3}}^{c_2} t_3(c_2)dc_2 - 2\pi\sqrt{-1}\alpha \tag{3.53}$$

on the curve (3.52). Then the numerical computation indicates that the two virtual turning points in question simultaneously hit the ordinary turning point at the zero of (3.53).

Next, let γ_3 be an (oriented) closed path with a base point $c_2 = \widetilde{\lambda}$ which goes around the zero of (3.53) in a counterclockwise manner and let λ_j $(j = 8, \cdots, 11)$ be points on this path γ_3 (cf. Figure 42). To observe how the configuration of the Stokes geometry for the AKT_{c_2} equation is changed when c_2 varies along γ_3, we see Figures 43, 45, 47, 49 and 51 that describe the Stokes geometry for the AKT_{c_2} equation with $c_2 = \widetilde{\lambda}, \lambda_8, \lambda_9, \lambda_{10}$ and λ_{11}, respectively, and their enlarged version around a turning point $x_1 = a_1(c_2)$, i.e., Figures 44, 46, 48, 50 and 52. We denote by v_1 and v_2 the two virtual turning points in question in Figure 43 and their analytic continuation along γ_3. Comparing these figures, particularly Figures 44 and 52, we find that the position of the two virtual turning points v_1 and v_2 is interchanged when c_2 varies along γ_3, that is, v_2 moves to the position of v_1 and v_1 moves to the position of v_2 after the analytic continuation along γ_3.

Then, with the aid of a computer, we observe that the two new Stokes curves emanating from v_1 and v_2 hit an ordinary turning point $a_1(c_2)$ on the curve (3.52) and,

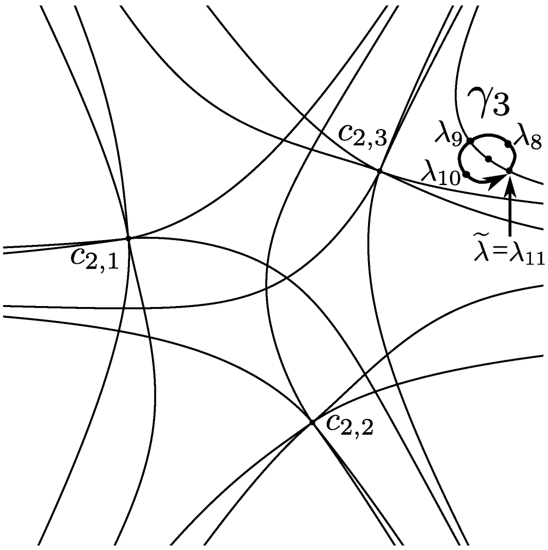

Figure 42. The path γ_3 and points λ_j $(j = 8, \cdots, 11)$ on it.

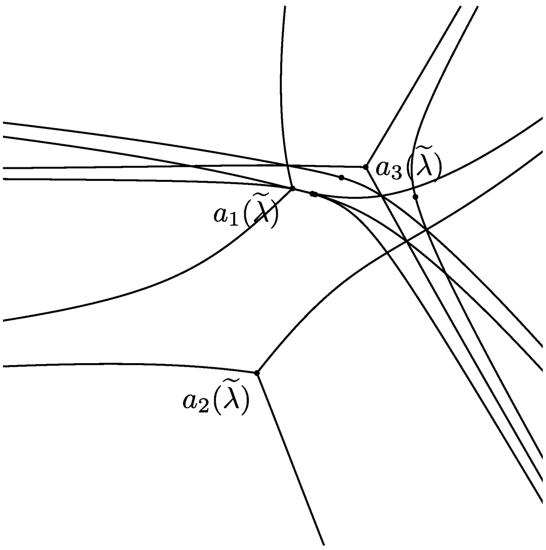

Figure 43. The Stokes geometry in question of the AKT$_{\widetilde{\lambda}}$ equation.

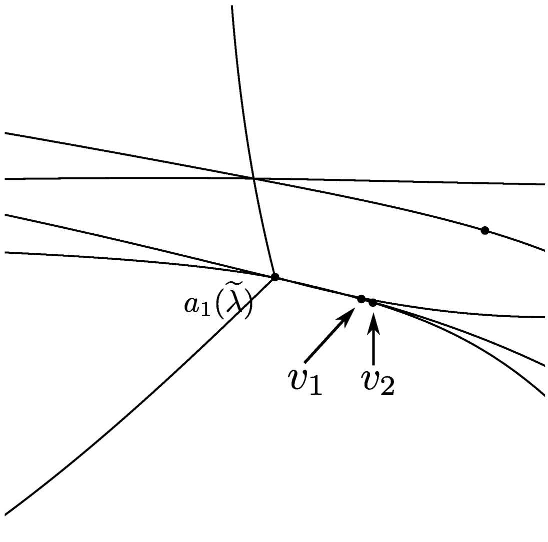

Figure 44. The enlarged version of Figure 43 around the turning point $x_1 = a_1(\widetilde{\lambda})$.

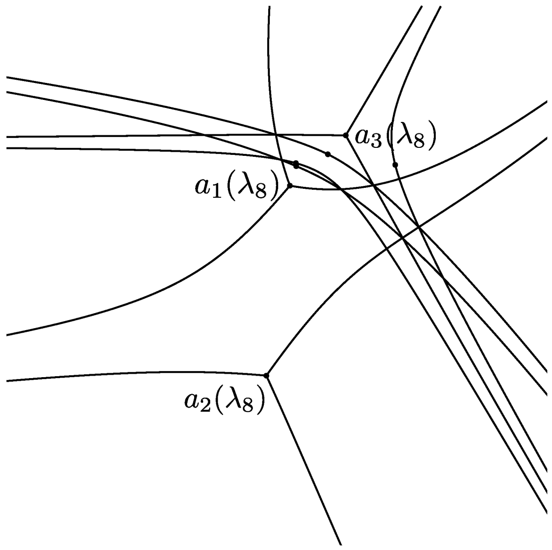

Figure 45. The Stokes geometry in question of the AKT_{λ_8} equation.

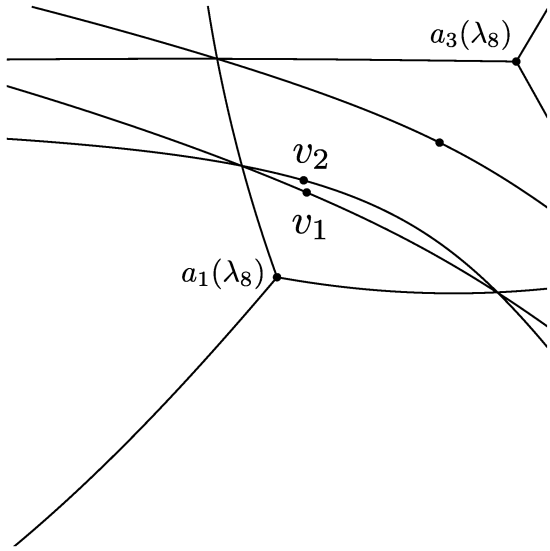

Figure 46. The enlarged version of Figure 45 around the turning point $x_1 = a_1(\lambda_8)$.

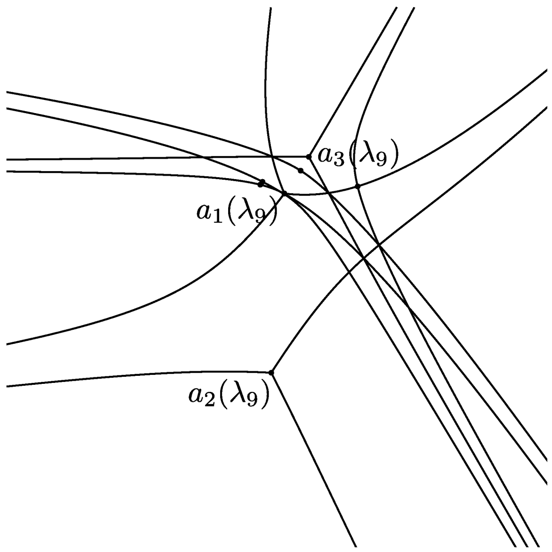

Figure 47. The Stokes geometry in question of the AKT_{λ_9} equation.

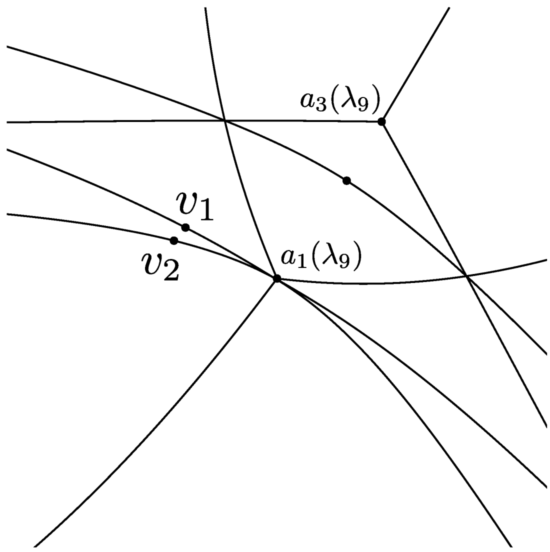

Figure 48. The enlarged version of Figure 47 around the turning point $x_1 = a_1(\lambda_9)$.

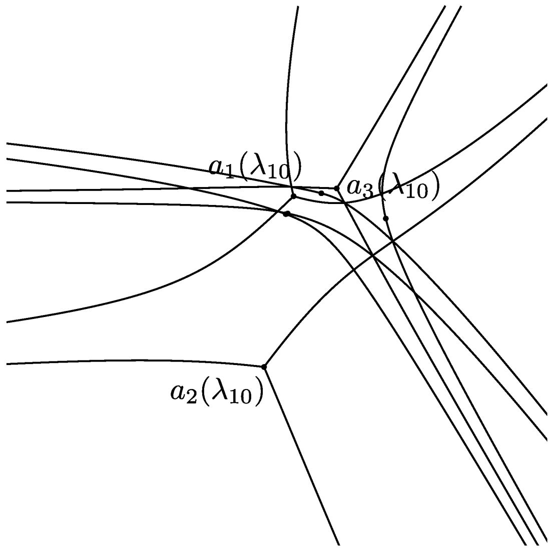

Figure 49. The Stokes geometry in question of the $\mathrm{AKT}_{\lambda_{10}}$ equation.

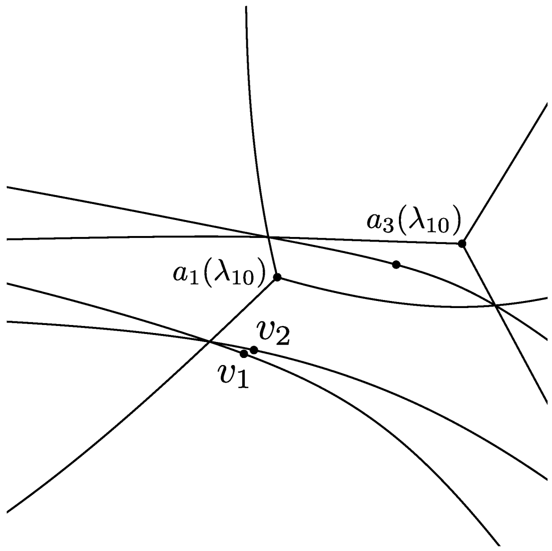

Figure 50. The enlarged version of Figure 49 around the turning point $x_1 = a_1(\lambda_{10})$.

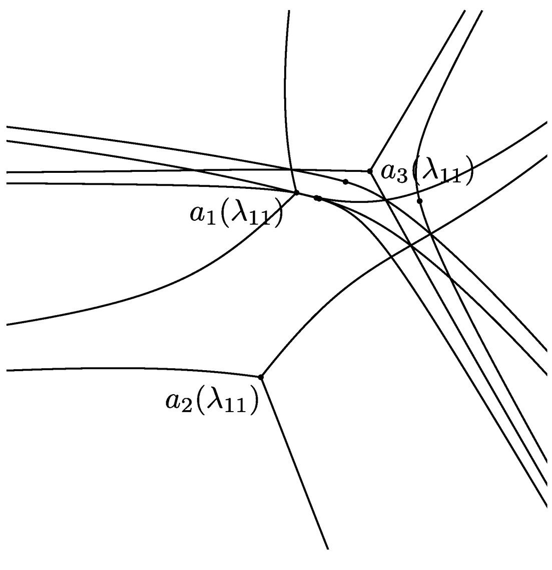

Figure 51. The Stokes geometry in question of the $\mathrm{AKT}_{\lambda_{11}}$ equation.

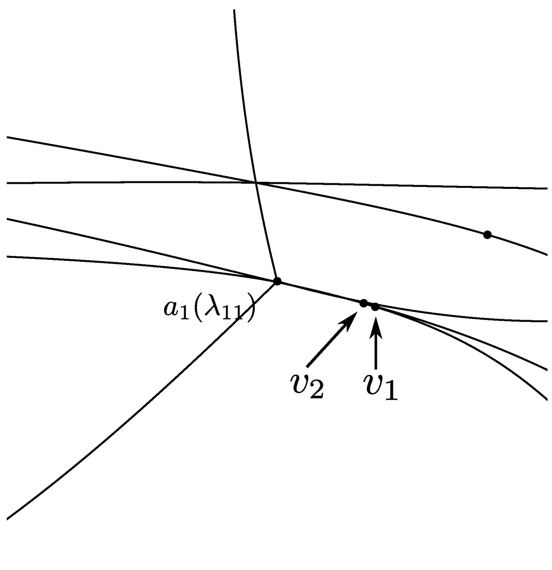

Figure 52. The enlarged version of Figure 51 around the turning point $x_1 = a_1(\lambda_{11})$.

in particular, the virtual turning points v_1 and v_2 themselves hit $a_1(c_2)$ at the zero of (3.53). Furthermore, the position of v_1 and v_2 is interchanged when c_2 varies around the zero of (3.53). Note that these geometric observations also have played an important role in proving Proposition 3.6, that is, proving that the virtual turning point v_2 and the new Stokes curve emanating from it are contained in the Stokes surface for the (1,4) hypergeometric system by using the fact that the other virtual turning point v_1 is contained in it.

§ 3.6. Virtual turning points of the AKT_{c_2} equation and the curve (3.51)

The results in the previous subsection indicate that the degeneracy of the Stokes geometry for the AKT_{c_2} equation in the sense that two virtual turning points hit an ordinary turning point occurs at the zero of (3.53) on the curve (3.52). In this subsection we continue to study similar degeneracies of the Stokes geometry for the AKT_{c_2} equation which are expected to occur on the curve (3.51).

Similarly to (3.53), we consider zeros of

(3.54)
$$\int_{c_{2,i}}^{c_2} t_i(c_2)dc_2 - 2\pi\sqrt{-1}\alpha k \ \text{ and } \ -\int_{c_{2,i}}^{c_2} t_{i-1}(c_2)dc_2 - 2\pi\sqrt{-1}\alpha k \quad (i = 1, 2, 3,\ k \in \mathbb{Z})$$

on the curve (3.51). Note that zeros of (3.54) with $k = 0$ coincide with $c_{2,i}$ $(i = 1, 2, 3)$. Note also that the zeros of (3.54) lie on the curves defined by

(3.55)
$$\Im\left(\overline{2\pi\sqrt{-1}\alpha}\int_{c_{2,i}}^{c_2} t_i(c_2)dc_2\right) = 0 \ \text{ and } \ \Im\left(\overline{2\pi\sqrt{-1}\alpha}\int_{c_{2,i}}^{c_2} t_{i-1}(c_2)dc_2\right) = 0 \quad (i = 1, 2, 3).$$

Figure 53 shows curves of (3.55) and the location of zeros of (3.54) with $k = 0, \pm 1, \pm 2$ in the c_2-plane. From each $c_{2,i}$ eight curves defined by (3.55) emanate and zeros of (3.54) are located on six of them (i.e., except for curves connecting $c_{2,i}$ each other). Having this fact in mind, we label the zeros of (3.54) for $k \neq 0$ as $p_i^{j,k}$ $(i = 1, 2, 3, j = 1, \cdots, 6, k \geq 1)$, as is shown in Figure 54.

On the other hand, it is claimed in [2] that the AKT_{c_2} equation has infinitely many virtual turning points and they are located on a family of certain real one-dimensional curves. In the current situation, as is shown in Figure 55, we see that the AKT_{c_2} equation with $c_2 = \lambda_1$ also has infinitely many virtual turning points and they lie on a family of curves defined by

$$\Im\left(\overline{2\pi\sqrt{-1}\alpha}\int_{a_i(c_2)}^{x_1} (\xi_i - \xi_{i+1})\, dx_1\right) = 0 \quad (i = 1, 2, 3). \tag{3.56}$$

As a matter of fact, the virtual turning points appearing in Figure 55 are obtained by

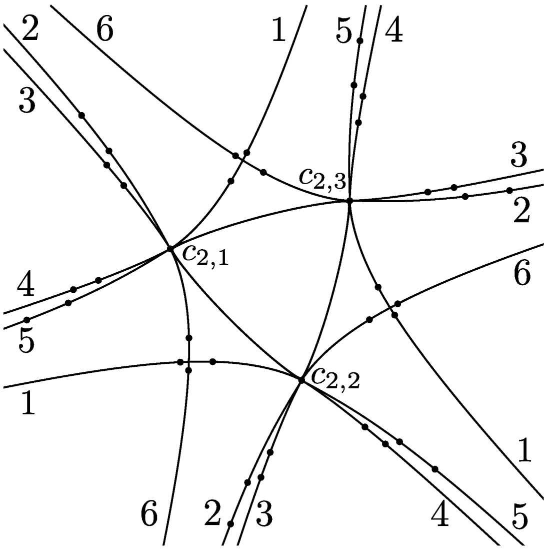

Figure 53. Curves of (3.55) and the location of zeros of (3.54) with $k = 0, \pm 1, \pm 2$.

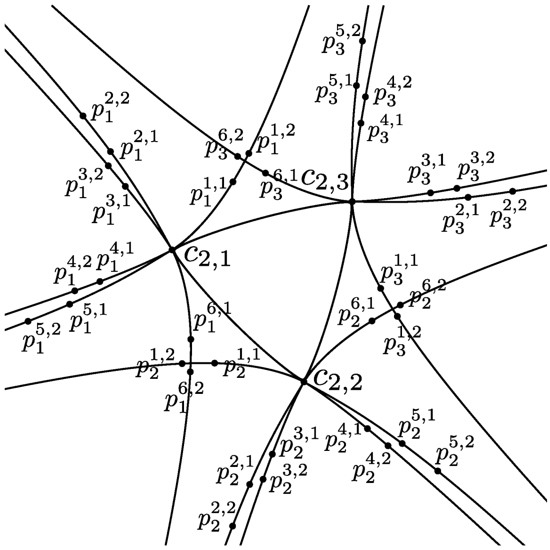

Figure 54. The labeling of zeros of (3.54).

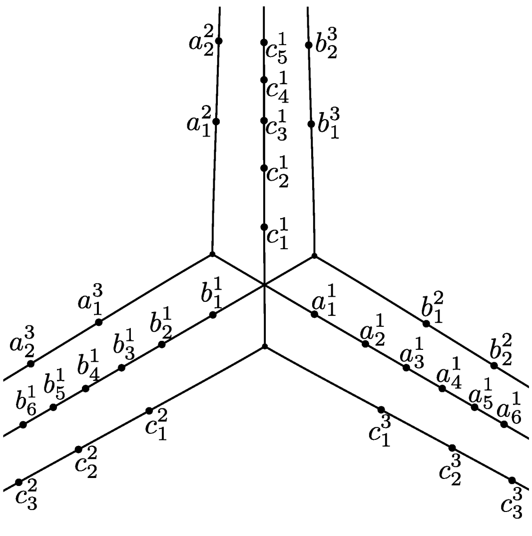

Figure 55. Virtual turning points of the AKT_{λ_1} equation and the curves (3.56).

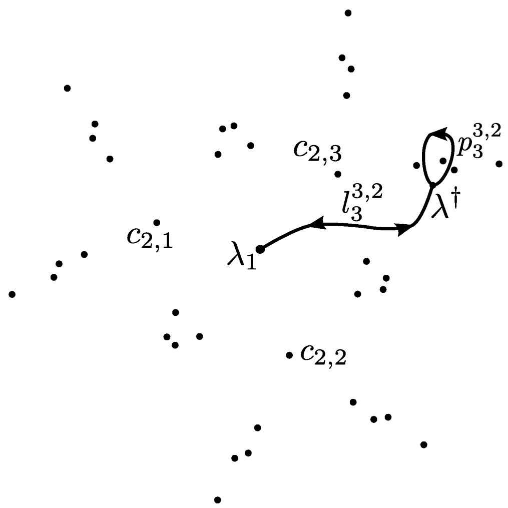

Figure 56. The closed path $l_3^{3,2}$.

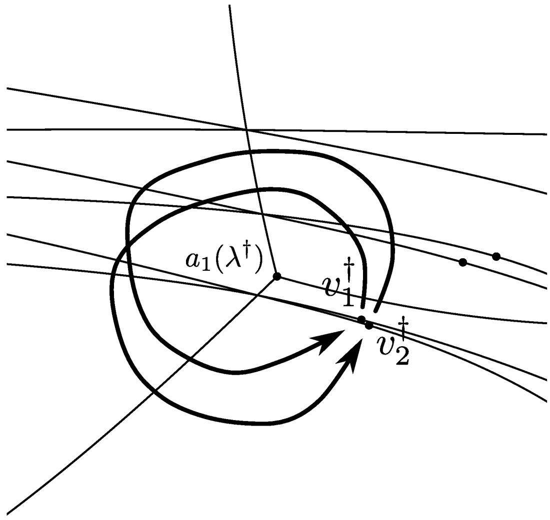

Figure 57. Two virtual turning points $v_1^\dagger$ and $v_2^\dagger$ at $c_2 = \lambda^\dagger$.

the relation

$$\int_{a_i(c_2)}^{x_1} (\xi_i - \xi_{i+1})\, dx_1 = 2\pi\sqrt{-1}\alpha k \quad (i = 1, 2, 3,\ k \in \mathbb{Z}) \tag{3.57}$$

which can be confirmed through the integral relation that defines each virtual turning point. In what follows, we label the virtual turning points appearing in Figure 55 as a_k^j, b_k^j and c_k^j $(j = 1, 2, 3, k \geq 1)$. (Cf. Figure 55.)

Now we consider the zero $p_3^{3,1}$, which is nothing but the zero of (3.53); let $l_3^{3,1}$ be a closed path starting from $c_2 = \lambda_1$ and going around $p_3^{3,1}$ defined by $\left(\gamma_2\big|_{[\lambda_1,\widetilde{\lambda}]}\right)^{-1} \circ \gamma_3 \circ \gamma_2\big|_{[\lambda_1,\widetilde{\lambda}]}$, where γ_2 is a path in Figure 25, γ_3 is a path in Figure 42 and $\gamma \circ \widetilde{\gamma}$ designates the composition of two paths γ and $\widetilde{\gamma}$. Then the analytic continuation of a_1^3 (resp. b_1^1) along $\gamma_2\big|_{[\lambda_1,\widetilde{\lambda}]}$ is given by v_2 (resp. v_1). Since it was observed in the preceding subsection that the position of v_1 and v_2 is interchanged after the analytic continuation along γ_3, we conclude that the position of two virtual turning points a_1^3 and b_1^1 is interchanged when c_2 varies along $l_3^{3,1}$. Note that a_1^3 and b_1^1 hit an ordinary turning point at $c_2 = p_3^{3,1}$ a center of the path γ_3. In a similar manner, let us take a zero $p_3^{3,2}$ of (3.54) and consider two virtual turning points a_2^3 and b_2^1 of the AKT_{λ_1} equation. With the aid of a computer, we observe that a_2^3 and b_2^1 hit an ordinary turning point at $c_2 = p_3^{3,2}$. Furthermore, letting $l_3^{3,2}$ be a closed path starting from λ_1 and going around $p_3^{3,2}$ shown in Figure 56, we also observe that the analytic continuation of a_2^3 (resp. b_2^1) from λ_1 to $\lambda^\dagger$ in Figure 56 along $l_3^{3,2}$ is given by $v_2^\dagger$ (resp. $v_1^\dagger$) in Figure 57 and that the position of $v_1^\dagger$ and $v_2^\dagger$ is interchanged after the analytic continuation around $p_3^{3,2}$ (cf.

Figure 57). Hence the position of a_2^3 and b_2^1 is interchanged when c_2 varies along $l_3^{3,2}$. In general, it is expected that for any $k \geq 1$ the position of two virtual turning points a_k^3 and b_k^1 is interchanged when c_2 varies along an appropriate closed path $l_3^{3,k}$ starting from λ_1 and going around $p_3^{3,k}$. In what follows we denote this property by

$$l_3^{3,k} : (a_k^3, b_k^1) \quad (\text{or} \quad l_3^{3,k}(a_k^3) = b_k^1, \quad l_3^{3,k}(b_k^1) = a_k^3). \tag{3.58}$$

More generally, we conjecture (and actually have numerically confirmed in part) that for any $i = 1, 2, 3,\ j = 1, \cdots, 6$ and $k \geq 1$ we can explicitly find a closed path $l_i^{j,k}$ starting from λ_1 and going around $p_i^{j,k}$ that satisfies

$$\begin{array}{lll} l_1^{1,k} : (b_k^2, a_{k+1}^1), & l_1^{2,k} : (b_k^2, c_k^2), & l_1^{3,k} : (b_k^3, c_k^1), \\ l_1^{4,k} : (c_k^2, b_k^1), & l_1^{5,k} : (c_k^3, b_k^3), & l_1^{6,k} : (c_k^3, a_{k+1}^1), \\ l_2^{1,k} : (c_k^2, b_{k+1}^1), & l_2^{2,k} : (c_k^2, a_k^2), & l_2^{3,k} : (c_k^3, a_k^1), \\ l_2^{4,k} : (a_k^2, c_k^1), & l_2^{5,k} : (a_k^3, c_k^3), & l_2^{6,k} : (a_k^3, b_{k+1}^1), \\ l_3^{1,k} : (a_k^2, c_{k+1}^1), & l_3^{2,k} : (a_k^2, b_k^2), & l_3^{3,k} : (a_k^3, b_k^1), \\ l_3^{4,k} : (b_k^2, a_k^1), & l_3^{5,k} : (b_k^3, a_k^3), & l_3^{6,k} : (b_k^3, c_{k+1}^1). \end{array} \tag{3.59}$$

Admitting (3.59), we can verify the following intriguing result:

(3.60) *Take any virtual turning point a_k^j, b_k^j or c_k^j $(j = 1, 2, 3, k \geq 1)$ of the AKT$_{\lambda_1}$ equation shown in Figure 55. Then it is obtained by the analytic continuation of a_1^1 along an appropriate closed path (in the c_2-plane) with the base point λ_1.*

This can be confirmed as follows: By the property of $l_1^{1,k-1}$ and $l_3^{4,k-1}$ in (3.59) we have

$$l_1^{1,k-1} \circ l_3^{4,k-1}(a_{k-1}^1) = l_1^{1,k-1}(b_{k-1}^2) = a_k^1, \tag{3.61}$$

where $l_1^{1,k-1} \circ l_3^{4,k-1}$ designates the composition of two closed paths $l_1^{1,k-1}$ and $l_3^{4,k-1}$. Hence we obtain

$$\left(l_1^{1,k-1} \circ l_3^{4,k-1}\right) \circ \left(l_1^{1,k-2} \circ l_3^{4,k-2}\right) \circ \cdots \circ \left(l_1^{1,1} \circ l_3^{4,1}\right)(a_1^1) = a_k^1. \tag{3.62}$$

In a similar way, we get

$$\left(l_2^{1,k-1} \circ l_1^{4,k-1}\right) \circ \left(l_2^{1,k-2} \circ l_1^{4,k-2}\right) \circ \cdots \circ \left(l_2^{1,1} \circ l_1^{4,1}\right)(b_1^1) = b_k^1, \tag{3.63}$$

$$\left(l_3^{1,k-1} \circ l_2^{4,k-1}\right) \circ \left(l_3^{1,k-2} \circ l_2^{4,k-2}\right) \circ \cdots \circ \left(l_3^{1,1} \circ l_2^{4,1}\right)(c_1^1) = c_k^1. \tag{3.64}$$

Thus a_k^1, b_k^1 and c_k^1 can be obtained by the analytic continuation of a_1^1, b_1^1 and c_1^1, respectively.

On the other hand, we have

$$l_3^{4,k}(a_k^1) = b_k^2, \quad l_2^{3,k}(a_k^1) = c_k^3, \tag{3.65}$$

$$l_1^{4,k}(b_k^1) = c_k^2, \quad l_3^{3,k}(b_k^1) = a_k^3, \tag{3.66}$$

$$l_2^{4,k}(c_k^1) = a_k^2, \quad l_1^{3,k}(c_k^1) = b_k^3. \tag{3.67}$$

Combining (3.62)-(3.64), we find that b_k^2 and c_k^3 (resp. c_k^2 and a_k^3, a_k^2 and b_k^3) can be obtained by the analytic continuation of a_1^1 (resp. b_1^1, c_1^1).

Furthermore, b_1^1 and c_1^1 can be obtained also by the analytic continuation of a_1^1. In fact, we have

$$l_3^{3,1} \circ l_2^{5,1} \circ l_2^{3,1}(a_1^1) = l_3^{3,1} \circ l_2^{5,1}(c_1^3) = l_3^{3,1}(a_1^3) = b_1^1, \tag{3.68}$$

$$l_2^{4,1} \circ l_3^{2,1} \circ l_3^{4,1}(a_1^1) = l_2^{4,1} \circ l_3^{2,1}(b_1^2) = l_2^{4,1}(a_1^2) = c_1^1. \tag{3.69}$$

Therefore the virtual turning points a_k^j, b_k^j and c_k^j of the AKT_{λ_1} equation can be obtained by the analytic continuation of a_1^1.

Thus we can expect that all the virtual turning points a_k^j, b_k^j and c_k^j ($j = 1, 2, 3, k \geq 1$) of the AKT_{λ_1} equation shown in Figure 55 should be obtained from one virtual turning point, say a_1^1, through the analytic continuation in the c_2-variable. This intriguing property is closely related to the degeneracy of the Stokes geometry for the AKT_{c_2} equation observed at the zeros of (3.54) on the curve (3.51). Recall that this degeneracy of the Stokes geometry also plays an important role in proving that virtual turning points of the AKT_{c_2} equation are contained in the Stokes surface for the (1,4) hypergeometric system (cf. Proposition 3.6). Hence the discussion in this subsection suggests that the virtual turning points a_k^j, b_k^j and c_k^j ($j = 1, 2, 3, k \geq 1$) of the AKT_{λ_1} equation and the new Stokes curves emanating there are all contained in the Stokes surface for the (1,4) hypergeometric system.

§ 4. Conclusion

In this paper, we study the two concrete holonomic systems, that is, the Pearcey system and the (1,4) hypergeometric system, from the viewpoint of the exact WKB analysis and, in particular, investigate their Stokes geometry.

We first study the Pearcey system with emphasis on its Stokes geometry and show that not only the ordinary Stokes curves for the BNR_{c_2} equation but also its new Stokes curve are contained in the Stokes surface for the Pearcey system. Proposition 2.7 plays an important role in our reasoning. (Proposition 3.5 also plays a similar and vital role in the study of the (1,4) hypergeometric system.)

In the study of the (1,4) hypergeometric system, we first show, in parallel with the results for the Pearcey system, that the (three) new Stokes curves relevant to Stokes phenomena for WKB solutions of the AKT_{c_2} equation, i.e., the restriction of the (1,4) hypergeometric system to $x_2 = c_2$, are contained in the Stokes surface for the (1,4) hypergeometric system. However, in the case of the (1,4) hypergeometric system, there exist infinitely many new Stokes curves and virtual turning points for the AKT_{c_2} equation and, as its consequence, a new type of degeneracy of the Stokes geometry for the AKT_{c_2} equation occurs. In §3.4, §3.5 and §3.6, by making use of such a degeneracy, we show that a new Stokes curve irrelevant to Stokes phenomena is also contained in the Stokes surface for the (1,4) hypergeometric system and, furthermore, investigate these degeneracy in detail. The results we have obtained with the aid of a computer strongly suggest that infinitely many virtual turning points of the AKT_{c_2} equation can be obtained from just one virtual turning point through the analytic continuation (with respect to the x_2-variable) and that they are all contained in the Stokes surface for the (1,4) hypergeometric system.

References

[1] T. Aoki, Toward the exact WKB analysis for holonomic systems, *RIMS Kôkyûroku*, No. 1433, 2005, pp. 1-8 (in Japanese).

[2] T. Aoki, T. Kawai and Y. Takei, New turning points in the exact WKB analysis for higher order ordinary differential equations, *Analyse algébrique des perturbations singulières, I, Méthodes résurgentes*, Hermann, 1994, pp. 69-84.

[3] H. L. Berk, W. M. Nevins and K. V. Roberts, New Stokes' line in WKB theory, *J. Math. Phys.*, **23** (1982), 988-1002.

[4] N. Honda, The geometric structure of a virtual turning point and the model of the Stokes geometry, *RIMS Kôkyûroku Bessatsu*, B10 (2008), 63-113.

[5] K. Okamoto and H. Kimura, On particular solutions of the Garnier systems and the hypergeometric functions of several variables, *Quarterly J. Math.*, **37** (1986), 61-80.

[6] T. Pearcey, The structure of an electromagnetic field in the neighbourhood of a cusp of a caustic, *Phil. Mag.*, **37** (1946), 311-317.

[7] S. Sasaki, On the role of virtual turning points in the deformation of higher order linear differential equation, I. and II., *RIMS Kôkyûroku*, No.1433, 2005, pp. 27-64 (I), 65-109 (II) (in Japanese).