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Parametric Stokes phenomena and Voros coefficients of the second Painlevé equation

By

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Abstract

In the author's paper [4] "parametric Stokes phenomena" occurring to 1-parameter (formal) solutions of the second Painlevé equation (P_{II}) in a certain region were analyzed, and explicit connection formulas describing the Stokes phenomena were derived. In this article we investigate parametric Stokes phenomena in other regions, and derive connection formulas by computing "P-Voros coefficients". We show that the connection formulas describing parametric Stokes phenomena are different depending on the regions where the independent variable lies. We also confirm the validity of these connection formulas from the view point of the isomonodromic deformation. Furthermore, we make a brief report for a result about the P-Voros coefficients of the third Painlevé equation in Appendix A.

§ 1. Introduction

The foundations of the theory of WKB analysis of Painlevé equations with a large parameter have been established by Aoki, Kawai and Takei. They introduced the notions of "P-turning points" and "P-Stokes curves" for Painlevé equations and discussed a WKB-theoretic transformation to the first Painlevé equation near a simple P-turning point ([1, 7, 8, 9]). (In this paper, in order to distinguish turning points (resp., Stokes curves) of Painlevé equations from those of linear differential equations, we use the terminology "P-turning points" (resp., "P-Stokes curves") for Painlevé equations, following [10].) Moreover, Takei also discussed a connection problem on P-Stokes curves of the first Painlevé equation ([15]) through the isomonodromic deformation method.

Painlevé equations have two or more P-turning points in general exept for the first Painlevé equation. For such equations, we can observe "*degeneration of P-Stokes*

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geometry" (i.e., two P-turning points are connected by a P-Stokes curve) when parameters contained in these equations take some special values. As we see in Section 2, the P-Stokes geometry of the second Painlevé equation with a large parameter $\eta > 0$

$$(P_{\mathrm{II}}) : \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c)$$

degenerates when the complex parameter c takes pure imaginary numbers. By the way, it is known that, in the case of the Weber equation, Stokes phenomena may occur for WKB solutions when such a kind of degeneration is observed in the Stokes geometry, i.e., there may be a gap in Borel sums of WKB solutions taken before and after the degeneration. (Cf. [14, 16]). We call such a kind of Stokes phenomena relevant to the degeneration of the Stokes geometry *parametric Stokes phenomena* since they happen when we discuss connection problems of the WKB solutions in the parameter space of the equation. Moreover, an explicit connection formula which describes parametric Stokes phenomena for the Weber equation is derived in [14, 16]. Note that a similar result for the Whittaker equation is obtained in [11].

Now, it is natural to expect that parametric Stokes phenomena may occur also in the case of the second Painlevé equaltion when the degeneration of P-Stokes geometry is observed. Parametric Stokes phenomena for 1-parameter solutions of (P_{II}) when t is on a certain region have already been analyzed in [4]. In this article, we analyze the parametric Stokes phenomena in other regions and derive explicit connection formulas that describe them. In addition we make a brief report on partial results about parametric Stokes phenomena of the third Painlevé equation of the type D_6 (in the sense of [12]).

The paper is organized as follows. In Section 2 we recall the definitions of P-turning points and P-Stokes curves. In Section 3, computing all *P-Voros coefficincets* of (P_{II}), we derive connection formulas describing parametric Stokes phenomena. We also confirm the validity of these connection formulas from the view point of the isomonodromic deformation of a second order linear differential equation (SL_{II}) in Section 4. We note that a part of these results are presented in author's previous article [4]. Parametric Stokes phenomena and the P-Voros coefficients for the third Painlevé equation are discussed in Appendix.

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§ 2. 1-parameter solutions and P-Stokes geometry of (P_{II})

In this article we assume that the parameter c contained in (P_{II}) is non-zero. It is well-known that (P_{II}) is equivalent to the following Hamiltonian system (H_{II}):

$$(H_{\mathrm{II}}): \begin{cases} \dfrac{d\lambda}{dt} = \eta\,\nu\,, \\ \dfrac{d\nu}{dt} = \eta(2\lambda^3 + t\lambda + c). \end{cases}$$

Our main interest consists in the analysis of *1-parameter solutions*, that is, formal solutions of (H_{II}) of the form:

$$\begin{cases} \lambda(t,c,\eta;\alpha) = \lambda^{(0)}(t,c,\eta) + \alpha\eta^{-1/2}\lambda^{(1)}(t,c,\eta)e^{\eta\phi_{\mathrm{II}}} + (\alpha\eta^{-1/2})^2\lambda^{(2)}(t,c,\eta)e^{2\eta\phi_{\mathrm{II}}} + \cdots\,, \\ \nu(t,c,\eta;\alpha) = \nu^{(0)}(t,c,\eta) + \alpha\eta^{-1/2}\nu^{(1)}(t,c,\eta)e^{\eta\phi_{\mathrm{II}}} + (\alpha\eta^{-1/2})^2\nu^{(2)}(t,c,\eta)e^{2\eta\phi_{\mathrm{II}}} + \cdots\,, \end{cases}$$

where α is a free parameter, $\lambda^{(k)}$ and $\nu^{(k)}$ are formal power series in η^{-1}:

$$\lambda^{(k)}(t,c,\eta) = \lambda_0^{(k)}(t,c) + \eta^{-1}\lambda_1^{(k)}(t,c) + \eta^{-2}\lambda_2^{(k)}(t,c) + \cdots\,,$$

$$\nu^{(k)}(t,c,\eta) = \nu_0^{(k)}(t,c) + \eta^{-1}\nu_1^{(k)}(t,c) + \eta^{-2}\nu_2^{(k)}(t,c) + \cdots\,,$$

and the phase function $\phi_{\mathrm{II}} = \phi_{\mathrm{II}}(t,c)$ is given by

$$\phi_{\mathrm{II}}(t,c) = \int^t \sqrt{6\lambda_0^{(0)}(t,c)^2 + t}\,dt.$$

Remark 1. In principle, we consider 1-parameter solutions for $\alpha \in \mathbb{C}$. However, as we will see in Theorem 3.5, we face to the situation where we have to extend the class of 1-parameter solutions when we discuss connection problems. Indeed, 1-parameter solutions for $\tilde{\alpha} = (1 + e^{2\pi i c\eta})\alpha$ ($\alpha \in \mathbb{C}$) appear there as a consequence of parametric Stokes phenomena, where we regard $\tilde{\alpha}$ as a holomorphic function of a complex variable η on an appropriate sectorial domain at $\eta = \infty$. However, such a kind of the extension of the class does not violate the validity of the following discussion.

See [4] for details of construction of 1-parameter solutions. Note that $(\lambda^{(0)}(t,c,\eta), \nu^{(0)}(t,c,\eta))$ itself is a formal power series solution of (H_{II}), which is called a 0-parameter solution. The leading term $\lambda_0^{(0)}(t,c)$ of a *0-parameter solution* is determined by the following algebraic equation:

$$2{\lambda_0^{(0)}}^3 + t\lambda_0^{(0)} + c = 0. \tag{2.1}$$

In what follows $\lambda_0^{(0)}$ is abbreviated to λ_0 for simplicity. In this article 1-parameter solutions are considered in a domain where the real part of ϕ_{II} is negative, i.e., $e^{\eta\phi_{\mathrm{II}}}$

is exponentially decaying when $\eta \to +\infty$. Here we note that we can find a similar kind of formal solutions, which is called transseries, in the study of nonlinear ordinary differential equations at an irregular singular point (e.g., [2]).

We know that if the first part $\alpha\eta^{-1/2}\lambda^{(1)}(t,c,\eta)e^{\eta\phi_{\rm II}}$ of $\lambda(t,c,\eta;\alpha)$ is denoted by $\tilde{\lambda}^{(1)}(t,c,\eta;\alpha)$, then we know that it satisfies the following second order linear differential equation

$$\frac{d^2\tilde{\lambda}^{(1)}}{dt^2} = \eta^2\Big(6\lambda^{(0)}(t,c,\eta)^2 + t\Big)\,\tilde{\lambda}^{(1)}, \tag{2.2}$$

which is the Fréchet derivative of $(P_{\rm II})$ at $\lambda = \lambda^{(0)}$. Thus $\tilde{\lambda}^{(1)}$ can be taken as a WKB solution (see [9, §2.1]) of (2.2) containing a free parameter α as an integration constant:

$$\begin{aligned}\tilde{\lambda}^{(1)} &= \alpha\frac{1}{\sqrt{R_{\rm odd}(t,c,\eta)}}\exp\Big(\int^t R_{\rm odd}(t,c,\eta)dt\Big) \\ &= \alpha\eta^{-1/2}(\lambda_0^{(1)}(t,c) + \eta^{-1}\lambda_1^{(1)}(t,c) + \eta^{-2}\lambda_2^{(1)}(t,c) + \cdots)e^{\eta\phi_{\rm II}}.\end{aligned} \tag{2.3}$$

Here $R_{\rm odd} = \eta R_{-1} + \eta^{-1}R_1 + \cdots$ is the odd part of a formal power series solution $R = \eta R_{-1} + R_0 + \eta^{-1}R_1 + \cdots$ of the Riccati equation associated with (2.2):

$$R^2 + \frac{dR}{dt} = \eta^2(6\lambda^{(0)}(t,c,\eta)^2 + t). \tag{2.4}$$

An important fact is that, once we fix a normalization (i.e., the lower endpoint and the path) of the integral of $R_{\rm odd}$ in (2.3), the coefficients $\lambda_\ell^{(k)}$ of the formal series $\lambda^{(k)}$ ($k \ge 2$, $\ell \ge 0$) are determined uniquely by some recursive relations. (The normalization of the integral in (2.3) will be specified in Section 3.)

Next, we recall the definition of P-turning points and P-Stokes curves.

Definition 2.1 ([9, Definition 4.5]). (i) A point $t = \tau$ is a *P-turning point* of a 1-parameter solution $\lambda(t,c,\eta;\alpha)$ if $6\big(\lambda_0(\tau)\big)^2 + \tau = 0$.
(ii) For a P-turning point $t = \tau$ of $\lambda(t,c,\eta;\alpha)$, a real one-dimensional curve defined by

$$\mathrm{Im}\int_\tau^t \sqrt{6\big(\lambda_0(t)\big)^2 + t}\,dt = 0$$

is called a *P-Stokes curve* of $\lambda(t,c,\eta;\alpha)$.

Although Definition 2.1 is stated with a fixed branch λ_0 of roots of (2.1), we may regard these notions are given on the Riemann surface of λ_0. See Remark 2 for details.

Since we assume that $c \neq 0$, there are three P-turning points at $t = \tau_j = -6\left(c/4\right)^{2/3}\omega^j$ ($\omega = e^{2\pi i/3}, j = 1,2,3$). Figure 1 $\sim$ 3 describe P-Stokes curves of $(P_{\rm II})$ near $\arg c = \pi/2$. These figures are drawn by Mathematica. We can observe that degeneration of P-Stokes geometry happens when $\arg c = \pi/2$. This degeneration can be confirmed by the following Proposition which was shown in [4].

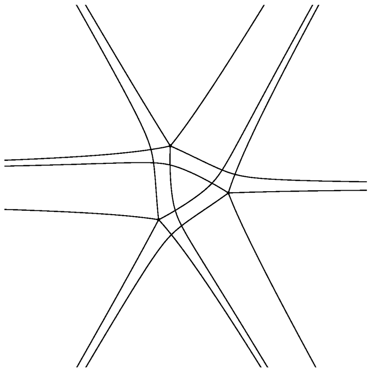

Figure 1. P-Stokes curves when $c = i + 0.25$.

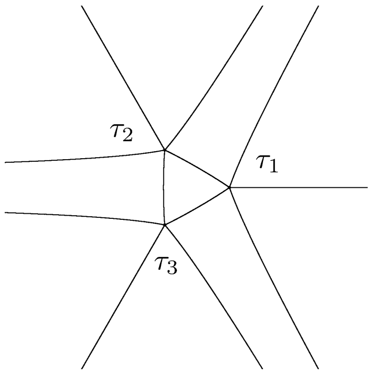

Figure 2. P-Stokes curves when $c = i$.

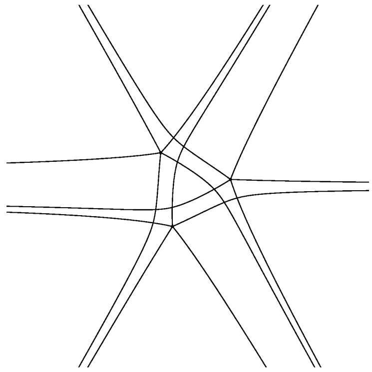

Figure 3. P-Stokes curves when $c = i - 0.25$.

Proposition 2.2 ([4]). *For a suitable branch of* $\sqrt{6\lambda_0^2 + t}$*, we have*

$$\int_{\tau_1}^{\tau_2} \sqrt{6\lambda_0(t,c)^2 + t}\, dt = 2\pi i c. \tag{2.5}$$

Note that, since (P_{II}) is invariant under the rotation of variables

$$t \mapsto e^{2\pi i/3} t, \ \ \lambda \mapsto e^{4\pi i/3} \lambda, \tag{2.6}$$

we also have

$$\int_{\tau_2}^{\tau_3} \sqrt{6\lambda_0(t,c)^2 + t}\, dt = \int_{\tau_3}^{\tau_1} \sqrt{6\lambda_0(t,c)^2 + t}\, dt = 2\pi i c.$$

In Section 3 we derive connection formulas for 1-parameter solutions of (P_{II}) which describe the parametric Stokes phenomenon relevant to the degeneration observed when $\arg c = \pi/2$.

Remark 2. Since P-turning points and P-Stokes curves are defined in terms of the algebraic function λ_0, it is natural to lift them onto the Riemann surface of λ_0. Figure 4 describes the lift of P-Stokes curves onto the Riemann surface of λ_0 when arg $c = \pi/2$. Wiggly lines, solid lines and dotted lines in Figure 4 represent cuts to define the Riemann surface of λ_0, P-Stokes curves on the sheet under consideration and P-Stokes curves on the other sheets, respectively. In Section 3 $\sim$ 4 we only consider the situation where arg c is sufficiently close to $\pi/2$. The parametric Stokes phenomenon occurring when t moves in sufficiently small neighborhood of t_0 in Figure 4 was analyzed in [4]. Parametric Stokes phenomena in other regions (for example, in a neighborhood of t_1 in Figure 4) are discussed in subsequent sections. Note that, since (P_{II}) has a rotational symmetry (2.6), it is sufficient to consider the problem on one of these three sheets in Figure 4.

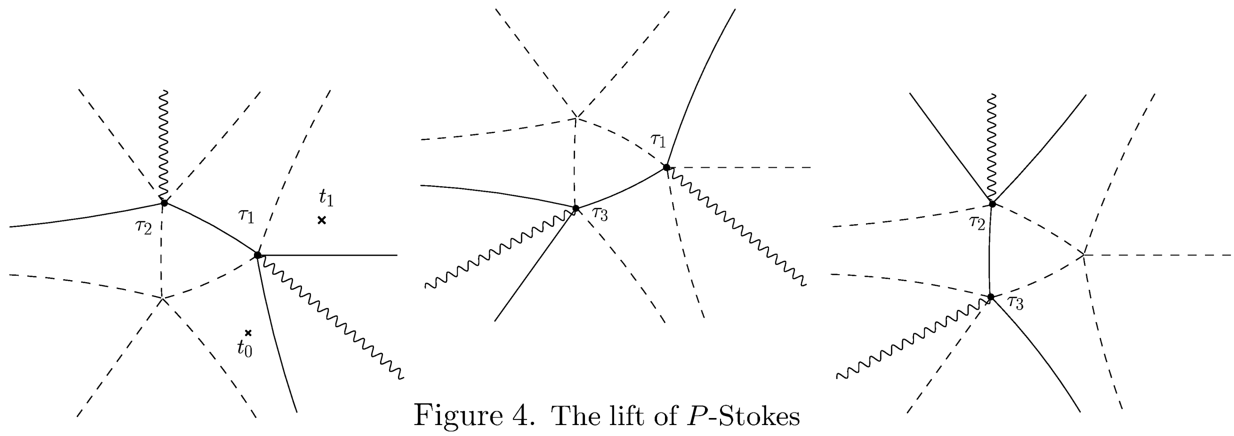

Figure 4. The lift of P-Stokes curves onto the Riemann surface of λ_0.

§ 3. Voros coefficients of ($P_{\rm II}$) and connection formulas

In this section, we first recall the definition of *Voros coefficients* of ($P_{\rm II}$) (or *P-Voros coefficients* for short), which was introduced in [4]. This is an analogue of Voros coefficients for linear ordinary differential equations (cf. [3, 16]). These series play an important role in the analysis of parametric Stokes phenomena.

P-Voros coefficients are formal power series of η^{-1} defined by the following integral:

$$
\begin{aligned}
(3.1)\qquad W(c,\eta) &= \int_{\tau}^{\infty} \big(R_{\rm odd}(t,c,\eta) - \eta R_{-1}(t,c)\big) dt \\
&= W_1(c)\eta^{-1} + W_3(c)\eta^{-3} + \cdots = \sum_{n=1}^{\infty} W_{2n-1}(c)\eta^{1-2n},
\end{aligned}
$$

where τ is a P-turning point. Since each coefficient R_{2k-1} of $R_{\rm odd}$ has a Puiseux expansion

$$(t-\tau)^{N/4} \sum_{n\ge 0} r_n\, (t-\tau)^{n/2} \quad (r_n \in \mathbb{C})$$

at each P-turning point τ with an odd integer N, integrals of $R_{\rm odd}$ from a P-turning point should be considered as a contour integral. P-Voros coefficients appear as the difference of the following two normalizations of 1-parameter solutions:

$$(3.2)\qquad \lambda_{\tau}^{(1)}(t,c,\eta) = e^{W}\, \lambda_{\infty}^{(1)}(t,c,\eta),$$

$$(3.3)\qquad \lambda_{\tau}(t,c,\eta;\alpha) = \lambda_{\infty}(t,c,\eta; e^{W}\alpha).$$

Here $\lambda_{\tau}(t,c,\eta;\alpha)$ (resp., $\lambda_{\infty}(t,c,\eta;\alpha)$) is the 1-parameter solution defined by using the following WKB solution of (2.2)

$$(3.4)\qquad \tilde{\lambda}_{\tau}^{(1)}(t,c,\eta;\alpha) = \alpha \frac{1}{\sqrt{R_{\rm odd}}} \exp\Big(\int_{\tau}^{t} R_{\rm odd} dt\Big)$$

$$\left(\text{resp.,}\quad \tilde{\lambda}^{(1)}_{\infty}(t,c,\eta;\alpha) = \alpha \frac{1}{\sqrt{R_{\text{odd}}}} \exp\left(\eta \int_{\tau}^{t} R_{-1} dt + \int_{\infty}^{t} (R_{\text{odd}} - \eta R_{-1}) dt\right)\right), \tag{3.5}$$

for the normalization of the first part $\tilde{\lambda}^{(1)}$. (As we noted in Section 2, 1-parameter solutions are uniquely determined once the normalization of $\tilde{\lambda}^{(1)}$ is fixed.) We call $\lambda_{\tau}(t,c,\eta;\alpha)$ (resp., $\lambda_{\infty}(t,c,\eta;\alpha)$) "the 1-parameter solution normalized at a P-turning point $t=\tau$" (resp., "the 1-parameter solution normalized at $t=\infty$"). And we use the following notations:

$$\lambda_{\tau}(t,c,\eta;\alpha) = \lambda^{(0)}(t,c,\eta) + \alpha\eta^{-1/2}\,\lambda^{(1)}_{\tau}(t,c,\eta)e^{\eta\phi_{\text{II}}} + (\alpha\eta^{-1/2})^2\,\lambda^{(2)}_{\tau}(t,c,\eta)e^{2\eta\phi_{\text{II}}} + \cdots,$$

$$\lambda_{\infty}(t,c,\eta;\alpha) = \lambda^{(0)}(t,c,\eta) + \alpha\eta^{-1/2}\,\lambda^{(1)}_{\infty}(t,c,\eta)e^{\eta\phi_{\text{II}}} + (\alpha\eta^{-1/2})^2\,\lambda^{(2)}_{\infty}(t,c,\eta)e^{2\eta\phi_{\text{II}}} + \cdots.$$

We also note that the function ϕ_{II} for above normalizations is given by

$$\phi_{\text{II}}(t,c) = \int_{\tau}^{t} \sqrt{6\lambda_0(t,c)^2 + t}\, dt. \tag{3.6}$$

Note that we should specify the integration paths in (3.4) and (3.5) in order to fix the normalization of $\tilde{\lambda}^{(1)}$ completely. Therefore P-Voros coefficients depends on the choice of the integration path as well. The goal of this section is to compute the P-Voros coefficients for all choices of the integration paths.

In order to state the main theorem, we need to fix the branch of the square root $R_{-1} = \sqrt{6\lambda_0^2 + t}$. Note that λ_0 has the following three possible asymptotic behaviors as t tends to ∞ since it satisfies the algebraic equation (2.1) of degree 3:

$$\lambda_0 = \pm\frac{i}{\sqrt{2}}t^{1/2} + O(t^{-1}), \tag{3.7}$$

$$\lambda_0 = -ct^{-1} + O(t^{-2}). \tag{3.8}$$

Figure 5 indicates the asymptotic behaviors of λ_0 when t tends to ∞ along P-Stokes curves. In Figure 5 we use the symbol ∞_A (resp., ∞_B) for an infinity such that λ_0 behaves like (3.7) (resp., (3.8)) when t tends to the infinity. In the same figures thick wiggly lines designate cuts for the determination of the branch of $\sqrt{6\lambda_0^2+t}$. Here we fix the branch of $\sqrt{6\lambda_0^2+t}$ such that "signs of P-Stokes curves" are assigned as in Figure 5; where the sign of a P-Stokes curve is defined as the sign of

$$\operatorname{Re}\int_{\tau}^{t} \sqrt{6\lambda_0^2 + t}\, dt,$$

where t is a point on the P-Stokes curve in question and τ is a P-turning point which the P-Stokes curve emanates from, and symbols $\oplus$ and $\ominus$ in Figure 5 represent the signs

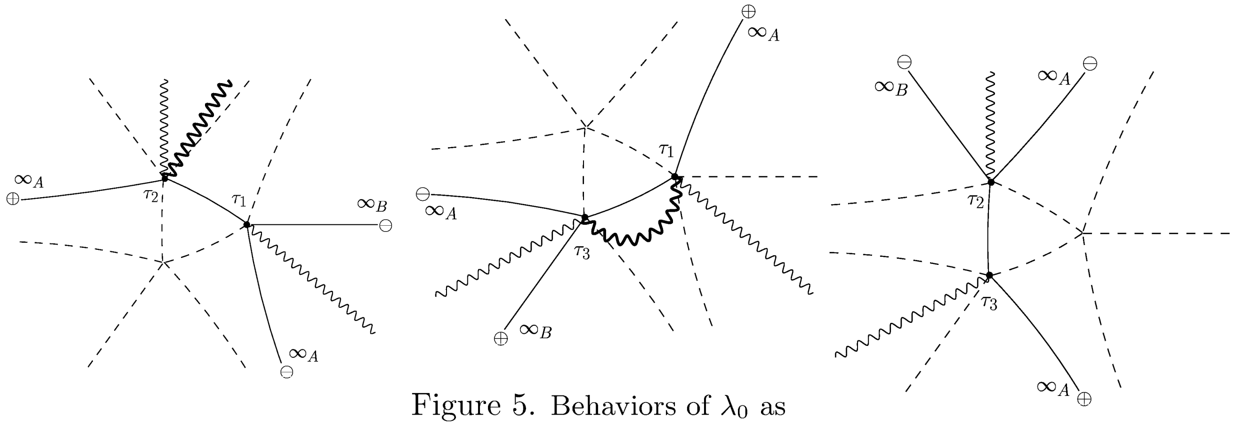

Figure 5. Behaviors of λ_0 as t tends to infinity.

of P-Stokes curves. (The sign does not depend on the point on the P-Stokes curve in question.) We use the branch fixed as above.

Let $W_{\tau_j,\infty_*}(c,\eta)$ ($j=1,2,3$, $*=A,B$) be the P-Voros coefficient defined by

$$W_{\tau_j,\infty_*}(c,\eta)=\int_{\tau_j}^{\infty_*}\big(R_{\mathrm{odd}}(t,c,\eta)-\eta R_{-1}(t,c)\big)dt, \tag{3.9}$$

where the integration path should be taken along a P-Stokes curve emanating from the P-turning point $t=\tau_j$. Then our main theorem is formulated as follows:

Theorem 3.1. *P-Voros coefficients $W_{\tau_j,\infty_*}(c,\eta)$ defined by (3.9) are represented explicitly as follws:*

$$W_{\tau_j,\infty_A}(c,\eta)=-\sum_{n=1}^{\infty}\frac{2^{1-2n}-1}{2n(2n-1)}B_{2n}(c\eta)^{1-2n}, \tag{3.10}$$

$$W_{\tau_j,\infty_B}(c,\eta)=0, \tag{3.11}$$

for all $j=1,2,3$. Here B_{2n} is the $2n$-th Bernoulli number defined by

$$\frac{z}{e^z-1}=1-\frac{1}{2}z+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n}. \tag{3.12}$$

Proof. It is convenient to prepare the following additional integrals Z_j in order to prove the theorem:

$$Z_j(c,\eta)=\frac{1}{2}\int_{\gamma_j}\big(R_{\mathrm{odd}}(t,c,\eta)-\eta R_{-1}(t,c)\big)dt \quad (j=1,2,3), \tag{3.13}$$

where the integration paths γ_j (j = 1,2,3) are shown in Figure 6. The dotted part of γ_2 represents a path on another sheet of the Riemann surface of $\sqrt{6\lambda_0^2+t}$. Actually,

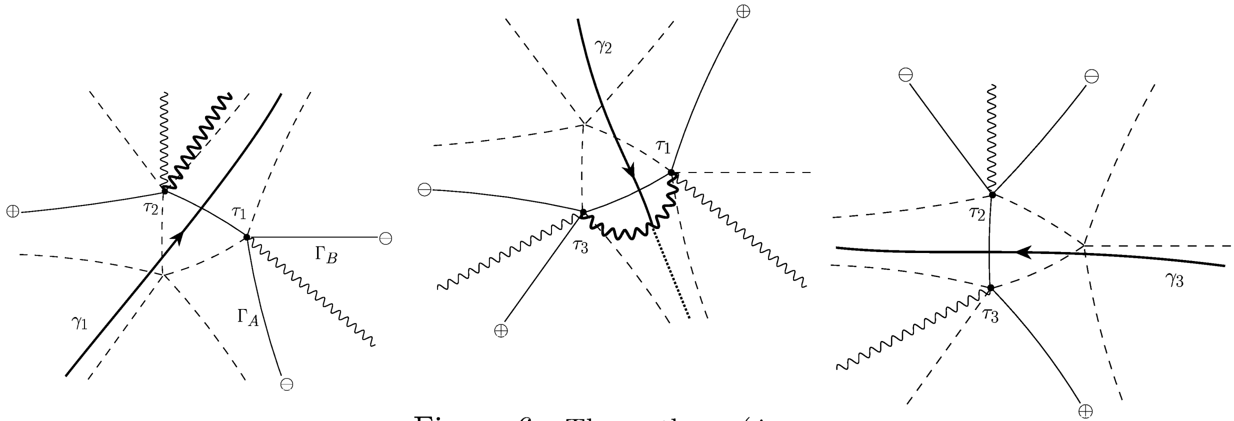

Figure 6. The path γ_j $(j = 1, 2, 3)$.

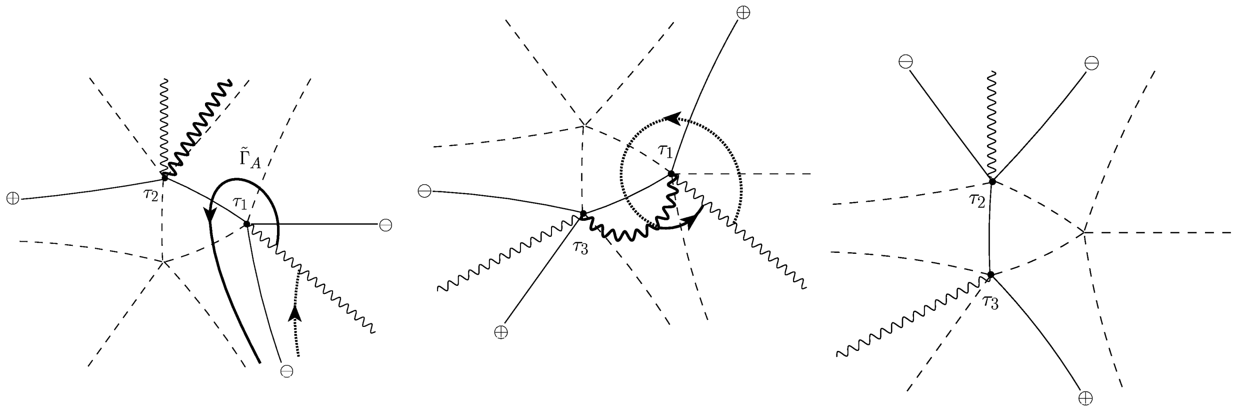

Figure 7. The path $\tilde{\Gamma}_A$.

since $(P_{\rm II})$ has the rotational symmetry (2.6), we can check that $Z_j(c,\eta) = Z_k(c,\eta)$ $(j, k \in \{1,2,3\})$.

First let us consider the P-Voros coefficient

$$W_{\tau_1,\infty_A}(c,\eta) = \int_{\Gamma_A} \big(R_{\rm odd}(t,c,\eta) - \eta R_{-1}(t,c)\big)dt \tag{3.14}$$

whose integration path is taken along the P-Stokes curve Γ_A in Figure 6 and we denote it by $W_{\Gamma_A}(c,\eta)$. As we noted in the begining of this section, the integral (3.14) should be considered as the following contour integral whose integration path $\tilde{\Gamma}_A$ is shown in Figure 7:

$$W_{\Gamma_A}(c,\eta) = \frac{1}{2}\int_{\tilde{\Gamma}_A} \big(R_{\rm odd}(t,c,\eta) - \eta R_{-1}(t,c)\big)dt.$$

Deforming the path of integration, we can easily see that

$$W_{\Gamma_A}(c,\eta) = -Z_1(c,\eta) - Z_2(c,\eta) = -2\,Z_1(c,\eta). \tag{3.15}$$

On the other hand, the P-Voros coefficient $W_{\Gamma_A}(c,\eta)$ has already computed in [4].

Theorem 3.2 ([4]). *The P-Voros coefficient* $W_{\Gamma_A}(c,\eta)$ *defined by* (3.14) *has the following explicit representation:*

$$W_{\Gamma_A}(c,\eta) = -\sum_{n=1}^{\infty} \frac{2^{1-2n}-1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n}, \tag{3.16}$$

where B_{2n} *is the 2n-th Bernoulli number defined by* (3.12).

This theorem was proved by solving a difference equation satisfied by $W_{\Gamma_A}(c,\eta)$. The difference equation was derived by using the Bäcklund transformation of (P_{II}) which induces the shift of parameter $c \mapsto c - \eta^{-1}$ (cf. [5]). The equalities (3.15) and (3.16) imply that

$$Z_j(c,\eta) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{1-2n}-1}{2n(2n-1)} B_{2n}(c\eta)^{1-2n} \quad (j=1,2,3). \tag{3.17}$$

Since each P-Voros coefficient can be expressed by a sum of $\pm Z_j$, we can compute all P-Voros coefficients. Therefore we can check (3.10) and (3.11) by direct computations. For example, let us consider the P-Voros coefficient

$$W_{\Gamma_B}(c,\eta) = \int_{\Gamma_B} \big(R_{\mathrm{odd}}(t,c,\eta) - \eta R_{-1}(t,c)\big) dt \tag{3.18}$$

whose integration path is taken along the P-Stokes curve Γ_B in Figure 6. We can easily see that

$$W_{\Gamma_B}(c,\eta) = -Z_1(c,\eta) + Z_2(c,\eta) = 0. \tag{3.19}$$

$\square$

Let $\mathcal{S}_{\arg c=\theta}[W(c,\eta)]$ be the Borel sum of a P-Voros coefficient $W(c,\eta)$ of (3.1) when $\arg c = \theta$; i.e.,

$$\mathcal{S}_{\arg c=\theta}[W(c,\eta)] = \int_0^{\infty} e^{-y\eta} W_B(c,y)\big|_{\arg c=\theta} dy, \tag{3.20}$$

where $W_B(c,y)$ is the Borel transform of $W(c,\eta)$ defined by

$$W_B(c,y) = \sum_{n=1}^{\infty} \frac{W_{2n-1}(c)}{(2n-2)!} y^{2n-2}, \tag{3.21}$$

and the path of the Laplace integral (3.20) is taken along the positive real axis. Using the expression (3.10), we can compute the Borel sums of the P-Voros coefficients explicitly when $\arg c = \pi/2 \pm \varepsilon$ for sufficiently small $\varepsilon > 0$.

Corollary 3.3 (cf.[16, §2],[4]). *Let $W_{\Gamma_A}(c,\eta)$ be the Voros coefficient defined by* (3.14) *(it is given by* (3.16) *explicitly). Then we have the following equalities.*

$$\mathcal{S}_{\arg c=\pi/2-\varepsilon}\big[W_{\Gamma_A}(c,\eta)\big] = -\log\frac{\Gamma(c\eta+1/2)}{\sqrt{2\pi}} + c\eta\,\big(\log(c\eta)-1\big), \tag{3.22}$$

$$\mathcal{S}_{\arg c=\pi/2+\varepsilon}\big[W_{\Gamma_A}(c,\eta)\big] = \log\frac{\Gamma(-c\eta+1/2)}{\sqrt{2\pi}} + c\eta\,\big(\log(c\eta)-1\big) - i\pi c\eta. \tag{3.23}$$

Therefore we have

$$\mathcal{S}_{\arg c=\pi/2-\varepsilon}\big[e^{W_{\Gamma_A}(c,\eta)}\big] = (1+e^{2\pi i c\eta})\,\mathcal{S}_{\arg c=\pi/2+\varepsilon}\big[e^{W_{\Gamma_A}(c,\eta)}\big]. \tag{3.24}$$

Now, using the equality (3.24), we can derive a connection formula which describes the parametric Stokes phenomenon for 1-parameter solutions at $t=t_0$ in Figure 4. (Note that this formula has already been obtained in [4].) Let $\lambda_{\tau_1}(t,c,\eta;\alpha)$ and $\lambda_{\infty_A}(t,c,\eta;\alpha)$ be the 1-parameter solutions whose first parts $\tilde{\lambda}^{(1)}_{\tau_1}$ and $\tilde{\lambda}^{(1)}_{\infty_A}$ are normalized along the paths in Figure 8 and 9 (i.e., integrals in (3.4) and (3.5) are taken along the paths in Figure 8 and 9.) respectively. For the generalized Borel summability of these 1-parameter solutions, the following theorem is proved recently by Kamimoto.

Theorem 3.4 ([6]). *Assume that the path of integration of R_{odd} in* (2.3) *never touches with any P-turning points and P-Stokes curves, and the real part of ϕ_{II} is negative. Then, the corresponding 1-parameter solution $\lambda(t,c,\eta;\alpha)$ is Borel summable in general sense as* [2]*; that is, the k-th formal series $\lambda^{(k)}(t,c,\eta)$ in $\lambda(t,c,\eta;\alpha)$ is Borel summable for each $k\geq 0$, and the generalized Borel sum of $\lambda(t,c,\eta;\alpha)$ defined by the infinite sum*

$$\mathcal{S}[\lambda(t,c,\eta;\alpha)] = \sum_{k\geq 0}(\alpha\eta^{-1/2})^k\mathcal{S}\big[\lambda^{(k)}(t,c,\eta)\big]e^{k\eta\phi_{\mathrm{II}}}$$

converges for sufficiently large $\eta>0$ and represents an analytic solution of (P_{II}). Here $\mathcal{S}[\lambda^{(k)}(t,c,\eta)]$ is the Borel sum of the formal power series $\lambda^{(k)}(t,c,\eta)$.

We use the same notation $\mathcal{S}_{\arg c=\theta}$ for generalized Borel sums of transseries when $\arg c=\theta$.

Due to Theroem 3.4, the 1-parameter solution λ_{∞_A} is Borel summable even in the case that the degeneration is happening since it is normalized along the path which is shown in Figure 9. Hence parametric Stokes phenomena do not occur for the 1-parameter solution λ_{∞_A}; that is,

$$\mathcal{S}_{\arg c=\pi/2-\varepsilon}\big[\lambda_{\infty_A}(t,c,\eta;\alpha)\big] = \mathcal{S}_{\arg c=\pi/2+\varepsilon}\big[\lambda_{\infty_A}(t,c,\eta;\tilde{\alpha})\big] \tag{3.25}$$

holds for $\alpha=\tilde{\alpha}$. Especially, the first part $\tilde{\lambda}^{(1)}_{\infty_A}$ of the 1-parameter solution λ_{∞_A} satisfies the following:

$$\mathcal{S}_{\arg c=\pi/2-\varepsilon}\big[\tilde{\lambda}^{(1)}_{\infty_A}(t,c,\eta;\alpha)\big] = \mathcal{S}_{\arg c=\pi/2+\varepsilon}\big[\tilde{\lambda}^{(1)}_{\infty_A}(t,c,\eta;\alpha)\big] \tag{3.26}$$

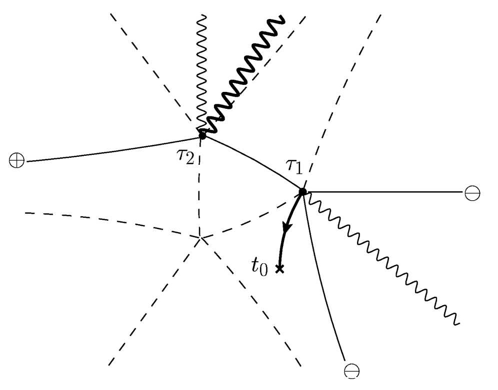

Figure 8. Integartion path from τ_1 to t (near t_0).

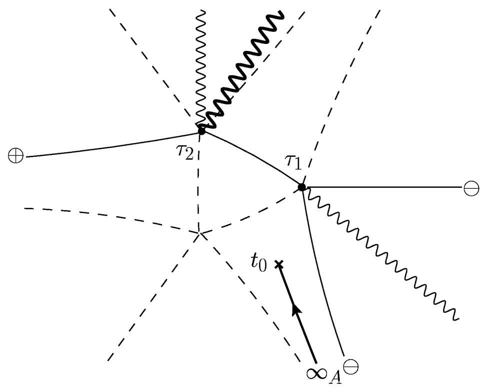

Figure 9. Integartion path from ∞ to t (near t_0).

On the other hand, we can not expect the generalized Borel summability for the 1-parameter solution λ_{τ_1} normalized at τ_1 with the path shown in Figure 10 when the degeneration is happening. However, since the equality $\lambda^{(k)}_{\tau_1}(t,c,\eta) = e^{kW_{\Gamma_A}(c,\eta)}\lambda^{(k)}_{\infty_A}(t,c,\eta)$ holds for each $k \geq 0$ by (3.3), the generalized Borel summability of λ_{τ_1} when arg $c = \pi/2 \pm \varepsilon$ follows from Theorem 3.4 and Corollary 3.3. For this 1-parameter solution, comparing the first part of the generalized Borel sums of λ_{τ_1} before and after the degeneration, we find

$$\mathcal{S}_{\arg c=\pi/2-\varepsilon}\big[\tilde{\lambda}^{(1)}_{\tau_1}(t,c,\eta;\alpha)\big] = \mathcal{S}_{\arg c=\pi/2+\varepsilon}\big[\tilde{\lambda}^{(1)}_{\tau_1}(t,c,\eta;\tilde{\alpha})\big]$$

holds for

$$\tilde{\alpha} = (1+e^{2\pi i c\eta})\,\alpha \tag{3.27}$$

due to (3.24) and (3.26). Thus we have the following non-trivial connection formula for the 1-parameter solution λ_{τ_1} at $t = t_0$.

Theorem 3.5 (Connection formula for the 1-parameter solution λ_{τ_1} at $t = t_0$, [4])**.** *Let ε be a sufficiently small positive number and let $\lambda_{\tau_1}(t,c,\eta;\alpha)$ be the 1-parameter solution normalized along the path which is shown in Figure 8. Then, the following relation holds:*

$$\mathcal{S}_{\arg c=\pi/2-\varepsilon}[\lambda_{\tau_1}(t,c,\eta;\alpha)] = \mathcal{S}_{\arg c=\pi/2+\varepsilon}[\lambda_{\tau_1}(t,c,\eta;\tilde{\alpha})]\big|_{\tilde{\alpha}=(1+e^{2\pi i c\eta})\,\alpha}. \tag{3.28}$$

This connection formula describes the discontinuous change of 1-parameter solution caused by the parametric Stokes phenomenon explicitly. This is an analogue of the result obtained in [14, 16] for the Weber equation.

Next let us consider a parametric Stokes phenomenon at another point. For example, here we discuss the case that t moves on a sufficiently small neighborhood of $t = t_1$ in Figure 4. In this case we have the following connection formula.

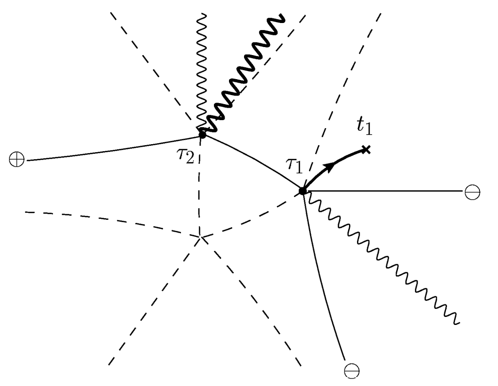

Figure 10. Integration path from τ_1 to t (near t_1).

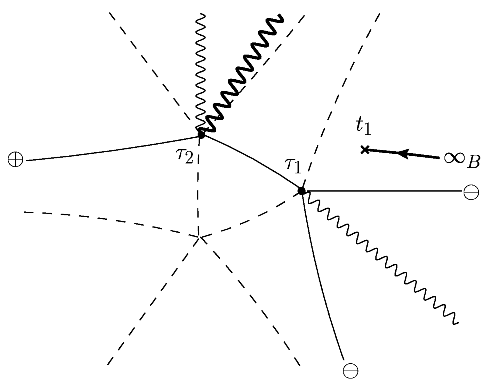

Figure 11. Integration path from ∞ to t (near t_1).

Theorem 3.6 (Connection formula for 1-parameter solutions at $t = t_1$). *Let ε be a sufficiently small positive number and $\lambda(t, c, \eta; \alpha)$ be the 1-parameter solution $\lambda_{\tau_1}(t, c, \eta; \alpha)$ normalized along the path in Figure 10, or the 1-parameter solution $\lambda_{\infty_B}(t, c, \eta; \alpha)$ normalized along the path in Figure 11. For the both cases, the following relation holds:*

$$\mathcal{S}_{\arg c=\pi/2-\varepsilon}[\lambda(t, c, \eta; \alpha)] = \mathcal{S}_{\arg c=\pi/2+\varepsilon}[\lambda(t, c, \eta; \tilde{\alpha})]\big|_{\tilde{\alpha}=\alpha}. \tag{3.29}$$

The formula (3.29) implies that no parametric Stokes phenomenon occurs for both $\lambda_{\tau_1}(t, c, \eta; \alpha)$ and $\lambda_{\infty_B}(t, c, \eta; \alpha)$ at $t = t_1$. Since the P-Voros coefficient W_{Γ_B} defined by (3.18) which represents the difference of $\lambda_{\tau_1}(t, c, \eta; \alpha)$ and $\lambda_{\infty_B}(t, c, \eta; \alpha)$ equals to 0 by Theorem 3.1, we have $\lambda_{\tau_1}(t, c, \eta; \alpha) = \lambda_{\infty_B}(t, c, \eta; \alpha)$ in a neighborhood of $t = t_1$. Furthermore, Theorem 3.4 implies that parametric Stokes phenomena do not occur for λ_{∞_B} because it is normalized along the path in Figure 11 which never touches with any P-turning points and P-Stokes curves. Thus we have the connection formula (3.29).

§ 4. Stokes multipliers of (SL_{II}) and connection formulas

In Section 3, we described parametric Stokes phenomena through the concrete representation of P-Voros coefficients and derived the relation for the parameters α and $\tilde{\alpha}$. (Cf., Theorem 3.5 and Theorem 3.6.) In this section we confirm the validity of the relation given in Theorem 3.6 from the view point of the isomonodromic deformation.

It is known that the second Painlevé equation (or Hamiltonian systems equivalent to it) represents the compatibility condition for a certain second oreder linear differential equation and its deformation equation (e.g., [5, 13]). Accordingly, the Hamiltonian system (H_{II}) with a large parameter η also represents the compatibility condition for the following Schrödinger equation (SL_{II}) and its deformation equation (D_{II}) ([7]):

$$\begin{cases} (SL_{\mathrm{II}}): \ \left(\dfrac{\partial^2}{\partial x^2} - \eta^2 Q_{\mathrm{II}}\right)\psi = 0, \\ (D_{\mathrm{II}}): \ \dfrac{\partial \psi}{\partial t} = A_{\mathrm{II}}\dfrac{\partial \psi}{\partial x} - \dfrac{1}{2}\dfrac{\partial A_{\mathrm{II}}}{\partial x}\psi, \end{cases}$$

where

$$Q_{\mathrm{II}} = x^4 + tx^2 + 2cx + 2K_{\mathrm{II}} - \eta^{-1}\frac{\nu}{x-\lambda} + \eta^{-2}\frac{3}{4(x-\lambda)^2},$$

$$K_{\mathrm{II}} = \frac{1}{2}\Big[\nu^2 - (\lambda^4 + t\lambda^2 + 2c\lambda)\Big], \quad A_{\mathrm{II}} = \frac{1}{2(x-\lambda)}.$$

The regular singular point $x = \lambda$ of (SL_{II}) is an apparent singular point because K_{II} has the above form. It is important that the monodromy data of (SL_{II}) computed by a fundamental system of solutions which satisfy both (SL_{II}) and (D_{II}) do not depend on t. (See [5, 13] for example.) Here monodromy data imply Stokes mulitipliers around $x = \infty$ in this case. Note that the system (SL_{II}) and (D_{II}) is obtained from the Jimbo-Miwa's Lax pair in [5, Appendix C].

Remark 3. In this section we only show the result for the case that t moves on a sufficiently small neighborhood of $t = t_1$ in Figure 4. See [4] for the computation of Stokes multipliers of (SL_{II}) when t moves near $t = t_0$ in Figure 4. We note that Stokes multipliers when t moves on other regions can be computed by the same way presented in [4].

First we will show the Stokes geometry of (SL_{II}). (See [9, Definition 2.4, Definition 2.6] for the definitions of turning points and Stokes curves.) (SL_{II}) has a double turning point at $x = \lambda_0$ and two simple turning points $x = a_1$ and a_2. Figure 12 $\sim$ 14 describe the Stokes curves of (SL_{II}) with t being sufficiently close to $t = t_1$ in Figure 4 and $\arg c$ varying near $\pi/2$. It is observed in Figure 13 that the Stokes geometry of (SL_{II}) does not degenerate when $\arg c = \pi/2$ even though the P-Stokes geometry degenerates. This is a big difference between the cases when t moves near t_0 and t_1. (See Remark 4.)

As is shown in [4], similarly to the construction of WKB solutions for linear equations, we can construct the following formal solutions $\psi_{\pm,\mathrm{IM}}$ which satisfy both (SL_{II}) and (D_{II}) with a 1-parameter solution $(\lambda(t,c,\eta;\alpha),\ \nu(t,c,\eta;\alpha))$ substituted into their coefficients:

$$\psi_{\pm,\mathrm{IM}} = e^{\pm U/2}\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left[\pm\left\{\eta\int_{a_1}^{x} S_{-1}dx + \int_{\infty}^{x}\big(S_{\mathrm{odd}} - \eta S_{-1}\big)dx\right\}\right]. \tag{4.1}$$

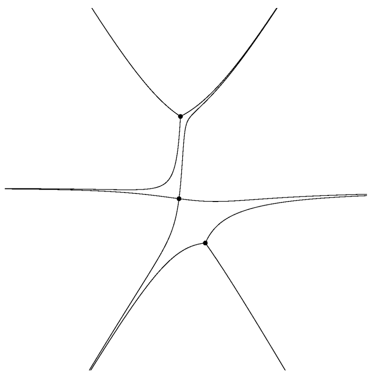

Figure 12. Stokes curves of (SL_{II}) at $t = t_1$ when $c = i + 0.25$.

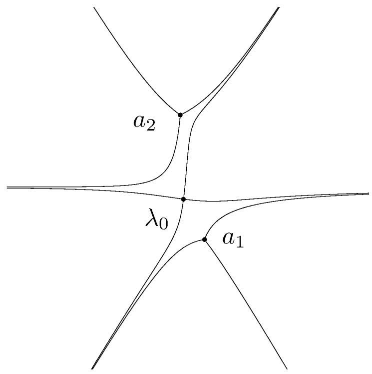

Figure 13. Stokes curves of (SL_{II}) at $t = t_1$ when $c = i$.

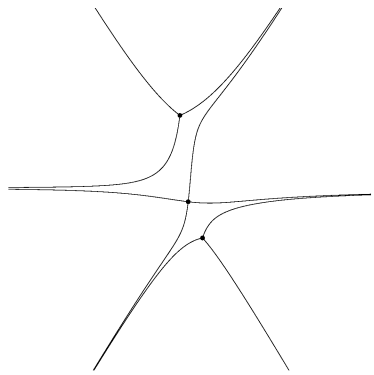

Figure 14. Stokes curves of (SL_{II}) at $t = t_1$ when $c = i - 0.25$.

Here $U = U(t, c, \eta; \alpha)$ is given by the following integral along the path in Figure 11

$$U = \eta \int_{\infty}^{t} \big(\lambda(t, c, \eta; \alpha) - \lambda_0(t, c)\big) dt, \tag{4.2}$$

$S_{\mathrm{odd}} = S_{\mathrm{odd}}(x, t, c, \eta; \alpha)$ is the odd part of a formal solution S of the Riccati equation

$$S^2 + \frac{\partial S}{\partial x} = \eta^2 Q_{\mathrm{II}}, \tag{4.3}$$

S_{-1} is the leading term of S_{odd} and a_1 is a simple turning point of (SL_{II}). See [4] for a construction of $\psi_{\pm,\mathrm{IM}}$. Here “IM” stands for Iso-Monodromic.

Now let us compute the Stokes multipliers of (SL_{II}) around $x = \infty$ by using $\psi_{\pm,\mathrm{IM}}$. Since a neighborhood of $x = \infty$ is divided into the six regions Ω_j and Ω'_j $(1 \leq j \leq 6)$ by Stokes curves as in Figures 15 and 16, we obtain six Stokes multipliers around $x = \infty$ for $\arg c = \pi/2 - \varepsilon$ and $\arg c = \pi/2 + \varepsilon$, respectively. Let $\mathfrak{s}_j = \mathfrak{s}_j(c, \eta; \alpha)$ (resp., $\mathfrak{s}'_j = \mathfrak{s}'_j(c, \eta; \alpha)$) be the Stokes multipliers corresponding to the analytic continuation from Ω_j to Ω_{j+1} (resp., from Ω'_j to Ω'_{j+1}) $(1 \leq j \leq 6$ and $\Omega_7 = \Omega_1)$. That is, for $1 \leq k \leq 3$, $\mathfrak{s}_{2k-1}$ and $\mathfrak{s}_{2k}$ are defined by

$$\begin{pmatrix} \psi_{+,\mathrm{IM}}^{\Omega_{2k}} \\ \psi_{-,\mathrm{IM}}^{\Omega_{2k}} \end{pmatrix} = \begin{pmatrix} 1 & \mathfrak{s}_{2k-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_{+,\mathrm{IM}}^{\Omega_{2k-1}} \\ \psi_{-,\mathrm{IM}}^{\Omega_{2k-1}} \end{pmatrix},$$

$$\begin{pmatrix} \psi_{+,\mathrm{IM}}^{\Omega_{2k+1}} \\ \psi_{-,\mathrm{IM}}^{\Omega_{2k+1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathfrak{s}_{2k} & 1 \end{pmatrix} \begin{pmatrix} \psi_{+,\mathrm{IM}}^{\Omega_{2k}} \\ \psi_{-,\mathrm{IM}}^{\Omega_{2k}} \end{pmatrix},$$

where $\psi_{\pm,\mathrm{IM}}^{\Omega_j}$ is the Borel sum of $\psi_{\pm,\mathrm{IM}}$ in the region Ω_j. These Stokes multipliers can be computed by using the “Voros' connection formula” (cf. [9, Theorem 2.23], [17]) on

Stokes curves emanating from simple turning points, and another connection formula on Stokes curves emanating from the double turning point $x = \lambda_0$. The latter connection formula is obtained through the "transformation theory" at the double turning point $x = \lambda_0$ established in [1], [8]. (See also [15].) We omit the details of the computation of the Stokes multipliers because they can be computed in a similar way presented in [4]. The results of the Stokes multipliers of (SL_{II}) computed by using the formal solutions $(\psi_{+,\mathrm{IM}}, \psi_{-,\mathrm{IM}})$ are as follows:

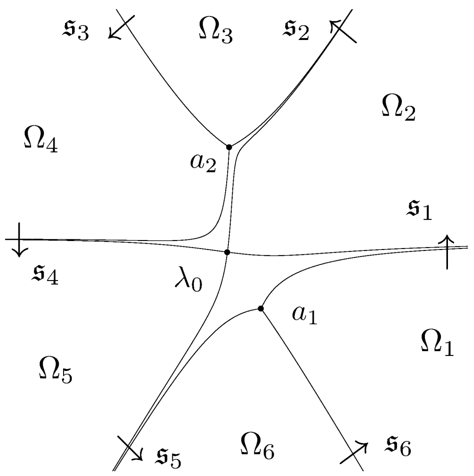

Figure 15. Stokes multipliers of (SL_{II}) when $\arg c = \pi/2 - \varepsilon$.

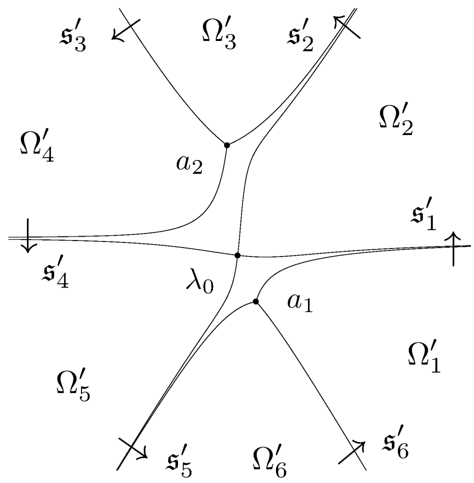

Figure 16. Stokes multipliers of (SL_{II}) when $\arg c = \pi/2 + \varepsilon$.

Stokes multipliers of (SL_{II}) at $t = t_1$. *Let $\lambda_{\tau_1}(t, c, \eta; \alpha)$ (resp., $\lambda_\infty(t, c, \eta; \alpha)$) be the 1-parameter solution of (P_{II}) normalized at $t = \tau_1$ (resp., $t = \infty$) whose normalization path is taken as in Figure 10 (resp., Figure 11).*
(i) If the 1-parameter solution $\lambda_\infty(t, c, \eta; \alpha)$ is substituted into the coefficients of (SL_{II}) and (D_{II}), then the corresponding Stokes multipliers $\mathfrak{s}_j = \mathfrak{s}_j(c, \eta; \alpha)$ and $\mathfrak{s}'_j = \mathfrak{s}'_j(c, \eta; \alpha)$ are given by the following:

$$(4.4)\qquad \begin{cases} \mathfrak{s}_1 = \mathfrak{s}'_1 = i\, e^{U-2V} + 2\sqrt{\pi}\,\alpha \\ \mathfrak{s}_2 = \mathfrak{s}'_2 = i\, e^{-2\pi i c\eta}\, e^{2V-U} \\ \mathfrak{s}_3 = \mathfrak{s}'_3 = i\, e^{2\pi i c\eta}\, e^{U-2V} \\ \mathfrak{s}_4 = \mathfrak{s}'_4 = i\, e^{2\pi i c\eta}\, e^{2V-U} - 2\sqrt{\pi}\,\alpha \\ \mathfrak{s}_5 = \mathfrak{s}'_5 = i\, e^{U-2V} \\ \mathfrak{s}_6 = \mathfrak{s}'_6 = i\, e^{2V-U}. \end{cases}$$

(ii) If the 1-parameter solution $\lambda_{\tau_1}(t, c, \eta; \alpha)$ is substituted into the coefficients of (SL_{II}) and (D_{II}), then the corresponding Stokes multipliers $\mathfrak{s}_j = \mathfrak{s}_j(c, \eta; \alpha)$ and $\mathfrak{s}'_j = \mathfrak{s}'_j(c, \eta; \alpha)$

are given by the following

$$\begin{cases} \mathfrak{s}_1 = \mathfrak{s}'_1 = i\,e^{U-2V} + 2\sqrt{\pi}\,\alpha\,e^{W_{\Gamma_B}} \\ \mathfrak{s}_2 = \mathfrak{s}'_2 = i\,e^{-2\pi ic\eta}\,e^{2V-U} \\ \mathfrak{s}_3 = \mathfrak{s}'_3 = i\,e^{2\pi ic\eta}\,e^{U-2V} \\ \mathfrak{s}_4 = \mathfrak{s}'_4 = i\,e^{2\pi ic\eta}\,e^{2V-U} - 2\sqrt{\pi}\,\alpha\,e^{W_{\Gamma_B}} \\ \mathfrak{s}_5 = \mathfrak{s}'_5 = i\,e^{U-2V} \\ \mathfrak{s}_6 = \mathfrak{s}'_6 = i\,e^{2V-U}. \end{cases} \tag{4.5}$$

Here α is the free parameter contained in the 1-parameter solution substituted into the coefficients of (SL_{II}) and (D_{II}), $U = U(t, c, \eta; \alpha)$ is given by (4.2), $V = V(t, c, \eta; \alpha)$ is the Voros coefficient of (SL_{II}) defined by

$$V(t, c, \eta; \alpha) = \int_{a_1}^{\infty} \big(S_{\mathrm{odd}}(x, t, c, \eta; \alpha) - \eta S_{-1}(x, t, c)\big)dx, \tag{4.6}$$

and $W_{\Gamma_B} = W_{\Gamma_B}(c, \eta)$ is the P-Voros coefficient defined by (3.18).

In the case of [4] (i.e., t moves near t_0) because the geometry of Stokes curves of (SL_{II}) changes discontinuously before and after $\arg c = \pi/2$, the expressions of Stokes multipliers also change. (See Remark 4 below.) However, in this case (i.e., t moves near t_1) the expressions of the Stokes multipliers $\mathfrak{s}_j$ and $\mathfrak{s}'_j$ coincide for all $1 \le j \le 6$ since the geometry of Stokes curves does not change before and after $\arg c = \pi/2$.

We also found the following fact about the Voros coefficient V of (SL_{II}).

Theorem 4.1. *The formal series $2V - U$ does not depend on t, and it is represented explicitly as follows:*

$$2V(t, c, \eta; \alpha) - U(t, c, \eta; \alpha) = 0. \tag{4.7}$$

Proof. First we note that V and U are expanded as

$$V(t, c, \eta; \alpha) = V^{(0)}(t, c, \eta) + \alpha\eta^{-1/2}V^{(1)}(t, c, \eta)e^{\eta\phi_{\mathrm{II}}} + (\alpha\eta^{-1/2})^2V^{(2)}(t, c, \eta)e^{2\eta\phi_{\mathrm{II}}} + \cdots,$$

$$U(t, c, \eta; \alpha) = U^{(0)}(t, c, \eta) + \alpha\eta^{-1/2}U^{(1)}(t, c, \eta)e^{\eta\phi_{\mathrm{II}}} + (\alpha\eta^{-1/2})^2U^{(2)}(t, c, \eta)e^{2\eta\phi_{\mathrm{II}}} + \cdots,$$

where $V^{(k)}(t, c, \eta)$ and $U^{(k)}(t, c, \eta)$ are formal power series in η^{-1} $(k \ge 0)$. As is shown in [4], differentiating (4.6) by t and using the equality

$$\frac{\partial}{\partial t}S_{\mathrm{odd}} = \frac{\partial}{\partial x}(A_{\mathrm{II}}S_{\mathrm{odd}}),$$

which is shown in [1], we can check that the formal series V satisfies

$$\frac{\partial}{\partial t}V = \frac{1}{2}\eta(\lambda - \lambda_0).$$

Therefore the formal series $2V - U$ does not depend on t. Especially, it does not contain exponential factors $e^{k\eta\phi}$ $(k \geq 1)$; i.e., $2V - U = 2V^{(0)} - U^{(0)}$ and it still does not depend on t. Moreover, it is also proved in [4] that the formal power series $V^{(0)}(t,c,\eta)$ satisfies the following difference equation (although it was considered in the case that t is sufficiently close to t_0 in [4], we also obtain the following difference equation in this case by completely the same method):

$$
\begin{aligned}
2V^{(0)}(t,c,\eta) - 2V^{(0)}(t,c-\eta^{-1},\eta) = {} & \frac{4}{3}\eta\big(\lambda_0(t,c)^3 - \lambda_0(t,c-\eta^{-1})^3\big) \\
& - c\eta \log\left\{\frac{2\lambda_0(t,c-\eta^{-1})^2 + t}{2\lambda_0(t,c)^2 + t}\right\} \\
& - \log\left\{\frac{2\lambda^{(0)}(t,c,\eta)^2 + t - 2\nu^{(0)}(t,c,\eta)}{2\lambda_0(t,c-\eta^{-1})^2 + t}\right\}.
\end{aligned} \tag{4.8}
$$

In order to evaluate $2V^{(0)} - U^{(0)}$, we take the limit $t \to \infty_B$ of (4.8) along the P-Stokes curve Γ_B in Figure 5. As in Figure 5 λ_0 behaves as (3.8) and we can easily check that

$$
\begin{aligned}
\lambda^{(0)}(t,c,\eta) &= -ct^{-1} + O(t^{-2}), \\
\nu^{(0)}(t,c,\eta) &= O(t^{-2}).
\end{aligned}
$$

Thus, taking the limit in the equality (4.8), we can see that the formal power series $V^{(0)}(\infty,c,\eta)$ satisfies

$$
2V^{(0)}(\infty,c,\eta) - 2V^{(0)}(\infty,c-\eta^{-1},\eta) = 0. \tag{4.9}
$$

On the other hand, as is shown in Appendix A of [4], $V^{(0)}(t,c,\eta)$ has the following homogeneity:

$$
V^{(0)}(r^{-2/3}t, r^{-1}c, r\eta) = V^{(0)}(t,c,\eta) \quad (r \neq 0).
$$

Therefore, the limit $V^{(0)}(\infty,c,\eta)$ satisfies

$$
V^{(0)}(\infty, r^{-1}c, r\eta) = V^{(0)}(\infty,c,\eta) \quad (r \neq 0).
$$

This implies that $V^{(0)}(\infty,c,\eta)$ is a formal power series of the form

$$
V^{(0)}(\infty,c,\eta) = \sum_{n\geq 1} v_n (c\eta)^{-n}, \tag{4.10}
$$

where $v_n \in \mathbb{C}$ is independent of c $(n \geq 1)$. Then it follows from (4.9) and (4.10) that $V^{(0)}(\infty,c,\eta) = 0$. Since $U^{(0)}(t,c,\eta)$ also tends to 0 in the limit $t \to \infty_B$ by definition, we have (4.7). $\square$

Now we derive the connection formulas describing the parametric Stokes phenomena at $t = t_1$ by using the explicit expressions of the Stokes multipliers of (SL_{II}) in the above lists. If the Borel sums of the 1-parameter solution $\lambda(t, c, \eta; \alpha)$ for arg $c = \pi/2 - \varepsilon$ and that of $\lambda(t, c, \eta; \tilde{\alpha})$ for arg $c = \pi/2 + \varepsilon$ coincide, then the corresponding Stokes multipliers represented by $\mathfrak{s}_j(c, \eta; \alpha)$ and $\mathfrak{s}'_j(c, \eta; \tilde{\alpha})$ of (SL_{II}) should coincide; that is,

$$\mathcal{S}_{\arg c=\pi/2-\varepsilon}[\mathfrak{s}_j(c, \eta; \alpha)] = \mathcal{S}_{\arg c=\pi/2+\varepsilon}[\mathfrak{s}'_j(c, \eta; \tilde{\alpha})] \quad (1 \leq j \leq 6) \tag{4.11}$$

should hold. Comparing the Borel sums of $\mathfrak{s}_j(c, \eta; \alpha)$ and $\mathfrak{s}'_j(c, \eta; \tilde{\alpha})$ given by (4.4), those are Stokes multipliers corresponding to the 1-parameter solution λ_∞, we find that the equalities (4.11) hold if and only if

$$\alpha = \tilde{\alpha}. \tag{4.12}$$

Similarly, in the case that $\lambda_{\tau_1}(t, c, \eta; \alpha)$ being substituted, we have the same conclusion as (4.12) because the P-Voros coefficient $W_{\Gamma_B}(c, \eta)$ equals to 0 by Theorem 3.1. The result (4.12) is consistent with (3.29); that is, parametric Stokes phenomena do not happen to both λ_{τ_1} and λ_∞. The relation $\alpha = \tilde{\alpha}$ obtained above is consistent with that in Theorem 3.6, and hence, it indicates the validity of Theorem 3.6.

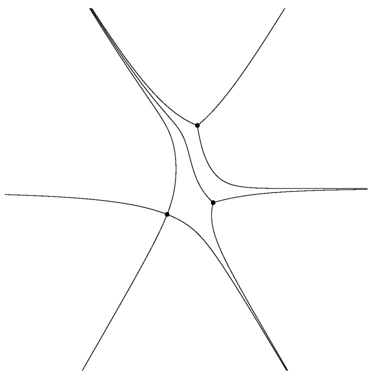

Figure 17. Stokes curves of (SL_{II}) at $t = t_0$ when $c = i + 0.25$.

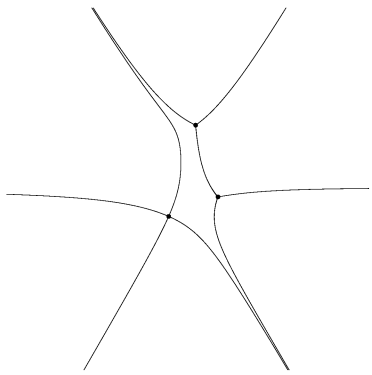

Figure 18. Stokes curves of (SL_{II}) at $t = t_0$ when $c = i$.

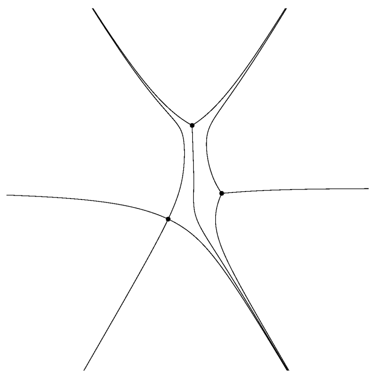

Figure 19. Stokes curves of (SL_{II}) at $t = t_0$ when $c = i - 0.25$.

Remark 4. As is shown in [4], if we consider a parametric Stokes phenomenon at $t = t_0$ through the computation of the Stokes multipliers of (SL_{II}), we can derive another connection formula (3.28). The reason is that, at $t = t_0$, we obtain the following explicit representation of the Voros coefficient V instead of Theorem 4.1:

$$2V(t, c, \eta; \alpha) - U(t, c, \eta; \alpha) = -\sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n} (c\eta)^{1-2n}, \tag{4.13}$$

where B_{2n} is the 2n-th Bernoulli number defined by (3.12). Furthermore, there is a difference in the Stokes geometry of (SL_{II}) between $t = t_0$ and t_1. Figure 17 $\sim$ 19 describes the Stokes curves of (SL_{II}) at $t = t_0$. In this case, the P-Stokes geometry of (P_{II}) and the Stokes geometry of (SL_{II}) degenerates simultaneously. This phenomenon does not observed at $t = t_1$.

Appendix A. Another example: the third Painlevé equation of type D_6

We will make a brief report on results about the Voros coefficients of the third Painlevé equation of the type D_6 (in the sense of [12]) with a large parameter η in the following form:

$$(P_{\mathrm{III}'}) : \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\Big(\frac{d\lambda}{dt}\Big)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \eta^2\Big(\frac{\lambda^3}{t^2} - \frac{c_\infty \lambda^2}{t^2} + \frac{c_0}{t} - \frac{1}{\lambda}\Big).$$

Here c_∞ and c_0 are complex parameters. We assume that they satisfy

$$c_\infty, c_0, c_\infty \pm c_0 \neq 0, \tag{A.1}$$

so that all P-turning points are simple. See [9, Definition 4.5] for the definitions of P-turning points and P-Stokes curves of $(P_{\mathrm{III}'})$.

We can also construct 1-parameter solutions of $(P_{\mathrm{III}'})$ in the following form:

$$\lambda(t, \mathbf{c}, \eta; \alpha) = \lambda^{(0)}(t, \mathbf{c}, \eta) + \alpha\eta^{-1/2}\lambda^{(1)}(t, \mathbf{c}, \eta)e^{\eta\phi_{\mathrm{III}'}} + (\alpha\eta^{-1/2})^2\lambda^{(2)}(t, \mathbf{c}, \eta)e^{2\eta\phi_{\mathrm{III}'}} + \cdots .$$

Here $\mathbf{c} = (c_\infty, c_0)$, α is a free parameter, $\lambda^{(k)}(t, \mathbf{c}, \eta) = \lambda_0^{(k)}(t, \mathbf{c}) + \eta^{-1}\lambda_1^{(k)}(t, \mathbf{c}) + \eta^{-2}\lambda_2^{(k)}(t, \mathbf{c}) + \cdots$ is a formal power series $(k \geq 0)$ and

$$\phi_{\mathrm{III}'} = \phi_{\mathrm{III}'}(t, \mathbf{c}) = \int^t \sqrt{\Delta(t, \mathbf{c})}\, dt, \quad \Delta(t, \mathbf{c}) = \frac{3{\lambda_0}^2}{t^2} - \frac{2c_\infty {\lambda_0}^2}{t^2} + \frac{1}{{\lambda_0}^2}, \tag{A.2}$$

where λ_0 (which is nothing but $\lambda_0^{(0)}$) is an algebraic function satisfying

$$\frac{{\lambda_0}^3}{t^2} - \frac{c_\infty {\lambda_0}^2}{t^2} + \frac{c_0}{t} - \frac{1}{\lambda_0} = 0. \tag{A.3}$$

Figures 20 $\sim$ 22 describe P-Stokes curves of $(P_{\mathrm{III}'})$ near $\mathbf{c} = (2, 2-i)$. We can observe that degeneration of P-Stokes geometry happens when $\mathbf{c} = (2, 2-i)$.

Similarly to (P_{II}), parametric Stokes phenomena relevant to the degeneration can be analyzed by investigating P-Voros coefficients of $(P_{\mathrm{III}'})$ defined by the following integral:

$$W(\mathbf{c}, \eta) = \int_\tau^\infty \big(R_{\mathrm{odd}}(t, \mathbf{c}, \eta) - \eta R_{-1}(t, \mathbf{c})\big)\, dt. \tag{A.4}$$

Here τ is a P-turning point of $(P_{\mathrm{III}'})$, $R_{\mathrm{odd}}(t, \mathbf{c}, \eta)$ is the odd part of a formal power series solution $R(t, \mathbf{c}, \eta)$ of the following equation:

$$R^2 + \frac{dR}{dt} = \Big(\frac{2}{\lambda^{(0)}}\frac{d\lambda^{(0)}}{dt} - \frac{1}{t}\Big)R - \Big(\frac{1}{\lambda^{(0)}}\frac{d\lambda^{(0)}}{dt}\Big)^2 + \eta^2\Big(\frac{3{\lambda^{(0)}}^2}{t^2} - \frac{2c_\infty\lambda^{(0)}}{t^2} + \frac{1}{{\lambda^{(0)}}^2}\Big), \tag{A.5}$$

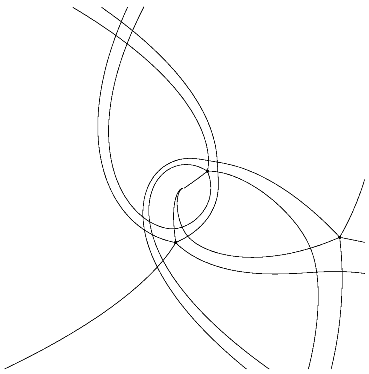

Figure 20. $\mathbf{c} = (2-0.1, 2-i)$.

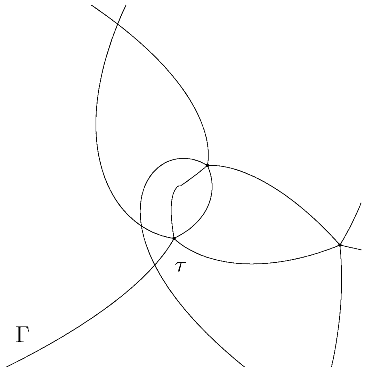

Figure 21. $\mathbf{c} = (2, 2-i)$.

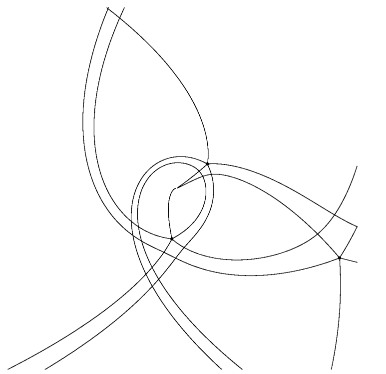

Figure 22. $\mathbf{c} = (2+0.1, 2-i)$.

which is the Riccati equation associated with the Fréchet derivative of $(P_{\mathrm{III}'})$ at the 0-parameter solution $\lambda^{(0)}(t, \mathbf{c}, \eta)$, and $R_{-1}(t, \mathbf{c}) = \sqrt{\Delta(t, \mathbf{c})}$ is the leading term of R_{odd}. (In this article we do not discuss about the choice of the branch of $R_{-1} = \sqrt{\Delta(t, \mathbf{c})}$.)

Theorem A.1. *P-Voros coefficient* (A.4) *integrated along the P-Stokes curve* Γ *in Figure 21 has the following explicit representation:*

$$W(\boldsymbol{c}, \eta) = \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n} \left(\frac{c_\infty - c_0}{2} \eta \right)^{1-2n}, \tag{A.6}$$

where B_{2n} is the 2n-th Bernoulli number defined by (3.12).

The explicit form (A.6) of the P-Voros coefficient closely resembles to (3.10). In this case the factor $(c_\infty - c_0)/2$ plays the role of c in the analysis of the parametric Stokes phenomenon of (P_{II}). Let $\lambda_\tau(t, \mathbf{c}, \eta; \alpha)$ be the 1-parameter solution normalized at P-turning point $t = \tau$, which is defined similar to the case of (P_{II}). Theorem A.1 and Corollary 3.3 imply that we have have the following connection formula for this 1-parameter solution:

Connection formula for the 1-parameter solution λ_τ. *Let ε be a sufficiently small positive number. If we take parameters as*

$$\tilde{\alpha} = (1 + e^{\pi i (c_\infty - c_0)\eta})\, \alpha, \tag{A.7}$$

then the generalized Borel sum of the 1-parameter solution $\lambda_\tau(t, \boldsymbol{c}, \eta; \alpha)$ when $\boldsymbol{c} = (2 - \varepsilon, 2 - i)$ and that of $\lambda_\tau(t, \boldsymbol{c}, \eta; \tilde{\alpha})$ when $\boldsymbol{c} = (2 + \varepsilon, 2 - i)$ coincide.

We have also computed explicit representations of P-Voros coefficients with all other choices of the integration paths in (A.4). The results will be presented in another article.

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