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Smooth-integrable and analytic-nonintegrable resonant Hamiltonians

By

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Abstract

This paper studies smooth integrability of an analytic-nonintegrable Hamiltonian system with resonance. In proving smooth integrability we construct first integrals as formal power series with an exponential factor, then we take the Borel sum of a formal integral with respect to the resonance variable. A first integral with an exponential factor was used in [3] or [5]. In this paper such factor is necessary because of a singular behavior caused by the resonance variables.

§ 1. Introduction

In [3] Bolsinov and Taimanov showed that there exists a Hamiltonian system related with geodesic flow on a Riemannian manifold which is C^∞-integrable and not C^ω-integrable. They also showed that non C^ω- integrability is closely related with the non Abelian property of the fundamental group of the manifold or with the monodromy of the geodesic flow. Then Gorni and Zampieri, [5] showed similar phenomena in the local analytic setting. In fact, their Hamiltonian has the form $H = -q_2p_2\partial_{q_1}r+(r^2+q_2\partial_{q_2}r)p_1$ in $\mathbb{R}^4$, where $r = q_1^2 + q_2^2$. Note that H has no linear part and has four resonance variables. In [3] and [5], the C^ω-nonintegrability was proved by geometrical arguments or elementary calculus, while C^∞-integrability was proved by concrete construction of smooth and non analytic first integrals.

Motivated by these results we will show that a similar phenomenon occurs for certain Hamiltonians with a pair of resonance variables. In fact we have C^ω- nonintegrability as well as smooth integrability of the Hamiltonian system. Our main point in

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this note is to introduce a method to show smooth integrability which does not depend on the concrete expression of a nonanalytic first integral. This is done in the following way. We first construct formal first integrals in a series containing an exponential factor. This idea comes from the fact that the first order system of n ordinary differential equations has an n-parameter family of formal solutions which are expanded in the series containing exponential factor. Moreover, an n-parameter family of formal solutions corresponds to n- functionally independent commuting formal first integrals. Next, for a formal first integral with an exponential factor, we give a meaning to the formal series by the Borel sum with respect to the resonance variable. We thus obtain functionally independent first integrals defined on some sector domain. In section 3 we will show smooth integrability as well as sectorial integrability by virtue of these summed first integrals.

This paper is organized as follows. In section 2 we construct functionally independent formal first integrals which are Borel summable in the resonance variable. In section 3 we study smooth integrability and sectorial integrability. In section 4 we briefly state C^ω- nonintegrability of our Hamiltonian system. The proof of the theorem in the last section will be published in [8].

§ 2. Construction of formal exponential series and summability

We write the variables in $\mathbb{R}^n$ or in $\mathbb{C}^n$ $(n \geq 2)$ in the form

$$(q_1, q_2, q_3, \cdots, q_n) = (q_1, q), \quad (p_1, p_2, p_3, \cdots, p_n) = (p_1, p)$$

in order to indicate the resonance variables q_1 and p_1. For an integer $\sigma \geq 1$ let $H = H_0 + H_1$ be the Hamiltonian function with H_0 and H_1 given, respectively, by

$$H_0 = q_1^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j, \tag{2.1}$$

$$H_1 = \sum_{j=2}^{n} q_j^2 B_j(q_1, q_1^{2\sigma} p_1, q), \tag{2.2}$$

where $B_j(q_1, s, t)$'s are holomorphic at the origin with respect to $(q_1, s, t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^{n-1}$. Consider the Hamiltonian system

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2, \ldots, n. \tag{2.3}$$

For the Hamitonian function H, define the Hamiltonian vector field χ_H by

$$\chi_H := \{H, \cdot\} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right), \tag{2.4}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket. We say that ϕ is the first integral of χ_H if $\chi_H\phi = 0$. We say that v is the formal integral of χ_H if $\chi_H v = 0$ as a formal power series. Eq. (2.3) is said to be C^ω-Liouville integrable if there exist first integrals $\phi_j \in C^\omega$ $(j = 1,\ldots,n)$ which are functionally independent on an open dense set and Poisson commuting, i.e., $\{\phi_j,\phi_k\} = 0$, $\{H,\phi_k\} = 0$. If $\phi_j \in C^\infty$ $(j = 1,\ldots,n)$, then we say C^∞- Liouville integrable.

We assume

$$B_j = B_j(q_1, q_1^{2\sigma}p_1, q) = B_{j,0}(q_1,q) + q_1^{2\sigma}p_1 B_{j,1}(q_1,q), \quad 2 \le j \le n, \tag{2.5}$$

where $B_{j,0}$ and $B_{j,1}$ are analytic at $q_1 = 0$, $q = 0$. Moreover, we suppose

$$\lambda_j \ (j = 2,3,\ldots,n) \text{ are linearly independent over } \mathbb{Z}. \tag{2.6}$$

Set

$$E_c \equiv E_c(q_1) := \exp\left(\frac{cq_1^{-2\sigma+1}}{(2\sigma-1)}\right) \tag{2.7}$$

and look for the formal first integral v in the form (cf. [1], [4])

$$v = \phi^{(\alpha)}(q_1,p_1,q,p)E^\alpha, \tag{2.8}$$

where $E^\alpha = E_{\lambda_2}^{\alpha_2}\cdots E_{\lambda_n}^{\alpha_n}$, and $\phi^{(\alpha)}(q_1,p_1,q,p)$ is a formal power series of q_1, q, p_1 and p. Denote by e_j the j-th unit vector, $(0,\cdots,1,\cdots 0)$ $(j = 2,3,\ldots,n)$. Then we have

Theorem 2.1. *Assume (2.5) and (2.6). Then χ_H has $2(n-1)$ functionally independent formal first integrals $\phi_j^{(0)}$ and $\phi^{(e_j)}E^{e_j}$ with $\alpha = 0$ and $\alpha = e_j$ in (2.8), respectively such that*

$$\phi_j^{(0)}(q_1,p_1,q,p) = p_j(q_j + A_j(q_1,p_1,q)) + \tilde{A}_j(q_1,p_1,q), \tag{2.9}$$

$$\phi^{(e_j)}(q_1,p_1,q,p) = p_j(q_j^2 + C_j(q_1,p_1,q)) + \tilde{C}_j(q_1,p_1,q), \tag{2.10}$$

and

$$A_j(q_1,p_1,q), \tilde{A}_j(q_1,p_1,q) = O(|q|^2), \quad C_j(q_1,p_1,q), \tilde{C}_j(q_1,p_1,q) = O(|q|^3), \tag{2.11}$$

when $q \to 0$ and where A_j, $\tilde{A}_j$, C_j and $\tilde{C}_j$ are polynomials of p_1 for $j = 2,3,\ldots,n$.

Although Theorem 2.1 can be proved by the similar method as in [7], we give the proof because the expressions of the formal first integrals like (2.9) and (2.10) are used in the next section.

Proof. By definition we have, for $\mathcal{L} := \{H_0,\cdot\}$ and $R := \{H_1,\cdot\}$,

$$\mathcal{L} = q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}p_1\frac{\partial}{\partial p_1} + \sum_{j=2}^{n}\lambda_j\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j}\right), \tag{2.12}$$

$$R = \sum_{j=2}^{n}\left(-2q_jB_j\frac{\partial}{\partial p_j}+q_j^2(\partial_{p_1}B_j)\frac{\partial}{\partial q_1}-q_j^2(\partial_{q_1}B_j)\frac{\partial}{\partial p_1}-q_j^2\nabla_qB_j\cdot\frac{\partial}{\partial p}\right). \tag{2.13}$$

By using the formula

$$\partial_{p_1}B_j = B_{j,1}q_1^{2\sigma}, \quad q_1^{2\sigma}(\partial/\partial q_1)E^\alpha = -(\sum_{j=2}^{n}\lambda_j\alpha_j)E^\alpha = -\langle\lambda,\alpha\rangle E^\alpha,$$

we have

$$\mathcal{L}(\phi^{(\alpha)}E^\alpha) = E^\alpha\left(q_1^{2\sigma}\frac{\partial}{\partial q_1}-2\sigma q_1^{2\sigma-1}p_1\frac{\partial}{\partial p_1}+\sum_{j=2}^{n}\lambda_j\left(q_j\frac{\partial}{\partial q_j}-p_j\frac{\partial}{\partial p_j}-\alpha_j\right)\right)\phi^{(\alpha)}, \tag{2.14}$$

and

$$R(\phi^{(\alpha)}E^\alpha) = E^\alpha\left(-\langle\lambda,\alpha\rangle\sum_{j=2}^{n}q_j^2B_{j,1}+R\right)\phi^{(\alpha)}. \tag{2.15}$$

It follows that if $v = E^\alpha\phi^{(\alpha)}$ is a formal first integral of χ_H, then $\phi^{(\alpha)}$ satisfies

$$\left(q_1^{2\sigma}\frac{\partial}{\partial q_1}-2\sigma q_1^{2\sigma-1}p_1\frac{\partial}{\partial p_1}+\sum_{j=2}^{n}\lambda_j\left(q_j\frac{\partial}{\partial q_j}-p_j\frac{\partial}{\partial p_j}-\alpha_j\right)\right)\phi^{(\alpha)} + \left(-\sum_{j=2}^{n}\langle\lambda,\alpha\rangle q_j^2B_{j,1}+R\right)\phi^{(\alpha)} = 0. \tag{2.16}$$

Expand $\phi^{(\alpha)}$ into the formal power series

$$\phi^{(\alpha)} = \sum_{\nu,k,\ell}\phi^{(\alpha)}_{\nu,k,\ell}(q_1)p_1^\nu p^k q^\ell, \tag{2.17}$$

then, insert the expansion into (2.16) and compare the coefficients of $p_1^\nu p^k q^\ell$. One can easily see that the first term of the left-hand side of (2.16) yields

$$\left(q_1^{2\sigma}\frac{\partial}{\partial q_1}-2\sigma q_1^{2\sigma-1}\nu+\lambda\cdot(\ell-k-\alpha)\right)\phi^{(\alpha)}_{\nu,k,\ell}(q_1). \tag{2.18}$$

Hence we obtain the recurrence relation like

$$\left(q_1^{2\sigma}\frac{\partial}{\partial q_1}-2\sigma q_1^{2\sigma-1}\nu+\lambda\cdot(\ell-k-\alpha)\right)\phi^{(\alpha)}_{\nu,k,\ell}(q_1) = F, \tag{2.19}$$

where F denotes terms which appear from the second term of the left-hand side of (2.16).

In order to get the detailed expression of F we first note

$$-2q_jB_j\frac{\partial}{\partial p_j}\phi^{(\alpha)} = -2B_j\sum \phi^{(\alpha)}_{\nu,k+e_j,\ell-e_j}(q_1)p_1^\nu p^k q^\ell(k_j+1). \tag{2.20}$$

Expand B_j into the power series of q and compare the coefficients of $p_1^\nu p^k q^\ell$ of the right-hand side. One can see that the terms containing $\phi^{(\alpha)}_{\nu,k+e_j,\mu}(q_1)$, $\mu \le \ell - e_j$ appear from (2.20). Similar terms appear from $q_j^2\nabla_q B_j\cdot\frac{\partial}{\partial p}\phi^{(\alpha)}$ and $q_j^2(\partial_{q_1}B_j)\frac{\partial}{\partial p_1}\phi^{(\alpha)}$. In the latter case there appear terms $\phi^{(\alpha)}_{\nu+1,k,\mu}(q_1)$ with $\mu\le \ell-2e_j$. In the same way one can see that there appear terms containing the quantities

$$\phi^{(\alpha)}_{\nu,k,\mu}(q_1),\quad q_1^{2\sigma}B_{j,1}\frac{\partial}{\partial q_1}\phi^{(\alpha)}_{\nu,k,\mu}(q_1),\quad \mu\le \ell-2e_j$$

from $-\lambda_j\alpha_j q_j^2B_{j,1}\phi^{(\alpha)}$ and $q_j^2q_1^{2\sigma}B_{j,1}\frac{\partial}{\partial q_1}\phi^{(\alpha)}$.

Let α be given. We will show that if α and ℓ satisfy $\ell-\alpha\notin\mathbb{Z}_+^{n-1}$, then $\phi^{(\alpha)}_{\nu,k,\ell}=0$ for all ν. Indeed, we have $\ell-k-\alpha\neq 0$ for every $k\in\mathbb{Z}_+^{n-1}$. We want to determine $\phi^{(\alpha)}_{\nu,k,\ell}(q_1)$. By the nonresonance condition (2.6) we have $\lambda\cdot(\ell-k-\alpha)\neq 0$ if and only if $\ell-k-\alpha\neq 0$. In the right-hand side of (2.20) there appear $\phi^{(\alpha)}_{\nu',k,\ell-\beta}$'s for which $\beta\ge 0$, $\beta\neq 0$. It follows that α and $\ell-\beta$ satisfy $\ell-\alpha-\beta\notin\mathbb{Z}_+^{n-1}$. Expand $\phi^{(\alpha)}_{\nu,k,\ell}$ into the formal power series of q_1 and insert it into (2.19). One easily sees that every coefficient is uniquely determined if the right-hand side F is known, i.e., $\phi^{(\alpha)}_{\nu',k,\ell-\beta}$'s $\beta\neq 0$ are given. Next we substitute $\phi^{(\alpha)}_{\nu',k,\ell-\beta}$ in F with the recurrence relations for $\phi^{(\alpha)}_{\nu',k,\ell-\beta}$'s which can be constructed similarly as $\phi^{(\alpha)}_{\nu,k,\ell}(q_1)$. By repeating the same argument, we finally arrive at the relation that the right-hand side of (2.20) vanishes, i.e., $F=0$ because we have $\ell-\beta\notin\mathbb{Z}_+^{n-1}$ after finite times of substitutions. Hence, by (2.19) we obtain the assertion.

We will construct the formal solutions in the cases $\alpha=0$ and $\alpha=e_j$ $(2\le j\le n)$ so that (2.9) and (2.10) are satisfied, respectively. First we consider the case $\alpha=0$. The argument is similar in the case $\alpha=e_j$.

We shall solve (2.19) inductively with respect to ℓ, $|\ell|=0,1,2,\ldots$. Let $\ell=0$. Because F in (2.19) vanishes, we have $\phi^{(0)}_{\nu,k,0}=0$ if $k\neq 0$. If $k=0$, then we have $(q_1\frac{\partial}{\partial q_1}-2\sigma\nu)\phi^{(0)}_{\nu,0,0}=0$. We have

$$\phi^{(0)}_{\nu,0,0}=c_\nu q_1^{2\sigma\nu}, \tag{2.21}$$

where c_ν is an arbitrary constant. By taking $c_\nu=0$ for $\nu=0,1,2,\ldots$, we have $\phi^{(0)}_{\nu,k,0}=0$ for all ν and k.

Let $|\ell| = 1$, $\ell \geq 0$. By the definition of F the nonvanishing term in F has the form $\phi^{(0)}_{\nu,k+e_i,\mu}$, $\mu \leq \ell - e_i$ for some i. Note that we have $|\mu| = 0$ since $|\ell| = 1$, $\ell \geq 0$, and hence $F = 0$ by induction. Let $|k| \geq |\ell|$. If $\ell \neq k$, then we have $\phi^{(0)}_{\nu,k,\ell} = 0$ for all ν. On the other hand, if $\ell = k$, then, by the same argument as in (2.21) we have $\phi^{(0)}_{\nu,k,k} = c^{(0)}_{\nu,k,k} q_1^{2\sigma\nu}$, where $c^{(0)}_{\nu,k,k}$ is an arbitrary constant. We choose $c^{(0)}_{\nu,k,k}$ so that $c^{(0)}_{0,e_j,e_j} = 1$ for some $2 \leq j \leq n$, and $c^{(0)}_{\nu,k,k} = 0$ if otherwise. Note that the nonvanishing term $\phi^{(0)}_{\nu,k,k}$ is given by $\phi^{(0)}_{0,e_j,e_j} = 1$. In the case $|k| < |\ell|$, we have $k = 0$, $\ell - k \neq 0$, and we can recursively determine $\phi^{(0)}_{\nu,k,\ell}$ as the formal power series of q_1.

Let $|\ell| = 2$ and consider the case $|k| \geq |\ell| = 2$. Then the corresponding terms in F of (2.19) vanish by definition and the inductive assumption. If $k \neq \ell$, then we have $\phi^{(0)}_{\nu,k,\ell} = 0$ for all ν, while, for $k = \ell$, we have $\phi^{(0)}_{\nu,k,\ell} = c^{(0)}_{\nu,k,\ell} q_1^{2\sigma\nu}$ for all ν. We set $c^{(0)}_{\nu,k,\ell} = 0$ and we obtain $\phi^{(0)}_{\nu,k,\ell} = 0$ for all ν and $|k| \geq |\ell|$. On the other hand, if $|k| < |\ell| = 2$, then one can determine $\phi^{(0)}_{\nu,k,\ell}$ from the recurrence relation. In order to see that $\phi^{(\alpha)}_{\nu,k,\ell} = 0$ if $k \neq e_j$ and $k \neq 0$ we note that the right-hand side F vanishes if $k \neq e_j$ and $k \neq 0$. Hence, by solving (2.19) we obtain $\phi^{(\alpha)}_{\nu,k,\ell} = 0$ if $k \neq e_j$ and $k \neq 0$. We can similarly argue by induction on $|\ell| \geq 3$ and show the existence of formal first integrals satisfying (2.9).

Next we consider the case $\alpha = e_j$ for some $2 \leq j \leq n$. We make similar argument as in the case $\alpha = 0$. Let $|\ell| = 0$. Then we have $\ell - \alpha = -e_j \notin \mathbb{Z}^{n-1}_+$. It follows that $\phi^{(\alpha)}_{\nu,k,0} = 0$ for all ν and k. Next we consider the case $|\ell| = 1$. If $\ell - \alpha \notin \mathbb{Z}^{n-1}_+$, then we have $\phi^{(\alpha)}_{\nu,k,0} = 0$. If $\ell - \alpha \in \mathbb{Z}^{n-1}_+$, then we have $\ell = \alpha$ since $|\ell| = |\alpha| = 1$. Because F consists of terms of the form $\phi^{(\alpha)}_{\nu,k+e_i,\mu}$, $\mu \leq \ell - e_i$ for some i, we have $F = 0$. Noting $\ell = \alpha$ we obtain $\phi^{(\alpha)}_{\nu,k,\alpha} = c^{(\alpha)}_{\nu,k,\alpha} q_1^{2\sigma\nu}$. Summing up the above we have

$$
\begin{aligned}
(2.22) \qquad \phi^{(\alpha)}_{\nu,k,\ell} &= 0 \quad (\nu = 0, 1, \ldots) \ \text{ if } \ \ell - \alpha - k \neq 0, |k| \geq |\ell - \alpha|, \\
\phi^{(\alpha)}_{\nu,k,\ell} &= c^{(\alpha)}_{\nu,k,\ell} q_1^{2\sigma\nu}, \quad (\nu = 0, 1, \ldots) \ \text{ if } \ \ell - \alpha - k = 0,
\end{aligned}
$$

where $c^{(\alpha)}_{\nu,k,\ell}$ is an arbitrary constant. Note that $\ell - \alpha - k = 0$ implies $k = 0$ by the above argument. Set $c^{(\alpha)}_{\nu,0,\alpha} = 0$ for every ν. Then we obtain $\phi^{(\alpha)}_{\nu,k,\ell} = 0$ for all ν, k and $|\ell| \leq 1$.

We consider the case $|\ell| = 2$. One may assume $\ell - \alpha \geq 0$. By arguing as in the previous case we have $F = 0$ and we obtain (2.22). Set $c^{(\alpha)}_{\nu,k,\ell} = 1$ for $\nu = 0$, $k = \alpha$, $\ell = 2\alpha$, while $c^{(\alpha)}_{\nu,k,\ell} = 0$ if otherwise.

We proceed as in the case $\alpha = 0$. Let $|\ell| = 3$. We have $\phi^{(\alpha)}_{\nu,k,\ell} = 0$ if $|k| \geq |\ell - \alpha|$ and $\ell - \alpha - k \neq 0$. If $\ell - \alpha - k = 0$, then the resonance term may appear. We set the arbitrary constants to be zero and we finally obtain $\phi^{(\alpha)}_{\nu,k,\ell} = 0$ if $|k| \geq |\ell - \alpha|$. On the other hand, if $|k| < |\ell - \alpha|$, then one can determine $\phi^{(\alpha)}_{\nu,k,\ell}$ recurrently. It should be noted that the same argument as in the case $\alpha = 0$ shows that $\phi^{(\alpha)}_{\nu,k,\ell} = 0$ if $k \neq e_j$

and $k \neq 0$. The general case $|\ell| \geq 3$ can be proved by induction on $|\ell|$. Hence we have (2.10). □

Summability of formal integrals.

We will give a meaning to the formal first integral constructed in the above. This is done by the Borel summation. As for the definition of the Borel sum and detailed argument we refer to [7] and [1]. For a given $\alpha \in \mathbb{Z}_+^n$ let $\phi^{(\alpha)}$ be the formal first integral constructed in the previous theorem. We define the set of singular directions S_0 associated with $\phi^{(\alpha)}$ by

$$\begin{aligned} S_0 := \{z \in \mathbb{C}; \exists \nu \geq 0,\ \exists k \geq 0,\ \exists \ell \geq 0, \text{ such that} \\ (2\sigma-1)z^{2\sigma-1} + \lambda \cdot (\ell - \alpha - k) = 0,\ \phi^{(\alpha)}_{\nu,k,\ell} \neq 0, \ell - \alpha - k \geq 0\} \setminus 0. \end{aligned} \tag{2.23}$$

Suppose that there exist a neighborhood Ω_0 of the origin and the convex cone $\Omega_1 \neq \emptyset$ with vertex at the origin such that the closure $\overline{S_0}$ of S_0 satisfies

$$\overline{S_0} \cap (\Omega_0 \cup \Omega_1) = \emptyset. \tag{2.24}$$

Then we have

Theorem 2.2. *Assume (2.5), (2.6) and (2.24) for some $\alpha \in \mathbb{Z}_+^n$. Let $v = E^\alpha \phi^{(\alpha)}$ be the formal first integral constructed in Theorem 2.1 which is a polynomial in p and p_1. Then v is $(2\sigma-1)$–summable in every direction of Ω_1 with respect to q_1. More precisely, for every $\xi \in \Omega_1$ there exists a neighborhood V_0 of the origin $q = 0$ such that v is analytic in $q \in V_0$ and $(2\sigma-1)$–summable with respect to q_1 in the direction ξ.*

This theorem was proved in [7]. Because the detailed proof is not used in this paper, we refer the detailed proof to [7], where $(2\sigma-1)$- summability of $\phi^{(\alpha)}$ in (2.8) and (2.17) for every α was proved. By this theorem we have the summability of formal first integrals except for directions in $\overline{S_0}$. Every direction in S_0 is singular in the sense that analytic continuation on the Borel plane of formal Borel transform of the formal first integral has singularities on S_0 in general.

We note that the Borel-summed formal first integrals are, indeed, first integrals where they are defined. For the sake of simplicity we use the same notation for the summed first integral in the sequel.

§ 3. Smooth integrability

Let $E^\alpha \phi^{(\alpha)}$ be the first integral given in Theorem 2.2. Assume (2.24) and define

$$\Sigma_{\phi^{(\alpha)}} := \left\{ z \in \mathbb{C};\ |\arg z - \arg \xi| < \frac{\pi}{2(2\sigma-1)},\ \xi \in \Omega_1 \right\}. \tag{3.1}$$

Then we have

Theorem 3.1. *Assume (2.5), (2.6) and (2.24) for some $\alpha \in \mathbb{Z}_+^n$. Then there exists an $\varepsilon_0 > 0$ such that $\phi^{(\alpha)}$ is holomorphic in*

$$\{(q_1, q, p_1, p); q_1 \in \Sigma_{\phi^{(\alpha)}},\ |q_1| < \varepsilon_0,\ p_1 \in \mathbb{C},\ p_j \in \mathbb{C},\ |q_j| < \varepsilon_0,\ j = 2, \ldots, n\}. \tag{3.2}$$

Moreover, there exists a sector $S_1 \subset \Sigma_{\phi^{(\alpha)}}$ such that $E^\alpha \phi^{(\alpha)}$ is C^∞ at $q_1 = 0$ when $q_1 \in S_1$, $q_1 \to 0$ and satisfies that all derivatives vanish at the origin.

Proof. Let $v = \phi^{(\alpha)} E^\alpha$ be the summed first integral. Because $\phi^{(\alpha)}$ is $(2\sigma - 1)$-summable in every direction in Ω_1, $\phi^{(\alpha)}$ is holomorphic in the domain (3.2). In order to show the smoothness of v at the origin we recall that every $\phi^{(\alpha)}$ is C^∞ when $q_1 \to 0$, $q_1 \in \Sigma_{\phi^{(\alpha)}}$ because $\phi^{(\alpha)}$ has an asymptotic expansion. On the other hand, in view of

$$E^\alpha = \exp\left(\frac{q_1^{-2\sigma+1}}{2\sigma - 1} \sum_{j=2}^{n} \lambda_j \alpha_j\right) \tag{3.3}$$

and noting that the opening of $\Sigma_{\phi^{(\alpha)}}$ is larger than $\pi/(2\sigma - 1)$, there exists a sector $S_1 \subset \Sigma_{\phi^{(\alpha)}}$ on which E^α together with its derivatives tends to zero when $q_1 \to 0$. Hence v is C^∞ when $q_1 \to 0$, $q_1 \in S_1$ as desired. □

Next we study the existence of commuting functionally independent first integrals in a sector. Let $\phi_j^{(0)}$ and $\phi^{(e_j)}$ $(j = 2, 3, \ldots, n)$ be the formal first integral constructed in Theorem 2.1 satisfying (2.9) and (2.10), respectively.

Theorem 3.2. *Suppose (2.5) and (2.6). Assume either that the convex hull of $\{\lambda_j; j = 2, 3, \ldots, n\}$ does not contain the origin or that S_0 is a finite set for $\phi_j^{(0)}$ and $\phi^{(e_j)}$ $(j = 2, 3, \ldots, n)$. Then, there exist a neighborhood U of the origin of $\mathbb{C}$, an $\varepsilon_0 > 0$ and a sector R with vertex at the origin such that χ_H has $2(n-1)$ functionally independent first integrals analytic on $q_1 \in R \cap U$, $p_1, p_i, q_i \in \mathbb{C}$, $|q_i| < \varepsilon_0$ $(i \geq 2)$ and smooth at the origin when $q_1 \to 0, q_1 \in R$ and p_1, p_i, q_i in some neighborhood of the origin. Moreover the first integrals $E^{e_j} \phi^{(e_j)}$ $(j = 2, 3, \ldots, n)$ and H are mutually commuting.*

Proof. We will show (2.24) for every $\alpha \geq 0$. If S_0 is a finite set for $\phi_j^{(0)}$ and $\phi^{(e_j)}$ $(j = 2, 3, \ldots, n)$, then one can choose the cone $\Omega_1 \neq \emptyset$ independent of j and a neighborhood of the origin Ω_0 so that (2.24) is verified. Assume that the convex hull of $\{\lambda_j; j = 2, 3, \ldots, n\}$ does not contain the origin. Let Γ_0 be the closed convex cone generated by λ_ν $(\nu = 2, \ldots, n)$. Without loss of generality we may assume that Γ_0 is contained in the set $\Re z > 0$. Hence we see that the set of points $\lambda \cdot (\ell - \alpha - k)$ for $\ell - \alpha - k \geq 0$ does not accumulate to the origin, where $\alpha = 0, e_j$ and k moves on a finite set. Therefore, since $\lambda \cdot (\ell - \alpha - k)$ is contained in Γ_0 for every ℓ, α, k such that

$\ell - \alpha - k \geq 0$, the definition of S_0 implies that one can choose the cone $\Omega_1 \neq \emptyset$ and a neighborhood of the origin Ω_0 independent of α satisfying (2.24). Moreover, we can take Ω_1 with opening greater than $\pi/(2\sigma - 1)$. It follows from Theorem 2.2 that the formal integrals $\phi_j^{(0)}$ and $\phi^{(e_j)}$ $(j = 2, 3, \ldots, n)$ are summable.

Let Γ_0' be the dual cone of Γ_0, namely the set of vectors z satisfying $\Re(z \cdot \eta) \geq 0$ for all $\eta \in \Gamma_0$. Note that Γ_0' is connected and contains positive real axis. Denote the image of $\Sigma_{\phi^{(\alpha)}}$ under the map $z \mapsto z^{2\sigma-1}$ by $\tilde{\Sigma}_{\phi^{(\alpha)}}$. We shall show that there exists $\tilde{\Sigma} \subset \tilde{\Sigma}_{\phi^{(\alpha)}}$ independent of α such that $\tilde{\Sigma} \cap (-\Gamma_0') \neq \emptyset$. Indeed, if the convex hull of $\{\lambda_j; j = 2, 3, \ldots, n\}$ does not contain the origin, then one can take the opening of $\tilde{\Sigma}$ sufficiently close to 2π. We next consider the case where S_0 is a finite set. In order to show $\tilde{\Sigma} \cap (-\Gamma_0') \neq \emptyset$ for some Ω_1 we note that one can take Ω_1 as an arbitrary cone outside a finitely many directions by the finiteness assumption of S_0. Because Γ_0' is an open cone, one can take Ω_1 as desired in view of the definition of $\Sigma_{\phi^{(\alpha)}}$.

We shall choose a sector R with vertex at the origin independent of j, $(j = 2, 3, \ldots, n)$ so that the integrals $E^{e_j}\phi^{(e_j)}$ and $\phi_j^{(0)}$, $(j = 2, 3, \ldots, n)$ are smooth on R with all derivatives vanishing at the origin when $q_1 \to 0$, $q_1 \in R$. We note that E^{α} with $\alpha = e_j$ $(j = 2, \ldots, n)$ are smooth at the origin when $q_1^{-2\sigma+1} \in -\Gamma_0'$. On the other hand $\phi^{(\alpha)}$ is smooth at the origin because it has an asymptotic expansion in $\Sigma_{\phi^{(\alpha)}}$. In order to show the smoothness of $\phi^{(\alpha)}E^{\alpha}$ for all α on R, we take $R \subset \Sigma_{\phi^{(\alpha)}}$ as an inverse image of some cone in $\tilde{\Sigma} \cap (-\Gamma_0')$ such that E^{α} and $\phi^{(\alpha)}$ are smooth on R. Note that we have the same assertion in the case where S_0 is a finite set because we have $\tilde{\Sigma} \cap (-\Gamma_0') \neq \emptyset$ for some Ω_1.

By virtue of (2.9) the first integrals $\phi_j^{(0)}$ $(j = 2, 3, \ldots, n)$ are functionally independent everywhere except for a set of measure zero. The same property holds for $E^{e_j}\phi^{(e_j)}$ $(j = 2, 3, \ldots, n)$ by (2.10). Hence we have the functional independentness of the first integrals $\phi_j^{(0)}$, $E^{e_j}\phi^{(e_j)}$ $(j = 2, 3, \ldots, n)$.

We will show the commutativity of first integrals $v^{(\alpha)} := E^{\alpha}\phi^{(\alpha)}$ and $v^{(\beta)} := E^{\beta}\phi^{(\beta)}$ for $\alpha = e_\nu$, $\beta = e_\mu$ $(\nu, \mu = 2, 3, \ldots, n)$. By the well-known indentity we have

$$\{H, \{v^{(\alpha)}, v^{(\beta)}\}\} + \{v^{(\beta)}, \{H, v^{(\alpha)}\}\} + \{v^{(\alpha)}, \{v^{(\beta)}, H\}\} = 0.$$

It follows that $\{H, \{v^{(\alpha)}, v^{(\beta)}\}\} = 0$. Set $w := \{v^{(\alpha)}, v^{(\beta)}\}$. Because w is Borel summable, w vanishes if its asymptotic expansion vanishes. By the definition of the formal solution in Theorem 2.1 the resonant term in $\phi^{(\alpha)}$ is given by $q^{2\alpha}p^{\alpha}$. Because w can be written in the form $E^{\alpha+\beta}\tilde{w}$ for some summable power series $\tilde{w}$, the resonant term in $\tilde{w}$ has the form $cq^{\nu+\alpha+\beta}p^{\nu}$ where $\nu = 0, \alpha, \beta, \alpha + \beta$, and c is a power series of q_1 and p_1. It follows that if the resonant term of w vanishes, then $\{H, w\} = 0$ implies $w = 0$ as desired.

We will calculate the resonant term in w. By definition we have

$$w = E^{\alpha}\frac{\partial}{\partial p_1}\phi^{(\alpha)}\frac{\partial}{\partial q_1}\left(E^{\beta}\phi^{(\beta)}\right) - E^{\beta}\frac{\partial}{\partial p_1}\phi^{(\beta)}\frac{\partial}{\partial q_1}\left(E^{\alpha}\phi^{(\alpha)}\right) + E^{\alpha+\beta}\nabla_p\phi^{(\alpha)}\cdot\nabla_q\phi^{(\beta)} - E^{\alpha+\beta}\nabla_p\phi^{(\beta)}\cdot\nabla_q\phi^{(\alpha)}. \tag{3.4}$$

We first consider the term corresponding to $\nu = 0$, $cq^{\alpha+\beta}$. In the first and the second terms of (3.4) the terms which do not contain the powers of p is $O(|q|^3)$ by (2.11). This proves that no term of the form $cq^{\alpha+\beta}$ appears because $|\alpha| = |\beta| = 1$. On the other hand, the terms which do not contain the powers of p in third and fourth terms of (3.4) is $O(|q|^4)$. Therefore no resonance term corresponding to $\nu = 0$ appears. Next we consider the case $\nu = \alpha + \beta$. In the third and fourth terms of (3.4) there appear no term containing $p^{\alpha+\beta}$. On the other hand, in view of the definition of $\phi^{(\alpha)}$ the resonant term in the first and the second terms of (3.4) with the form $cq^{2(\alpha+\beta)}p^{\alpha+\beta}$ vanishes by virtue of (2.10) and (2.11). We consider the case $\nu = \alpha$ or $\nu = \beta$. In the first and the second terms of (3.4) the term which contains p^{α} or p^{β} is $O(|q|^3)$. Hence no resonance term appears. Consider the third term of (3.4). If $\alpha = e_j$ and $\beta = e_k$, $(j \neq k)$, then only j-th component of the homogeneous degree 2 part of $\nabla_p\phi^{(\alpha)}$ does not vanish, while the j-th component of the homogeneous degree 2 part of $\nabla_q\phi^{(\beta)}$ vanishes. Hence we see that the resonance term in $\nabla_p\phi^{(\alpha)}\cdot\nabla_q\phi^{(\beta)}$ vanishes. One can argue similarly for the fourth term of (3.4) and show that no resonant term appears from the fourth term. This proves the assertion and we have proved the theorem. $\square$

We will study the smooth integrability on a line with respect to the resonance variable.

Theorem 3.3. *Suppose (2.5) and (2.6). Assume either the convex hull of $\{\lambda_j; j = 2, 3, \ldots, n\}$ does not contain the origin or that $\sigma = 1$ and S_0 is a finite set for $\phi_j^{(0)}$ $(j = 2, 3, \ldots, n)$. Then there exists a ray R emenating from the origin such that the Hamiltonian vector field χ_H is smoothly integrable in a neighborhood of the origin when $q_1 \in R \cup (-R) \cup \{0\}$, $p_1, p_i, q_i \in \mathbb{C}$.*

Proof. We continue to use the same notation as in the preceding theorem. Let $\phi_i^{(0)}$, $(i = 2, 3, \ldots, n)$ be the formal first integrals given in Theorem 2.1. By the same argument as in the preceding theorem, they are summable. We want to show that there exists a ray R such that $\phi_i^{(0)}$ is smooth in some neighborhood of the origin of $q_1 \in R \cup (-R) \cup \{0\}$ and $p_1, p_i, q_i \in \mathbb{C}$. Let S_0 be the union of all singular directions of $\phi_j^{(0)}$, $(j = 2, 3, \ldots, n)$. If S_0 is a finite set, then, by virtue of $\sigma = 1$ and the definition, $\cap_j \Sigma_{\phi_j^{(0)}}$ has the opening greater than π and contains $R \cup (-R)$ for some ray R. Hence one can choose R as desired. If the convex hull of $\{\lambda_j; j = 2, 3, \ldots, n\}$ does not contain the origin, then one can easily show that the opening of $\cap_j \tilde{\Sigma}_{\phi_j^{(0)}}$ is greater than π. Therefore, by considering the inverse image of $\cap_j \tilde{\Sigma}_{\phi_j^{(0)}}$ under the map $z \mapsto z^{2\sigma-1}$ and by the summability of $\phi_j^{(0)}$ we can choose R as desired.

By using the same argument as in the preceding theorem we can show the commutativity of $\phi_j^{(0)}$, $(j = 2, 3, \ldots, n)$. We next show the functional independentness of $\phi_j^{(0)}$ $(j = 2, 3, \ldots, n)$ and H. Note that the functional independentness of $\phi_j^{(0)}$'s follows from the definitions of the formal power series expansion and their summability. Indeed, by virtue of (2.9) and (2.11) and the independentness of p_j's, we have the functional independentness of $\phi_j^{(0)}$'s.

We will show the functional independentness of $\phi_j^{(0)}$ $(j = 2, \ldots, n)$ and H on an open dense set. Suppose that there exists a smooth function $F \equiv F(y, X) \in C^\infty$, $y = (y_2, \ldots, y_n)$ such that $F(\phi_2^{(0)}, \ldots, \phi_n^{(0)}, H) \equiv 0$ with q_1 and q being in some sector with vertex at the origin and in some neighborhood of the origin, respectively, and $p_1 \in \mathbb{C}$, $p \in \mathbb{C}^{n-1}$. If there exists an open set Ω such that $\frac{\partial F}{\partial X} \equiv 0$ for all $X \in \Omega$, then F is independent of X in Ω. On the other hand, we have $H = q_1^{2\sigma} p_1 + \varepsilon$, where ε can be made arbitrarily small by taking q sufficiently small and p_1, q_1 and p bounded. We fix $q_1 \neq 0$. Then there exists an open set in $\mathbb{C}$ such that if p_1 is on the open set and q stays in a sufficiently small neighborhood of the origin, then we have $H \in \Omega$. It follows that $F(\phi_2^{(0)}, \ldots, \phi_n^{(0)}, 0) \equiv 0$ on such a set because F is independent of X. This contradicts the functional independentness of $\phi_j^{(0)}$'s.

Next we assume that $\frac{\partial F}{\partial X}(y^{(0)}, X^{(0)}) \neq 0$ for some $y^{(0)}$ and $X^{(0)}$. We may assume that there exists a neighborhood of $(y^{(0)}, X^{(0)})$, $W_1 \times W_2$ such that the nonvanishing relation holds for $(y, X) \in W_1 \times W_2$. By virtue of (2.9) and (2.11) one can choose open sets Ω_1 and Ω_2 in some neighborhood of the origin of $0 \in \mathbb{C}^{n-1}$ and in $\mathbb{C}^{n-1}$, respectively such that if $q \in \Omega_1$ and $p \in \Omega_2$, then $\phi^{(0)} = (\phi_2^{(0)}, \ldots, \phi_n^{(0)})$ is in W_1. By the similar argument as above one can show that $H \in W_2$. Then we have $\frac{\partial F}{\partial X}(\phi^{(0)}, H) \neq 0$. Hence, by the implicit function theorem there exists a smooth function G such that $H = G(\phi_2^{(0)}, \ldots, \phi_n^{(0)})$.

Let $q_1 \neq 0$ and fix it. We have $\frac{\partial H}{\partial p_1} = q_1^{2\sigma} + \frac{\partial \varepsilon}{\partial p_1}$. One can make $\frac{\partial \varepsilon}{\partial p_1}$ arbitrarily small if we take q sufficiently small. Therefore there exists $C > 0$ independent of q such that if q is in some neighborhood of the origin, then we have $\left|\frac{\partial H}{\partial p_1}\right| \geq C > 0$. On the other hand, we have

$$\frac{\partial H}{\partial p_1} = \frac{\partial G}{\partial p_1} = \sum_{\nu=2}^{n} \frac{\partial G}{\partial y_\nu}(\phi^{(0)}) \frac{\partial \phi_\nu^{(0)}}{\partial p_1}.$$

By (2.9) and (2.11) the quantity $\frac{\partial \phi_\nu^{(0)}}{\partial p_1}$ tends to zero when q tends to zero. This is a contradiction because the left-hand side is bounded from the below by a nonzero constant independent of q. Hence we have the assertion. This ends the proof. □

§ 4. C^ω-nonintegrability

In this section we briefly state the C^ω-nonintegrability of our Hamiltonian system. Although the fact is not necessary in the proof of the preceding theorems, it implies that our Hamiltonian system has a similar property like those in [3] and [5].

Let $H := H_0 + H_1$ with H_0 and H_1 given, respectively, by

$$H_0 = A(q_1^{2\sigma} p_1) + \sum_{j=2}^{n} \lambda_j q_j p_j, \tag{4.1}$$

$$H_1 = -\sum_{j=2}^{n} p_j^2 B_j(q_1, q_1^{2\sigma} p_1, p), \tag{4.2}$$

where $A(s)$ and $B_j(q_1, s, t)$ are holomorphic at the origin with respect to $(q_1, s, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}$, and

$$A(0) = 0, \quad A'(0) = 1. \tag{4.3}$$

Then we have

Theorem 4.1. *Assume that the equation*

$$q_1^{2\sigma} \frac{dv}{dq_1} - 2\lambda_k v = B_k(q_1, 0, 0) \tag{4.4}$$

has no analytic solution v at the origin for $k = 2, 3, \ldots, n$. Moreover, suppose (2.6). Then, for any C^ω-first integral u of χ_H there exists an analytic function $\phi(z)$ of one variable in some neighborhood of the origin $z = 0$ such that $u = \phi(H)$. Especially χ_H is not C^ω-Liouville integrable.

Remark 1. (i) Nonexistence of an analytic solution to (4.4) corresponds to the non Abelian property of the fundamental group introduced in [3]. We can also prove that the condition holds if and only if the monodromy of an analytic continuation of every solution of (4.4) along a path encircling the origin does not vanish. (cf. Lemma 6 of [6]). We remark that if B_k is a polynomial, then the set of B_k satisfying the condition is an open dense set.

(ii) Theorem 4.1 also holds for H_1 given by (2.2) by exchanging the variables p and q.

Example 4.2. Let $\sigma \in \mathbb{N}$, $c \in \mathbb{C} \setminus \{0\}$. Consider the Hamiltonian

$$H_1 := \sum_{j=2}^{n} B_j(q_1) p_j^2, \tag{4.5}$$

where $B_j(q_1)$ is an analytic function in some neighborhood of $q_1 = 0$. Similar Hamiltonian was considered in [3] in relation with a geodesic flow. Let $A(s) = s + cs^2$ in H_0 and define $H := H_0 + H_1$. Assume (2.6) and suppose that (4.4) has no analytic solution for every k. Then H satisfies the conditions of Theorem 4.1. Hence χ_H is not C^ω-Liouville integrable.

The proof of Theorem 4.1 will be published in [8].

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